RELATIONS (XII, R. S. AGGARWAL)

EXERCISE 1A (Pg.No.: 16)

1. Find the domain and range of the relation $R = \{(-1, 1), (1, 1), (2, 4), (-2, 4)\}$

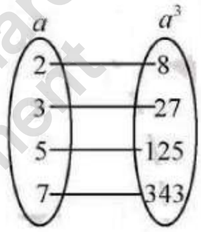
Sol. $dom(R) = \{-1, 1, -2, 2\}$, range $(R) = \{1, 4\}$

2. Let R = {(a, a^3): a is a Prime Number less than 5}
Find the range of R.

Sol. Let A = { a : a is a prime number less then 5}
$\Rightarrow$ A = {2, 3}
Now, R = { (2, 8), (3, 27)}
By the definition of range R. Range(R) = {8, 27} Ans.

3. Let $R = \{(a, a^3) : a \text{ is a prime number less than } 10\}$.
Find (i) R (ii) dom(R) and (iii) range (R)

Sol. a is a prime number less than 10
$a = 2, 3, 5, 7$
$R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$
dom$(R) = \{2, 3, 5, 7\}$, range $(R) = \{8, 27, 125, 343\}$

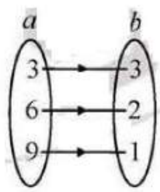


4. Let R = {x, y) : x + 2y = 8} be a Relation on N.
Write range of R.

Sol $\because$ x + 2y = 8
$\Rightarrow$ x = 8 – 2y
Putting y = 1, we have x = 6 Putting y = 2, we have x = 4 Putting y = 3, we have x = 2
Here, R = { (2, 3), (4, 2), (6, 1) } Range (R) = (3, 2, 6)

5. List the elements of each of the following relations. Find the domain and range in each case.
$R_2 = \{(a, b) : a \in N, b \in N \text{ and } a + 3b = 12\}$

Sol. $R_2\{(a, b) : a \in N, b \in N \text{ and } a + 3b = 12\}$
$R_2 = \{(3, 3), (6, 2), (9, 1)\}$,
dom$(R_2) = \{3, 6, 9\}$,
range $(R_2) = \{3, 2, 1\}$



6. Let $R = \{(a,b) : b = |a-1|, a \in Z \text{ and } |a| < 3\}$
Find the domain and range of R

Sol. Let, A = {a : a ε z and |a| < 3}
$\Rightarrow$ A = {-2, -1, 0, 1, 2}
$\because$ R = {(a, b) : b = |a – 1|, a ε z and |a| < 3}
$\Rightarrow$ R = {(-2, 3), (-1, 2), (0, 1), (1, 0), (2, 1)}

Clearly domain (R) = { -2, -1, 0, 1, 2}
Range(R) = {0, 1, 2, 3}

7. Let $R=\left\{\left(a,\frac{1}{a}\right): a\in N \text{ and } 1<a<5\right\}$

Find the domain and range of R

Sol. Let A = {a: a∈N, & 1 < a < 5}

⇒ A = {2, 3, 4}

$\therefore R=\left\{\left(2,\frac{1}{2}\right),\left(3,\frac{1}{3}\right),\left(4,\frac{1}{4}\right)\right\}$

Domain = {2, 3, 4} Range = $\left\{\frac{1}{2},\frac{1}{3},\frac{1}{4}\right\}$

8. Let $R=\{(a,b): a,b\in N \text{ and } b=a+5, a<4\}$

Find the domain and range of R

Sol. Let, A = {a : a ∈ N & a < 4}

⇒ A = {1, 2, 3} ∴ R = {(1, 6), (2, 7), (3, 8)}

Clearly, Domain = {1, 2, 3}and Range = {6, 7, 8}

9. Let S be the set of all sets and let $R=\{(A,B): A\subset B\}$, i.e. A is proper subset of B. show that R is

(i) transitive (ii) not reflexive (iii) not symmetric

Sol. (i) transitivity: -

Let, A, B and C ∈ S, such that (A, B) & (B, C) ∈ R.

Let, A, B and C ∈ S, such that (A, B) & (B, C) ∈ R.

∵ (A, B) ∈ R ⇒ A ⊂ B ......(i) (B, C) ∈ R ⇒ B ⊂ C ......(ii)

From (i) and (ii), we have A ⊂ C ⇒ (A, C) ∈ R

Thus, R is a transitive Relation S.

