

DIFFERENTIATION (XII, R. S. AGGARWAL)

EXERCISE 10A (Pg.No.: 370)

Differentiate each of the following w.r.t.x:

1. $y = \sin 4x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\sin 4x)}{d(4x)} \times \frac{d(4x)}{dx} \Rightarrow \frac{dy}{dx} = \cos 4x \cdot 4 \quad \therefore \frac{dy}{dx} = 4 \cos 4x$$

2. $y = \cos 5x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\cos 5x)}{d(5x)} \times \frac{d(5x)}{dx} \Rightarrow \frac{dy}{dx} = -\sin 5x \cdot 5 \quad \therefore \frac{dy}{dx} = -5 \sin 5x$$

3. $y = \tan 3x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\tan 3x)}{d(3x)} \times \frac{d(3x)}{dx} \Rightarrow \frac{dy}{dx} = \sec^2 3x \cdot 3 \quad \therefore \frac{dy}{dx} = 3 \sec^2 3x$$

4. $y = \cos(x^3)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\cos(x^3)\}}{d(x^3)} \times \frac{d(x^3)}{dx} \Rightarrow \frac{dy}{dx} = -\sin(x^3) \cdot 3x^2 \quad \therefore \frac{dy}{dx} = -3x^2 \sin(x^3)$$

5. $y = \cot^2 x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\cot^2 x)}{d(\cot x)} \times \frac{d(\cot x)}{dx} \Rightarrow \frac{dy}{dx} = 2 \cot x (-\operatorname{cosec}^2 x) \quad \therefore \frac{dy}{dx} = -2 \cot x \operatorname{cosec}^2 x$$

6. $y = \tan^3 x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\tan^3 x)}{d(\tan x)} \times \frac{d(\tan x)}{dx} \Rightarrow \frac{dy}{dx} = 3 \tan^2 x \sec^2 x$$

7. $y = \cot(\sqrt{x})$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\cot(\sqrt{x})\}}{d(\sqrt{x})} \times \frac{d(\sqrt{x})}{dx} \Rightarrow \frac{dy}{dx} = -\operatorname{cosec}^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \quad \therefore \frac{dy}{dx} = -\frac{\operatorname{cosec}^2(\sqrt{x})}{2\sqrt{x}}$$

8. $y = \sqrt{\tan x}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\sqrt{\tan x})}{d(\tan x)} \times \frac{d(\tan x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\tan x}} \sec^2 x \therefore \frac{dy}{dx} = \frac{\sec^2 x}{2\sqrt{\tan x}}$$

9. $y = (5 + 7x)^6$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d\{(5+7x)^6\}}{d(5+7x)} \times \frac{d(5+7x)}{dx} \Rightarrow \frac{dy}{dx} = 6(5+7x)^{6-1} (0+7) \\ \Rightarrow \frac{dy}{dx} &= 6(5+7x)^5 \cdot 7 \therefore \frac{dy}{dx} = 42(5+7x)^5 \end{aligned}$$

10. $y = (3 - 4x)^5$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d\{(3-4x)^5\}}{d(3-4x)} \times \frac{d(3-4x)}{dx} \Rightarrow \frac{dy}{dx} = 5(3-4x)^{5-1} \left\{ \frac{d(3)}{dx} - \frac{d(4x)}{dx} \right\} \\ \Rightarrow \frac{dy}{dx} &= 5(3-4x)^4 \{0-4\} \therefore \frac{dy}{dx} = -20(3-4x)^4 \end{aligned}$$

11. $y = (2x^2 - 3x + 4)^5$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d\{(2x^2-3x+4)^5\}}{d(2x^2-3x+4)} \times \frac{d(2x^2-3x+4)}{dx} \Rightarrow \frac{dy}{dx} = 5(2x^2-3x+4)^{5-1} \left\{ \frac{d(2x^2)}{dx} - \frac{d(3x)}{dx} + \frac{d(4)}{dx} \right\} \\ \Rightarrow \frac{dy}{dx} &= 5(2x^2-3x+4)^4 \left\{ 2 \frac{d(x^2)}{dx} - 3 \frac{d(x)}{dx} + 0 \right\} \Rightarrow \frac{dy}{dx} = 5(2x^2-3x+4)^4 (2.2x-3.1) \\ \Rightarrow \frac{dy}{dx} &= 5(2x^2-3x+4)^4 (4x-3) \therefore \frac{dy}{dx} = (20x-15)(2x^2-3x+4)^4 \end{aligned}$$

12. $y = (ax^2 + bx + c)^6$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d\{(ax^2+bx+c)^6\}}{d(ax^2+bx+c)} \times \frac{d(ax^2+bx+c)}{dx} \\ \Rightarrow \frac{dy}{dx} &= 6(ax^2+bx+c)^5 \left\{ a \frac{d(x^2)}{dx} + b \frac{d(x)}{dx} + 0 \right\} \Rightarrow \frac{dy}{dx} = 6(ax^2+bx+c)^5 \{a.2x+b.1\} \\ \Rightarrow \frac{dy}{dx} &= 6(ax^2+bx+c)^5 (2ax+b) \end{aligned}$$

13. $y = \frac{1}{(x^2 - 3x + 5)^3}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{(x^2-3x+5)^{-3}\}}{d(x^2-3x+5)} \times \frac{d(x^2-3x+5)}{dx} \Rightarrow \frac{dy}{dx} = -3(x^2-3x+5)^{-3-1} \left\{ \frac{d(x^2)}{dx} - \frac{d(3x)}{dx} + \frac{d(5)}{dx} \right\}$$

$$\Rightarrow \frac{dy}{dx} = -3(x^2 - 3x + 5)^{-4} (2x - 3 + 0) \Rightarrow \frac{dy}{dx} = -3(x^2 - 3x + 5)^{-4} (2x - 3) \therefore \frac{dy}{dx} = \frac{-3(2x - 3)}{(x^2 - 3x + 5)^4}$$

14. $y = \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d\left(\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\right)}{d\left(\frac{a^2 - x^2}{a^2 + x^2}\right)} \times \frac{d\left(\frac{a^2 - x^2}{a^2 + x^2}\right)}{dx} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{2\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}} \left[\frac{(a^2 + x^2) \frac{d(a^2 - x^2)}{dx} - (a^2 - x^2) \frac{d(a^2 + x^2)}{dx}}{(a^2 + x^2)^2} \right] \\ \Rightarrow \frac{dy}{dx} &= \frac{\sqrt{a^2 + x^2}}{2\sqrt{a^2 - x^2}} \left[\frac{(a^2 + x^2)(0 - 2x) - (a^2 - x^2)(0 + 2x)}{(a^2 + x^2)\sqrt{a^2 + x^2}\sqrt{a^2 - x^2}} \right] \\ \Rightarrow \frac{dy}{dx} &= \frac{2x(-a^2 - x^2 - a^2 + x^2)}{2\sqrt{a^2 - x^2}(a^2 + x^2)^{3/2}} \Rightarrow \frac{dy}{dx} = \frac{-2a^2x}{\sqrt{(a^2 - x^2)(a^2 + x^2)^{3/2}}} \Rightarrow \frac{dy}{dx} = \frac{-2a^2x}{(a^2 + x^2)^{3/2}(a^2 - x^2)^{1/2}} \end{aligned}$$

15. $y = \sqrt{\frac{1 + \sin x}{1 - \sin x}}$

Sol. Given, $y = \sqrt{\frac{1 + \sin x}{1 - \sin x}} = \sqrt{\frac{(1 + \sin x)(1 + \sin x)}{(1 - \sin x)(1 + \sin x)}} \Rightarrow y = \sqrt{\frac{(1 + \sin x)^2}{1 - \sin^2 x}} = \frac{1 + \sin x}{\cos x} \Rightarrow y = \sec x + \tan x$

$$\therefore \frac{dy}{dx} = \sec x \tan x + \sec^2 x = \sec x (\tan x + \sec x)$$

16. $y = \cos^2(x^3)$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d\{\cos^2(x^3)\}}{d\{\cos(x^3)\}} \times \frac{d\{\cos(x^3)\}}{d(x^3)} \times \frac{d(x^3)}{dx} \Rightarrow \frac{dy}{dx} = 2\cos(x^3) \cdot (-\sin x^3) \cdot 3x^2 \\ \Rightarrow \frac{dy}{dx} &= -3x^2 \cdot \{2\cos(x^3) \cdot \sin(x^3)\} \Rightarrow \frac{dy}{dx} = -3x^2 \sin(2x^3) \therefore \frac{dy}{dx} = -3x^2 \sin(2x^3) \end{aligned}$$

