Problem Set 7 Solutions

Phil313Q

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Show your work as much as possible for partial credit. When symbolizing propositions (or events) with capital letters, provide a symbolization key. You may use a calculator to convert fractions to decimals. When giving answers in decimals, round to the nearest hundredth (e.g. round 0.3323 to 0.33). Circle your final answers clearly.

1. Suppose you roll two, fair, 6-sided dice.

   What's the probability that one of the die will have a greater number of pips facing up than the other die?

   There are 36 possibilities total, and one die has more pips than the other whenever you don't roll doubles, leaving 30 possibilities left:

   ~~(1,1)~~ (1,2) (1,3) (1,4) (1,5) (1,6)
   (2,1) ~~(2,2)~~ (2,3) (2,4) (2,5) (2,6)
   (3,1) (3,2) ~~(3,3)~~ (3,4) (3,5) (3,6)
   (4,1) (4,2) (4,3) ~~(4,4)~~ (4,5) (4,6)
   (5,1) (5,2) (5,3) (5,4) ~~(5,5)~~ (5,6)
   (6,1) (6,2) (6,3) (6,4) (6,5) ~~(6,6)~~

   So, the answer is $\boxed{\frac{30}{36} \approx 0.83}$

2. A card is drawn from a standard deck of fifty-two cards which has been well shuffled seven times. What is the probability that the card is:

   (a) Either a black card or a red face (Jack, Queen, King) card?

   B = draw a black card
   R = draw a red face card
   $Pr(B) = \frac{26}{52}$
   $Pr(R) = \frac{6}{52}$
   $Pr(B \vee R) = Pr(B) + Pr(R) = \frac{26}{52} + \frac{6}{52} = \boxed{\frac{32}{52} \approx 0.62}$

   (b) Either an even number (2 through 10) or a diamond card?

   E = draw an even card
   D = draw a non-even diamond card (including faces)
   $Pr(E) = \frac{5\times4}{52} = \frac{20}{52}$
   $Pr(D) = \frac{13-5}{52} = \frac{8}{52}$
   $Pr(E \vee D) = Pr(E) + Pr(D) = \frac{20}{52} + \frac{8}{52} = \boxed{\frac{28}{52} \approx 0.54}$

3. When two cards are drawn in succession from a standard pack of cards, what are the probabilities of drawing:

   (a) two red cards, with replacement?

   R_n = draw a red card on the n^{th} draw
   $Pr(R_1) = \frac{26}{52}$
   $Pr(R_2|R_1) = \frac{26}{52}$
   $Pr(R_1 \& R_2) = Pr(R_1) \times Pr(R_2|R_1) = (\frac{26}{52})^2 = \boxed{\frac{676}{2704} = 0.25}$

   (b) two red cards, without replacement?

   R_n = draw a red card on the n^{th} draw
   $Pr(R_1) = \frac{26}{52}$
   $Pr(R_2|R_1) = \frac{25}{51}$
   $Pr(R_1 \& R_2) = Pr(R_1) \times Pr(R_2|R_1) = \frac{26}{52} \times \frac{25}{51} = \boxed{\frac{650}{2652} \approx 0.25}$

4. Suppose that 20% of our oranges come from Guatemala, 50% come from Hawaii, and 30% come from Ecuador. 3% of oranges from Guatemala contain worms, 2% of oranges from Hawaii contain worms, and 4% of oranges from Ecuador contain worms. An orange is selected randomly and contains worms.

   What is the probability that this orange did NOT come from Hawaii?

   G = the orange came from Guatemala
   H = the orange came from Hawaii
   E = the orange came from Ecuador
   W = the orange contains worms
   $Pr(G) = 0.20$
   $Pr(H) = 0.50$
   $Pr(E) = 0.30$
   $Pr(W|G) = 0.03$
   $Pr(W|H) = 0.02$
   $Pr(W|E) = 0.04$

$$
\begin{aligned}
Pr(H|W) = \frac{Pr(H\&W)}{Pr(W)} &= \frac{Pr(H)Pr(W|H)}{Pr((H\&W)\vee(G\&W)\vee(E\&W))} \\
&= \frac{Pr(H)Pr(W|H)}{Pr(H)Pr(W|H)+Pr(G)Pr(W|G)+Pr(E)Pr(W|E)} \\
&= \frac{.5\times.02}{(.5\times.02)+(.2\times.03)+(.3\times.04)} \\
&= \frac{.01}{(.01)+(.006)+(.012)} = \frac{.01}{.028}
\end{aligned}
$$

$$Pr(\sim H|W) = 1 - Pr(H|W) = 1 - \frac{.01}{.028} = \boxed{\frac{.018}{.028} \approx 0.64} \quad (1)$$