(ii) Non – reflexive ; →

∵ A ⊄ A ⇒ (A, A) ∈ R

thus, R is non reflexive.

(iii) Non. Symmetric :→

Let, A ⊂ B

∴ (A, B) ∈ R

But, B ⊄ A R

∴ (B, A) ∉ R ∵ (A, B) ∈ R & (B, A) ∉ R ∴ R is non – symmetric

10. Let A be the set of all points in a plane and let O be the origin. Show that the relation R, defined by $R=\{(P,Q): OP=OQ\}$ is an equivalence relation.

Sol. Let O denote the origin in the given plane. Then, $R=\{(P,Q): OP=OQ\}$.

We observe the following properties of relation R:

Reflexivity : For any point P inset A, we have $OP=OP$

$\Rightarrow (P,P)\in R$. Thus, $(P,P)\in R$ for all $P\in A$. So, R is reflexive.

Symmetric : Let P and Q be two points inset A such that $(P,Q)\in R$

$\Rightarrow OP=OQ \Rightarrow OQ=OP \Rightarrow (Q,P)\in R$.

Thus, $(P,Q)\in R \Rightarrow (Q,P)\in R$ for $P,Q\in A$. So, R is symmetric.

Transitivity : Let P, Q and S be three points in set A such that $(P, Q) \in R$ and $(Q, S) \in R$

$\Rightarrow OP = OQ$ and $OQ = OS$ $\Rightarrow OP = OS$ $\Rightarrow (P, S) \in R$

So, R is transitive. Hence, R is an equivalence relation.

Let P be a fixed point in set A and Q be a point in set A such that $(P, Q) \in R$. Then, $(P, Q) \in R$.

$\Rightarrow OP = OQ$ $\Rightarrow Q$ moves in the plane in such a way that its distance from the origin.

$O(0, 0)$ is always same and is equal to OP.

$\Rightarrow$ Locus of Q is circle with centre at the origin and radius OP.

Hence, the set of all points related to P in the circle passing through P with origin O as centre.

11. On the set S of all real numbers, define a relation $R = \{(a, b) : a \le b\}$

Show that R is (i) reflexive (ii) transitive (iii) not symmetric

Sol. (i) Reflexivity : $\rightarrow$

Let, a be an arbitrary element on S.

$\because a \le a \Rightarrow (a, a) \in R$ thus, R is reflective.

(ii) Transitivity : $\rightarrow$

Let, a, b & c $\in$ s such that (a, b) and (b, c) $\in$ s

$\because (a, b) \in R \Rightarrow a \le b$ ....(i) and $(b, c) \in R \Rightarrow b \le c$ ...(ii)

from (i) and (ii) we have $a \le c \Rightarrow (a, c) \in R$

Thus, R is transitive.

(iii) Non symmetry : $\rightarrow$

$\because 5 \le 6 \Rightarrow (5, 6) \in R$

$But,\ 6 \nleq 5 \Rightarrow (6, 5) \in R$

thus, R is non symmetric

12. Let $A = \{1, 2, 3, 4, 5, 6\}$ and let $R = \{(a, b) : a, b \in A$ and $b = a + 1\}$

Show that R is (i) not reflexive (ii) not symmetric and (iii) not transitive

Sol. In roster form,

R = {(1, 2), (2, 3), (3, 4), (4, 5) (5, 6)}

Non – reflective:$\rightarrow$

$\because (1, 1) \in R$, but $1 \in A$ thus, R is non-reflective.

Non – symmetric :$\rightarrow$ $\because (1, 2) \in R$ but, $(2, 1) \notin R$

thus, R is non-symmetric.

Non transitive:$\rightarrow$ $\because (1, 2) \in R$ & $(2, 3) \in R$

But, $(1, 3) \notin R$ thus, R is non- transitive

EXERCISE 1B (Pg. No.: 18)

1. Define a relation on a set. What do you mean by the domain and range of a relation?

Sol. Relation in a set :- A relation R in a set A is a subset of $A\times A$.

Thus, R is a relation is a set $A \Leftrightarrow R\subseteq A\times A$, if $(a, b)\in R$, then we say that a is related to b and write, aRb. If $(a, b)\notin R$, then we say that a is not related to b and write $a \not R b$.