17. $y = \sec^3(x^2 + 1)$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d\{\sec^3(x^2 + 1)\}}{d\{\sec(x^2 + 1)\}} \times \frac{d\{\sec(x^2 + 1)\}}{d(x^2 + 1)} \times \frac{d(x^2 + 1)}{dx} \\ \Rightarrow \frac{dy}{dx} &= 3\sec^2(x^2 + 1) \cdot \sec(x^2 + 1) \tan(x^2 + 1) \cdot \left[\frac{d(x^2)}{dx} + \frac{d(1)}{dx} \right] \end{aligned}$$

$$\Rightarrow \frac{dy}{dx} = 3 \sec^3(x^2 + 1) \tan(x^2 + 1)(2x + 0) \quad \therefore \frac{dy}{dx} = 6x \sec^3(x^2 + 1) \tan(x^2 + 1)$$

18. $y = \sqrt{\cos 3x}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\sqrt{\cos 3x})}{d(\cos 3x)} \times \frac{d(\cos 3x)}{d(3x)} \times \frac{d(3x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\cos 3x}} \cdot (-\sin 3x) \cdot 3$$

$$\therefore \frac{dy}{dx} = \frac{-3 \sin 3x}{2\sqrt{\cos 3x}}$$

19. $y = \sqrt[3]{\sin 2x}$

Sol. $y = (\sin 2x)^{\frac{1}{3}}$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{(\sin 2x)^{\frac{1}{3}}\}}{d(\sin 2x)} \times \frac{d(\sin 2x)}{d(2x)} \times \frac{d(2x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{3} (\sin 2x)^{\frac{1}{3}-1} \cos 2x \cdot 2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{3} (\sin 2x)^{-\frac{2}{3}} \cos 2x \quad \therefore \frac{dy}{dx} = \frac{2 \cos 2x}{3 (\sin 2x)^{\frac{2}{3}}}$$

20. $y = \sqrt{1 + \cot x}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\sqrt{1 + \cot x})}{d(1 + \cot x)} \times \frac{d(1 + \cot x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{1 + \cot x}} \left\{ \frac{d(1)}{dx} + \frac{d(\cot x)}{dx} \right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{1 + \cot x}} (0 - \operatorname{cosec}^2 x) \quad \therefore \frac{dy}{dx} = -\frac{\operatorname{cosec}^2 x}{2\sqrt{1 + \cot x}}$$

21. $y = \operatorname{cosec}^3\left(\frac{1}{x^2}\right)$

Sol. $y = \operatorname{cosec}^3(x^{-2})$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\operatorname{cosec}^3(x^{-2})\}}{d\{\operatorname{cosec}(x^{-2})\}} \times \frac{d\{\operatorname{cosec}(x^{-2})\}}{d(x^{-2})} \times \frac{d(x^{-2})}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 3 \operatorname{cosec}^{3-1}(x^{-2}) \{-\operatorname{cosec}(x^{-2}) \cot(x^{-2})\} (-2x^{-2-1})$$

$$\Rightarrow \frac{dy}{dx} = 6 \operatorname{cosec}^2(x^{-2}) \operatorname{cosec}(x^{-2}) \cot(x^{-2}) \cdot x^{-3} \quad \therefore \frac{dy}{dx} = \frac{6 \operatorname{cosec}^3\left(\frac{1}{x^2}\right) \cdot \cot\left(\frac{1}{x^2}\right)}{x^3} = \frac{6}{x^3} \operatorname{cosec}^3 \frac{1}{x^2} \cot \frac{1}{x^2}$$

22. $y = \sqrt{\sin(x^3)}$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\sqrt{\sin(x^3)}\}}{d\{\sin(x^3)\}} \times \frac{d\{\sin(x^3)\}}{d(x^3)} \times \frac{d(x^3)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\sin(x^3)}} \cdot \cos(x^3) \cdot 3x^2 \quad \therefore \frac{dy}{dx} = \frac{3x^2 \cos(x^3)}{2\sqrt{\sin(x^3)}}$$

23. $y = \sqrt{x \sin x}$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(\sqrt{x \sin x})}{d(x \sin x)} \times \frac{d(x \sin x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x \sin x}} \cdot \left\{ x \frac{d(\sin x)}{dx} + \sin x \frac{d(x)}{dx} \right\} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{2\sqrt{x \sin x}} \{ x \cos x + \sin x \cdot 1 \} \quad \therefore \frac{dy}{dx} = \frac{x \cos x + \sin x}{2\sqrt{x \sin x}} \end{aligned}$$

24. $y = \sqrt{\cot(\sqrt{x})}$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d\{\sqrt{\cot(\sqrt{x})}\}}{d\{\cot(\sqrt{x})\}} \times \frac{d\{\cot(\sqrt{x})\}}{d(\sqrt{x})} \times \frac{d(\sqrt{x})}{dx} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{2\sqrt{\cot(\sqrt{x})}} \{-\operatorname{cosec}^2(\sqrt{x})\} \cdot \frac{1}{2\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{-\operatorname{cosec}^2(\sqrt{x})}{4\sqrt{x}\sqrt{\cot(\sqrt{x})}} \end{aligned}$$

25. $y = \cot^3(x^2)$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d\{\cot^3(x^2)\}}{d\{\cot(x^2)\}} \times \frac{d\{\cot(x^2)\}}{d(x^2)} \times \frac{d(x^2)}{dx} \Rightarrow \frac{dy}{dx} = 3\cot^2(x^2) \cdot \{-\operatorname{cosec}^2(x^2)\} \cdot 2x \\ \Rightarrow \frac{dy}{dx} &= -6x \cot^2(x^2) \operatorname{cosec}^2(x^2) \end{aligned}$$

26. $y = \cos(\sin \sqrt{ax+b})$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d\{\cos(\sin \sqrt{ax+b})\}}{d\{\sin(\sqrt{ax+b})\}} \times \frac{d\{\sin(\sqrt{ax+b})\}}{d(\sqrt{ax+b})} \times \frac{d(\sqrt{ax+b})}{d(ax+b)} \times \frac{d(ax+b)}{dx} \\ \Rightarrow \frac{dy}{dx} &= \frac{-\sin(\sin \sqrt{ax+b}) \cdot \cos(\sqrt{ax+b}) \cdot a}{2\sqrt{ax+b}} \quad \therefore \frac{dy}{dx} = -\frac{a \sin(\sin \sqrt{ax+b}) \cdot \cos(\sqrt{ax+b})}{2\sqrt{ax+b}} \end{aligned}$$

27. $y = \sqrt{\operatorname{cosec}(x^3+1)}$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(\sqrt{\operatorname{cosec}(x^3+1)})}{d\{\operatorname{cosec}(x^3+1)\}} \times \frac{d\{\operatorname{cosec}(x^3+1)\}}{d(x^3+1)} \times \frac{d(x^3+1)}{dx} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{2\sqrt{\operatorname{cosec}(x^3+1)}} \{-\operatorname{cosec}(x^3+1) \cdot \cot(x^3+1)\} \cdot \left\{ \frac{d(x^3)}{dx} + \frac{d(1)}{dx} \right\} \end{aligned}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\operatorname{cosec}(x^3+1) \cot(x^3+1) \cdot (3x^2+0)}{2\sqrt{\operatorname{cosec}(x^3+1)}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-3x^2 \sqrt{\operatorname{cosec}(x^3+1)} \sqrt{\operatorname{cosec}(x^3+1)} \cdot \cot(x^3+1)}{2\sqrt{\operatorname{cosec}(x^3+1)}}$$

$$\therefore \frac{dy}{dx} = -\frac{3x^2}{2} \sqrt{\operatorname{cosec}(x^3+1)} \cdot \cot(x^3+1)$$

28. $y = \sin 5x \cdot \cos 3x$

Sol. $y = \frac{1}{2}(2 \sin 5x \cdot \cos 3x) \Rightarrow y = \frac{1}{2}\{\sin(5x+3x) + \sin(5x-3x)\} \Rightarrow y = \frac{1}{2}(\sin 8x + \sin 2x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{1}{2} \left\{ \frac{d(\sin 8x)}{d(8x)} \times \frac{d(8x)}{dx} + \frac{d(\sin 2x)}{d(2x)} \times \frac{d(2x)}{dx} \right\} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(\cos 8x \cdot 8 + \cos 2x \cdot 2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(4 \cos 8x + \cos 2x)}{2} \therefore \frac{dy}{dx} = 4 \cos 8x + \cos 2x$$

