Ex 3.1

Q1

Function = Let A and B be two non-empty sets. A relation f from A to B, i.e., a sub-set of $A \times B$, is called a function (or a mapping or a map) from A to B, if

(i) for each $a \in A$ there exists $b \in B$ such that $(a, b) \in f$

(ii) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$

If $(a, b) \in f$, then 'b' is called the image of 'a' under f

If a function f is expresed as the set of ordered pairs, the domain f is the set of all first components of members of f and the range of f is the set of second components of members of f.

Q2

Function = Let A and B be two non-empty sets. Then a function 'f' from set A to set B is a rule or method or correspondence which associates elements of set A to elements of set b such that:

(i) all elements of set A are associated to element in set B.

(ii) an element of set A is associated to a unique element in set B.

In other words, a function 'f' from a set A to set B associates each element of set A to a unique element of set b.

Q3

Function is a type of relation. But in a function no two ordered pairs have the same first element. For eg: R_1 and R_2 are two relations.

Clearly, R_1 is a function, but R_2 is not a function because two ordered pairs $(1, 2)$ and $(1, 4)$ have the same first element.

This means every function is a relation but every relation is not a function.

Q4

We have,

$$f(x) = x^2 - 2x - 3$$

Now,

$$\begin{aligned} f(-2) &= (-2)^2 - 2(-2) - 3 \\ &= 4 + 4 - 3 \\ &= 5 \end{aligned}$$

$$\begin{aligned} f(-1) &= (-1)^2 - 2(-1) - 3 \\ &= 1 + 2 - 3 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(-0) &= (-0)^2 - 2 \times 0 - 3 \\ &= -3 \end{aligned}$$

$$\begin{aligned} f(1) &= (1)^2 - 2 \times 1 - 3 \\ &= 1 - 2 - 3 \\ &= -4 \end{aligned}$$

$$\begin{aligned} f(2) &= (2)^2 - 2 \times 2 - 3 \\ &= 4 - 4 - 3 \\ &= -3 \end{aligned}$$

(a) $\text{Rang}(f) = \{-4, -3, 0, 5\}$

(b) Clearly, pre-images of $6, -3$ and 5 is ϕ, $\{0,2\}$, -2 respectively.

Q5

We have,

$$f(x) = \begin{cases} 3x - 2, & x < 0 \\ 1, & x = 0 \\ 4x + 1, & x > 0 \end{cases}$$

Now,

$$f(1) = 4 \times 1 + 1 = 5,$$

$$f(-1) = 3 \times (-1) - 2 = -3 - 2 = -5,$$

$$f(0) = 1,$$

and, $f(2) = 4 \times 2 + 1 = 9$

$\therefore$ $f(1) = 5,\quad f(-1) = -5,$

$f(0) = 1,\quad f(2) = 9,$

Q6

We have,

$f(x) = x^2$ ---(i)

(a) clearly range of $f = R^+$ (set of all real numbers greater than or equal to zero)

(b) we have,

$\{x : f(x) = 4\}$

$\Rightarrow$ $f(x) = 4$ ---(ii)

Using equation (i) and equation (ii), we get

$x^2 = 4$

$\Rightarrow$ $x = \pm 2$

$\therefore$ $\{x : f(x) = 4\} = \{-2, 2\}$

(c) $\{y : f(y) = -1\}$

$\Rightarrow$ $f(y) = -1$ ---(iii)

Clearly, $x^2 \neq -1$ or $x^2 \geq 0$

$\Rightarrow$ $f(y) \neq -1$

$\therefore$ $\{y : f(y) = -1\} = \phi$

Q7

We have,

$f = R^+ \to R$

and $f(x) = \log_e x$ ---(i)

(a) Now,

$f = R^+ \to R$

$\therefore$ the image set of the domain of $f = R$

(b) Now,

$\{x : f(x) = -2\}$

$\Rightarrow$ $f(x) = -2$ ---(ii)

Using equation (i) and equation (ii), we get

$\log_e x = -2$

$\Rightarrow$ $x = e^{-2}$ $[\because \log_a b = c \Rightarrow b = a^c]$

$\therefore$ $\{x : f(x) = -2\} = \{e^{-2}\}$

(c) Now,

$f(xy) = \log_e(xy)$ $[f(x) = \log_e x]$

$= \log_e x + \log_e y$ $[\because \log mn = \log m + \log n]$

$f(x) + f(y)$

$\therefore$ $f(xy) = f(x) + f(y)$

Yes, $f(xy) = f(x) + f(y)$.

Q8

(a) we have,

$\{(x, y) = y = 3x, x \in \{1,2,3\}, y \in \{3,6,9,12\}\}$

Putting $x = 1,2,3$ in $y = 3x$, we get

$y = 3,6,9$ respectively

$\therefore \quad R = \{(1,3), (2,6), (3,9)\}$

Yes, it is a function.

(b) we have,

$\{(x, y) : y > x + 1, x = 1,2$ and $y = 2,4,6\}$

Putting $x = 1,2$ in $y > x + 1$, we get

$y > 2, y > 3$ respectively.

$\therefore \quad R = \{(1,4), (1,6), (2,4), (2,6)\}$

It is not a function from A to B because two ordered pairs in R have the same first element.

(c) we have,

$\{(x, y) = x + y = 3, x, y \in \{0,1,2,3\}\}$

Now,

$y = 3 - x$

Putting $x = 0,1,2,3$, we get

$y = 3,2,1,0$ respectively

$\therefore \quad R = \{(0,3), (1,2), (2,1), (3,0)\}$

Yes, this relation is a function.

Q9

We have,

$f : R \to R$ and $g : c \to c$

$\therefore$ Domain $(f) = R$ and Domain $(g) = c$

$\therefore$ Domain $(f) \neq$ Domain $(g) = c$

$\therefore$ $f(x)$ and $g(x)$ are not equal functions.

Q10

(i) We have,

$$f(x) = x^2$$

Range of $f(x) = R^+$ (set of all real numbers greater than or equal to zero)

$$= \{x \in R \mid x \geq 0\}$$

(ii) We have,

$$g(x) = \sin x$$

Range of $g(x) = \{x \in R : -1 \leq x \leq 1\}$

(iii) We have,

$$h(x) = x^2 + 1$$

Range of $h(x) = \{x \in R : x \geq 1\}$

Q11

(a) We have,

$$f_1 = \{(1,1),\ (2,11),\ (3,1),\ (4,15)\}$$

f_1 is a function from X to Y.