[1] Proof of the first equality in this line, noting first that $((H\vee \sim H)\&W)$ is truth-functionally equivalent to W:
$Pr((H\vee\sim H)\&W) = Pr(W)$

[*Since H, G, and E are mutually exclusive and jointly exhaustive, you could also have solved for* $Pr(G \vee E|W) = Pr(G|W) + Pr(E|W)$]

5. In Smuggler's Airport, 60% of suitcases contain illegal drugs. Spot, the drug dog, is trained to bark at illegal drugs and sit otherwise. Spot correctly barks/sits 90% of the time. A random suitcase is selected for screening, and Spot barks at it.

   (a) What is the probability that the suitcase contains illegal drugs, given that Spot barked at it?

   I = the suitcase has illegal drugs in it
   B = Spot barks at the suitcase
   $Pr(I) = 0.60$
   $Pr(\sim I) = 0.40$
   $Pr(B|I) = 0.90$
   $Pr(B| \sim I) = 0.10$

   $$Pr(I|B) = \frac{Pr(I\&B)}{Pr(B)} = \frac{Pr(I)Pr(B|I)}{Pr((B\&I)\vee(B\&\sim I))} = \frac{Pr(I)Pr(B|I)}{Pr(B\&I)+Pr(B\&\sim I)}$$

   $$= \frac{Pr(I)Pr(B|I)}{Pr(I)Pr(B|I)+Pr(\sim I)Pr(B|\sim I)}$$

   $$= \frac{.6\times.9}{(.6\times.9)+(.4\times.1)}$$

   $$= \frac{.54}{.54+.04} \boxed{\frac{.54}{.58} \approx 0.93}$$

6. Three urns labeled 'A', 'B', and 'C' each contain 10 colored balls with the following distribution:

   Urn A: 5 red, 5 green
   Urn B: 2 red, 8 green
   Urn C: 0 red, 10 green

   An urn is chosen at random with the following odds:

   Urn A: 50% chance of being chosen
   Urn B: 25% chance of being chosen
   Urn C: 25% chance of being chosen

   (a) Two balls are drawn from this urn *with replacement*. Both are red. **What is the probability that we have urn A?**

   A = we pulled from urn A
   B = we pulled from urn B
   C = we pulled from urn C
   R_n= we pulled red on the n^{th} pull

$Pr((H\&W) \vee (\sim H\&W)) = Pr(W)$
$Pr(H\&W) + Pr(\sim H\&W) = Pr(W)$
$\frac{Pr(H\&W)}{Pr(W)} + \frac{Pr(\sim H\&W)}{Pr(W)} = \frac{Pr(W)}{Pr(W)}$
$Pr(H|W) + Pr(\sim H|W) = 1$
$Pr(\sim H|W) = 1 - Pr(H|W) \qquad \square$

$$
\begin{aligned}
Pr(A|R_1\&R_2)) = \frac{Pr(A\&R_1\&R_2)}{Pr(R_1\&R_2)} &= \frac{Pr(A\&R_1)Pr(R_2|A\&R_1)}{Pr([A\&R_1\&R_2]\vee[B\&R_1\&R_2]\vee[C\&R_1\&R_2])} \\
&= \frac{Pr(A)Pr(R_1|A)Pr(R_2|A\&R_1)}{Pr(A\&R_1\&R_2)+Pr(B\&R_1\&R_2)+Pr(C\&R_1\&R_2)} \qquad (2) \\
&= \frac{Pr(A)Pr(R_1|A)Pr(R_2|A\&R_1)}{Pr(A)Pr(R_1|A)Pr(R_2|A\&R_1)+Pr(B)Pr(R_1|B)Pr(R_2|B\&R_1)+\mathbf{0}} \\
&= \frac{\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}}{(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2})+(\frac{1}{4}\times\frac{1}{5}\times\frac{1}{5})} \\
&= \frac{\frac{1}{8}}{\frac{1}{8}+\frac{1}{100}} = \boxed{\frac{1}{1.08} \approx 0.93}
\end{aligned}
$$

(b) Two balls are drawn from this urn *without replacement*. Both are green. **What is the probability that we have urn C?**

