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TEXAS
The University of Texas at Austin

PHIL 313Q: Inductive Logic

ch. 5 of “Probability and Inductive Logic” by Ian Hacking

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Conditional Probability

- We've been dealing mostly with "categorical" probability: probability with no assumptions
  - e.g. the probability of drawing an ace on the 1st card
  - **Pr(A_1)**
- Now we'll focus on "conditional" probability: probability on one event *on the condition of another*
  - e.g. the probability of drawing an ace on the 2nd card *on the condition that* we drew an ace on the 1st card
  - **Pr(A_2 | A_1)**
- We already saw that **Pr(A & B)** = **Pr(A)** x **Pr(B | A)**.
- So, **Pr(B | A)** = **Pr(A & B)** / **Pr(A)**, *if Pr(A) > 0*
- Note that, also, **Pr(A & B)** = **Pr(B & A)** = **Pr(B)** x **Pr(A | B)**
- So, **Pr(A | B)** = **Pr(B & A)** / **Pr(B)**

Conditional Probability

You basically have 4 equations at your disposal:

(1) **Pr(A & B) = Pr(A) x Pr(B | A)**
(2) **Pr(A & B) = Pr(B) x Pr(A | B)**

(3) **Pr(A | B) = Pr(A & B) / Pr(B)** *from (2)*
(4) **Pr(B | A) = Pr(A & B) / Pr(A)** *from (1)*

In conditional probability problems, you'll be able to infer from the setup, or calculate, some of these terms in order to solve for the others.

Conditional Probability (Examples)

Take a fair die. What's the probability of rolling a 6 assuming you roll an even?
Let $\mathbf{R_n}$ = roll *n*, **E** = roll even
What's $\mathbf{Pr(R_6 \mid E)}$?

Intuitively, half (3) of the faces are even, and 6 is one of them. So, ⅓.

$\mathbf{Pr(R_6 \mid E) = Pr(R_6 \;\&\; E) / Pr(E)}$
$\mathbf{= Pr(R_6) / Pr(E)}$ <- this requires insight. R_6 and E are NOT mutually exclusive, so I can't multiply their probabilities to get $\mathbf{Pr(R_6 \;\&\; E)}$
$= \frac{1}{6} / \frac{1}{2}$
$= \frac{1}{3}$

Checks out!

Conditional Probability (Examples)

Let **M** = roll a 1 or a prime number
Let **E** = roll an even number
Let $\mathbf{R}_{n}$ = roll an *n*

What's **Pr(E | M)**?

Intuitively, there are 4 ways to get **M** (1, 2, 3, 5), one of which (2) is even. So, ¼.

Pr(E | M) = **Pr(E & M)** / **Pr(M)**
= **Pr($\mathbf{R}_2$)** / (4/6)   <- this requires insight. R_6 and E are NOT mutually exclusive
= ⅙ / (4/6) = ¼

Checks out!

Conditional Probability (Examples)

Take a standard 52-card, shuffled deck. You're dealt a card at random.
Let **R** = dealt red, **C** = dealt clubs, **A** = dealt ace

Suppose you're told you've been dealt either red or clubs, i.e. **R v C**
What's **Pr(A | R v C)**?

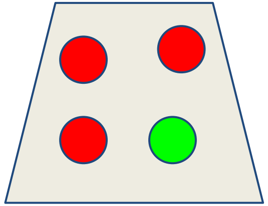
Intuitively: 26 red + 13 clubs = 39 red-or-club cards. 3 are aces. So, 3/39 = 1/13.

Pr(A | R v C) = **Pr(A & (R v C))** / **Pr(R v C)**
= (3/52) / (39/52) <- this requires insight, but I could have added Pr(R) and Pr(C) to get Pr(R v C), since those are mutually exclusive
= 3/39 = 1/13

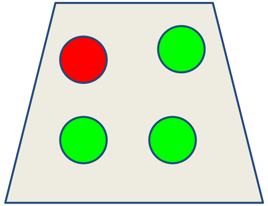
Checks out!

Compounding Events (Again)

Urn 1



Urn 2



Setup: flip a fair coin. If Heads, pull from Urn 1. If Tails, pull from Urn 2.

R = Pull a red ball, **H** = Flip heads, **T** = Flip tails

What is **Pr(R)**?

Pr(R) = **Pr((H&R) v (T&R))**

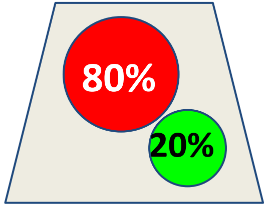
= **Pr(H&R)** + **Pr(T&R)**

= [**Pr(H)** x **Pr(R | H)**] + [**Pr(T) x Pr(R | T)**]

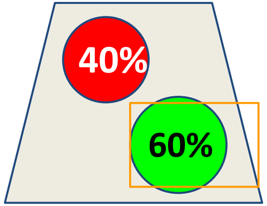
= [$\frac{1}{2}$ x $\frac{3}{4}$] + [$\frac{1}{2}$ x $\frac{1}{4}$] = $\frac{1}{8}$ + $\frac{3}{8}$ = 4/8 = $\frac{1}{2}$