Domain and range of a relation. Let R be a relation in a set A. Then, the set of all first co-ordinate of element of R is called the domain of R, written as $\text{dom}(R)$ and the set of all second co-ordinate of R is called the range of R, write as range (R).

2. Let A be the set of all triangles in a plane show that the relation $R=\{(\Delta_1,\Delta_2):\Delta_1\sim\Delta_2\}$ is an equivalence relation on A

Sol. Reflectivity:→

Let, Δ be an arbitrary element on A.

$\because \Delta\sim\Delta \Rightarrow (\Delta, \Delta) \in R\ \forall\ \Delta \in R$ thus, R is reflective

Symmetricity :→

Let, Δ_1 and $\Delta_2 \in$ A, such that $(\Delta_1, \Delta_2) \in R$

$\because (\Delta_1, \Delta_2) \in R \Rightarrow \Delta_1 \sim \Delta_2$

$\Rightarrow \Delta_2 \sim \Delta_1 \Rightarrow (\Delta_2, \Delta_1) \in R$

thus, R is symmetric relation.

Transitivity : →

Let, Δ_1, Δ_2 and $\Delta_3 \in$ a such that, $(\Delta_1, \Delta_2$ & $(\Delta_2, \Delta_3) \in R$

$\because (\Delta_1, \Delta_2) \in R \Rightarrow \Delta_1 \sim \Delta_2$ .....(i)

& $(\Delta_2, \Delta_3) \in R \Rightarrow \Delta_2 \sim \Delta_3$ .....(ii)

From (i) and (ii) we have

$\Delta_1 \sim \Delta_3 \Rightarrow (\Delta_1, \Delta_3) \in R$

thus, R is transitive.

3. Let $R=\{(a, b):a, b\in Z$ and $(a+b)$ is even$\}$. Show that R is an equivalence relation on Z.

Sol. $R=\{(a, b):a, b\in Z$ and $a+b$ is even$\}$

Reflexive : Let $a\in Z$ then $a+a=2a$, which is even. $\therefore$ $(a, a)\in R$, hence it is reflexive.

Symmetric : Let $(a, b)\in Z$ then $a+b=\text{even} \Rightarrow b+a=\text{even}$. $\therefore$ (b, a) also belongs to R.

$(a, b)\in R \Rightarrow (b, a)\in R$. Here R is symmetric also.

Transitive : Let (a, b) and $(b, c)\in R$ then $a+b=\text{even}=2k$ and $b+c=\text{even}=2r$

Adding then, $a+2b+c=2k+2r \Rightarrow a+c=2(k+r-b) \Rightarrow a+c=\text{even} \quad \therefore (a, c)\in R$

Hence, it is transitive also.

$\therefore$ Since, R is reflexive, symmetric and transitive. Hence, R is an equivalence.

4. Let $R=\{(a, b):a, b\in Z$ and $(a-b)$ is divisible by $5\}$. Show that R is an equivalence relation on Z.

Sol. $(a, b)\in R\Leftrightarrow a-b$ is divisible by 5.

(i) Reflexive : All $(a, a)\in R$ as $a-a=0$ which is divisible by 5. Hence, R is reflexive.

(ii) Symmetric : If $(a, b)\in R$, then $a-b$ is divisible by 5.

$\therefore\ a-b=5k$ and $(b-a)=-5k \quad \therefore\ (b-a)$ is also divisible by 5.

$\therefore\ (b, a)\in R \quad \therefore\ R$ is symmetric also.

(iii) Transitive : If $(a,\ b)$ and $(b,c)\in R$ then we must have according to definition of R

$a-b$ is divisible by 5. $\therefore\ a-b=5m$ and $b-c$ is divisible by 5

$b-c=5n$

Adding then we get, $a-c=5(m+n)$

$\therefore\ (a-c)$ is also divisible by 5. Hence, $(a, c)\in R$, hence the given ration is transitive also.

$\therefore\ R$ is reflexive, symmetric and transitive. Hence R is an equivalence relation.

5. Show that the relation R defined on the set $A=\{1, 2, 3, 4, 5\}$, given by $R=\{(a, b):|a-b| \text{ is even}\}$ is an equivalence relation.

Sol. We have, $R=\{(a, b):|a-b| \text{ is even}\}$, where $a, b\in A=\{1, 2, 3, 4\}$

We observe the following proposition of relation R:

Reflexivity : For any $a\in A$, we have $|a-a|=0$, which is even.