G_n = we pulled green on the n^{th} pull

$$
\begin{aligned}
Pr(C|G_1\&G_2)) = \frac{Pr(C\&G_1\&G_2)}{Pr(G_1\&G_2)} &= \frac{Pr(C\&G_1)Pr(G_2|C\&G_1)}{Pr([A\&G_1\&G_2]\vee[B\&G_1\&G_2]\vee[C\&G_1\&G_2])} \\
&= \frac{Pr(C)Pr(G_1|C)Pr(G_2|C\&G_1)}{Pr(A\&G_1\&G_2)+Pr(B\&G_1\&G_2)+Pr(C\&G_1\&G_2)} \\
&= \frac{Pr(C)Pr(G_1|C)Pr(G_2|C\&G_1)}{Pr(A)Pr(G_1|A)Pr(G_2|A\&G_1)+Pr(B)Pr(G_1|B)Pr(G_2|B\&G_1)+Pr(C)Pr(G_1|C)Pr(G_2|C\&G_1)} \\
&= \frac{(\frac{1}{4}\times1\times1)}{(\frac{1}{2}\times\frac{1}{2}\times\frac{4}{9})+(\frac{1}{4}\times\frac{4}{5}\times\frac{7}{9})+(\frac{1}{4}\times1\times1)} = \frac{.25}{.\overline{11}+.1\overline{55}+.25} = \boxed{\frac{.25}{.51\overline{55}} \approx 0.48}
\end{aligned}
$$

[Will also accept 0.49 to accommodate stronger rounding.]

7. There's a 10% chance of rain in Austin, and you have to decide between taking I-35 or Mopac to go to your friend's house.

   Mopac will take at least 1 hour no matter what, plus EITHER (i) 3 additional hours, if it rains OR (ii) 1 additional hours, if it doesn't rain.

   I-35 will take at least 3 hours no matter what, plus EITHER (i) 1 additional hours, if it rains OR (ii) 0 additional hours, if it doesn't rain.

   What's the expected travel time (expected value) of traveling by each freeway?

   $Exp(Mopac) = 1 + (3 \times .1) + (1 \times .9) = \boxed{\text{2.2 hours}}$
   $Exp(I35) = 3 + (1 \times .1) + (0 \times .9) = \boxed{\text{3.1 hours}}$

8. Suppose we run a lottery with 1,250 tickets of equal chance of being drawn. Tickets cost \$2.35 each. Two tickets will be drawn in succession in order to determine two winners. The first ticket drawn will win \$2,000, and the second ticket drawn will win \$1,000. (When a ticket is drawn once, it cannot be drawn again.)

   (a) If you buy exactly two tickets, what's the probability that you will win both prizes?

   F = you win the first prize

[2] Note that since urn C has no red balls, $Pr(C\&R_1\&R_2) = 0$

$S =$ you win the second prize
$Pr(F\&S) = Pr(F)Pr(S|F) = \frac{2}{1250} \times \frac{1}{1249} = \boxed{\frac{2}{1{,}561{,}250} \approx 0.00}$

(b) What is the expected value of buying a ticket in this lottery?
$Exp(ticket) = (1 \times -2.35) + (\frac{1}{1250} \times 2,000) + (\frac{1}{1250} \times 1,000)$
$= -2.35 + 1.6 + .8 = \boxed{\$0.05}$

9. Suppose the Prisoner's Dilemma is played with the following payout structure. If both prisoners choose PEACE, they each get 3 points. If both prisoners choose WAR, they both get -2 points. If they choose differently, the prisoner who chooses WAR gets 5 points, and the prisoner who chooses PEACE gets -5 points.

(b) Construct a decision matrix to prove that choosing WAR dominates choosing PEACE.
Without loss of generality, take A's perspective:

-	B Chooses Peace	B Chooses War
A Chooses Peace	+3 to A	-5 to A
A Chooses War	+5 to A	-2 to A

(a) Assuming that the prisoners reason similarly and choose the same option 80% of the time, how much more or less expected value is there for a prisoner to choose PEACE instead of WAR?
$Exp(Peace) = (.8 \times 3) + (.2 \times -5) = 2.4 - 1 = 1.4$
$Exp(War) = (.8 \times -2) + (.2 \times 5) = -1.6 + 1 = -0.6$
Under these assumptions, choosing Peace yields $(1.4 - (-0.6)) =$ $\boxed{\text{2 more points of expected value}}$ than choosing War.