29. $y = \sin 2x \cdot \sin x$

Sol. $y = \frac{1}{2}(2 \sin 2x \cdot \sin x) \Rightarrow y = \frac{1}{2}\{\cos(2x-x) - \cos(2x+x)\} \Rightarrow y = \frac{1}{2}\{\cos x - \cos 3x\}$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{1}{2} \left\{ \frac{d(\cos x)}{dx} - \frac{d(\cos 3x)}{d(3x)} \cdot \frac{d(3x)}{dx} \right\} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\{-\sin x + \sin 3x \cdot 3\} \therefore \frac{dy}{dx} = \frac{3}{2} \sin 3x - \frac{1}{2} \sin x$$

30. $y = \cos 4x \cdot \cos 2x$

Sol. $y = \frac{1}{2}(2 \cos 4x \cdot \cos 2x) \Rightarrow y = \frac{1}{2}\{\cos(4x+2x) + \cos(4x-2x)\} \Rightarrow y = \frac{1}{2}(\cos 6x + \cos 2x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{1}{2} \left\{ \frac{d(\cos 6x)}{d(6x)} \times \frac{d(6x)}{dx} + \frac{d(\cos 2x)}{d(2x)} \times \frac{d(2x)}{dx} \right\} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(-\sin 6x \cdot 6 - \sin 2x \cdot 2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{2}(-3 \sin 6x - \sin 2x) \therefore \frac{dy}{dx} = -(3 \sin 6x + \sin 2x)$$

Find $\frac{dy}{dx}$, when :

31. $y = \sin \left(\frac{1+x^2}{1-x^2} \right)$

Sol. Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d \left\{ \sin \left(\frac{1+x^2}{1-x^2} \right) \right\}}{d \left(\frac{1+x^2}{1-x^2} \right)} \times \frac{d \left(\frac{1+x^2}{1-x^2} \right)}{dx}$

$$\begin{aligned}\Rightarrow \frac{dy}{dx} &= \cos\left(\frac{1+x^2}{1-x^2}\right) \left\{ \frac{(1-x^2) \frac{d(1+x^2)}{dx} - (1+x^2) \frac{d(1-x^2)}{dx}}{(1-x^2)^2} \right\} \\ \Rightarrow \frac{dy}{dx} &= \cos\left(\frac{1+x^2}{1-x^2}\right) \left\{ \frac{(1-x^2)(0+2x) - (1+x^2)(0-2x)}{(1-x^2)^2} \right\} \\ \Rightarrow \frac{dy}{dx} &= 2x \cos\left(\frac{1+x^2}{1-x^2}\right) \cdot \left\{ \frac{1-x^2+1+x^2}{(1-x^2)^2} \right\} \Rightarrow \frac{dy}{dx} = \frac{4x}{(1-x^2)^2} \cos\left(\frac{1+x^2}{1-x^2}\right)\end{aligned}$$

32. $y = \frac{\sin x + x^2}{\cot 2x}$

Sol. $y = \frac{\sin x}{\cot 2x} + \frac{x^2}{\cot 2x} \Rightarrow y = \sin x \cdot \tan 2x + x^2 \cdot \tan 2x$

Differentiating both sides with respect to x

$$\begin{aligned}\frac{dy}{dx} &= \left\{ \sin x \cdot \frac{d(\tan 2x)}{d(2x)} \times \frac{d(2x)}{dx} + \tan 2x \cdot \frac{d(\sin x)}{dx} \right\} + \left\{ x^2 \cdot \frac{d(\tan 2x)}{d(2x)} \cdot \frac{d(2x)}{dx} + \tan 2x \cdot \frac{d(x^2)}{dx} \right\} \\ \Rightarrow \frac{dy}{dx} &= \sin x \cdot \sec^2(2x) \cdot 2 + \tan 2x \cdot \cos x + x^2 \sec^2 2x \cdot 2 + \tan 2x \cdot 2x \\ \Rightarrow \frac{dy}{dx} &= 2 \sec^2(2x) (\sin x + x^2) + \tan 2x (\cos x + 2x) \\ \Rightarrow 2(\sin x + x^2) \sec^2 2x &+ (\cos x + 2x) \tan 2x\end{aligned}$$

33. If $y = \frac{\cos x - \sin x}{\cos x + \sin x}$, prove that $\frac{dy}{dx} + y^2 + 1 = 0$

Sol. We have, $y = \frac{\cos x - \sin x}{\cos x + \sin x}$

Dividing Numerator and Denominator by $\cos x$, we get

$$\begin{aligned}y &= \frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}} \Rightarrow y = \frac{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}} \Rightarrow y = \frac{1 - \tan x}{1 + \tan x} \\ \Rightarrow y &= \frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \cdot \tan x} \Rightarrow y = \tan\left(\frac{\pi}{4} - x\right) \quad \dots (i)\end{aligned}$$

Differentiating both sides of (i) with respect to x , $\frac{dy}{dx} = \frac{d\left\{\tan\left(\frac{\pi}{4} - x\right)\right\}}{d\left(\frac{\pi}{4} - x\right)} \times \frac{d\left(\frac{\pi}{4} - x\right)}{dx}$

$$\Rightarrow \frac{dy}{dx} = \sec^2\left(\frac{\pi}{4} - x\right) \{0 - 1\} \Rightarrow \frac{dy}{dx} = -\sec^2\left(\frac{\pi}{4} - x\right)$$

L.H.S. $\frac{dy}{dx} + y^2 + 1 = -\sec^2\left(\frac{\pi}{4} - x\right) + \tan^2\left(\frac{\pi}{4} - x\right) + 1 = -\sec^2\left(\frac{\pi}{4} - x\right) + \sec^2\left(\frac{\pi}{4} - x\right) = 0 = \text{R.H.S}$

Hence proved

34. If $y = \frac{(\cos x + \sin x)}{(\cos x - \sin x)}$, Prove that $\frac{dy}{dx} = \sec^2\left(x + \frac{\pi}{4}\right)$

Sol. We have, $y = \frac{\cos x + \sin x}{\cos x - \sin x}$, dividing Numerator and Denominator by $\cos x$

$$y = \frac{\frac{\cos x + \sin x}{\cos x}}{\frac{\cos x - \sin x}{\cos x}} \Rightarrow y = \frac{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}} \Rightarrow y = \frac{1 + \tan x}{1 - \tan x}$$

$$\Rightarrow y = \frac{\tan \frac{\pi}{4} + \tan x}{1 - \tan \frac{\pi}{4} \cdot \tan x} \Rightarrow y = \tan\left(\frac{\pi}{4} + x\right) \quad \dots (i)$$

Differentiating both sides of (i) w.r.t. x

$$\frac{dy}{dx} = \frac{d\left\{\tan\left(\frac{\pi}{4} + x\right)\right\}}{d\left(\frac{\pi}{4} + x\right)} \times \frac{d\left(\frac{\pi}{4} + x\right)}{dx} \Rightarrow \frac{dy}{dx} = \sec^2\left(\frac{\pi}{4} + x\right) \cdot \left\{\frac{d(\pi/4)}{dx} + \frac{d(x)}{dx}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \sec^2\left(\frac{\pi}{4} + x\right) \{0 + 1\} \therefore \frac{dy}{dx} = \sec^2\left(\frac{\pi}{4} + x\right)$$

EXERCISE 10B (Pg.no.: 379)

Differentiate each of the following w.r.t. x :

1. (i) e^{4x} (ii) e^{-5x} (iii) e^{x^3}

Sol. (i) $y = e^{4x}$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(e^{4x})}{d(4x)} \times \frac{d(4x)}{dx} \Rightarrow \frac{dy}{dx} = e^{4x} \cdot 4 \therefore \frac{dy}{dx} = 4e^{4x}$$

(ii) $y = e^{-5x}$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(e^{-5x})}{d(-5x)} \times \frac{d(-5x)}{dx} \Rightarrow \frac{dy}{dx} = e^{-5x} (-5) \therefore \frac{dy}{dx} = -5e^{-5x}$$

(iii) $y = e^{x^3}$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(e^{x^3})}{d(x^3)} \times \frac{d(x^3)}{dx} \Rightarrow \frac{dy}{dx} = e^{x^3} \cdot 3x^2$$

2. (i) $e^{\frac{2}{x}}$ (ii) $e^{\sqrt{x}}$ (iii) $e^{-2\sqrt{x}}$

Sol. (i) $y = e^{\frac{2}{x}}$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left(e^{\frac{2}{x}}\right)}{d\left(\frac{2}{x}\right)} \times \frac{d\left(\frac{2}{x}\right)}{dx} \Rightarrow \frac{dy}{dx} = e^{\frac{2}{x}} \cdot 2 \cdot \frac{d\left(\frac{1}{x}\right)}{dx} \Rightarrow \frac{dy}{dx} = 2e^{\frac{2}{x}} \left(-\frac{1}{x^2}\right) \therefore \frac{dy}{dx} = \frac{-2e^{\frac{2}{x}}}{x^2}$$

