BINARY OPERATIONS (XII, R.S. AGARWAL)

EXERCISE 3-A [Pg. No. 74]

1. Let * be a binary operation on the set I of all integers, defined by $a*b=3a+4b-2$. Find the value of $4*5$

Sol. $a*b = 3a + 4b - 2$

$\Rightarrow 4*5 = 3\times 4 + 4\times 5 - 2 = 30$

2. The binary operation * on R is defined by $a*b=2a+b$. Find $(2*3)*4$

Sol. Given, $a*b = 2a + b \ \forall$ a and b $\in$ R.

Now, $(2*3)*4$

$=(2\times 2+3)*4 = 12*4 = 2\times 12 + 4 = 28$

3. Let * be a binary operation on the set of al nonzero real numbers, defined by $a*b=\frac{ab}{5}$. Find the value of x given that $2*(x*5)=10$

Sol. Given:- $a*b=\frac{ab}{5}$ $\qquad \because 2*(x*5)=10$

$\Rightarrow 2*\left(\frac{x\times 5}{5}\right)=10 \Rightarrow 2*x=10 \Rightarrow \frac{2\times x}{5}=10 \Rightarrow x=25$

4. Let $x: R\times R\to R$ be a binary operation given by $a*b=a+4b^2$. Then, compute $(-5)*(2*0)$

Sol. Given:- $a*b = a+4b^2$

Now, $(-5)*(2*0)$

$=(-5)*(2+4\times 0^2) = -5*2 = -5+4\times(-2)^2 = 11$

5. Let * be a binary operation on the set Q of all rational numbers given as $a*b=(2a-b)^2$ for all $a, b\in Q$. Find $3*5$ and $5*3$. is $3*5=5*3$?

Sol. Given :$\rightarrow a*b=(2a-b)^2$

Now, $3*5=(2\times 3-5)^2=1$

And $5*3=(2\times 5-3)^2=49$

Here, $3*5\neq 5*3$

6. Let * be a binary operation on N given by $a*b=l$ cm of a and b. Find the value of $20*16$

Is * (i) commutative (ii) associative ?

Sol. Given : $\rightarrow a*b$ = L.C.M of a and b

$\therefore 20*16$ = L.C.M of 20 and 16 = 80.

Commutatively:$\rightarrow$

Let, a and b $\in$ N

$\because$ L.C.M at a and b = L.C.M of b and a

$\Rightarrow (a*b)=(b*a)\ \forall$ a & b $\in$ N

Hence, * is commutative.

Associatively : $\rightarrow$

$(a*b)*c$

= {L.C.M of a and b} * c

= L.C.M of [{L.C.M of a and b} and C]= L.C.M of [L.C.M of a, b and c]

= L.C.M of [a and {L.C.M of b and c}]= a * { L.C.M of b and c}

= a * (b * c)

Here, a * (b * C) = (a * b) * c $\forall$ a, b $\in$ c $\in$ N Hence, * is associative.

7. If * be the binary operation on the set Z of all integers defined by $a*b=\left(a+3b^2\right)$, find 2*4

Sol. Given: $\rightarrow$ a * b = a + $3b^2$

$\therefore$ 2 * 4 = 2 + 3 $\times$ $(4)^2$ = 50

8. show that * on Z^+ defined by $a*b=|a-b|$ is not binary operation

Sol. On Z^+, * is defined by $a*b=|a-b|$, it is seen that for $a, a\in Z^+$.

$a*a=|a-a|=0\notin Z^+$, hence * is not a binary operation

9. Let * on Z^+ defined by $a*b=a^b$ is neither commutative nor associative

Sol. **Commutativity :** Let $a, b\in N$, then $a*b=a^b$ and $b*a=b^a$.

a^b and b^a are not equal for every $a, b\in N$.

$\Rightarrow a*b\neq b*a \quad \Rightarrow$ * is not commutative only.

Associativity : Let $a, b, c\in N$, then

$(a*b)*c=a^b*c=\left(a^b\right)^c=a^{bc}$ ...(1)

and $a*(b*c)=a*b^c=(a)^{b^c}$ ...(2)

From (1) & (2), $(a*b)*c\neq a*(b*c) \Rightarrow$ * is associative on N.

