OCT 2022

TEXAS
The University of Texas at Austin

PHIL 313Q: Inductive Logic

ch. 4 of “Probability and Inductive Logic” by Ian Hacking

Alex Rausch
PhD Candidate, The University of Texas at Austin

Propositions & Events

- We have language to assign probabilities to two kinds of things: (i) propositions and (ii) events
  - ***Propositions*** are statements, assertions, or conjectures
    - e.g. that you'll have a car accident this year
    - e.g. that it will thunderstorm tonight
    - propositions are *true* or *false*, like the sentences of SL and PL
    - "That it will thunderstorm tonight is probable."
  - ***Events*** are happenings, occurrences, or things that take time
    - e.g. a car accident involving you this year
    - e.g. a thunderstorm tonight
    - events *occur* or *do not occur*
    - "A thunderstorm tonight is probable."

Proposition Notation

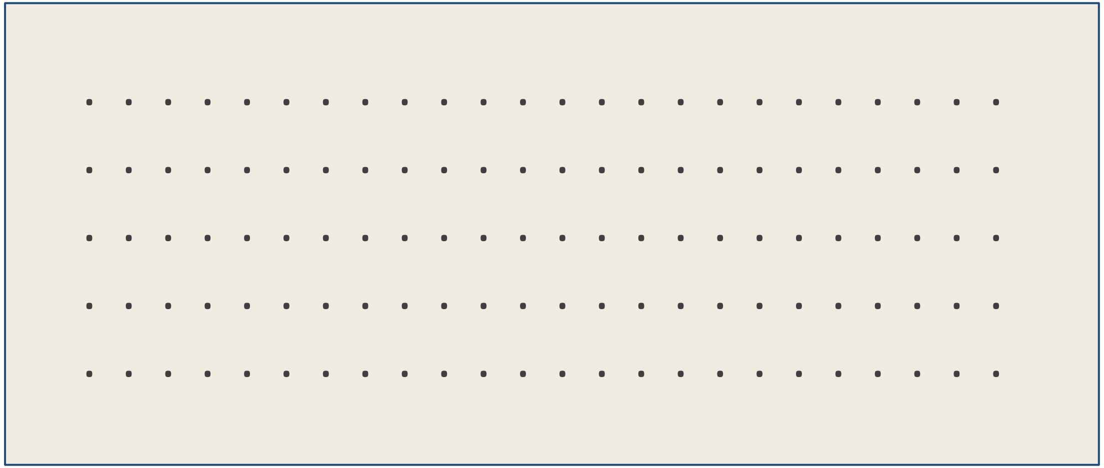
- We'll use capital letters, like in sentential logic (SL), to symbolize propositions
  - that a thunderstorm will occur tonight = **T**
  - that you will have a car accident this year = **C**
- We'll use *some* of the logical connectives of SL, too, namely:
  - Disjunction: ' **v** '
  - Conjunction: ' **&** '
  - Negation: ' **~** '
- These combine to form sentences, like in SL
  - **(O v C)**
  - **~O**
  - **(O & C) v F**

Event Notation

- While logicians tend to use proposition-language, statisticians tend to use event-language
- They use capital letters to symbolize events
  - a thunderstorm tonight = **T**
  - a car accident this year = **C**
- But the logical connectives don't make much sense – what would the following event even be: **(T & C)**
- So, statisticians use *set theory notation* instead of *logical connectives*
  - **(T U C)** instead of **(T v C)**
  - **(T ∩ C)** instead of **(T & C)**
  - **T'** instead of **~T**