Conditional Probability (Examples)

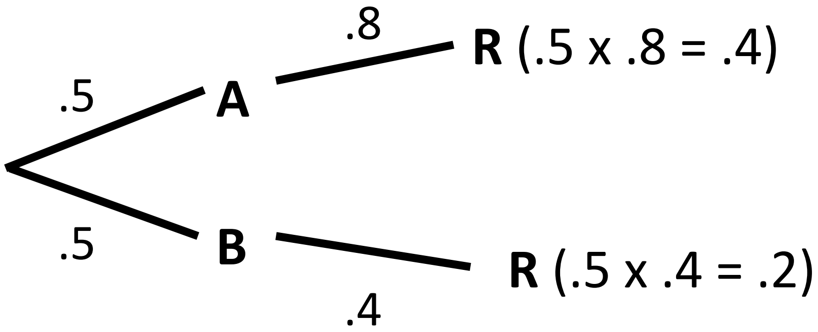
Urn A



Urn B



Setup: Pick a ball at random. Then replace.
R = Pull a red ball, **A** = Pull from A, **B** = Pull from B
What is **Pr(A | R)**?

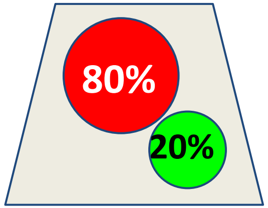


Intuitively: these are mutually exclusive paths,
so **P(R) = .4 + .2 = .6**
60% chance of pulling red.
60% = 40% from A + 20% from B
So, 40/60 Rs are from A.
So, P(A | R) = 40/60 = ⅔

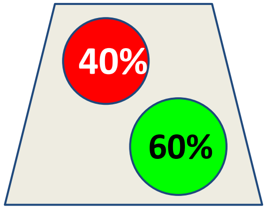
$\Pr(A \,\&\, B) = \Pr(A) \times \Pr(B \mid A)$ $\Pr(A \mid B) = \Pr(A\&B) / \Pr(B)$

Conditional Probability (Examples)

Urn A



Urn B



Setup: Pick a ball at random. Then replace.

R = Pull a red ball, **A** = Pull from A, **B** = Pull from B

Notice **Pr(R | A)** = .8, **Pr(R | B)** = .4, and **Pr(A)** = **Pr(B)** = .5

Suppose you pull a red ball. What's the probability it came from A?

$$\begin{aligned}
\Pr(A \mid R) &= \Pr(A\&R) / \Pr(R) \\
&= [\, \Pr(R \mid A) \times P(A) \,] / \Pr((A \,\&\, R) \vee (B \,\&\, R)) \\
&= [\, \Pr(R \mid A) \times P(A) \,] / [\, \Pr(A \,\&\, R) + \Pr(B \,\&\, R) \,] \\
&= [\, .8 \times .5 \,] / [\, \Pr(R \mid A) \times P(A) + \Pr(R \mid B) \times P(B) \,] \\
&= .4 / [\, .4 + .2 \,] = .4/.6 = \tfrac{2}{3}
\end{aligned}$$

Conditional Probability (Examples)

Acme (supplies 60%)	Bolt (supplies 40%)
96% reliable	72% reliable

What's the probability that a random shock absorber is reliable?

A = Shock absorber chosen randomly is from Acme

B = Shock absorber chosen randomly is from Bolt

R = Shock absorber chosen randomly is reliable

Pr(R) = Pr((A&R) v (B&R)) = Pr(A&R) + Pr(B&R)

= Pr(A)Pr(R | A) + Pr(B)Pr(R | B)

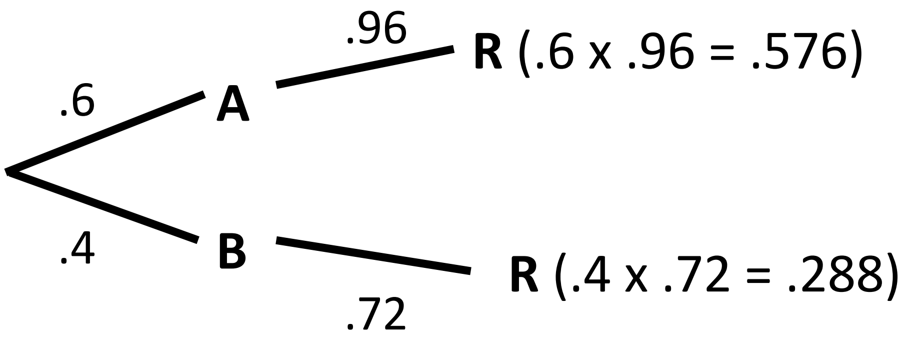
= [.6 x .96] + [.4 x .72] = .864

Pr(A & B) = Pr(A) x Pr(B | A) **Pr(A | B) = Pr(A&B) / Pr(B)**

Conditional Probability (Examples)

<u>Acme (supplies 60%)</u> <u>Bolt (supplies 40%)</u>

96% reliable	72% reliable



Mutually exclusive paths, so chance of R is .576 + .288 = .864

Conditional Probability (Examples)

Steroid Team (80% juice) Clean Team (20% juice)

Setup: Coach sends a team randomly. Then the team that goes gets a member tested. Suppose the team-member tests positive for steroids. What's the chance the coach sent the Steroid Team?

