Ex 4.1

Inverse Trigonometric Functions Ex 4.1 Q1.

Let $\tan^{-1}\left(-\sqrt{3}\right) = y$. Then, $\tan y = -\sqrt{3} = -\tan\frac{\pi}{3} = \tan\left(-\frac{\pi}{3}\right)$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{3}\right)$ is $-\sqrt{3}$.

Therefore, the principal value of $\tan^{-1}\left(\sqrt{3}\right)$ is $-\frac{\pi}{3}$.

Concept Insight:
The range for $\tan^{-1}$ is same as $\sin^{-1}$ except that it is an open interval, as $\tan(-\pi/2)$ and $\tan(\pi/2)$ are not defined. So the method of finding principal value is same as $\sin^{-1}$ given in the first problem. Also note that $\tan(-x) = -\tan x$.

Let $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = y$. Then, $\cos y = -\frac{1}{\sqrt{2}} = -\cos\left(\frac{\pi}{4}\right) = \cos\left(\pi-\frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$.

We know that the range of the principal value branch of $\cos^{-1}$ is $[0,\pi]$ and $\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$.

Let $\text{cosec}^{-1}\left(-\sqrt{2}\right) = y$. Then, $\text{cosec}\, y = -\sqrt{2} = -\text{cosec}\left(\frac{\pi}{4}\right) = \text{cosec}\left(-\frac{\pi}{4}\right)$.

We know that the range of the principal value branch of cosec^{-1} is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$ and $\text{cosec}\left(-\frac{\pi}{4}\right) = -\sqrt{2}$.

Therefore, the principal value of $\text{cosec}^{-1}\left(-\sqrt{2}\right)$ is $-\frac{\pi}{4}$.

We know that for any $x \in [-1, 1]$, $\cos^{-1} x$ represents angle in $[0, \pi]$

$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ = an angle in $[0, \pi]$ whose cosine is $\left(-\frac{\sqrt{3}}{2}\right)$

$$= \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\therefore \quad \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

We know that, for any $x \in R$, $\tan^{-1} x$ represents an angle in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x.

So,

$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ = An angle in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ whose tangest is $\frac{1}{\sqrt{3}}$

$$= \frac{\pi}{6}$$

$$\therefore \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

We know that, for $x \in R$, $sec^{-1}x$ represents an angle in $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$sec^{-1}\left(-\sqrt{2}\right)$ = An angle in $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ whose secant is $\left(-\sqrt{2}\right)$

$$= \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

$$sec^{-1}\left(-\sqrt{2}\right) = \frac{3\pi}{4}.$$

We know that, for any $x \in R$, $cot^{-1}x$ represents an angle in $(0, \pi)$

$cot^{-1}\left(-\sqrt{3}\right)$ = An angle in $(0, \pi)$ whose contangent is $\left(-\sqrt{3}\right)$

$$= \pi - \frac{\pi}{6}$$

$$= \frac{5\pi}{6}$$

$$\therefore cot^{-1}\left(-\sqrt{3}\right) = \frac{5\pi}{6}.$$

We know that, for any $x \in R$, $\sec^{-1} x$ represents an angle in $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$\sec^{-1}(2)$ = An angle is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ whose secant is 2

$$= \frac{\pi}{3}$$

$$\therefore \sec^{-1}(2) = \frac{\pi}{3}.$$

We know that, for any $x \in R$. $\text{cosec}^{-1}x$ is an angle in $\left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

$cosec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ = An angle is $\left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ whose cosecant is $\left(\frac{2}{\sqrt{3}}\right)$

$$= \frac{\pi}{3}$$

$$\therefore cosec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{3}.$$

Inverse Trigonometric Functions Ex 4.1 Q2.

Let $\cos^{-1}\left(\frac{1}{2}\right)=x$. Then, $\cos x=\frac{1}{2}=\cos\left(\frac{\pi}{3}\right)$.

$\therefore \cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$

Let $\sin^{-1}\left(\frac{1}{2}\right)=y$. Then, $\sin y=\frac{1}{2}=\sin\left(\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$

$\therefore \cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}+\frac{2\pi}{6}=\frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}$

$\therefore \sin^{-1}\frac{1}{2}=\frac{\pi}{6}$

$\therefore \tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]=\tan^{-1}\left[2\cos\left(2\times\frac{\pi}{6}\right)\right]$

$=\tan^{-1}\left[2\cos\frac{\pi}{3}\right]=\tan^{-1}\left[2\times\frac{1}{2}\right]$

$=\tan^{-1}1=\frac{\pi}{4}$

Concept Insight:

Solve the innermost bracket first, so first find the principal value of $\sin^{-1}(1/2)$

Let $\tan^{-1}(1)=x$. Then, $\tan x=1=\tan\frac{\pi}{4}$.

$\therefore \tan^{-1}(1)=\frac{\pi}{4}$

Let $\cos^{-1}\left(-\frac{1}{2}\right)=y$. Then, $\cos y=-\frac{1}{2}=-\cos\left(\frac{\pi}{3}\right)=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\left(\frac{2\pi}{3}\right)$.