(b) We have,

$$f_2 = \{(1,1),\ (2,7),\ (3,5)\}$$

f_2 is not a function from X to Y because there is an element $4 \in x$ which is not associated to any element of Y.

(c) We have,

$$f_3 = \{(1,5),\ (2,9),\ (3,1),\ (4,5),\ (2,11)\}$$

f_3 is not a function from X to Y because an element $2 \in x$ is associated to two elements 9 and 11 in Y.

Q12

We have,

$f(x)$ = highest prime factor of x.

$\therefore \quad 12 = 3 \times 4,$

$13 = 13 \times 1,$

$14 = 7 \times 2,$

$15 = 5 \times 3,$

$16 = 2 \times 8,$

$17 = 17 \times 1$

$\therefore \quad f = \{(12,3),\ (13,13),\ (14,7),\ (15,5),\ (16,2),\ (17,17)\}$

$\therefore$ Range$(f) = \{3,13,7,5,2,17\}$

Q13

We know that,

if $f : A \to 13$

such that $y \in 3$. Then,

$f^{-1}(y) = \{x \in A : f(x) = y\}$. In other words, $f^{-1}(y)$ is the set of pre-images of y.

Let $f^{-1}\{17\} = x$. Then, $f(x) = 17$

$\Rightarrow \quad x^2 + 1 = 17$

$\Rightarrow \quad x^2 = 17 - 1 = 16$

$\Rightarrow \quad x = \pm 4$

Let $f^{-1}\{-3\} = x$. Then, $f(x) = -3$

$\Rightarrow \quad x^2 + 1 = -3$

$\Rightarrow \quad x^2 = -3 - 1 = -4$

$\Rightarrow \quad x = \sqrt{-4}$

$\therefore \quad f^{-1}\{-3\} = \theta$

Q14

We have,

$A = \{p, q, r, s\}$ and $B = \{1, 2, 3\}$

(a) Now,

$R_1 = \{(p, 1), (q, 2), (r, 1), (s, 2)\}$

R_1 is a function

(b) Now,

$R_2 = \{(p, 1), (q, 1), (r, 1), (s, 1)\}$

R_2 is a function

(c) Now,

$R_3 = \{(p, 1), (q, 2), (p, 2), (s, 3)\}$

R_3 is not a function because an element $p \in A$ is associated to two elements 1 and 2 in B.

(d) Now,

$R_4 = \{(p, 2), (q, 3), (r, 2), (s, 2)\}$

R_4 is a function.

Q15

We have,

$f(n)$ = the highest prime factor of n.

Now,

$9 = 3 \times 3,$

$10 = 5 \times 2,$

$11 = 11 \times 1,$

$12 = 3 \times 4,$

$13 = 13 \times 1$

$\therefore \quad f = \{(9, 3), (10, 5), (11, 11), (12, 3), (13, 13)\}$

Clearly, range $(f) = \{3, 5, 11, 13\}$

Q16

We have,

$$f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 3x, & 3 \le x \le 10 \end{cases}$$

and, $$g(x) = \begin{cases} x^2, & 0 \le x \le 2 \\ 3x, & 2 \le x \le 10 \end{cases}$$

Now, $f(3) = (3)^2 = 9$ and $f(3) = 3 \times 3 = 9$

and, $g(2) = (2)^2 = 4$ and $g(2) = 3 \times 2 = 6$

We observe that $f(x)$ takes unique value at each point in its domain $[0,10]$. However $g(x)$ does not takes unique value at each point in its domain $[0,10]$.

Hence, $g(x)$ is not a function.

Q17

Given $f(x) = x^2$

$$f(1.1) = 1.21$$
$$f(1) = 1$$
$$\frac{f(1.1) - f(1)}{(1.1) - 1} = \frac{1.21 - 1}{1.1 - 1}$$
$$= \frac{0.21}{0.1}$$
$$= 2.1$$

Q18

$f : X \to R$ given by $f(x) = x^3 + 1$

$f(-1) = (-1)^3 + 1 = -1 + 1 = 0$

$f(0) = (0)^3 + 1 = 0 + 1 = 1$

$f(3) = (3)^3 + 1 = 27 + 1 = 28$

$f(9) = (9)^3 + 1 = 81 + 1 = 82$

$f(7) = (7)^3 + 1 = 343 + 1 = 344$

Set of ordered pairs are $\{(-1,0), (0,1), (3,28), (9,82), (7,344)\}$

Ex 3.2

Q1

We have,

$$f(x) = x^2 - 3x + 4$$

Now,

$$f(2x+1) = (2x+1)^2 - 3(2x+1) + 4$$
$$= 4x^2 + 1 + 4x - 6x - 3 + 4$$
$$= 4x^2 - 2x + 2$$

It is given that

$$f(x) = f(2x+1)$$
$$\Rightarrow \quad x^2 - 3x + 4 = 4x^2 - 2x + 2$$
$$\Rightarrow \quad 0 = 4x^2 - x^2 - 2x + 3x + 2 - 4$$
$$\Rightarrow \quad 3x^2 + x - 2 = 0$$
$$\Rightarrow \quad 3x^2 + 3x - 2x - 2 = 0$$
$$\Rightarrow \quad 3x(x+1) - 2(x+1) = 0$$
$$\Rightarrow \quad (x+1)(3x-2) = 0$$
$$\Rightarrow \quad x + 1 = 0 \quad \text{or} \quad 3x - 2 = 0$$
$$\Rightarrow \quad x = -1 \quad \text{or} \quad x = \frac{2}{3}$$

Q2

We have,

$$f(x) = (x-a)^2(x-b)^2$$

Now,

$$f(a+b) = (a+b-a)^2(a+b-b)^2$$
$$= b^2a^2$$
$$\Rightarrow \quad f(a+b) = a^2b^2$$

Q3

We have,

$$y = f(x) = \frac{ax-b}{bx-a}$$

$$\Rightarrow \quad y = \frac{ax-b}{bx-a}$$

$$\Rightarrow \quad y(bx-a) = ax-b$$

$$\Rightarrow \quad xyb - ay = ax - b$$

$$\Rightarrow \quad xyb - ax = ay - b$$

$$\Rightarrow \quad x(by-a) = ay-b$$

$$\Rightarrow \quad x = \frac{ay-b}{by-a}$$

$$\Rightarrow \quad x = f(y)$$

Hence, proved

Q4

We have,

$$f(x) = \frac{1}{1-x}$$

Now,

$$f\{f(x)\} = f\left\{\frac{1}{1-x}\right\}$$

$$= \frac{1}{1 - \frac{1}{1-x}}$$

$$= \frac{1}{\frac{1-x-1}{1-x}}$$

$$= \frac{1-x}{-x}$$

$$= \frac{x-1}{x}$$

$$\therefore \quad f[f\{x\}] = f\left\{\frac{x-1}{x}\right\}$$

$$= \frac{1}{1 - \left(\frac{x-1}{x}\right)}$$

$$= \frac{1}{\frac{x-x+1}{x}}$$

$$= \frac{x}{1}$$

$$= x$$

$\therefore \quad f[f(x)] = x$ Hence, proved.