(ii) $y = e^{\sqrt{x}}$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(e^{\sqrt{x}})}{d(\sqrt{x})} \times \frac{d(\sqrt{x})}{dx} \Rightarrow \frac{dy}{dx} = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \therefore \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

(iii) $y = e^{-2\sqrt{x}}$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(e^{-2\sqrt{x}})}{d(-2\sqrt{x})} \times \frac{d(-2\sqrt{x})}{dx} \Rightarrow \frac{dy}{dx} = e^{-2\sqrt{x}} (-2) \frac{d(\sqrt{x})}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^{-2\sqrt{x}} (-2) \frac{1}{2\sqrt{x}} \therefore \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

3. (i) $e^{\cot x}$ (ii) $e^{-\sin 2x}$ (iii) $e^{\sqrt{\sin x}}$

Sol. (i) $y = e^{\cot x}$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(e^{\cot x})}{d(\cot x)} \times \frac{d(\cot x)}{dx} \Rightarrow \frac{dy}{dx} = e^{\cot x} (-\operatorname{cosec}^2 x) \therefore \frac{dy}{dx} = -e^{\cot x} \cdot \operatorname{cosec}^2 x$$

(ii) $y = e^{-\sin 2x}$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(e^{-\sin 2x})}{d(-\sin 2x)} \times \frac{d(-\sin 2x)}{d(2x)} \times \frac{d(2x)}{dx} \Rightarrow \frac{dy}{dx} = e^{-\sin 2x} (-\cos 2x) 2 \therefore \frac{dy}{dx} = -2e^{-\sin 2x} \cos 2x$$

(iii) $y = e^{\sqrt{\sin x}}$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(e^{\sqrt{\sin x}})}{d(\sqrt{\sin x})} \times \frac{d(\sqrt{\sin x})}{d(\sin x)} \times \frac{d(\sin x)}{dx} \Rightarrow \frac{dy}{dx} = e^{\sqrt{\sin x}} \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x \therefore \frac{dy}{dx} = \frac{\cos x \cdot e^{\sqrt{\sin x}}}{2\sqrt{\sin x}}$$

4. (i) $\tan(\log x)$ (ii) $\log \sec x$ (iii) $\log \sin \frac{x}{2}$

Sol. (i) $y = \tan(\log x)$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\tan(\log x)\}}{d(\log x)} \times \frac{d(\log x)}{dx} \Rightarrow \frac{dy}{dx} = \sec^2(\log x) \cdot \frac{1}{x} \therefore \frac{dy}{dx} = \frac{\sec^2(\log x)}{x}$$

(ii) $y = \log(\sec x)$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\log(\sec x)\}}{d(\sec x)} \times \frac{d(\sec x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sec x} \cdot \sec x \cdot \tan x \Rightarrow \frac{dy}{dx} = \tan x$$

(iii) $y = \log\left(\sin \frac{x}{2}\right)$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left\{\log\left(\sin \frac{x}{2}\right)\right\}}{d\left(\sin \frac{x}{2}\right)} \times \frac{d\left(\sin \frac{x}{2}\right)}{d\left(\frac{x}{2}\right)} \times \frac{d\left(\frac{x}{2}\right)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sin \frac{x}{2}} \cos\left(\frac{x}{2}\right) \cdot \frac{1}{2} \frac{dx}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \cot\left(\frac{x}{2}\right)$$

5. (i) $\log_3 x$ (ii) 2^{-x} (iii) 3^{x+2}

Sol. (i) $y = \log_3 x$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\log_3 x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{x \log 3} \quad \left[\therefore \frac{d(\log_a x)}{dx} = \frac{1}{x \log a} \right]$$

(ii) $y = 2^{-x}$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(2^{-x})}{d(-x)} \times \frac{d(-x)}{dx} \Rightarrow \frac{dy}{dx} = 2^{-x} \log 2 \cdot (-1) \therefore \frac{dy}{dx} = -2^{-x} \log 2$$

(iii) $y = 3^{x+2}$, Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(3^{x+2})}{d(x+2)} \times \frac{d(x+2)}{dx} \Rightarrow \frac{dy}{dx} = 3^{x+2} \log 3 \cdot \left\{ \frac{d(x)}{dx} + \frac{d(2)}{dx} \right\} \\ &\Rightarrow \frac{dy}{dx} = 3^x \cdot 3^2 \log 3 \cdot \{1+0\} \therefore \frac{dy}{dx} = 9 \cdot 3^x \log 3 \end{aligned}$$

6. (i) $\log\left(x + \frac{1}{x}\right)$ (ii) $\log \sin 3x$ (iii) $\log(x + \sqrt{1+x^2})$

Sol. (i) $y = \log\left(x + \frac{1}{x}\right)$, Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d\left\{\log\left(x + \frac{1}{x}\right)\right\}}{d\left(x + \frac{1}{x}\right)} \times \frac{d\left(x + \frac{1}{x}\right)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{x + \frac{1}{x}} \left\{ \frac{d(x)}{dx} + \frac{d\left(\frac{1}{x}\right)}{dx} \right\} \\ &\Rightarrow \frac{dy}{dx} = \frac{x}{x^2 + 1} \left\{ 1 + \left(-\frac{1}{x^2}\right) \right\} \Rightarrow \frac{dy}{dx} = \frac{x}{x^2 + 1} \left\{ 1 - \frac{1}{x^2} \right\} \Rightarrow \frac{dy}{dx} = \frac{x}{x^2 + 1} \left\{ \frac{x^2 - 1}{x^2} \right\} \therefore \frac{dy}{dx} = \frac{(x^2 - 1)}{x(x^2 + 1)} \end{aligned}$$

(ii) $y = \log(\sin 3x)$, Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\log(\sin 3x)\}}{d(\sin 3x)} \times \frac{d(\sin 3x)}{d(3x)} \times \frac{d(3x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sin 3x} \cdot \cos 3x \cdot 3 \Rightarrow \frac{dy}{dx} = 3 \cot 3x$$

(iii) $y = \log(x + \sqrt{x^2 + 1})$, Differentiating both sides with respect to x ,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d\{\log(x + \sqrt{x^2 + 1})\}}{d(x + \sqrt{x^2 + 1})} \times \frac{d(x + \sqrt{x^2 + 1})}{dx} \\ &\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \left\{ \frac{d(x)}{dx} + \frac{d(\sqrt{x^2 + 1})}{d(x^2 + 1)} \times \frac{d(x^2 + 1)}{dx} \right\} \\ &\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \left\{ 1 + \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x \right\} \Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \left\{ \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \right\} \therefore \frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}} \end{aligned}$$

7. $y = e^{\sqrt{x}} \cdot \log x$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= e^{\sqrt{x}} \cdot \frac{d(\log x)}{dx} + \log x \cdot \frac{d(e^{\sqrt{x}})}{d(\sqrt{x})} \times \frac{d(\sqrt{x})}{dx} \Rightarrow \frac{dy}{dx} = e^{\sqrt{x}} \cdot \frac{1}{x} + \log x \cdot e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \\ &\Rightarrow \frac{dy}{dx} = e^{\sqrt{x}} \left(\frac{1}{x} + \frac{\log x}{2\sqrt{x}} \right) \therefore \frac{dy}{dx} = e^{\sqrt{x}} \left\{ \frac{2 + \sqrt{x} \log x}{2x} \right\} \end{aligned}$$

8. $y = \log \sin(\sqrt{x^2 + 1})$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\log(\sin \sqrt{x^2 + 1})\}}{d(\sin \sqrt{x^2 + 1})} \times \frac{d\{\sin(\sqrt{x^2 + 1})\}}{d(\sqrt{x^2 + 1})} \times \frac{d(\sqrt{x^2 + 1})}{d(x^2 + 1)} \times \frac{d(x^2 + 1)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin(\sqrt{x^2 + 1})} \cdot \cos(\sqrt{x^2 + 1}) \cdot \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x \quad \therefore \frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 1}} \cdot \cot(\sqrt{x^2 + 1})$$

9. $y = e^{2x} \sin 3x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = e^{2x} \frac{d(\sin 3x)}{d(3x)} \times \frac{d(3x)}{dx} + \sin 3x \frac{d(e^{2x})}{d(2x)} \times \frac{d(2x)}{dx} \Rightarrow \frac{dy}{dx} = e^{2x} \cdot \cos 3x \cdot 3 + \sin 3x \cdot e^{2x} \cdot 2$$

$$\therefore \frac{dy}{dx} = e^{2x} (3 \cos 3x + 2 \sin 3x)$$

10. $y = e^{3x} \cos 2x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(e^{3x} \cos 2x)}{dx} \Rightarrow \frac{dy}{dx} = e^{3x} \frac{d(\cos 2x)}{d(2x)} \times \frac{d(2x)}{dx} + \cos 2x \frac{d(e^{3x})}{d(3x)} \times \frac{d(3x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^{3x} (-\sin 2x) \cdot 2 + \cos 2x \cdot e^{3x} \cdot 3 \quad \therefore \frac{dy}{dx} = e^{3x} (-2 \sin 2x + 3 \cos 2x)$$

