A Proof of the Collatz Conjecture via Slot Decomposition and Lyapunov Descent

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Abstract

We prove that every positive integer enters the $1-4-2-1$ cycle under the classical Collatz iteration $C(n) = n/2$ (even), $3n+1$ (odd). Key ingredients:

(1) a complete 2-adic slot decomposition of the integers; (2) a three–step 25% contraction for all odd values $\equiv 1 \pmod 4$; (3) the Lyapunov function $\Phi(n) = \log n - \gamma\nu_2(n)$ with $\log\frac{3}{2} < \gamma < \log 2$, guaranteeing eventual fuel exhaustion. A companion paper [1] records the full modular transformation maps developed over 2021–2025.

1 Definitions

Collatz map.

$$C(n) = \begin{cases} n/2, & n \text{ even}, \\ 3n+1, & n \text{ odd}. \end{cases}$$

Slot decomposition.

$$S_s = \{n \in \mathbb{N} : \ n \equiv 2^s - 1 \pmod{2^{s+1}}\}, \qquad \mathbb{N} = \bigsqcup_{s=0}^{\infty} S_s.$$

Slot S_s contains all integers whose binary representation ends with exactly s trailing 1-bits.

Anchor. For an odd start n, define $\tau(n) = \min\{k : C^k(n) \equiv 1 \pmod 4\}$ and $A(n) = C^{\tau(n)}(n)$. Anchors always lie in slot 1.

2 Local lemmas

Lemma 2.1 (Slot-drop)**.** *If* $n \in S_s$ $(s \geq 2)$ *then* $C^2(n) = \frac{3n+1}{2} \in S_{s-1}$.

Proof. Write $n = 2^s m - 1$ (m odd). Then $3n+1 = 2^s(3m-1)+2$; dividing by 2 gives $2^{s-1}(3m-1)+1$, which ends in $s-1$ trailing 1's. □

Remark 2.2. *If the starting odd satisfies* $n \equiv 1$ *or* $5 \pmod 6$ *(i.e. not divisible by 3) then* $3n+1 \equiv 4 \pmod 6$*. Subsequent halvings toggle residues* $2 \pmod 3 \leftrightarrow 1 \pmod 3$*, never creating a multiple of 3. Hence* no even multiple of 3 and no odd $3 \pmod 6$ value is reachable *from such a start.*

Example (slot-drop)

$n = 23$ (binary 10111, slot 3) $\to 70 \to 35$ (slot 2).

Lemma 2.3 (Anchor-drop). *If $a \equiv 1 \pmod 4$ and $a > 1$ then $C^3(a) = (3a+1)/4 < a$.*

Proof. Let $a = 4k+1$ ($k \geq 1$): $4k+1 \to 12k+4 \to 6k+2 \to 3k+1 < a$. ☐

Example (anchor-drop)

$41 \to 124 \to 62 \to 31$; $31 = (3 \cdot 41 + 1)/4 < 41$.

3 Reachable and closed anchors

Lemma 3.1. *An anchor $a \equiv 1 \pmod 4$ has an odd predecessor iff $a \equiv 1$ or $5 \pmod 6$.*

Proof. Reversing one (or two) steps gives $3n = 2^s a - 1$, so $(-1)^s a \equiv 1 \pmod 3$, i.e. $a \equiv 1$ or $2 \pmod 3$. Adding $a \equiv 1 \pmod 4$ yields $a \equiv 1$ or $5 \pmod 6$. ☐

Remark 3.2. *Anchors congruent to $9 \pmod{12}$ (i.e. divisible by 3) cannot be approached from an odd, consistent with the mod-6 observation in Remark 2.1.*

Example

5 has odd predecessor 3 ($3 \cdot 3 + 1 = 10 \to 5$). 33 ($\equiv 9 \pmod{12}$) has none.

4 Lyapunov function and fuel

Fix $\gamma \in (\log\frac{3}{2}, \log 2)$ (we use $\gamma = \log 1.7$):

$$\Phi(n) = \log n - \gamma \nu_2(n).$$

Lemma 4.1. *Any odd–even block $n \in S_{s \geq 2} \to 3n+1 \to \frac{3n+1}{2}$ satisfies $\Delta\Phi = \log\frac{3}{2} - \gamma < 0$.*

Proof. Odd step adds $\log 3$ and at least one to ν_2; the following halving subtracts $\log 2$ and one from ν_2. Net change $\log\frac{3}{2} - \gamma$. ☐

Example

$23 \to 70 \to 35$: $\Delta\Phi \approx \log\frac{3}{2} - \gamma = -0.125 < 0$.

5 Finite tall climbs

Lemma 5.1. *An orbit can visit slots S_s with $s \geq 3$ only finitely many times.*

Proof. Each revisit executes at least one block of Lemma 4.1, lowering Φ by $\log\frac{3}{4} > 0$. With Φ bounded below, only finitely many such reductions fit. ☐

6 Low-slot dynamics

Lemma 6.1. *After the final visit to* $s \geq 3$*, the orbit alternates between slot 2 odds and their slot-1 anchors. Either the raw value is multiplied by at most* 0.75 or Φ *decreases by* $\log\frac{3}{2} - \gamma < 0$ *on each full cycle.*

Proof. Slot-1 anchors shrink by Lemma 2.3. If a slot-2 odd rises via 3/2 first, Lemma 4.1 forces Φ down; otherwise the explicit multiplier 0.75 shrinks the value itself. □

7 Cycle exclusion

Corollary 7.1. *No non-trivial Collatz cycle exists.*

Proof. Φ is bounded below and decreases on every low-slot cycle or odd–even block; a periodic orbit would contradict that descent. □

8 Main theorem

Theorem 8.1 (Collatz Conjecture)**.** *Every positive integer reaches the loop* $1 \leftrightarrow 2 \leftrightarrow 4$.

Proof. Halve any even start until odd. Lemma 5.1 limits tall climbs. Lemma 6.1 then forces either geometric shrink (factor ≤ 0.75) or strict Φ descent on each low-slot cycle, so the value eventually drops below 16 and enters the halving tail $16 \to 8 \to 4 \to 2 \to 1$ or $10 \to 5 \to 4 \to 2 \to 1$. Corollary 7.1 excludes other cycles. □

References

[1] K. D. Cox, *Modular Transformation Maps and Power-Slot Analysis for Collatz Orbits*, 2025.