Q5

We have,

$$f(x) = \frac{x+1}{x-1}$$

Now,

$$f[f(x)] = f\left(\frac{x+1}{x-1}\right)$$

$$= \frac{\left(\frac{x+1}{x-1}\right)+1}{\left(\frac{x+1}{x-1}\right)-1}$$

$$= \frac{\frac{x+1+x-1}{x-1}}{\frac{x+1-1(x-1)}{x-1}}$$

$$= \frac{\frac{2x}{x-1}}{\frac{x+1-x+1}{x-1}}$$

$$= \frac{2x}{2}$$

$$= x$$

$\therefore \quad f[f(x)] = x$ Hence, proved.

Q6

We have,

$$f(x) = \begin{cases} x^2, & \text{when } x < 0 \\ x, & \text{when } 0 \le x < 1 \\ \frac{1}{x}, & \text{when } x \ge 1 \end{cases}$$

(a) $f(1/2) = \frac{1}{2}$

(b) $f(-2) = (-2)^2 = 4$

(c) $f(1) = \frac{1}{1} = 1$

(d) $f(\sqrt{3}) = \frac{1}{\sqrt{3}}$

(e) $f(\sqrt{-3}) =$ does not exist because $\sqrt{-3} \notin$ domain(f).

Q7

We have,

$$f(x) = x^3 - \frac{1}{x^3} \qquad \text{---(i)}$$

Now,

$$f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 - \frac{1}{\left(\frac{1}{x}\right)^3}$$

$$= \frac{1}{x^3} - \frac{1}{\frac{1}{x^3}}$$

$$\Rightarrow \quad f\left(\frac{1}{x}\right) = \frac{1}{x^3} - x^3 \qquad \text{---(ii)}$$

Adding equation (i) and equation (ii), we get

$$f(x) + f\left(\frac{1}{x}\right) = \left(x^3 - \frac{1}{x^3}\right) + \left(\frac{1}{x^3} - x^3\right)$$

$$= x^3 - \frac{1}{x^3} + \frac{1}{x^3} - x^3$$

$$= 0$$

$$\therefore \quad f(x) + f\left(\frac{1}{x}\right) = 0 \qquad \text{Hence, proved.}$$

Q8

We have,

$$f(x) = \frac{2x}{1+x^2}$$

Now,

$$f(\tan\theta) = \frac{2(\tan\theta)}{1+\tan^2\theta}$$

$$= \sin 2\theta \qquad \left[\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$\therefore \quad f(\tan\theta) = \sin 2\theta \qquad \text{Hence, proved.}$$

Q9

i. $: \frac{x-1}{x+1}$

$$f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}-1}{\frac{1}{x}+1} = \frac{\frac{1-x}{x}}{\frac{1+x}{x}} = \frac{1-x}{1+x} = -f(x)$$

ii. $f(x) = \frac{x-1}{x+1}$

$$f\left(-\frac{1}{x}\right) = \frac{-\frac{1}{x}-1}{-\frac{1}{x}+1} = \frac{\frac{-1-x}{x}}{\frac{-1+x}{x}} = \frac{-1-x}{-1+x} = -\frac{1}{\frac{1+x}{x-1}} = -\frac{1}{f(x)}$$

Q10

We have,

$$f(x) = (a - x^n)^{1/n},\ a > 0$$

Now,

$$f(f(x)) = f(a - x^n)^{1/n}$$
$$= \left[a - \left\{(a - x^n)^{1/n}\right\}^n\right]^{1/n}$$
$$= \left[a - (a - x^n)\right]^{1/n}$$
$$= \left[a - a + x^n\right]^{1/n}$$
$$= (x^n)^{1/n}$$
$$= (x)^{n \times \frac{1}{n}}$$
$$= x$$

$\therefore \quad f(f(x)) = x$ Hence, proved.

Q11

We have,

$$af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \qquad \text{---(i)}$$

$$\Rightarrow \quad af\left(\frac{1}{x}\right) + bf(x) = \frac{1}{\frac{1}{x}} - 5$$

$$= x - 5$$

$$\Rightarrow \quad af\left(\frac{1}{x}\right) + bf(x) = x - 5 \qquad \text{---(ii)}$$

Adding equations (i) and (ii), we get

$$af(x) + bf(x) + bf\left(\frac{1}{x}\right) + af\left(\frac{1}{x}\right) = \frac{1}{x} - 5 + x - 5$$

$$\Rightarrow \quad (a+b)f(x) + f\left(\frac{1}{x}\right)(a+b) = \frac{1}{x} + x - 10$$

$$\Rightarrow \quad f(x) + f\left(\frac{1}{x}\right) = \frac{1}{a+b}\left[\frac{1}{x} + x - 10\right] \qquad \text{---(iii)}$$

Subtracting equation (ii) from equation (i), we get

$$af(x) - bf(x) + bf\left(\frac{1}{x}\right) - af\left(\frac{1}{x}\right) = \frac{1}{x} - 5 - x + 5$$

$$\Rightarrow \quad (a-b)f(x) - f\left(\frac{1}{x}\right)(a-b) = \frac{1}{x} - x$$

$$\Rightarrow \quad f(x) - f\left(\frac{1}{x}\right) = \frac{1}{a-b}\left[\frac{1}{x} - x\right]$$