11. $y = e^{-5x} \cot 4x$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(e^{-5x} \cot 4x)}{dx} \Rightarrow \frac{dy}{dx} = e^{-5x} \frac{d(\cot 4x)}{d(4x)} \times \frac{d(4x)}{dx} + \cot 4x \frac{d(e^{-5x})}{d(-5x)} \times \frac{d(-5x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^{-5x} (-\operatorname{cosec}^2 4x) \cdot 4 + \cot 4x \cdot e^{-5x} \cdot (-5) \quad \therefore \frac{dy}{dx} = -e^{-5x} (5 \cot 4x + 4 \operatorname{cosec}^2 4x)$$

12. $y = e^x \log(\sin 2x)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = e^x \frac{d\{\log(\sin 2x)\}}{d(\sin 2x)} \times \frac{d(\sin 2x)}{d(2x)} \times \frac{d(2x)}{dx} + \log(\sin 2x) \times \frac{d(e^x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^x \cdot \frac{1}{\sin 2x} \cdot \cos 2x \cdot 2 + \log(\sin 2x) \cdot e^x \quad \therefore \frac{dy}{dx} = e^x \{2 \cot 2x + \log(\sin 2x)\}$$

13. $y = \log(\operatorname{cosec} x - \cot x)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\log(\operatorname{cosec} x - \cot x)\}}{d(\operatorname{cosec} x - \cot x)} \times \frac{d(\operatorname{cosec} x - \cot x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x} \left\{ \frac{d(\operatorname{cosec} x)}{dx} - \frac{d(\cot x)}{dx} \right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x} \{-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x\} \Rightarrow \frac{dy}{dx} = \frac{\operatorname{cosec} x (\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)} \quad \therefore \frac{dy}{dx} = \operatorname{cosec} x$$

14. $y = \log \left(\sec \frac{x}{2} + \tan \frac{x}{2} \right)$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d \left\{ \log \left(\sec \frac{x}{2} + \tan \frac{x}{2} \right) \right\}}{d \left(\sec \frac{x}{2} + \tan \frac{x}{2} \right)} \times \frac{d \left(\sec \frac{x}{2} + \tan \frac{x}{2} \right)}{dx} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\sec \frac{x}{2} + \tan \frac{x}{2}} \left\{ \frac{d \left(\sec \frac{x}{2} \right)}{d \left(\frac{x}{2} \right)} \times \frac{d \left(\frac{x}{2} \right)}{dx} + \frac{d \left(\tan \frac{x}{2} \right)}{d \left(\frac{x}{2} \right)} \times \frac{d \left(\frac{x}{2} \right)}{dx} \right\} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\sec \frac{x}{2} + \tan \frac{x}{2}} \left\{ \sec \frac{x}{2} \tan \frac{x}{2} \cdot \frac{1}{2} + \sec^2 \frac{x}{2} \cdot \frac{1}{2} \right\} \Rightarrow \frac{dy}{dx} = \frac{\frac{1}{2} \sec \frac{x}{2} \left\{ \tan \frac{x}{2} + \sec \frac{x}{2} \right\}}{\left\{ \sec \frac{x}{2} + \tan \frac{x}{2} \right\}} \therefore \frac{dy}{dx} = \frac{1}{2} \sec \frac{x}{2} \end{aligned}$$

15. $y = \sqrt{\frac{1+e^x}{1-e^x}}$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d \left(\sqrt{\frac{1+e^x}{1-e^x}} \right)}{d \left(\frac{1+e^x}{1-e^x} \right)} \times \frac{d \left(\frac{1+e^x}{1-e^x} \right)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2 \sqrt{\frac{1+e^x}{1-e^x}}} \left\{ \frac{(1-e^x) \frac{d(1+e^x)}{dx} - (1+e^x) \frac{d(1-e^x)}{dx}}{(1-e^x)^2} \right\} \\ \Rightarrow \frac{dy}{dx} &= \frac{\sqrt{1-e^x}}{2 \sqrt{1+e^x}} \left\{ \frac{(1-e^x)(0+e^x) - (1+e^x)(0-e^x)}{(1-e^x) \sqrt{1-e^x} \sqrt{1+e^x}} \right\} \Rightarrow \frac{dy}{dx} = \frac{e^x(1-e^x+1+e^x)}{2 \sqrt{1+e^x} \sqrt{1-e^x} (1-e^x)} \\ \therefore \frac{dy}{dx} &= \frac{2e^x}{(1-e^x) \sqrt{1+e^x}} \end{aligned}$$

16. $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d \left(\frac{e^x + e^{-x}}{e^x - e^{-x}} \right)}{d \left(\frac{e^x + e^{-x}}{e^x - e^{-x}} \right)} \Rightarrow \frac{dy}{dx} = \frac{(e^x - e^{-x}) \frac{d(e^x + e^{-x})}{dx} - (e^x + e^{-x}) \frac{d(e^x - e^{-x})}{dx}}{(e^x - e^{-x})^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x - e^{-x})^2} \Rightarrow \frac{dy}{dx} = \frac{(e^{2x} + e^{-2x} - 2) - (e^{2x} + e^{-2x} + 2)}{(e^x - e^{-x})^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{e^{2x} + e^{-2x} - 2 - e^{2x} - e^{-2x} - 2}{(e^x - e^{-x})^2} \therefore \frac{dy}{dx} = \frac{-4}{(e^x - e^{-x})^2} \end{aligned}$$

17. $y = xe^{\sqrt{\sin x}}$

Sol. Differentiating both sides with respect to x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(xe^{\sqrt{\sin x}})}{dx} \Rightarrow \frac{dy}{dx} = x \cdot \frac{d(e^{\sqrt{\sin x}})}{d(\sqrt{\sin x})} \times \frac{d(\sqrt{\sin x})}{d(\sin x)} \times \frac{d(\sin x)}{dx} + e^{\sqrt{\sin x}} \cdot \frac{d(x)}{dx} \\ \Rightarrow \frac{dy}{dx} &= x \cdot e^{\sqrt{\sin x}} \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x + e^{\sqrt{\sin x}} \cdot 1 \quad \therefore \frac{dy}{dx} = e^{\sqrt{\sin x}} \left\{ \frac{x \cos x}{2\sqrt{\sin x}} + 1 \right\}\end{aligned}$$

18. $y = e^{\sin x} \cdot \sin(e^x)$

Sol. Differentiating both sides with respect to x

$$\begin{aligned}\frac{dy}{dx} &= e^{\sin x} \cdot \frac{d\{\sin(e^x)\}}{d(e^x)} \times \frac{d(e^x)}{dx} + \sin(e^x) \cdot \frac{d(e^{\sin x})}{d(\sin x)} \times \frac{d(\sin x)}{dx} \\ \Rightarrow \frac{dy}{dx} &= e^{\sin x} \cdot \cos(e^x) \cdot e^x + \sin(e^x) \cdot e^{\sin x} \cdot \cos x \quad \therefore \frac{dy}{dx} = e^{\sin x} \cdot \{e^x \cos(e^x) + \cos x \sin e^x\}\end{aligned}$$

19. $y = e^{\sqrt{1-x^2}} \tan x$

Sol. Differentiating both sides with respect to x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(e^{\sqrt{1-x^2}} \tan x)}{dx} \Rightarrow \frac{dy}{dx} = e^{\sqrt{1-x^2}} \frac{d(\tan x)}{dx} + \tan x \cdot \frac{d(e^{\sqrt{1-x^2}})}{d(\sqrt{1-x^2})} \times \frac{d(\sqrt{1-x^2})}{d(1-x^2)} \times \frac{d(1-x^2)}{dx} \\ \Rightarrow \frac{dy}{dx} &= e^{\sqrt{1-x^2}} \cdot \sec^2 x + \tan x \cdot e^{\sqrt{1-x^2}} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot (0-2x) \\ \Rightarrow \frac{dy}{dx} &= e^{\sqrt{1-x^2}} \sec^2 x + \tan x \cdot \frac{e^{\sqrt{1-x^2}} \cdot (-2x)}{2\sqrt{1-x^2}} \quad \therefore \frac{dy}{dx} = e^{\sqrt{1-x^2}} \left\{ \sec^2 x - \frac{x \tan x}{\sqrt{1-x^2}} \right\}\end{aligned}$$