Behind the Scenes

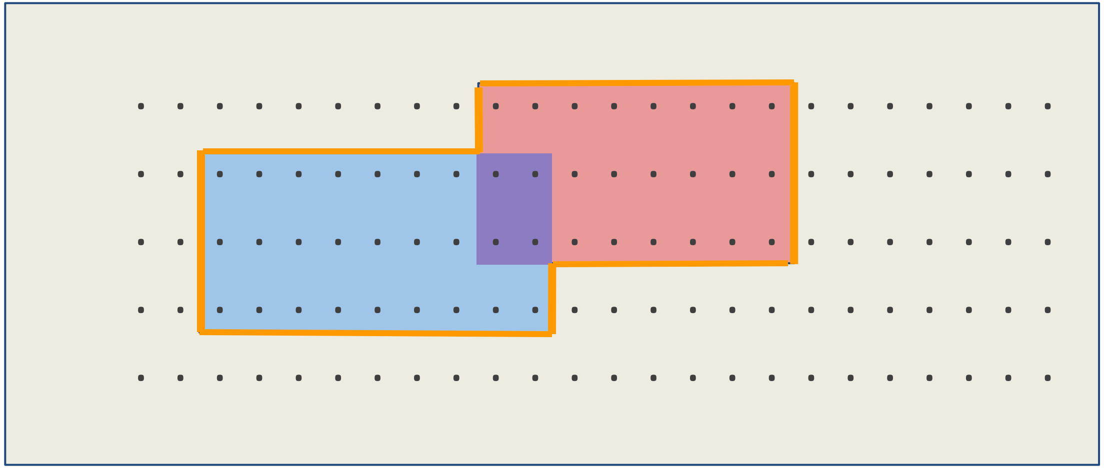
- What’s really happening here is that there are just two ways of talking about the same underlying reality



- The dots represent “possible worlds”, or complete ways the world could be
- In some worlds, you have a car accident this year. In others, you don’t.
- In some worlds, it will thunderstorms tonight. In others, it doesn’t.

Behind the Scenes: Propositions

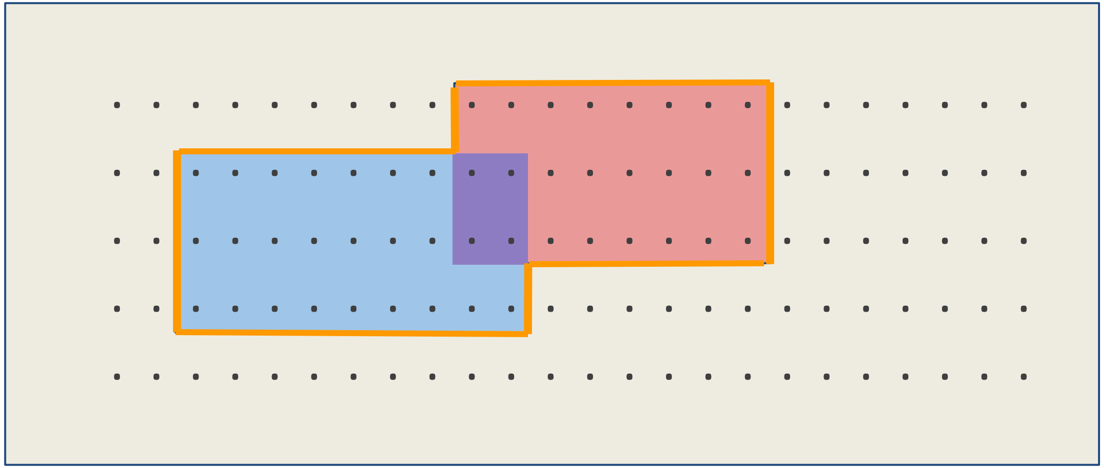
- **T** : It will thunderstorm tonight
- **C** : You will have a car accident this year



- **(T & C)** : It will thunderstorm tonight AND you will have a car accident this year
- **(T v C)** : It will thunderstorm tonight OR you will have a car accident this year
- **~T** : It will not thunderstorm tonight

Behind the Scenes: Events

- **T** = set of worlds where a thunderstorm tonight occurs
- **C** = set of worlds where a car accident this year occurs



- **(T ∩ C)** = set of worlds in both **T** and **C** (i.e., their “intersection”)
- **(T U C)** = set of worlds in either **T** or **C** (i.e., their “union”)
- **T’** = set of worlds NOT in **T**

Notation: Probability

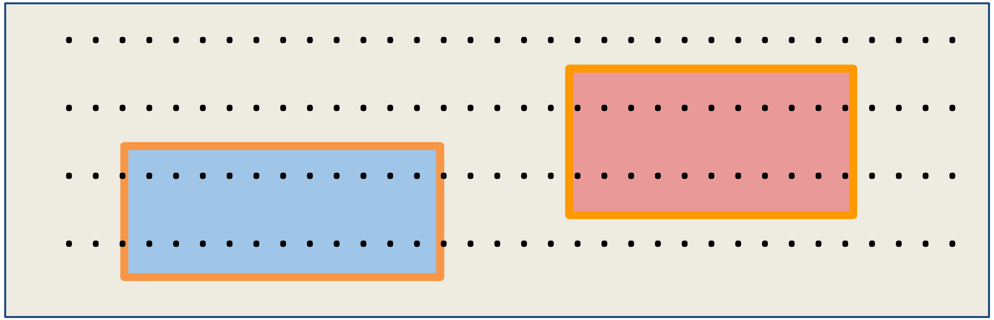
- The probability that there will be a thunderstorm tonight = the probability of a thunderstorm tonight = **Pr(T)**
  - **Pr(~T)**, **Pr(T&C)**, **Pr(TvC)**, **Pr[T&(Tv~C)]** all make sense, too
- Probabilities lie between 0 and 1
  - So, for any proposition **φ**, 0 ≤ **Pr(φ)** ≤ 1