10. Let $a*b=1\text{cm }(a,b)$ for all values of $a, b\in N$

(i) Find $(12*16)$

(ii) show that * is commutative on N

(iii) Find the identity element in N

(iv) Find all invertible elements in N

Sol. (i) 12 * 16 = L.C.M (12, 16)= 48

(ii) Let, a & b $\in$ N

a * b = L.C.M (a, b)

= L.C.M(b, a)= b* a $\forall$ a, b $\in$ N

Hence, * is commutative.

(iii) Let, C be the identity element

$\because$ c is the identity element $\therefore$ c * a = a $\forall$ a $\in$ N

$\Rightarrow$ L.C.M (8, 9) = a$\Rightarrow$ C = 1

(iv) Let, a be an orbitoary element on N

Now, a * b = 1 {for a^{-1} = b

$\Rightarrow$ L.C.M (a, b) = 1$\Rightarrow$ a = b = 1

$\Rightarrow$ 1 is the only element in N, Which is invertible.

11 Let Q^+ be the set of all positive rational numbers

(i) show that the operation * on Q^+ defined by $a*b=\frac{1}{2}(a+b)$ is a binary operation

(ii) show that * is commutative

(iii) show that * is not associative

Sol. (i) Let, a & b $\in$ Q^+

$\Rightarrow (a+b) \in Q^+ \Rightarrow \frac{1}{2}(a+b) \in Q^+ \Rightarrow a * b \in Q^+$

Hence, * on Q^+ is a binary operation.

(ii) Let, a & b $\in Q^+$

$a * b = \frac{1}{2}(a+b) = \frac{1}{2}(b+a) = b * a \ \forall a \ \& \ b \in Q^+$

Hence, * is commutative on Q^+

(iii) $10 * (2 * 6) = 10 * \frac{1}{2}(2+6) = 10 * 4 = \frac{10+4}{2} = 7$

$(10 * 2) * 6 = \frac{1}{2}(10+2) * 6 = 6 * 6 = \frac{6+6}{2} = 6$

$\because 10 * (2 * 6) \neq (10 * 2) * 6 \ \therefore$ * is not associative.

12. Show that the set $A = (-1, 0, 1)$ is not closed for addition.

Sol. We have, $1 \in A$, $1 \in A$ and $1+1 = 2 \notin A$. Hence, A is not closed for addition.

13. * on $R-\{-1\}$, defined by $(a*b) = \frac{a}{(b+1)}$ is neither commutative nor associative

Sol. **Commutativity :** Let $a, b \in R-\{-1\}$, then $a*b = \frac{a}{b+1}$ and $b*a = \frac{b}{a+1}$

$\Rightarrow a*b \neq b*a \quad \Rightarrow$ * is not commutative on $R-\{-1\}$.

Associativity : Let $a, b, c \in R-\{-1\}$, then

$$(a*b)*c = \left(\frac{a}{b+1}\right)*c = \frac{\frac{a}{b+1}}{c+1} = \frac{a}{(b+1)(c+1)} \quad \ldots(1)$$

$$\text{and } a*(b*c) = a*\left(\frac{b}{c+1}\right) = \frac{a}{\frac{b}{c+1}+1} = \frac{a(c+1)}{b+c+1} \quad \ldots(2)$$

From (1) and (2), $(a*b)*c \neq a*(b*c) \quad \Rightarrow$ * is not associative on $R-\{-1\}$

14. For all $a, b \in R$, we defined $a*b = |a-b|$

Show that * is commutative but not associative

Sol. Let, a & b $\in$ R.