20. $y = \frac{e^x}{1+\cos x}$

Sol. Differentiating both sides with respect to x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\left(\frac{e^x}{1+\cos x}\right)}{dx} \Rightarrow \frac{dy}{dx} = \frac{(1+\cos x) \frac{d(e^x)}{dx} - e^x (1+\cos x)}{(1+\cos x)^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{(1+\cos x) \cdot e^x - e^x (0-\sin x)}{(1+\cos x)^2} \quad \therefore \frac{dy}{dx} = \frac{e^x (1+\cos x + \sin x)}{(1+\cos x)^2}\end{aligned}$$

21. $y = x^3 e^x \cos x$

Sol. Differentiating both sides with respect to x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^3 e^x \cos x)}{dx} \Rightarrow \frac{dy}{dx} = x^3 \cdot \frac{d(e^x \cos x)}{dx} + (e^x \cos x) \frac{d(x^3)}{dx} \\ \Rightarrow \frac{dy}{dx} &= x^3 \cdot \{e^x \sin x + \cos x \cdot e^x\} + e^x \cos x \cdot 3x^2 \Rightarrow \frac{dy}{dx} = e^x x^2 (-x \sin x + x \cos + 3 \cos x)\end{aligned}$$

22. $y = e^{x \cos x}$

Sol. Differentiating both sides with respect to x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(e^{x \cos x})}{d(x \cos x)} \times \frac{d(x \cos x)}{dx} \Rightarrow \frac{dy}{dx} = e^{x \cos x} \left\{ x \frac{d(\cos x)}{dx} + \cos x \cdot \frac{d(x)}{dx} \right\} \\ \Rightarrow \frac{dy}{dx} &= e^{x \cos x} \cdot \{-x \sin x + \cos x \cdot 1\} \quad \therefore \frac{dy}{dx} = e^{x \cos x} (\cos x - x \sin x)\end{aligned}$$

EXERCISE 10C (Pg.No.: 388)

Differentiate each of the following w.r.t.x:

1. $y = \cos^{-1}(2x)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\cos^{-1}(2x)\}}{d(2x)} \times \frac{d(2x)}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-(2x)^2}} \cdot 2 \frac{dx}{dx} \quad \therefore \frac{dy}{dx} = \frac{-2}{\sqrt{1-4x^2}}$$

2. $y = \tan^{-1}(x^2)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\tan^{-1}(x^2)\}}{d(x^2)} \times \frac{d(x^2)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{1+(x^2)^2} \cdot 2x \quad \therefore \frac{dy}{dx} = \frac{2x}{1+x^4}$$

3. $y = \sec^{-1}(\sqrt{x})$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\sec^{-1}(\sqrt{x})\}}{d(\sqrt{x})} \times \frac{d(\sqrt{x})}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x}\sqrt{(\sqrt{x})^2-1}} \cdot \frac{1}{2\sqrt{x}} \quad \therefore \frac{dy}{dx} = \frac{1}{2x\sqrt{x-1}}$$

4. $y = \sin^{-1}\left(\frac{x}{a}\right)$

Sol. Differentiating both sides with respect to x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\left\{\sin^{-1}\left(\frac{x}{a}\right)\right\}}{d\left(\frac{x}{a}\right)} \times \frac{d\left(\frac{x}{a}\right)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{1}{a} \cdot \frac{d(x)}{dx} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \cdot \frac{1}{a} \cdot 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}} \cdot \frac{1}{a} \Rightarrow \frac{dy}{dx} = \frac{a}{\sqrt{a^2-x^2}} \cdot \frac{1}{a} \quad \therefore \frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}}\end{aligned}$$

5. $y = \tan^{-1}(\log x)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\tan^{-1}(\log x)\}}{d(\log x)} \times \frac{d(\log x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{1+(\log x)^2} \cdot \frac{1}{x} \quad \therefore \frac{dy}{dx} = \frac{1}{x\{1+(\log x)^2\}}$$

6. $y = \cot^{-1}(e^x)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\cot^{-1}(e^x)\}}{d(e^x)} \times \frac{d(e^x)}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{1+(e^x)^2} \cdot e^x \quad \therefore \frac{dy}{dx} = -\frac{e^x}{1+e^{2x}}$$

7. $y = \log(\tan^{-1} x)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\log(\tan^{-1} x)\}}{d(\tan^{-1} x)} \times \frac{d(\tan^{-1} x)}{dx} \quad \therefore \frac{dy}{dx} = \frac{1}{\tan^{-1} x} \cdot \frac{1}{(1+x^2)}$$

8. $y = \cot^{-1}(x^3)$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\cot^{-1}(x^3)\}}{d(x^3)} \times \frac{d(x^3)}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{1+(x^3)^2} \cdot 3x^2 \quad \therefore \frac{dy}{dx} = \frac{-3x^2}{1+x^6}$$

9. $y = \sin^{-1}(\cos x)$

Sol. Given $y = \sin^{-1}(\cos x) = \sin^{-1} \sin\left(\frac{\pi}{2} - x\right) = \frac{\pi}{2} - x$

Differentiating both sides with respect to x , we get, $\frac{dy}{dx} = 0 - 1 = -1$.

10. $y = (1+x^2) \tan^{-1} x$

Sol. Differentiating both sides with respect to x , $\frac{dy}{dx} = (1+x^2) \frac{d(\tan^{-1} x)}{dx} + \tan^{-1} x \frac{d(1+x^2)}{dx}$
 $\Rightarrow \frac{dy}{dx} = (1+x^2) \cdot \frac{1}{1+x^2} + \tan^{-1} x \cdot (0+2x) \Rightarrow \frac{dy}{dx} = 1 + 2x \tan^{-1} x$

11. $y = \tan^{-1}(\cot x)$

Sol. Given $y = \tan^{-1}(\cot x) = \tan^{-1} \tan\left(\frac{\pi}{2} - x\right) = \frac{\pi}{2} - x$

Differentiating both sides with respect to x , we get, $\frac{dy}{dx} = 0 - 1 = -1$

12. $y = \log(\sin^{-1} x^4)$

Sol. Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d\{\log(\sin^{-1} x^4)\}}{d(\sin^{-1} x^4)} \times \frac{d(\sin^{-1} x^4)}{d(x^4)} \times \frac{d(x^4)}{dx}$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{\sin^{-1} x^4} \cdot \frac{1}{\sqrt{1-(x^4)^2}} \cdot 4x^3 \Rightarrow \frac{dy}{dx} = \frac{4x^3}{\sin^{-1}(x^4) \cdot \sqrt{1-x^8}}$

13. $y = \{\cot^{-1}(x^2)\}^3$

Sol. Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left[\{\cot^{-1}(x^2)\}^3\right]}{d\{\cot^{-1}(x^2)\}} \times \frac{d\{\cot^{-1}(x^2)\}}{d(x^2)} \times \frac{d(x^2)}{dx} \Rightarrow \frac{dy}{dx} = 3\{\cot^{-1}(x^2)\}^2 \cdot \left\{-\frac{1}{1+(x^2)^2}\right\} \cdot 2x$$

$$\therefore \frac{dy}{dx} = \frac{-6x}{1+x^4} \{\cot^{-1}(x^2)\}^2 = \frac{-6x(\cot^{-1} x^2)^2}{(1+x^4)}$$

14. $y = \tan^{-1}(\cos \sqrt{x})$

Sol. Differentiating both sides with respect to x

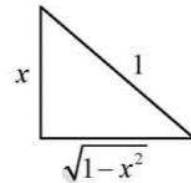
$$\frac{dy}{dx} = \frac{d\{\tan^{-1}(\cos\sqrt{x})\}}{d(\cos\sqrt{x})} \times \frac{d\{\cos(\sqrt{x})\}}{d(\sqrt{x})} \times \frac{d(\sqrt{x})}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+(\cos\sqrt{x})^2} \cdot -\sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \therefore \frac{dy}{dx} = \frac{-\sin(\sqrt{x})}{2\sqrt{x}(1+\cos^2\sqrt{x})}$$

15. $y = \tan(\sin^{-1} x)$

Sol. Let $\sin^{-1} x = t$, $\sin t = x$... (i)

$$= \tan t = \frac{x}{\sqrt{1-x^2}} \Rightarrow t = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$