S = coach sent Steroid Team, **C** = coach sent Clean Team, **U** = tested positive

Pr(S | U) = **Pr(S & U) / Pr(U)** =

= **Pr(S)Pr(U|S)** / **Pr((S&U) v (C&U))**

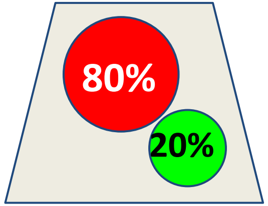
= (.5 x .8) / (**Pr(S)**x**Pr(U | S)** + **Pr(C)**x**Pr(U | C)**)

= .4 / [(.5 x .8) + (.5 x .2)]

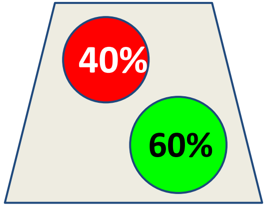
= .4 / (.4+.1) = .4/.5 = 80%

Conditional Probability (Examples)

Urn A



Urn B



Setup: Flip a coin. If heads, draw from Urn A twice with replacement. If tails, draw from Urn B twice with replacement.

R_n = Pull a red ball on n^{th} draw, **A** = Pulled from Urn A (heads), **B** = Pulled from Urn B (tails)

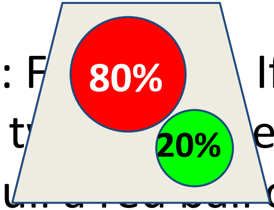
What is **$Pr(A \mid R_1 \& R_2)$**?

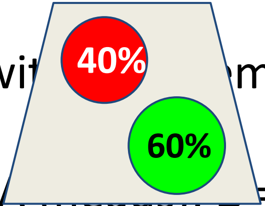
$$= Pr(A \& R_1 \& R_2) / Pr(R_1 \& R_2)$$
$$= Pr(A \& R_1)*Pr(R_2 \mid A \& R_1) / (Pr(A \& R_1 \& R_2) + Pr(B \& R_1 \& R_2))$$
$$= Pr(A)Pr(R_1 \mid A) * .8 / (Pr(A \& R_1 \& R_2) + Pr(B \& R_1 \& R_2))$$
$$= (½)(.8) * .8 / (Pr(A \& R_1 \& R_2) + Pr(B \& R_1 \& R_2))$$
$$= .32 / (.4 + Pr(B \& R_1)Pr(R_2 \mid B \& R_1))$$

Urn A

$Pr(A \& B) = Pr(A) \times Pr(B \mid A)$ $Pr(A \mid B) = Pr(A\&B) / Pr(B)$

Conditional Probability (Examples)





Setup: F If heads, draw from Urn A twice wi ment. If tails, draw from Urn B t eplacement.

$\mathbf{R}_n$ = Pu on n^{th} draw, **A** = Pulled from Urn = Pulled from Urn B (tails)

What is $\mathbf{Pr(A \mid R_1 \& R_2)}$?

$= Pr(A \& R_1 \& R_2) / Pr(R_1 \& R_2)$

$= Pr(A \& R_1)*Pr(R_2 \mid A \& R_1) / (Pr(A \& R_1 \& R_2) + Pr(B \& R_1 \& R_2))$

$= Pr(A)Pr(R_1 \mid A) * .8 / (Pr(A \& R_1 \& R_2) + Pr(B \& R_1 \& R_2))$

$= (½)(.8) * .8 / (Pr(A \& R_1 \& R_2) + Pr(B \& R_1 \& R_2))$

$= .32 / (.32 + Pr(B \& R_1)Pr(R_2 \mid B \& R_1))$

$= .32 / (.32 + Pr(B)Pr(R_1 \mid B).4)$

$= .32 / (.32 + (½)(.4)(.4))$

$= .32 / .32 + .08 = .32 / .4 = .8$

Group Practice

- see slide #7, and imagine running two consecutive trials. How would you think about the probability of the following event: *pulling a red ball on the first trial and a green ball on the second trial*? (cf. #3 on Pset 5)
  - Since there is replacement between trials, the probability of these (independent) compound event is the probability of the first multiplied by the probability of the second.
- *p56.* #3
  - **T** = Randomly selected child lives in Triangle
  - **P** = Randomly selected child tests positive
  - **Pr(T) = .02**, **Pr(P | T)** = .14, **Pr(P | ~T)** = .01
  - (a) **Pr(T & P)** = **Pr(T)Pr(P | T)** = .02 x .14 = **.0028**
  - (b) **Pr(P)** = **Pr((P&T) v (P&~T))** = **Pr(P&T)** + **Pr(P&~T)** = .0028 + **Pr(~T)Pr(P | ~T)** = .0028 + (.98 x .01) = .0028 + .0098 = **.0126**
  - (c) **Pr(T | P)** = **Pr(T & P)/Pr(P)** = .0028 / .0126 = **2/9** (or about .22…)