$a * b = |a - b|$

$= |b - a| = b * a$

$\therefore$ * is commutative. $(2 * 3) * 4 = |2 - 3| * 4$

$= 1 * 4 = |1 - 4| = 3$

and, $2 * (3 * 4) = 2 * (3 - 4)$

$= 2 * 1 = |2 - 1| = 1$

$\because (2 * 3) * 4 \neq 2 * (3 * 4) \ \therefore$ * is not associative

15. For all $a, b \in N$, we defined $a*b = a^3 + b^3$

Show that * is commutative but not associative

Sol. Let, a and b $\in$ N

$a * b = a^3 + b^3$

$= b * a \ \forall a, b \in N$

$\therefore$ * is commutative. $(1 * 2) * 3 = (1^3 + 2^3) * 3$

$= 9 * 3 = 9^3 + 3^3 = 729 + 27 = 756$

$1 * (2 * 3) = 1 * (2^2 + 3^2)$

$= 1 * 35 = 1^3 + 35^3 = 1 + 42875 = 42876$

$\because (1 * 2) * 3 \neq 1 * (2 * 3)$ $\therefore$ f is not associative

16. Let X be a non-empty set and * be a binary operation on $P(X)$, the power set of X, defined by $A * B = A \cap B$ for all $A, B \in P(X)$.

(i) Find the identity element in (PX) (ii) Show that X is the only invertible element in $P(X)$.

Sol. (i) Since $A \cap X = A$ for all A in $P(X)$. $\therefore$ X is the identity element.

(ii) Let A be invertible in $P(X)$ and let B be its inverse.

Then, $A \cap B = X$. This is possible only when $A = B = X$.

$\therefore$ X is the only invertible element in $P(X)$ and its inverse in X.

17. A binary operation * on the set $\{0, 1, 2, 3, 4, 5\}$ is defined as $a * b = \begin{cases} a+b; & if\ a+b<6 \\ a+b-6 & if\ a+b \geq 6 \end{cases}$

Show that 0 is the identity for this operation and each element a has an inverse $(6-a)$

Sol. Here, $a * 0 = a + 0 \begin{cases} \because a < 6 \\ \therefore a + 0 < 6 \end{cases}$

$\Rightarrow a * 0 = 0$

Hence, O is the identity element

Inverse element of a : $\rightarrow$

Let, $b = a^{-1}$

$\therefore a * b = 0$

$\Rightarrow a + b = 0$ if $a + b < 6$

or, $a + b - 6 = 0$ if $a + b \geq 6$

If, $a + b = 0 \Rightarrow a = -b$

clearly, If $a = 0$ then, $a^{-1} = 0$

If, $a + b - 6 = 0$ then, $b = 6 - a$

$\therefore a^{-1} = 6 - a, \ \forall a \in \{1, 2, 3, 4, 5\}$

EXERCISE 3-B [Pg.No. 78]

1. Define * on N by $m * n = \text{lcm}(m, n)$. Show that * is a binary operation which is commutative as well as associative.

Sol. Let m and $n \in N$, then $m * n = \text{lcm}(m, n) = \text{lcm}(n, m) = n * m$

Hence, * is commutative binary operation.

and $(m * n) * p = (\text{lcm}(m, n)) * p = $ lcm of (lcm of (m, n) and p) = lcm of (m, n, p)

and $m * (n * p) = m * (\text{lcm of } (n, p)) = $ lcm of (m and lcm of (n, p)) = lcm of (m, n, p)

$\therefore$ $(m * n) * p = m * (n * p)$. Hence, the operation is associative.

2. Define * on Z by $a * b = a - b + ab$. Show that * is a binary operation on Z which is neither commutative nor associative.

Sol. Commutativity : Let us take two elements 1 and 2 of Z.

Then, $1*2=1-2+1\times 2=1$ and $2*1=2-1+2\times 1=3$. $1*2\neq 2*1$

Hence, the binary operation is not commutative.

Associativity : $2, 3, 4\in Z$

$$(2*3)*4=(2-3+2\times 3)*4=5*4=5-4+5\times 4=21$$

and $2*(3*4)=2*(3-4+12)=2*11=2-11+2\times 11=13$

$\Rightarrow (2*3)*4\neq(2*3)*4$. Hence, binary operation is not associative.

3. Define $*$ on Z by $a*b=a+b-ab$. Show that $*$ is a binary operation on Z which is commutative as well as associative.

Sol. Let $a, b\in Z$, then by definition $a*b=a+b-ab$ and $b*a=b+a-ba$

Hence, $a*b=b*a$.

For Associativity : Let $a, b, c\in Z$.