Adding equations (iii) and (iv), we get

$$2f(x) = \frac{1}{a+b}\left[\frac{1}{x} + x - 10\right] + \frac{1}{a-b}\left[\frac{1}{x} - x\right]$$

$$\Rightarrow \quad 2f(x) = \frac{(a-b)\left[\frac{1}{x} + x - 10\right] + (a+b)\left[\frac{1}{x} - x\right]}{(a+b)(a-b)}$$

$$\Rightarrow \quad 2f(x) = \frac{\frac{a}{x} + ax - 10a - \frac{b}{x} - bx + 10b + \frac{a}{x} - ax + \frac{b}{x} - bx}{a^2 - b^2}$$

$$\Rightarrow \quad 2f(x) = \frac{\frac{2a}{x} - 10a + 10b - 2bx}{a^2 - b^2}$$

$$\Rightarrow \quad f(x) = \frac{1}{a^2 - b^2} \times \frac{1}{2}\left[\frac{2a}{x} - 10a + 10b - 2bx\right]$$

$$= \frac{1}{a^2 - b^2}\left[\frac{a}{x} - 5a + 5b - bx\right]$$

$$\therefore \quad f(x) = \frac{1}{a^2 - b^2}\left[\frac{a}{x} - bx - 5a + 5b\right]$$

$$= \frac{1}{a^2 - b^2}\left[\frac{a}{x} - bx\right] - \frac{5(a-b)}{a^2 - b^2}$$

$$= \frac{1}{a^2 - b^2}\left[\frac{a}{x} - bx\right] - \frac{5(a-b)}{(a-b)(a+b)}$$

$$= \frac{1}{a^2 - b^2}\left[\frac{a}{x} - bx\right] - \frac{5}{a+b}$$

Ex 3.3

Q1

We have,

$$f(x) = \frac{1}{x}$$

Clearly, $f(x)$ assumes real values for all real values for all x except for the values of $x = 0$

Hence, Domain$(f) = R - \{0\}$

We have,

$$f(x) = \frac{1}{x-7}$$

Clearly, $f(x)$ assumes real values for all real values for all x except for the values of x satisfying $x - 7 = 0$ i.e., $x = 7$

Hence, Domain$(f) = R - \{7\}$

We have,

$$f(x) = \frac{3x-2}{x+1}$$

We observe that $f(x)$ is a rational function of x as $\frac{3x-2}{x+1}$ is a rational expression.

Clearly, $f(x)$ assumes real values for all x except for the values of x for which $x + 1 = 0$ i.e., $x = -1$

Hence, Domain $= R - \{-1\}$

We have,

$$f(x) = \frac{2x+1}{x^2-9}$$

$$= \frac{2x+1}{(x^2-3^2)}$$

$$= \frac{2x+1}{(x-3)(x+3)} \qquad [\because a^2 - b^2 = (a-b)(a+b)]$$

We observe that $f(x)$ is a rational function of x as $\frac{2x+1}{x^2-9}$ is a rational expression.

Clearly, $f(x)$ assumes real values for all x except for all those values of x for which $x^2 - 9 = 0$ i.e., $x = -3, 3$

Hence, Domain$(f) = R - \{-3, 3\}$.

We have,

$$f(x) = \frac{x^2 + 2x + 1}{x^2 - 8x + 12}$$
$$= \frac{x^2 + 2x + 1}{x^2 - 6x - 2x + 12}$$
$$= \frac{x^2 + 2x + 1}{x(x-6) - 2(x-6)}$$
$$= \frac{x^2 + 2x + 1}{(x-6)(x-2)}$$

Clearly, $f(x)$ is a rational function of x as $\frac{x^2 + 2x + 1}{x^2 - 8x + 12}$ is a rational expression in x.

We observe that $f(x)$ assumes real values for all x except for all those values of x for which $x^2 - 8x + 12 = 0$ i.e., $x = 2, 6$

$\therefore$ Domain$(f) = R - \{2, 6\}$

Q2

(i) We have,

$$f(x) = \sqrt{x-2}$$

Clearly, $f(x)$ assumes real values, if

$$x - 2 \geq 0$$

$\Rightarrow \quad x \geq 2$

$\Rightarrow \quad x \in [2, \infty)$

Hence, Domain$(f) = [2, \infty]$

(ii) We have,

$$f(x) = \frac{1}{\sqrt{x^2-1}}$$

Clearly, $f(x)$ assumes real values, if

$$x^2 - 1 > 0$$

$\Rightarrow \quad (x-1)(x+1) > 0 \qquad [\because a^2 - b^2 = (a-b)(a+b)]$

$\Rightarrow \quad x < -1$ or $x > 1$

$\Rightarrow \quad x \in (-\infty, -1) \cup (1, \infty)$

Hence, domain$(f) = (-\infty, -1) \cup (1, \infty)$

(iii) We have,

$$f(x) = \sqrt{9-x^2}$$

Clearly, $f(x)$ assumes real values, if

$$9 - x^2 \geq 0$$

$\Rightarrow \quad 9 \geq x^2$

$\Rightarrow \quad x^2 \leq 9$

$\Rightarrow \quad -3 \leq x \leq 3$

$\Rightarrow \quad x \in [-3, 3]$

Hence, domain$(f) = [-3, 3]$

(iv) We have,

$$f(x) = \sqrt{\frac{x-2}{3-x}}$$

Clearly, $f(x)$ assumes real values, if

$x - 2 \geq 0$ and $3 - x > 0$

$\Rightarrow$ $x \geq 2$ and $3 > x$

$\Rightarrow$ $x \in [2, 3]$

Hence, domain$(f) = [2, 3)$.

Q3

We have,

$$f(x) = \frac{ax+b}{bx-a}$$

We observe that $f(x)$ is a rational function of x as $\frac{ax+b}{bx-a}$ is a rational expression.

Clearly, $f(x)$ assumes real values for all x except for the values of x for which

$bx - a = 0$ i.e., $bx = a$

$\Rightarrow \quad x = \frac{a}{b}$

$\therefore$ Domain$(f) = R - \left\{\frac{a}{b}\right\}$

Range of f: Let $f(x) = y$

$\Rightarrow \quad \frac{ax+b}{bx-a} = y$

$\Rightarrow \quad ax + b = y(bx - a)$

$\Rightarrow \quad ax + b = bxy - ax$

$\Rightarrow \quad b + ay = bxy - ax$

$\Rightarrow \quad b + ay = x(by - a)$

$\Rightarrow \quad \frac{b+ay}{b-ay} = x$

$\Rightarrow \quad x = \frac{b+ay}{by-a}$

Clearly, x will take real value for all $x \in R$ except for

$by - a = 0$

$\Rightarrow \quad by = a$

$\Rightarrow \quad y = \frac{a}{b}$

$\therefore$ Range$(f) = R - \left\{\frac{a}{b}\right\}$.