$\therefore\ (a, a)\in R$ for all $a\in A$. So, R is reflexive.

Symmetry : Let $(a, b)\in R \Rightarrow |a-b|$ is even $\Rightarrow |b-a|$ is even $\Rightarrow (b, a)\in R$

Thus, $(a, b)\in R \Rightarrow (b, a)\in R$. So, R is symmetric.

Transitivity : Let $(a, b)\in R$ and $(b, c)\in R$. Then, $(a, b)\in R$ and $(b, c)\in R$

$\Rightarrow |a-b|$ is even and $|b-c|$ is even

$\Rightarrow$ (a and b both are even or both are odd) and (b and c both are even or both are odd).

Now two cases arise :

Case I : When b is even in this case, $(a, b)\in R$ and $(b, c)\in R$

$\Rightarrow |a-b|$ is even and $|b-c|$ is even $\Rightarrow a$ is even and c is even $\quad [\because b$ is even$]$

$\Rightarrow |a-c|$ is even $\Rightarrow (a, c)\in R$

Case II : When b is odd in this case, $(a, b)\in R$ and $(b, c)\in R$

$\Rightarrow |a-b|$ is even and $|b-c|$ is even $\Rightarrow a$ is odd and c is odd $\quad [\because b$ is odd$]$

$\Rightarrow |a-c|$ is even $\Rightarrow (a, c)\in R$. Thus, $(a, b)\in R$ and $(b, c)\in R \Rightarrow (a, c)\in R$

So, R is transitive. Hence, R is an equivalence relation.

6. Show that the relation R on $N\times N$, defined by $(a, b)R(c, d)\Leftrightarrow a+d=b+c$ is an equivalence relation.

Sol. We observe the following proposition of relation R:

Reflexivity : Let (a, b) be an arbitrary element of $N\times N$

Then, $(a, b)\in N\times N \Rightarrow (a, b)\in N \Rightarrow a+b=b+a$

$\Rightarrow (a, b)R(a, b)$. Thus, $(a, b)R(a, b)$ for all $(a, b)\in N\times N$. So, R is reflexive on $N\times N$.

Symmetry : Let $(a, b), (c, d)\in N\times N$ be such that $(a, b)R(c, d)$. Then, $(a, b)R(c, d)$

$\Rightarrow a+d=b+c \Rightarrow c+b=d+a \quad$ [By commutativity of order on N]

$\Rightarrow (c, d)R(a, b)$, Thus $(a, b)R(c, d) \Rightarrow (c, d)R(a, b)$ for all $(a, b), (c, d)\in N\times N$

So, R is symmetric on $N \times N$.

Transitivity : Let $(a, b), (c, d), (e, f) \in N \times N$ such that $(a, b) R (c, d)$ and $(c, d) R (e, f)$

Then, $(a, b) R (c, d) \Rightarrow a + d = b + c$, $(c, d) R (e, f) \Rightarrow c + f = d + e$

$\Rightarrow (a + d) + (c + f) = (b + c) + (d + e) \Rightarrow a + f = b + e \Rightarrow (a, b) R (e, f)$

Thus, $(a, b) R (c, d)$ and $(c, d) R (e, f) \Rightarrow (a, b) R (e, f)$ for all $(a, b), (c, d), (e, f) \in N \times N$

So, R is transitive on $N \times N$.

Hence, R being reflexive, symmetric and transitive is an equivalence relation on $N \times N$.

7. Let S be the set of all real numbers and let $R = \{(a, b) : a, b \in S \text{ and } a = \pm b\}$

Show that R is an equivalence relation on S.

Sol. As $a = \pm b$ and $a^2 = b^2$ have same meaning.

$\therefore$ Given relation R becomes $R = \{(a, b) : a, b \in S \text{ and } a^2 = b^2\}$

Now, $(a, a) \in R \because a^2 = a^2$ is true. $\therefore R$ is reflexive and if $(a, b) \in R$

$\Rightarrow a^2 = b^2 \Rightarrow b^2 = a^2 \Rightarrow (b, a) \in R \quad \therefore R$ is symmetric

Also, if $(a, b) \in R$ and $(b, c) \in R \Rightarrow a^2 = b^2$ and $b^2 = c^2 \Rightarrow a^2 = c^2 \Rightarrow (a, c) \in R$

$\therefore R$ is transitive.