- A *necessarily true proposition* has probability 1.
- An *event that must occur* has probability 1.
- We stipulate that the symbol **Ω** ("omega") stands for an arbitrary necessarily true proposition or event that must occur
    - So, **Pr(Ω)** = 1

Addition (Mutual Exclusivity)

- Two propositions are **mutually exclusive** iff they can't both be true at once
    - i.e., the set containing them is not consistent
    - i.e., **Pr(P&Q)** = 0
    - e.g. "The 2nd coin-flip lands heads" and "The 2nd coin-flip lands tails"

- If P and Q are mutually exclusive, then **Pr(P) + Pr(Q)** = **Pr(PvQ)**



Addition (Example)

Take a fair, 6-sided die.

Let **E** = the die lands with an even number of pips up

For $1 \leq n \leq 6$, let $\mathbf{R}_n$ = the die lands with exactly n pips up

e.g. $\mathbf{R}_1$ = the die lands with exactly 1 pip up

What's **Pr(E)**?

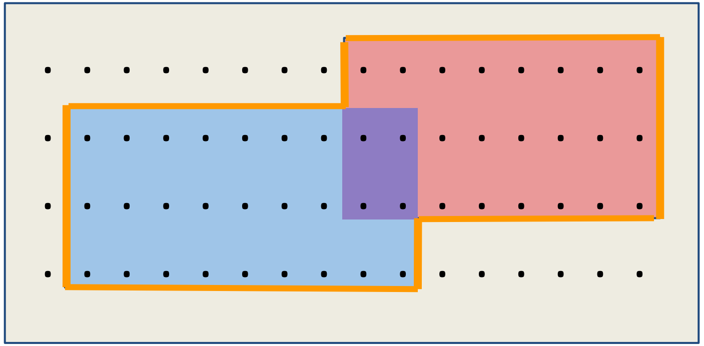
$$\mathbf{Pr(E) = Pr(R_2 \vee R_4 \vee R_6)}$$
$$\mathbf{= Pr(R_2) + Pr(R_4) + Pr(R_6)}$$

<- only because R_2, R_4, and R_6 are mutually exclusive!

$$\mathbf{= 1/6 + 1/6 + 1/6}$$
$$\mathbf{= 3/6 = 1/2}$$

Addition (Mutual Exclusivity)

- Why **Pr(P)** + **Pr(Q)** ≠ **Pr(P v Q)**, if **P** and **Q** are not mutually exclusive:
  - You will "double-count" their overlap
  - As a result, you could get **Pr(P v Q)** > 1 (not possible)



Addition (Tangent/Hate)

- “The probabilities of mutually exclusive propositions *add up,*” (so, the probabilities of NOT mutually exclusive propositions don’t add up…)
- But probabilities are just numbers between 0 and 1, and any numbers between 0 and 1 can “add up”.
- Suppose **P** and **Q** are NOT mutually exclusive, and **Pr(P)**=.75 and **Pr(Q)**=.75
- Then **Pr(P)** + **Pr(Q)** = 1.5
- That’s correct, but it doesn’t tell you anything meaningful
    - that is, **Pr(P)** + **Pr(Q) = 1.5 ≠ Pr(PvQ)**

- **“The probabilities of mutually exclusive propositions *add up* <u>*to the probability that at least one of them will be true*</u>” is better**

Addition (Teaser)

Actually, for *any* two propositions **P** and **Q**,

Pr(P v Q) = Pr(P) + Pr(Q) - Pr(P & Q)

- We just subtract any possible “double counting” we did by adding **Pr(P)** and **Pr(Q)**
- If **P** and **Q** are mutually exclusive, then **Pr(P & Q) = 0**.