We have,

$$f(x) = \frac{ax-b}{cx-d}$$

We observe that $f(x)$ is a rational function of x as $\frac{ax-b}{cx-d}$ is a rational expression.

Clearly, $f(x)$ assumes real values for all x except for all those values of x for which

$cx - d = 0$ i.e., $cx = d$

$\Rightarrow \quad x = \frac{d}{c}$

$\therefore$ Domain$(f) = R - \left\{\frac{d}{c}\right\}$

Range: Let $f(x) = y$

$\Rightarrow \quad \frac{ax-b}{cx-d} = y$

$\Rightarrow \quad ax - b = y(cx - d)$

$\Rightarrow \quad ax - b = cxy - dy$

$\Rightarrow \quad dy - b = cxy - 9x$

$\Rightarrow \quad dy - b = x(cy - a)$

$\Rightarrow \quad \frac{dy-b}{cy-a} = x$

Clearly, x assumes real values for all y except

$$cy - a = 0 \text{ i.e., } y = \frac{a}{c}$$

Hence, range$(f) = R - \left\{\frac{a}{c}\right\}$

We have,

$$f(x) = \sqrt{x-1}$$

Clearly, $f(x)$ assumes real values, if

$$x - 1 \geq 0$$

$\Rightarrow \quad x \geq 1$

$\Rightarrow \quad x \in [1, \infty)$

Hence, domain$(f) = [1, \infty)$

Range: For $x \geq 1$, we have,

$$x - 1 \geq 0$$

$\Rightarrow \quad \sqrt{x-1} \geq 0$

$\Rightarrow \quad f(x) \geq 0$

Thus, $f(x)$ takes all real values greater than zero.

Hence, range$(f) = [0, \infty)$

We have,

$$f(x) = \sqrt{x-3}$$

Clearly, $f(x)$ assumes real values, if

$$x - 3 \geq 0$$

$\Rightarrow \quad x \geq 3$

$\Rightarrow \quad x \in [3, \infty)$

Hence, domain$(f) = [3, \infty)$

Range: For $x \geq 3$, we have,

$$x - 3 \geq 0$$

$\Rightarrow \quad \sqrt{x-3} \geq 0$

$\Rightarrow \quad f(x) \geq 3$

Thus, $f(x)$ takes all real values greater than zero.

Hence, range$(f) = [0, \infty)$

We have,

$$f(x) = \frac{x-2}{2-x}$$

Domain of f: Clearly, $f(x)$ is defined for all $x \in R$ except for which

$$2 - x \neq 0 \text{ i.e., } x \neq 2$$

Hence, domain$(f) = R - \{2\}$

Range of f: Let $f(x) = y$

$\Rightarrow \quad \frac{x-2}{2-x} = y$

$\Rightarrow \quad \frac{-1(2-x)}{2-x} = y$

$\Rightarrow \quad -1 = y$

$\Rightarrow \quad y = -1$

$\therefore$ Range$(f) = \{-1\}$

We have,

$$f(x) = |x-1|$$

Clearly, $f(x)$ is defined for all $x \in R$

$\Rightarrow \quad$ Domain$(f) = R$

Range: Let $f(x) = y$

$\Rightarrow \quad |x-1| = y$

$\Rightarrow \quad f(x) \geq 0 \; \forall \; x \in R$

It follows from the above relation that y takes all real values greater or equal to zero.

$\therefore$ Range$(f) = [0, \infty)$

As |x| is defined for all real numbers, its domain is R and range is only negative numbers because, |x| is always positive real number for all real numbers and -|x| is always negative real numbers.

In order to have F(x) has defined value, term inside square root should always be greater than or equal to zero which gives domain as $-3 \leq x \leq 3$

Where as Range of above function is limited to [0 , 3]

Ex 3.4

Q1

We have,

$f(x) = x^3 + 1$ and $g(x) = x + 1$

Now,

$f + g : R \to R$ given by $(f+g)(x) = x^3 + x + 2$

$f - g : R \to R$ given by $(f-g)(x) = x^3 + 1 - (x+1)$

$= x^3 - x$

$cf : R \to R$ given by $(cf)(x) = c(x^3 + 1)$

$fg : R \to R$ given by $(fg)(x) = (x^3+1)(x+1)$

$= x^4 + x^3 + x + 1$

$\frac{1}{f} : R - \{-1\} \to \text{R}$ given by $\left(\frac{1}{f}\right)(x) = \frac{1}{x^3+1}$

$\frac{f}{g} : R - \{-1\} \to \text{R}$ given by $\left(\frac{f}{g}\right)(x) = \frac{(x+1)(x^2-x+1)}{x+1}$

$= x^2 - x + 1$

We have,

$f(x) = \sqrt{x-1}$ and $g(x) = \sqrt{x+1}$

Now,

$f + g : (1, \infty) \to R$ defined by $(f+g)(x) = \sqrt{x-1} + \sqrt{x+1}$,

$f - g : (1, \infty) \to R$ defined by $(f-g)(x) = \sqrt{x-1} - \sqrt{x+1}$,

$cf : (1, \infty) \to R$ defined by $(cf)(x) = c\sqrt{x-1}$,

$fg : (1, \infty) \to R$ defined by $(fg)(x) = (\sqrt{x-1})(\sqrt{x+1})$

$= \sqrt{x^2 - 1}$

$\frac{1}{f} : (1, \infty) \to \text{R}$ defined by $\left(\frac{1}{f}\right)(x) = \frac{1}{\sqrt{x-1}}$

$\frac{f}{g} : (1, \infty) \to \text{R}$ defined by $\left(\frac{f}{g}\right)(x) = \sqrt{\frac{x-1}{x+1}}$

Q2

We have,

$f(x) = 2x + 5$ and $g(x) = x^2 + x$

We observe that $f(x) = 2x + 5$ is defined for all $x \in R$.