Now, $(a*b)*c=(a+b-ab)*c=a+b-ab+c-(a+b-ab)c$

$$=a+b+c-ab-bc-ca+abc \qquad \ldots(1)$$

and $a*(b*c)=a*(b+c-bc)=a+b+c-bc-a(b+c-bc)$

$$=a+b+c-ab-bc-ca+abc \qquad \ldots(2)$$

Hence, by (1) and (2), $(a*b)*c=\vec{a}\times(\vec{b}*c)$

Hence, the binary operation on Z is associative.

4. Consider a binary operation on $Q-\{1\}$, defined by $a*b=a+b-ab$

(i) Find the identity element in $Q-\{1\}$ (ii) Show that each $a\in Q-\{1\}$ has its inverse

Sol. (i) Let e be the identity element.

Then, $a * e = a \ \forall a \in Q\{1\}$

$\Rightarrow a+e-ae=a \quad \Rightarrow e(1-a)=0 \quad \Rightarrow e=0\in Q-\{1\}$

Now, $a * 0 = a + 0 = a$

and, $0 * a = 0 + a = a$

thus, o is the identity element in $Q-\{1\}$

(ii) Let $a\in Q-\{1\}$ and Let, $a^{-1}=b$,

Now, $a * b = 0$

$$\Rightarrow a+b-ab=0 \Rightarrow a=ab-b \Rightarrow a=(a-1)\cdot b \Rightarrow b=\frac{a}{a-1}\in Q-\{1\} \Rightarrow a^{-1}=\frac{a}{a-1}\in Q-\{1\}$$

Hence, each $a\in Q-\{1\}$ has its inverse.

5. Let Q_0 be the set of all non-zero rational numbers. Let $*$ be a binary operation on Q_0, defined by $a*b=\frac{ab}{4}$ for all $a, b\in Q_0$.

(i) Show that $*$ is commutative and associative (ii) Find the identity element in Q_0

(iii) Find the inverse of an element a in Q_0

Sol. Let $a*b=\frac{ab}{4}$

(i) For all $a, b, c\in Q_0$, we have $a*b=\frac{ab}{4}=\frac{ba}{4}=b*a$ and $(a*b)*c=\frac{ab}{4}*c=\frac{\frac{ab}{4}*c}{4}=\frac{(ab)c}{16}$

Also, $a*(b*c)=a*\frac{bc}{4}=\frac{a\left(\frac{bc}{4}\right)}{4}=\frac{a(bc)}{16}$. But $(ab)c=a(bc)$. Hence, $(a*b)*c=a*(b*c)$

(ii) Let e be the identity element and let $a\in Q_0$. Then $a*e=a \Rightarrow \frac{ae}{4}=a \Rightarrow e=4$

$\therefore$ 4 is the identity element in Q.

(iii) Let $a\in Q_0$ a and let its inverse be b, then, $a*b=e \Rightarrow \frac{ab}{4}=4 \Rightarrow b=\frac{16}{a}\in Q_0$

Thus, each $a\in Q_0$ has $\frac{16}{a}$ as its inverse.

6. On the set Q^+ of all positive rational numbers, define an operation * on Q^+ by $a*b=\frac{ab}{2}$ for all $a,b\in Q^+$

Show that

(i) * is a binary operation on Q^+ (ii) * is commutative (iii) * is associative

Find the identity element in Q^+ for *

What is the inverse of $a\in Q^+$?

Sol. Let, a and b $\in Q^+$ $\because$ a and b $\in Q^+$

$\Rightarrow a\cdot b\in Q^+ \Rightarrow \frac{a\cdot b}{2}\in Q^+ \Rightarrow a*b\in Q^+$

Hence, * is binary operation on Q^+

Commutatively :→

Let, a and b $\in Q^+$

Now, $a*b=\frac{ab}{2}=\frac{ba}{2}=b*a\ \forall\ a\ \&\ b\ Q\ Q^+$

Hence, * is commutative on Q^+

Associativity:→

Let, a, b and c $\in Q^+$

Now, $a*(b*c)=a*\left(\frac{bc}{2}\right)$

$=\frac{a\left(\frac{ab}{2}\right)}{2}=\frac{abc}{4}$ and, $(a*b)*c=\frac{ab}{2}*c$