Multiplication (Independence)

- two propositions are **independent** iff the truth of one does not make the truth of the other any more or less probable
  - e.g. "The first card drawn is red" and "The first card drawn is an ace"
- Drawing a red card doesn't change the odds of drawing an ace (1/13)
  - Drawing an ace doesn't change the odds of drawing a red card (½)
- if **A** and **B** are independent, then **Pr(A)** x **Pr(B)** = **Pr(A & B)**

Example

Let **A** = the die rolls a 1 on the 1st roll

Let **B** = the die rolls a 2 on the 2nd roll

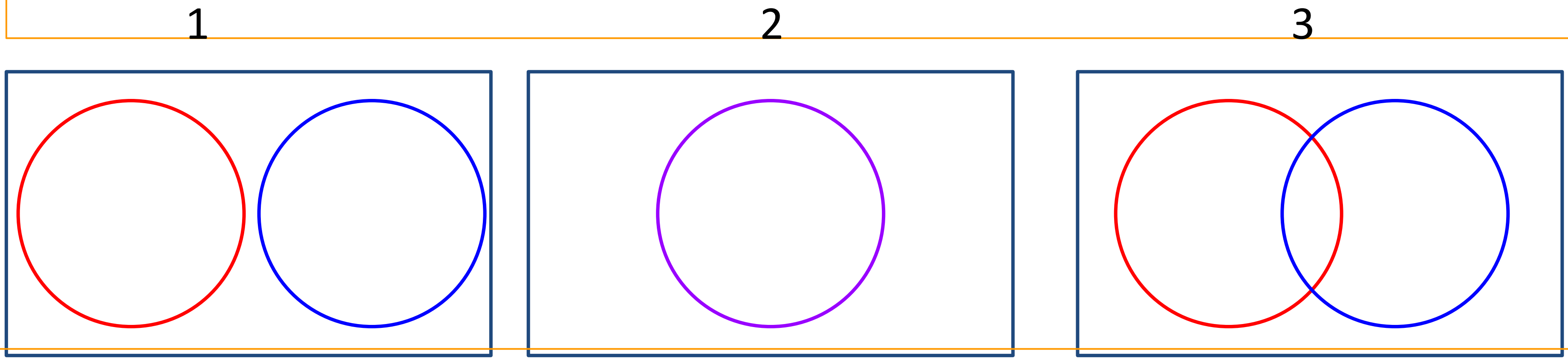
Pr(A&B) = Pr(A) x Pr(B) *<- only because A and B are independent!*

= ⅙ x ⅙

= 1/36

Multiplication (Independence)

- But wait…consider this "proof" that *no two* propositions are independent!
- For any **A** and **B**, one of these three scenarios obtains:

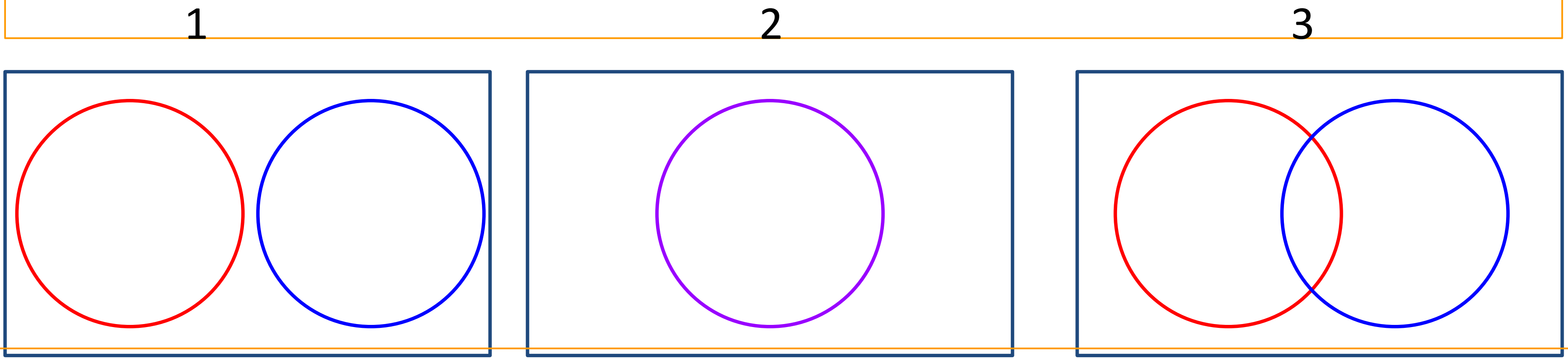


In 1, if A is true, then B must be false. So A and B are not independent.
In 2, if A is true, then B must be true. So A and B are not independent.
In 3, if A is true, then the probability of B changes. So A and B are not independent.