$\therefore \cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$

Let $\sin^{-1}\left(-\frac{1}{2}\right)=z$. Then, $\sin z=-\frac{1}{2}=-\sin\left(\frac{\pi}{6}\right)=\sin\left(-\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

$\therefore \tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

$=\frac{\pi}{4}+\frac{2\pi}{3}-\frac{\pi}{6}$

$=\frac{3\pi+8\pi-2\pi}{12}=\frac{9\pi}{12}=\frac{3\pi}{4}$

$\tan^{-1}\left(\sqrt{3}\right) = \text{Angle in } \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \text{ whose tangent is } \sqrt{3}$

$= \frac{\pi}{3}$

$\sec^{-1}(-2) = \text{An angle in } [0, \pi] - \left\{\frac{\pi}{2}\right\} \text{ whose secant is } (-2)$

$= \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

$\operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right) = \text{An angle in } \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\} \text{ whose cosecant is } \left(\frac{2}{\sqrt{3}}\right)$

$= \frac{\pi}{3}$

Hence,

$\tan^{-1}\sqrt{3} - \sec^{-1}(-2) + \operatorname{cosec}^{-1}\frac{2}{\sqrt{3}}$

$= \frac{\pi}{3} - \frac{2\pi}{3} + \frac{\pi}{3}$

$= 0$

$\therefore \tan^{-1}\sqrt{3} - \sec^{-1}\left(-\sqrt{2}\right) + \operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right) = 0$

Let $\cos^{-1}\left(\frac{1}{2}\right) = x$. Then, $\cos x = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$

$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Let $\sin^{-1}\left(-\frac{1}{2}\right) = y$. Then, $\sin y = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right)$

$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\therefore \cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{3} - \left(-\frac{2\pi}{6}\right) \quad = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

Inverse Trigonometric Functions Ex 4.1 Q3.

Let $\sin^{-1}\left(\frac{1}{2}\right) = x$. Then, $\sin x = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$

$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = y$. Then, $\sin y = \frac{1}{\sqrt{2}} = \sin\left(\frac{\pi}{4}\right)$

$\therefore \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$\therefore \sin^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{6} - \frac{2\pi}{4} = \frac{\pi}{6} - \frac{\pi}{2} = \frac{\pi - 3\pi}{6} = -\frac{\pi}{3}$

Let $\sin^{-1}\left(-\frac{1}{2}\right) = x$. Then, $\sin x = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right)$

$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

Let $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = y$. Then, $\cos y = \frac{-\sqrt{3}}{2} = \cos\left(\pi - \frac{\pi}{6}\right)$

$\therefore \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

$\therefore \sin^{-1}\left(-\frac{1}{2}\right) + 2\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -\frac{\pi}{6} + \frac{10\pi}{6} = \frac{-\pi + 10\pi}{6} = \frac{9\pi}{6} = \frac{3\pi}{2}$

Let $\tan^{-1}(-1) = x$. Then, $\tan x = -1 = -\tan\left(\frac{\pi}{4}\right) = \tan\left(\pi - \frac{\pi}{4}\right)$

$\therefore \tan^{-1}(-1) = \frac{3\pi}{4}$

Let $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = y$. Then, $\cos y = \frac{-1}{\sqrt{2}} = -\cos\left(\frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right)$

$\therefore \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

$\therefore \tan^{-1}(-1) + \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4} + \frac{3\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$

Let $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = x$. Then, $\sin x = -\frac{\sqrt{3}}{2} = -\sin\left(\frac{\pi}{3}\right) = \sin\left(\pi - \frac{\pi}{3}\right)$

$\therefore \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3}$

Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$. Then, $\cos y = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$

$\therefore \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

$\therefore \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} + \frac{\pi}{6} = \frac{4\pi + \pi}{6} = \frac{5\pi}{6}$

Let $\tan^{-1}\left(\sqrt{3}\right) = x$. Then, $\tan x = \sqrt{3} = \tan\left(\frac{\pi}{3}\right)$

$\therefore \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

Let $\sec^{-1}(-2) = y$. Then, $\sec y = -2 = \sec\left(\pi - \frac{\pi}{3}\right)$

$\therefore \sec^{-1}(-2) = \frac{2\pi}{3}$

$\therefore \tan^{-1}\left(\sqrt{3}\right) - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = \frac{\pi - 2\pi}{3} = -\frac{\pi}{3}$

Ex 4.2

Inverse Trigonometric Functions Ex 4.2 Q1

8.i

$$= \cos^{-1}\left(\frac{a\cos\theta + b}{a + b\cos\theta}\right) \qquad \left\{\text{Since } \cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right\}$$

$= \text{RHS}$

Hence,

$$2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right) = \cos^{-1}\left(\frac{a\cos\theta + b}{a + b\cos\theta}\right)$$