So, domain$(f) = R$

Clearly $g(x) = x^2 + x$ is defined for all $x \in R$

So, domain $(g) = R$

$\therefore$ Domain$(f) \cap$ Domain$(g) = R$

(i) Clearly, $(f+g) : R \to R$ is given by

$$\begin{aligned}(f+g)(x) &= f(x) + g(x)\\ &= 2x + 5 + x^2 + x\\ &= x^2 + 3x + 5\end{aligned}$$

Domain$(f+g) = R$

(ii) We find that $f - g : R \to R$ is defined as

$$\begin{aligned}(f-g)(x) &= f(x) - g(x)\\ &= 2x + 5 - (x^2 + x)\\ &= 2x + 5 - x^2 - x\\ &= -x^2 + x + 5\end{aligned}$$

Domain$(f-g) = R$

(iii) We find that $fg : R \to R$ is given by

$$\begin{aligned}(fg)(x) &= f(x) \times g(x)\\ &= (2x+5) \times (x^2 + x)\\ &= 2x^3 + 2x^2 + 5x^2 + 5x\\ &= 2x^3 + 7x^2 + 5x\end{aligned}$$

$$\text{Domain}(fg) = R$$

(iv) We have,

$$g(x) = x^2 + x$$

$\therefore \quad f(x) = 0 \Rightarrow x^2 + x = 0$

$\Rightarrow \quad x(x+1) = 0$

$\Rightarrow \quad x = 0 \text{ or, } x = -1$

So, $\text{domain}\left(\frac{f}{g}\right) = \text{domain}(f) \cap \text{domain}(g) - \{x : g(x) = 0\}$

$$= R - \{-\phi, 0\}$$

We find that, $\frac{f}{g} : R - \{-1, 0\} \to R$ is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x+5}{x^2+x}$

$$\text{Domain}\left(\frac{f}{g}\right) = R - \{-1, 0\}$$

Q3

We have,

$$f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x - 1, & 0 < x \le 2 \end{cases}$$

Now,

$$f(|x|) = |x| - 1, \text{ where } -2 \le x \le 2$$

and $$|f(x)| = \begin{cases} 1, & -2 \le x \le 0 \\ -(x-1), & 0 \le x \le 1 \\ (x-1), & 1 \le x \le 2 \end{cases}$$

$\therefore \quad g(x) = f(|x|) + |f(x)|$

$$= \begin{cases} -x & -2 \le x \le 0 \\ 0, & 0 < x < 1 \\ 2(x-1), & 1 \le x \le 2 \end{cases}$$

Q4

We have,

$f(x) = \sqrt{x+1}$ and $g(x) = \sqrt{9-x^2}$

We observe that $f(x) = \sqrt{x+1}$ is defined for all $x \geq -1$

So, $\text{domain}(f) = [-1, \infty]$

Clearly, $g(x) = \sqrt{9-x^2}$ is defined for

$9 - x^2 \geq 0 \Rightarrow x^2 - 9 \leq 0$

$\Rightarrow \quad x^2 - 3^2 \leq 0$

$\Rightarrow \quad (x-3)(x+3) \leq 0$

$\Rightarrow \quad x \in [-3, 3]$

$\therefore \quad \text{domain}(g) = [-3, 3]$

Now,

$$\text{domain}(f) \cap \text{domain}(g) = [-1, \infty] \cap [-3, 3] = [-1, 3]$$

$f + g : [-1, 3] \rightarrow R$ is given by $(f+g)(x) = f(x) + g(x) = \sqrt{x+1} + \sqrt{9-x^2}$

We have,

$f(x) = \sqrt{x+1}$ and $g(x) = \sqrt{9-x^2}$

We observe that $f(x) = \sqrt{x+1}$ is defined for all $x \geq -1$

So, $\text{domain}(f) = [-1, \infty]$

Clearly, $g(x) = \sqrt{9-x^2}$ is defined for

$9 - x^2 \geq 0 \Rightarrow x^2 - 9 \leq 0$

$\Rightarrow \quad x^2 - 3^2 \leq 0$

$\Rightarrow \quad (x-3)(x+3) \leq 0$

$\Rightarrow \quad x \in [-3, 3]$

$\therefore \quad \text{domain}(g) = [-3, 3]$

Now,

$$\text{domain}(f) \cap \text{domain}(g) = [-1, \infty] \cap [-3, 3] = [-1, 3]$$

$g - f : [-, 3] \rightarrow R$ is given by $(g-f)(x) = g(x) - f(x) = \sqrt{9-x^2} - \sqrt{x+1}$

We have,

$$f(x) = \sqrt{x+1} \text{ and } g(x) = \sqrt{9-x^2}$$

We observe that $f(x) = \sqrt{x+1}$ is defined for all $x \geq -1$

So, $\text{domain}(f) = [-1, \infty)$

Clearly, $g(x) = \sqrt{9-x^2}$ is defined for

$$9 - x^2 \geq 0 \Rightarrow x^2 - 9 \leq 0$$

$\Rightarrow \quad x^2 - 3^2 \leq 0$

$\Rightarrow \quad (x-3)(x+3) \leq 0$

$\Rightarrow \quad x \in [-3, 3]$

$\therefore \quad \text{domain}(g) = [-3, 3]$

Now,

$$\text{domain}(f) \cap \text{domain}(g) = [-1, \infty) \cap [-3, 3]$$
$$= [-1, 3]$$

$fg : [-, 3] \rightarrow R$ is given by $(fg)(x) = f(x) \times g(x) = \sqrt{x+1} \times \sqrt{9-x^2}$

$$= \sqrt{9 + 9x - x^2 - x^3}$$

We have,

$$f(x) = \sqrt{x+1} \text{ and } g(x) = \sqrt{9-x^2}$$

We observe that $f(x) = \sqrt{x+1}$ is defined for all $x \geq -1$

So, $\text{domain}(f) = [-1, \infty]$

Clearly, $g(x) = \sqrt{9-x^2}$ is defined for

$$9 - x^2 \geq 0 \Rightarrow x^2 - 9 \leq 0$$

$\Rightarrow \quad x^2 - 3^2 \leq 0$

$\Rightarrow \quad (x-3)(x+3) \leq 0$

$\Rightarrow \quad x \in [-3, 3]$

$\therefore \quad \text{domain}(g) = [-3, 3]$

Now,

$$\text{domain}(f) \cap \text{domain}(g) = [-1, \infty] \cap [-3, 3]$$
$$= [-1, 3]$$