Where's the flaw in this "proof?"

Multiplication (Independence)

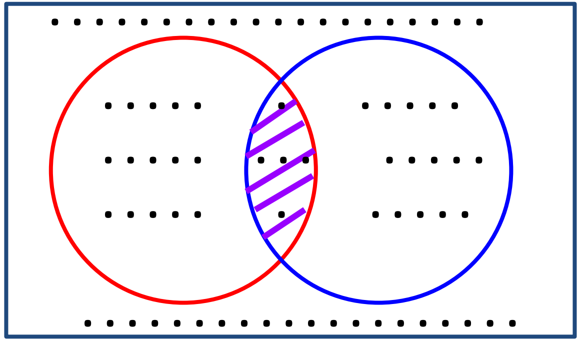
- But wait…consider this "proof" that *all* propositions are independent!
- For any **A** and **B**, one of these three scenarios obtains:



In 1, if A is true, then B must be false. So A and B are not independent.
In 2, if A is true, then B must be true. So A and B are not independent.
~~In 3, if A is true, then the probability of B changes. So A and B are not independent.~~

If A is true, then the probability of B *MIGHT stay the same*!

Multiplication (Independence)



Example
Let there be 80 dots total.
Let **A** contain 20 dots.
Let **B** contain 20 dots.
Let **A&B** contain 5 dots.

Pr(A) = 20/80 = ¼ **Pr(B)** = 20/80 = ¼ **Pr(B | A)** = 5/20 = ¼

Pr(A&B) = **Pr(A)** x **Pr(B | A)**
= **Pr(A)** x **Pr(B)** *<- only because Pr(B | A) = Pr(B)*
= 1/16

Pr(A&B) = 5/80 = 1/16

Odd Question #4

To throw a total of 7 with a pair of dice, you have to get a 1 and a 6, or a 2 and a 5, or a 3 and a 4.

To throw a total of 6 with a pair of dice, you have to get a 1 and a 5, or a 2 and a 4, or a 3 and another 3.

With two fair dice, you would expect:

(a) To throw 7 more frequently than 6
(b) To throw 6 more frequently than 7
(c) To throw 6 and 7 equally often

Odd Question #4

Assuming tosses are independent, there are 36 possible outcomes:

[1,1]	[2,1]	[3,1]	[4,1]	[5,1]	[6,1]
[1,2]	[2,2]	[3,2]	[4,2]	[5,2]	[6,2]
[1,3]	[2,3]	[3,3]	[4,3]	[5,3]	[6,3]
[1,4]	[2,4]	[3,4]	[4,4]	[5,4]	[6,4]
[1,5]	[2,5]	[3,5]	[4,5]	[5,5]	[6,5]
[1,6]	[2,6]	[3,6]	[4,6]	[5,6]	[6,6]

There are 6 *mutually exclusive* ways to throw a 7, at 1/36 probability each.

So **Pr(throw 7 with 2 dice) = Pr([1,6] v [2,5] v [3,4] v [4,3] v [5,2] v [6,1])**

= **Pr([1,6]) + Pr([2,5]) + Pr([3,4]) + Pr([4,3]) + Pr([5,2]) + Pr([6,1])**<- only cuz M.E!

= 1/36 + 1/36 + 1/36 + 1/36 + 1/36 + 1/36 = 6/36 = 1/6

Odd Question #4

Assuming tosses are independent, there are 36 possible outcomes:

[1,1]	[2,1]	[3,1]	[4,1]	[5,1]	[6,1]
[1,2]	[2,2]	[3,2]	[4,2]	[5,2]	[6,2]
[1,3]	[2,3]	[3,3]	[4,3]	[5,3]	[6,3]
[1,4]	[2,4]	[3,4]	[4,4]	[5,4]	[6,4]
[1,5]	[2,5]	[3,5]	[4,5]	[5,5]	[6,5]
[1,6]	[2,6]	[3,6]	[4,6]	[5,6]	[6,6]

There are 5 *mutually exclusive* ways to throw a 6, at 1/36 probability each.

So **Pr(throw 6 with 2 dice) = Pr([1,5] v [2,4] v [3,3] v [4,2] v [5,1])**

= Pr([1,5]) + Pr([2,4]) + Pr([3,3]) + Pr([4,2]) + Pr([5,1])<- only cuz M.E!