8. Let S be the set of all points in a plane and let R be a relation in S defined by $R = \{(A, B) : d(A, B) < 2 \text{ units}\}$, where $d(A, B)$ is the distance between the points A and B. Show that R is reflexive and symmetric but not transitive.

Sol. **(i) Reflexive :** $d(A, A) < 2 \Rightarrow (A, A) \in R$

(ii) Symmetric : $(A, B) \in R \Rightarrow d(A, B) < 2 \Rightarrow d(B, A) < 2 \quad [\because d(B, A) = d(A, B)]$

$\Rightarrow (B, A) \in R$

(iii) Transitive : Consider points $A(0, 0)$, $B(1.5, 0)$, $C(3, 0)$

Then, $d(A, B) = 1.5$, $d(B, C) = 1.5$ and $d(A, C) = 3$.

$\therefore d(A, C) < 2$ is not true.

Hence, R is reflexive and symmetric but not transitive.

9. Let S be the set of all real numbers. Show that the relation $R = \{(a, b) : a^2 + b^2 = 1\}$ is symmetric but neither reflexive nor transitive.

Sol. $S = \{(a, b) : a^2 + b^2 = 1\}$

Reflexive : The given relation S is on the set of real numbers.

Let $a \in R$ then $a^2 + a^2 \neq 1$ for $a = 2, 3...$ Hence, S is not reflexive.

Symmetric : Let $(a, b) \in R$ then we must have $a^2 + b^2 = 1$

$\therefore (b, a) \in R$, Hence $(a, b) \in R \Rightarrow (b, a) \in R$. $\therefore S$ is symmetric.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$.

$\because (\cos 30°, \sin 30°) \in R$ and $(\sin 30°, \cos 30°) \in R$, but $(\cos 30°, \cos 30°) \notin R$

Hence, S is not transitive. $\therefore S$ is symmetric but neither reflexive nor transitive

10. Let $R = \{(a, b) : a = b^2\}$ for all $a, b \in N$. Show that R satisfies none of reflexivity, symmetric and transitivity.

Sol. We have, $R = \{(a, b) : a = b^2\}$ where $a, b \in N$

Reflexivity : We observe that $2 \neq (2)^2 \Rightarrow 2$ is not related to 2, i.e., $(2, 2) \notin R$

So, R is not reflexive

Symmetry : We observed that $4 = (2)^2$ $(4, 2) \in R$ but $(2, 4) \notin R$ $[\because 4^2 \neq 2]$

So R is not symmetric.

Transitive : Clearly, $(16, 4) \in R$ and $(4, 2) \in R$ but $(16, 2) \notin R$. Hence not transitive.

11. Show that the relation $R = \{(a, b) : a > b\}$ on N is transitive but neither reflexive nor symmetric.

Sol. We have, $R = \{(a, b) : a > b\}$, where $a, b \in R$.

Reflexivity : For any $a \in R$, we have, $(a > b)$

$\Rightarrow (a, b) \notin R$ for all $a \notin R$ $\Rightarrow$ so is not reflexive.

Symmetric : We observe that $(3, 4) \in R$ but $(4, 3) \notin R$. So, R is not symmetric.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$. Then, $(a, b) \in R$.

$\Rightarrow a > b$ and $b > c$ $\Rightarrow a > c$ $\Rightarrow (a, c) \in R$

So, R is transitive. Hence, R is transitive but neither reflexive nor symmetric.

12. Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$. Show that R is reflexive but neither symmetric nor transitive.

Sol. Since $1, 2, 3 \in A$ and $(1, 1), (2, 2), (3, 3) \in R$ is for each $a \in A$, $(a, a) \in R$.

So, R is reflexive. We observe that $(1, 2) \in R$ but $(2, 1) \notin R$. So, R is not symmetric.

Also, $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$. So, R is not transitive.

13. Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (3, 2)\}$. Show that R is reflexive and transitive but not symmetric.

Sol. $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (3, 2)\}$

As $(1, 1), (2, 2), (3, 3), (4, 4) \in R$ is reflexive and we observe that $(1, 2) \in R$ but $(2, 1) \in R$.

So, R is not symmetric. Also, $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \in R$. So, R is transitive.