We have, $g(x) = \sqrt{9-x^2}$

$\therefore \quad 9 - x^2 = 0 \Rightarrow x^2 - 9 = 0$

$\Rightarrow \quad (x-3)(x+3) = 0$

$\Rightarrow \quad x = \pm 3$

So, $\text{domain}\left(\frac{f}{g}\right) = [-1, 3] - [-3, 3] = [-1, 3]$

$\therefore \quad \frac{f}{g} : [-1, 3] \rightarrow R$ is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x+1}}{\sqrt{9-x^2}}$

We have,

$$f(x) = \sqrt{x+1} \text{ and } g(x) = \sqrt{9 - x^2}$$

We observe that $f(x) = \sqrt{x+1}$ is defined for all $x \geq -1$

So, $\text{domain}(f) = [-1, \infty]$

Clearly, $g(x) = \sqrt{9 - x^2}$ is defined for

$$9 - x^2 \geq 0 \Rightarrow x^2 - 9 \leq 0$$

$\Rightarrow \quad x^2 - 3^2 \leq 0$

$\Rightarrow \quad (x-3)(x+3) \leq 0$

$\Rightarrow \quad x \in [-3, 3]$

$\therefore \quad \text{domain}(g) = [-3, 3]$

Now,

$$\text{domain}(f) \cap \text{domain}(g) = [-1, \infty] \cap [-3, 3] = [-1, 3]$$

We have,

$$f(x) = \sqrt{x+1}$$

$\therefore \quad \sqrt{x+1} = 0$

$\Rightarrow \quad x + 1 = 0$

$\Rightarrow \quad x = -1$

So, $\text{domain}\left(\frac{g}{f}\right) = [-1, 3] - \{-1\}$

$$= [-1, 3]$$

$\therefore \quad \frac{g}{f} : [-1, 3] \to R$ is given by $\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{9 - x^2}}{\sqrt{x+1}}$

We have,

$$f(x)=\sqrt{x+1} \text{ and } g(x)=\sqrt{9-x^2}$$

We observe that $f(x)=\sqrt{x+1}$ is defined for all $x \ge -1$

So, $\text{domain}(f)=[-1,\infty]$

Clearly, $g(x)=\sqrt{9-x^2}$ is defined for

$$9-x^2 \ge 0 \Rightarrow x^2-9 \le 0$$

$\Rightarrow \quad x^2-3^2 \le 0$

$\Rightarrow \quad (x-3)(x+3) \le 0$

$\Rightarrow \quad x \in [-3,3]$

$\therefore \quad \text{domain}(g)=[-3,3]$

Now,

$$\text{domain}(f) \cap \text{domain}(g)=[-1,\infty] \cap [-3,3] = [-1,3]$$

$2f-\sqrt{5}g : [-,3] \to R$ defined by $(2f-\sqrt{5}g)(x)=2\sqrt{x+1}-\sqrt{5}\sqrt{9-x^2}$

$$=2\sqrt{x+1}-\sqrt{45-5x^2}.$$

We have,

$$f(x)=\sqrt{x+1} \text{ and } g(x)=\sqrt{9-x^2}$$

We observe that $f(x)=\sqrt{x+1}$ is defined for all $x \ge -1$

So, $\text{domain}(f)=[-1,\infty]$

Clearly, $g(x)=\sqrt{9-x^2}$ is defined for

$$9-x^2 \ge 0 \Rightarrow x^2-9 \le 0$$

$\Rightarrow \quad x^2-3^2 \le 0$

$\Rightarrow \quad (x-3)(x+3) \le 0$

$\Rightarrow \quad x \in [-3,3]$

$\therefore \quad \text{domain}(g)=[-3,3]$

Now,

$$\text{domain}(f) \cap \text{domain}(g)=[-1,\infty] \cap [-3,3] = [-1,3]$$

$f^2+7f : [-1,\infty] \to R$ defined by $(f^2+7f)(x)=f^2(x)+7f(x)$ $\quad [\because D(f)=[-1,\infty]]$

$$=\left(\sqrt{x+1}\right)^2+7\sqrt{x+1}$$

$$=x+1+7\sqrt{x+1}$$

We have,

$$f(x) = \sqrt{x+1} \text{ and } g(x) = \sqrt{9-x^2}$$

We observe that $f(x) = \sqrt{x+1}$ is defined for all $x \geq -1$

So, $\text{domain}(f) = [-1, \infty]$

Clearly, $g(x) = \sqrt{9-x^2}$ is defined for

$$9 - x^2 \geq 0 \Rightarrow x^2 - 9 \leq 0$$

$\Rightarrow \quad x^2 - 3^2 \leq 0$

$\Rightarrow \quad (x-3)(x+3) \leq 0$

$\Rightarrow \quad x \in [-3, 3]$

$\therefore \quad \text{domain}(g) = [-3, 3]$

Now,

$$\text{domain}(f) \cap \text{domain}(g) = [-1, \infty] \cap [-3, 3]$$
$$= [-1, 3]$$

We have,

$$g(x) = \sqrt{9-x^2}$$

$\therefore \quad 9 - x^2 = 0 \Rightarrow x^2 - 9 = 0$

$\Rightarrow \quad (x-3)(x+3) = 0$

$\Rightarrow \quad x = \pm 3$

So, $\text{domain}\left(\frac{1}{g}\right) = [-3, 3] - \{-3, 3\}$

$$= (-3, 3)$$

$\therefore \quad \frac{5}{g} = (-3, 3) \rightarrow R$ defined by $\left(\frac{5}{g}\right)(x) = \frac{5}{\sqrt{9-x^2}}$

Q5

We have,

$f(x) = \log_e(1-x)$

and $g(x) = [x]$

$f(x) = \log_e(1-x)$ is defined, if $1 - x > 0$

$\Rightarrow \quad 1 > x$

$\Rightarrow \quad x < 1$

$\Rightarrow \quad x \in (-\infty, 1)$

$\therefore$ Domain$(f) = (-\infty, 1)$

$g(x) = [x]$ is defined for all $x \in R$

$\therefore$ Domain$(g) = R$

$\therefore$ Domain$(f) \cap R$ Domain$(g) = (-\infty, 1) \cap R$

$= (-\infty, 1)$

(i) $f + g : (-\infty, 1) \to R$ defined by $(f+g)(x) = f(x) + g(x)$

$= \log_e(1-x) + [x]$

(ii) $fg : (-\infty, 1) \to R$ defined by $(fg)(x) = f(x) \times g(x)$

$= \log_e(1-x) \times [x]$

$= [x] \log_e(1-x)$

(iii) $g(x) = [x]$

$\therefore \quad [x] = 0$

$\Rightarrow \quad x \in (0, 1)$

So, domain$\left(\frac{f}{g}\right)$ = domain$(f) \cap$ domain$(g) - \{x : g(x) = 0\}$