Putting value of t in equation (i), $y = \tan\left(\tan^{-1}\frac{x}{\sqrt{1-x^2}}\right) \Rightarrow y = \frac{x}{\sqrt{1-x^2}}$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left(\frac{x}{\sqrt{1-x^2}}\right)}{dx} \Rightarrow \frac{dy}{dx} = \frac{\left(\sqrt{1-x^2}\right)\frac{d(x)}{dx} - x\frac{d(\sqrt{1-x^2})}{d(1-x^2)} \times \frac{d(1-x^2)}{dx}}{(\sqrt{1-x^2})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left\{\sqrt{1-x^2} \cdot 1 - x \cdot \frac{1}{2\sqrt{1-x^2}}(-2x)\right\}}{(1-x^2)} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-x^2}\sqrt{1-x^2} + x^2}{\sqrt{1-x^2} \cdot 1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-x^2+x^2}{(1-x^2)\sqrt{1-x^2}} \therefore \frac{dy}{dx} = \frac{1}{(1-x^2)^{3/2}}$$

16. $y = e^{\tan^{-1}(\sqrt{x})}$

Sol. Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d\{e^{\tan^{-1}(\sqrt{x})}\}}{d\{\tan^{-1}(\sqrt{x})\}} \times \frac{d\{\tan^{-1}(\sqrt{x})\}}{d(\sqrt{x})} \times \frac{d(\sqrt{x})}{dx}$

$$\Rightarrow \frac{dy}{dx} = e^{\tan^{-1}(\sqrt{x})} \cdot \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{e^{\tan^{-1}(\sqrt{x})}}{2\sqrt{x}(1+x)}$$

17. $y = \sqrt{\sin^{-1}(x^2)}$

Sol. Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d(\sqrt{\sin^{-1} x^2})}{d(\sin^{-1} x^2)} \times \frac{d(\sin^{-1} x^2)}{d(x^2)} \times \frac{d(x^2)}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\sin^{-1} x^2}} \cdot \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x \therefore \frac{dy}{dx} = \frac{x}{\sqrt{\sin^{-1}(x^2)} \sqrt{1-x^4}}$$

18. If $y = \sin^{-1}(\cos x) + \cos^{-1}(\sin x)$ prove that $\frac{dy}{dx} = -2$

Sol. Given $y = \sin^{-1}(\cos x) + \cos^{-1}(\sin x)$

$$\Rightarrow y = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - x\right)\right\} + \cos^{-1}\left\{\cos\left(\frac{\pi}{2} - x\right)\right\} \Rightarrow y = \frac{\pi}{2} - x + \frac{\pi}{2} - x \Rightarrow y = \pi - 2x$$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\pi)}{dx} - \frac{d(2x)}{dx} \Rightarrow \frac{dy}{dx} = 0 - 2 \therefore \frac{dy}{dx} = -2$$

19. Prove that $\frac{d}{dx}\{2x \cdot \tan^{-1} x - \log(1+x^2)\} = 2 \tan^{-1} x$

Sol. $y = 2x \cdot \tan^{-1} x - \log(1+x^2)$, Differentiating both sides with respect to x

$$\begin{aligned} \frac{dy}{dx} &= \left[2x \frac{d(\tan^{-1} x)}{dx} + \tan^{-1} x \frac{d(2x)}{dx} \right] - \left[\frac{d\{\log(1+x^2)\}}{d(1+x^2)} \times \frac{d(1+x^2)}{dx} \right] \\ \Rightarrow \frac{dy}{dx} &= \left(2x \cdot \frac{1}{1+x^2} + \tan^{-1} x \cdot 2 \right) - \left(\frac{1}{1+x^2} \cdot 2x \right) \therefore \frac{dy}{dx} = 2 \tan^{-1} x \end{aligned}$$

EXERCISE 10D (Pg.No.: 402)

1. $y = \sin^{-1}\left(\sqrt{\frac{1-\cos x}{2}}\right)$

Sol. $y = \sin^{-1}\left\{\sqrt{\frac{2\sin^2 \frac{x}{2}}{2}}\right\} \Rightarrow y = \sin^{-1}\left(\sin \frac{x}{2}\right) \Rightarrow y = \frac{x}{2}$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d\left(\frac{x}{2}\right)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \frac{d(x)}{dx} \therefore \frac{dy}{dx} = \frac{1}{2}$

2. $y = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$

Sol. $y = \tan^{-1}\left(\frac{2\sin \frac{x}{2} \cos \frac{x}{2}}{2\cos^2 \frac{x}{2}}\right) \Rightarrow y = \tan^{-1}\left(\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}\right) \Rightarrow y = \tan^{-1}\left(\tan \frac{x}{2}\right) \Rightarrow y = \frac{x}{2}$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left(\frac{x}{2}\right)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \frac{d(x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \cdot 1 \therefore \frac{dy}{dx} = \frac{1}{2}$$

3. $y = \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

Sol. $y = \cot^{-1}\left(\frac{2\cos^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}}\right) \Rightarrow y = \cot^{-1}\left(\frac{\cos \frac{x}{2}}{\sin \frac{x}{2}}\right) \Rightarrow y = \cot^{-1}\left(\cot \frac{x}{2}\right) \Rightarrow y = \frac{x}{2}$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d\left(\frac{x}{2}\right)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \frac{d(x)}{dx} \therefore \frac{dy}{dx} = \frac{1}{2}$

4. $y = \cot^{-1} \left(\sqrt{\frac{1+\cos x}{1-\cos x}} \right)$

Sol. $y = \cot^{-1} \left(\sqrt{\frac{2\cos^2 \frac{x}{2}}{2\sin^2 \frac{x}{2}}} \right) \Rightarrow y = \cot^{-1} \left(\cot \frac{x}{2} \right) \Rightarrow y = \frac{x}{2}$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d\left(\frac{x}{2}\right)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \frac{d(x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2}$

5. $y = \tan^{-1} \left(\frac{\cos x + \sin x}{\cos x - \sin x} \right)$

Sol. $y = \tan^{-1} \left(\frac{\cos x + \sin x}{\cos x - \sin x} \right)$, Dividing Numerator and Denominator by $\cos x$

$$\Rightarrow y = \tan^{-1} \left(\frac{\frac{\cos x + \sin x}{\cos x}}{\frac{\cos x - \sin x}{\cos x}} \right) \Rightarrow y = \tan^{-1} \left(\frac{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}} \right) \Rightarrow y = \tan^{-1} \left(\frac{1 + \tan x}{1 - \tan x} \right)$$

$$\Rightarrow y = \tan^{-1} \left(\frac{\tan \frac{\pi}{4} + \tan x}{1 - \tan \frac{\pi}{4} \tan x} \right) \Rightarrow y = \tan^{-1} \left\{ \tan \left(\frac{\pi}{4} + x \right) \right\} \Rightarrow y = \frac{\pi}{4} + x$$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d(\pi/4)}{dx} + \frac{d(x)}{dx} \Rightarrow \frac{dy}{dx} = 0 + 1 \therefore \frac{dy}{dx} = 1$

6. $y = \cot^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right)$

Sol. $y = \cot^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right)$, Dividing Numerator and Denominator by $\cos x$

$$\Rightarrow y = \cot^{-1} \left(\frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}} \right) \Rightarrow y = \cot^{-1} \left(\frac{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}} \right) \Rightarrow y = \cot^{-1} \left(\frac{1 - \tan x}{1 + \tan x} \right)$$

$$\Rightarrow y = \cot^{-1} \left\{ \frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \tan x} \right\} \Rightarrow y = \cot^{-1} \left\{ \tan \left(\frac{\pi}{4} - x \right) \right\} \Rightarrow y = \cot^{-1} \left[\cot \left\{ \frac{\pi}{2} - \left(\frac{\pi}{4} - x \right) \right\} \right]$$

$$\Rightarrow y = \frac{\pi}{2} - \frac{\pi}{4} + x \Rightarrow y = \frac{\pi}{4} + x$$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d(\pi/4)}{dx} + \frac{d(x)}{dx} \Rightarrow \frac{dy}{dx} = 0 + 1 \therefore \frac{dy}{dx} = 1$

7. $y = \cot^{-1} \left(\sqrt{\frac{1 + \cos 3x}{1 - \cos 3x}} \right)$

Sol. $y = \cot^{-1} \left(\sqrt{\frac{1 + \cos 3x}{1 - \cos 3x}} \right) \Rightarrow y = \cot^{-1} \left(\sqrt{\frac{2 \cos^2 \frac{3x}{2}}{2 \sin^2 \frac{3x}{2}}} \right)$
 $\Rightarrow y = \cot^{-1} \left(\sqrt{\cot^2 \frac{3x}{2}} \right) \Rightarrow y = \cot^{-1} \left(\cot \frac{3x}{2} \right) \Rightarrow y = \frac{3x}{2}$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d\left(\frac{3}{2}x\right)}{dx} \Rightarrow \frac{dy}{dx} = \frac{3}{2} \frac{d(x)}{dx} \therefore \frac{dy}{dx} = \frac{3}{2}$