$=\frac{\frac{ab}{2}\cdot c}{2}=\frac{abc}{4}$

$\because (a*b)*c=a*(b*c), \forall\ a, b\ \&\ c\in Q^+$

Hence, * is associative on Q^+

Identity element:-

Let, $e\in Q^+$ b the identity element $\therefore a*e=a$ and $e*a=a$

$\Rightarrow \frac{a\cdot e}{2}=a$ and $\frac{e\cdot a}{2}=a \Rightarrow e=2\in Q^+$

Hence, e = 2 is the identity element on Q^+

Inverse of a :-

Let , $a^{-1} = b$

Now, $a * b = 2 \Rightarrow \frac{a \cdot b}{2} = 2 \Rightarrow b = \frac{4}{a} \Rightarrow a^{-1} = \frac{4}{a}$

7. Let Q^+ be the set of all positive rational numbers.

(i) Show that the operation * on Q^+ defined by $a * b = \frac{1}{2}(a+b)$ is a binary operation

(ii) Show that * is commutative (iii) Show that * is not associative

Sol. **(i)** On Q^+, * defined by $a * b = \frac{a+b}{2}$

It is seen that for each $a, b \in Q^+$, there is a unique element $\frac{a+b}{2}$ in Q^+.

This means that * carries each pair (a, b) to a unique element $a * b = \frac{a+b}{2}$ in Q^+.

Therefore, * is a binary operation.

(ii) Commutative : $a * b = \frac{a+b}{2} = \frac{b+a}{2} = b * a$, $a * b = b * a$, which shows * is commutative.

(iii) Associative : $(a * b) * c = \left(\frac{a+b}{2}\right) * c = \frac{\left(\frac{a+b}{2}\right) + c}{2} = \frac{a+b+2c}{4}$

$a * (b * c) = a * \left(\frac{b+c}{2}\right) = \frac{a + \left(\frac{b+c}{2}\right)}{2} = \frac{2a+b+c}{4}$

Now, $\frac{a+b+2c}{4} \neq \frac{2a+b+c}{4} \Rightarrow (a * b) * c \neq a * (b * c)$, hence, * is not associative.

8. Let Q be the set of all rational numbers. Define an operation * on $Q - \{-1\}$ by $a * b = a + b + ab$.

Show that :

(i) * is a binary operation on $Q - \{-1\}$ (ii) * is commutative

(iii) * is associative (iv)zero is the identity element in $Q - \{-1\}$ for *

(v) $a^{-1} = \left(\frac{-a}{1+a}\right)$, where $a \in Q - \{-1\}$

Sol. **(i)** On Q^+, * is defined by $a * b = a + b + ab$, it is seen that for each $a, b \in Q^+$ these is unique element $a + b + ab$ in Q^+ This means that * carries each pair (a, b) to a unique element $a * b = a + b + ab$ in Q^+. Therefore, * is a binary operation.

Note that $a + b + ab = -1$ is not possible.

$\because$ If $a + b + ab = -1 \Rightarrow (1+a)(1+b) = 0 \Rightarrow a = -1$ or $b = -1$, which is not possible.

$\because$ Both cannot be –1.

(ii) Commutativity : For any $a, b \in R - \{-1\}$,

We have $a * b = a + b + ab$ and $b * a = b + a + ba$ and as $a + b + ab = b + a + ba$

$\therefore$ $a * b = b * a$. So, * is commutative on $R - \{-1\}$.

(iii) Associativity : For any $a, b, c \in R - \{-1\}$ we have $(a * b) * c = (a + b + ab) * c$

$(a*b)*c=(a+b+ab)*c=(a+b+ab)c$

$\Rightarrow\ (a*b)*c=a+b+c+ab+bc+ac+abc$ ...(1)

and $a*(b*c)=a*(b+c+bc)$

$a*(b*c)=a+(b+c+bc)+a(b+c+bc)=a+b+c+ab+bc+ca+abc$ ...(2)

From (1) and (2), we have, $(a*b)*c=a*(b*c)$ for all $a, b, c\in R-\{-1\}$

So, * is associative on $R-\{-1\}$.