$= (-\infty, 0)$

$\therefore \quad \frac{f}{g} : (-\infty, 0) \to R$ defined by $\left(\frac{f}{g}\right)(x) = \frac{\log_e(1-x)}{[x]}$

(iv) We have,

$$f(x) = \log_e(1-x)$$

$$\Rightarrow \quad \frac{1}{f(x)} = \frac{1}{\log_e(1-x)}$$

$\therefore \quad \frac{1}{f(x)}$ is defined if $\log_e(1-x)$ is defined and $\log_e(1-x) \neq 0$

$$\Rightarrow \quad 1-x > 0 \quad \text{and} \quad 1-x \neq 0$$

$$\Rightarrow \quad x < 1 \quad \text{and} \quad x \neq 0$$

$$\Rightarrow \quad x \in (-\infty, 0) \cup (0,1)$$

$$\therefore \quad \text{domain}\left(\frac{g}{f}\right) = (-\infty, 0) \cup (0,1)$$

$\frac{g}{f} : (-\infty, 0) \cup (0,1) \to R$ defined by $\left(\frac{g}{f}\right)(x) = \frac{[x]}{\log_e(1-x)}$

Now,

$$(f+g)(-1) = f(-1) + g(-1)$$

$$= \log_e(1-(-1)) + [-1]$$

$$= \log_e 2 - 1$$

$$\Rightarrow \quad (f+g)(-1) = \log_e 2 - 1$$

(v) $fg(0) = \log_e(1-0) \times [0]$

$$= 0$$

(vi) $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$ = does not exist

(vii) $\left(\frac{g}{f}\right)\left(\frac{1}{2}\right) = \dfrac{\left[\frac{1}{2}\right]}{\log_e\left(1-\frac{1}{2}\right)} = 0$

Q6

We have,

$$f(x) = \sqrt{x+1},\ g(x) = \frac{1}{x}$$

and $h(x) = 2x^2 - 3$

Clearly, $f(x)$ is defined for $x + 1 \geq 0$

$\Rightarrow \quad x \geq -1$

$\Rightarrow \quad x \in [-1, \infty]$

$\therefore$ Domain$(f) = [-1, \infty]$

$g(x)$ is defined for $x \neq 0$

$\Rightarrow \quad x \in R - \{0\}$

and, $h(x)$ is defined for all $x \in R$

$\therefore$ Domain$(f) \cap$ Domain$(g) \cap$ Domain$(h) = [-1, \infty] - \{0\}$

Clearly,

$2f + g - h : [-1, \infty] - \{0\} \to R$ is given by

$$(2f + g - h)(x) = 2f(x) + g(x) - h(x)$$

$$= 2\sqrt{x+1} + \frac{1}{x} - 2x^2 + 3$$

$$\therefore \quad (2f + g - h)(1) = 2\sqrt{1+1} + \frac{1}{1} - 2 \times (1)^2 + 3$$

$$= 2\sqrt{2} + 1 - 2 + 3$$

$$= 2\sqrt{2} + 4 - 2$$

$$= 2\sqrt{2} + 2$$

and, $(2f + g - h)(0)$ does not exist, it is not lies in the domain $x \in [-1, \infty] - \{0\}$.

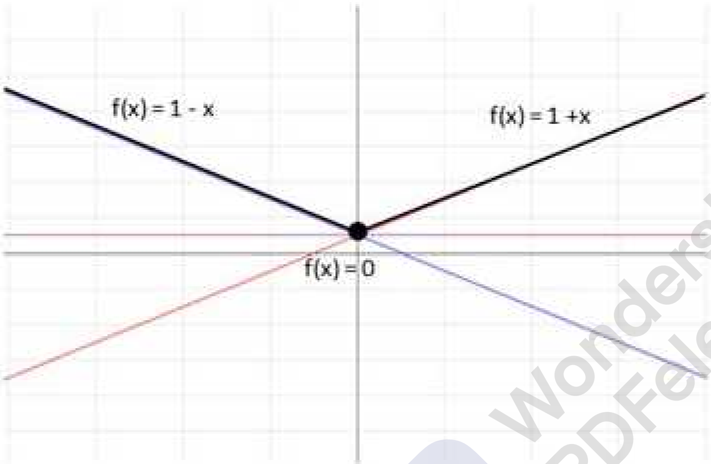
Q7

Let,

$$y = f(x) = \begin{cases} 1 - x, & x < 0 \\ 1, & x = 0 \\ x + 1, & x > 0 \end{cases}$$

The graph of f(x) for x <0 is the part of the line y = 1-x that lies to the left of origin.
The graph of f(x) for x >0 is the part of the line y = 1+x that lies to the right of origin.
For x = 0, the graph of f(x) represents the point (0,1)

The graph of f(x) is shown below.



Q8

$f: R \to R$ *defined by* $(f+g)(x) = 3x - 2$

$f: R \to R$ *defined by* $(f-g)(x) = -x + 4$

$f: R - \left\{\frac{3}{2}\right\} \to R$ *defined by* $\frac{f}{g}(x) = \frac{x+1}{2x-3}$

Q9

$f+g:[0,\infty) \to R$ *defined by* $(f+g)(x) = \sqrt{x} + x;$

$f-g:[0,\infty) \to R$ *defined by* $(f-g)(x) = \sqrt{x} - x;$

$fg:[0,\infty) \to R$ *defined by* $(fg)(x) = x^{3/2};$

$\frac{f}{g}:[0,\infty) \to R$ *defined by* $\left(\frac{f}{g}\right)(x) = \frac{1}{\sqrt{x}};$

Q10

$(f+g): R \to [0, \infty)$ defined by $(f+g)(x) = x^2 + 2x + 1 = (x+1)^2$

$(f-g): R \to R$ defined by $(f-g)(x) = x^2 - 2x - 1$

$(fg): R \to R$ defined by $(fg)(x) = 2x^3 + x^2$

$\left(\frac{f}{g}\right): R \to R$ defined by $\left(\frac{f}{g}\right)(x) = \frac{x^2}{2x+1}$