= 1/36 + 1/36 + 1/36 + 1/36 + 1/36 = 5/36

Odd Question #4

To throw a total of 7 with a pair of dice, you have to get a 1 and a 6, or a 2 and a 5, or a 3 and a 4.

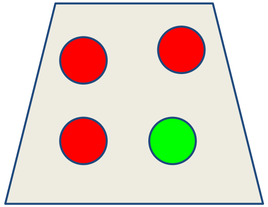
To throw a total of 6 with a pair of dice, you have to get a 1 and a 5, or a 2 and a 4, or a 3 and another 3.

With two fair dice, you would expect:

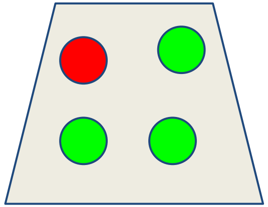
(a) To throw 7 more frequently than 6 (6/36 > 5/36)
~~(b) To throw 6 more frequently than 7~~
~~(c) To throw 6 and 7 equally often~~

Compounding Events

<u>Urn 1</u>



<u>Urn 2</u>



Setup: flip a fair coin. If Heads, pull from Urn 1. If Tails, pull from Urn 2.

$\mathbf{R_1}$ = Pull red from Urn 1, $\mathbf{R_2}$ = Pull red from Urn 2, **H** = Flip heads, **T** = Flip tails

What is **Pr(pull red)**?

Pr(Pull red) = $\mathbf{Pr((H\&R_1) \vee (T\&R_2))}$

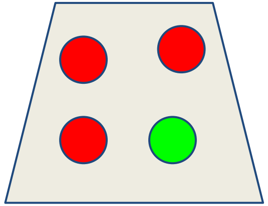
= $\mathbf{Pr(H\&R_1) + Pr(T\&R_2)}$ *<- only because (H&R) and (T&R) are mutually exclusive!*

= $\mathbf{[Pr(H)} \times \mathbf{Pr(R_1)] + [Pr(T)} \times \mathbf{Pr(R_2)]}$ *<- only because H and R1 (T and R2)* are independent

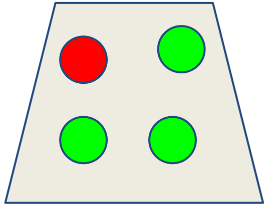
$= [\ \tfrac{1}{2} \times \tfrac{3}{4}\ ] + [\ \tfrac{1}{2} \times \tfrac{1}{4}\ ] = \tfrac{3}{8} + \tfrac{1}{8} = 4/8 = \tfrac{1}{2}$

Compounding Events

Urn 1



Urn 2



What if we flip the coin, then pull 2 balls *with replacement*?

Let $\mathbf{R_1}$ = Red on 1st pull, $\mathbf{R_2}$ = Red on 2nd pull

What's $\mathbf{Pr(R_1\ \&\ R_2)}$?

X = Flip heads (½), then pull red from Urn 1 (¾), replace, then pull red from Urn 1 (¾)

Y = Flip tails (½), then pull red from Urn 2 (¼), replace, then pull red from Urn 2 (¼)

$\mathbf{Pr(R_1\ \&\ R_2)} = \mathbf{Pr(X \vee Y)}$

$= \mathbf{Pr(X) + Pr(Y)}$ *<- only because X and Y are mutually exclusive!*

$= [\ \frac{1}{2} \times \frac{3}{4} \times \frac{3}{4}\ ] + [\ \frac{1}{2} \times \frac{1}{4} \times \frac{1}{4}\ ] = 9/32 + 1/32 = 10/32 = 5/16$

Recap

- Two propositions are **<u>mutually exclusive</u>** iff they can't both be true at once
- If **A** and **B** are mutually exclusive, then **Pr(A) + Pr(B)** = **Pr(A v B)**
- This follows from **Pr(A v B)** = **Pr(A)** + **Pr(B)** - **Pr(A & B)**, *where **Pr(A & B)** = 0*

- Two propositions are **<u>independent</u>** iff the truth of one does not make the truth of the other any more or less probable
- If **A** and **B** are independent, then **Pr(A)** x **Pr(B)** = **Pr(A & B)**
- This follows from **Pr(A & B) = Pr(A) x Pr(B | A)**, *where **Pr(B | A)** = **Pr(B)***

Group Exercises

Probability & Inductive Logic p. 45, #2–4