(iv) Existence of identity : Let e be the identity element. Then, $a*e=a=e*a$ for all $a\in R-\{-1\}$

$\Rightarrow a+e+ae=a$ and $e+a+ea=a$ for all $a\in R-\{-1\}$

$\Rightarrow e(1+a)=0$ for all $a\in R-\{-1\}$ $\Rightarrow$ $e=0$

Also, $o\in R-\{-1\}$. So, o is the identity element for * defined on $R-\{-1\}$.

Existence of inverse : Let $a\in R-\{-1\}$ and let b be the inverse of a.

Then, $a*b=e=b*a \Rightarrow a*b=e$

$\Rightarrow a+b+ab=0$ [$\because$ Identity element is o]

$\Rightarrow b=\frac{-a}{a+1}$, Since, $a\in R-\{-1\}$ [$\because a\neq -1 \Rightarrow a+1\neq 0$, hence, $\frac{-a}{a+1}$ is defined]

Hence, every element of $R-\{-1\}$ is invertible and the inverse of an element a is $\frac{-a}{a+1}$.

9. Let $A=N\times N$. Define * on A by $(a, b)*(c, d)=(a+c, b+d)$

Show that :

(i) A is closed for *

(ii) * is commutative

(iii) * is associative

(iv) identity element does not exist in A

Sol. (i) Let $(a, b)\in A$ and $(c, d)\in A$, then $a, b, c, d\in N$

$(a, b)*(c, d)=(a+c, b+d)\in A$ [$\because a+c\in N, b+d\in N$]

$\therefore$ A is closed for *.

(ii) Commutativity : Let $(a, b), (c, d)\in A$,

then $(a, b)*(c, d)=(a+c, b+d)$ and $(c, d)*(a, b)=(c+a, d+b)$

$\because$ $a+c=c+a$ and $b+d=d+b$ for all $a, b, c, d\in N$

$\therefore$ $(a+c, b+d)=(c+a, d+b)$ for all $a, b, c, d\in N$

$\Rightarrow (a, b)*(c, d)=(c, d)*(a, b)$ for all $(a, b), (c, d)\in N\times N=A$

(iii) Associativity : For any $(a, b), (c, d), (e, f)\in A$, we have, '*'

$\{(a, b)*(c, d)\}*(e, f)=(a+c, b+d)*(e, f)=((a+c)+e, (b+d)+f)$

$=(a+(c+e), b+(d+f))$ [$\because$ Addition is associative on N]

$=(a, b)*(c+e, d+f)=(a, b)*\{(c, d)*(e, f)\}$

So, '*' is associative on A.

(iv) Let (x, y) be the identity element in A. Then $(a, b)*(x, y)=(a, b)$ for all $(a, b)\in A$

$\Rightarrow (a+x, b+y)=(a, b)$ for all $(a, b)\in A$ $\Rightarrow a+x=a, b=b+y$ for all $a, b\in N$

$\Rightarrow x=0,\ y=0$, clearly $(0,0)\in A$. Hence, identity element does not exist in A.

10. Let $A=\{1,-1,i,-i\}$ be the set of four 4^{th} roots of unity. Prepare the composition table for multiplication on A and show that

(i) A is closed for multiplication
(ii) multiplication is associative on A
(iii) multiplication is commutative on A
(iv) 1 is the multiplicative identity
(v) every element in A has its multiplicative inverse

Sol.

•	1	–1	i	–i
1	1	–1	i	–i
–1	–1	1	–i	i
i	i	–i	–1	1
–i	–i	i	1	–1

(i) Clearly every element of table belongs to the set A. Hence A is closed for multiplication.

(ii) Clearly $a(bc)=(ab)c$ is satisfied for all $a,b,c\in A$. Hence multiplication is Associative.

(iii) Clearly the table is symmetrical about the diagonal line.
Hence multiplication is commutative for A.

(iv) As $1.1=1,\ 1.(-1)=-1,\ 1.i=i$ and $1.(-i)=-i$ and $1.1=1,\ -1.1=-1,\ i.1=i$ and $-i.1=-i$.
Hence 1 is multiplicative identity.

(v) As 1 is present in every row and columns of product.
Hence every element in A has its multiplicative inverse.