8. $y = \sec^{-1} \left(\frac{1 + \tan^2 x}{1 - \tan^2 x} \right)$

Sol. $y = \sec^{-1} \left(\frac{1 + \tan^2 x}{1 - \tan^2 x} \right) \Rightarrow y = \sec^{-1} \left(\frac{1}{\frac{1 - \tan^2 x}{1 + \tan^2 x}} \right) \Rightarrow y = \sec^{-1} \left(\frac{1}{\cos 2x} \right)$
 $\Rightarrow y = \sec^{-1} (\sec 2x) \Rightarrow y = 2x$

Differentiating both sides with respect to x

$\frac{dy}{dx} = \frac{d(2x)}{dx} \Rightarrow \frac{dy}{dx} = 2 \frac{d(x)}{dx} \Rightarrow \frac{dy}{dx} = 2 \cdot 1 \therefore \frac{dy}{dx} = 2$

9. $y = \sin^{-1} \left(\frac{1 - \tan^2 x}{1 + \tan^2 x} \right)$

Sol. $y = \sin^{-1} \left(\frac{1 - \tan^2 x}{1 + \tan^2 x} \right) \Rightarrow y = \sin^{-1} (\cos 2x) \Rightarrow y = \sin^{-1} \left\{ \sin \left(\frac{\pi}{2} - 2x \right) \right\} \Rightarrow y = \frac{\pi}{2} - 2x$

Differentiating both sides with respect to x

$\Rightarrow \frac{dy}{dx} = \frac{d\left(\frac{\pi}{2}\right)}{dx} - \frac{d(2x)}{dx} \Rightarrow \frac{dy}{dx} = 0 - 2 \therefore \frac{dy}{dx} = -2$

10. $y = \operatorname{cosec}^{-1} \left(\frac{1 + \tan^2 x}{2 \tan x} \right)$

Sol. $y = \operatorname{cosec}^{-1} \left(\frac{1 + \tan^2 x}{2 \tan x} \right) \Rightarrow y = \operatorname{cosec}^{-1} \left(\frac{1}{\frac{2 \tan x}{1 + \tan^2 x}} \right)$

$\Rightarrow y = \operatorname{cosec}^{-1} \left(\frac{1}{\sin 2x} \right) \Rightarrow y = \operatorname{cosec}^{-1} (\operatorname{cosec} 2x) \Rightarrow y = 2x$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d(2x)}{dx} \Rightarrow \frac{dy}{dx} = 2 \frac{dx}{dx} \therefore \frac{dy}{dx} = 2$

11. $y = \cot^{-1}(\operatorname{cosec} x + \cot x)$

Sol. $y = \cot^{-1}(\operatorname{cosec} x + \cot x) \Rightarrow y = \cot^{-1}\left(\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right) \Rightarrow y = \cot^{-1}\left(\frac{1 + \cos x}{\sin x}\right)$

$$\Rightarrow y = \cot^{-1}\left(\frac{2 \cos^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}}\right) \Rightarrow y = \cot^{-1}\left(\frac{\cos \frac{x}{2}}{\sin \frac{x}{2}}\right) \Rightarrow y = \cot^{-1}\left(\cot \frac{x}{2}\right) \Rightarrow y = \frac{x}{2}$$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d(x/2)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \cdot \frac{d(x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2}$

12. $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$

Sol. $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x) \Rightarrow y = \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - x\right)\right\} + \cot^{-1}\left\{\cot\left(\frac{\pi}{2} - x\right)\right\}$

$$\Rightarrow y = \frac{\pi}{2} - x + \frac{\pi}{2} - x \Rightarrow y = \pi - 2x$$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d(\pi)}{dx} - \frac{d(2x)}{dx} \Rightarrow \frac{dy}{dx} = 0 - 2 \therefore \frac{dy}{dx} = -2$

13. $y = \sin^{-1}(\sqrt{1-x^2})$

Sol. $y = \sin^{-1}(\sqrt{1-x^2})$, Let $x = \cos \theta \therefore \theta = \cos^{-1}(x)$

$$\Rightarrow y = \sin^{-1}(\sqrt{1-\cos^2 \theta}) \Rightarrow y = \sin^{-1}(\sqrt{\sin^2 \theta}) \Rightarrow y = \sin^{-1}(\sin \theta) \Rightarrow y = \theta$$

Putting the value of θ , $y = \cos^{-1}(x)$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d\{\cos^{-1}(x)\}}{dx} \therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

14. $y = \sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right)$

Sol. $y = \sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right)$, Let $x = \cos \theta \therefore \theta = \cos^{-1}(x) \Rightarrow y = \sin^{-1}\left(\sqrt{\frac{1-\cos \theta}{2}}\right)$

$$\Rightarrow y = \sin^{-1}\left(\sqrt{\frac{2 \sin^2 \theta / 2}{2}}\right) \Rightarrow y = \sin^{-1}\left(\sin \frac{\theta}{2}\right) \Rightarrow y = \frac{\theta}{2}$$

Putting the value of $\theta \Rightarrow y = \frac{1}{2} \cos^{-1}(x)$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{1}{2} \frac{d\{\cos^{-1}(x)\}}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{2\sqrt{1-x^2}}$

15. $y = \cos^{-1} \left(\sqrt{\frac{1+x}{2}} \right)$

Sol. $y = \cos^{-1} \left(\sqrt{\frac{1+x}{2}} \right)$, Let $x = \cos \theta \therefore \theta = \cos^{-1}(x)$

$$\Rightarrow y = \cos^{-1} \left(\sqrt{\frac{1+\cos \theta}{2}} \right) \Rightarrow y = \cos^{-1} \left(\sqrt{\frac{2 \cos^2 \frac{\theta}{2}}{2}} \right)$$

$$\Rightarrow y = \cos^{-1} \left(\sqrt{\cos^2 \theta / 2} \right) \Rightarrow y = \cos^{-1} (\cos \theta / 2) \Rightarrow y = \theta / 2$$

Putting the value of θ , $y = \frac{1}{2} \cos^{-1}(x)$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{1}{2} \frac{d(\cos^{-1} x)}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{2\sqrt{1-x^2}}$

16. $y = \cos^{-1} (\sqrt{1-x^2})$

Sol. $y = \cos^{-1} (\sqrt{1-x^2})$, Let $x = \sin \theta \therefore \theta = \sin^{-1} x$

$$\Rightarrow y = \cos^{-1} (\sqrt{1-\sin^2 \theta}) \Rightarrow y = \cos^{-1} (\sqrt{\cos^2 \theta}) \Rightarrow y = \cos^{-1} (\cos \theta), y = \theta$$

Putting the value of θ , $y = \sin^{-1} x$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d(\sin^{-1} x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

17. $y = \sin^{-1} (2x\sqrt{1-x^2})$

Sol. $y = \sin^{-1} (2x\sqrt{1-x^2})$, Let $x = \sin \theta \therefore \theta = \sin^{-1}(x)$

$$y = \sin^{-1} (2 \sin \theta \sqrt{1-\sin^2 \theta}) \Rightarrow y = \sin^{-1} (2 \sin \theta \cdot \cos \theta) \Rightarrow y = \sin^{-1} (\sin 2\theta) \Rightarrow y = 2\theta$$

Putting the value of θ , $y = 2 \sin^{-1}(x)$

Differentiating both sides with respect to x , $\frac{dy}{dx} = 2 \frac{d(\sin^{-1} x)}{dx} \Rightarrow \frac{dy}{dx} = 2 \cdot \frac{1}{\sqrt{1-x^2}} \therefore \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

18. $y = \sin^{-1} (3x-4x^3)$

Sol. $y = \sin^{-1} (3x-4x^3)$, Let $x = \sin \theta \therefore \theta = \sin^{-1}(x) \Rightarrow y = \sin^{-1} (3 \sin \theta - 4 \sin^3 \theta)$

$$\Rightarrow y = \sin^{-1} (\sin 3\theta) \Rightarrow y = 3\theta$$

Putting the value of θ , $y = 3 \sin^{-1} x$

Differentiating both sides with respect to x , $\frac{dy}{dx} = 3 \frac{d(\sin^{-1} x)}{dx}$

$$\Rightarrow \frac{dy}{dx} = 3 \cdot \frac{1}{\sqrt{1-x^2}} \therefore \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$$

19. $y = \sin^{-1}(1 - 2x^2)$

Sol. $y = \sin^{-1}(1 - 2x^2)$, Let $x = \sin \theta \therefore \theta = \sin^{-1} x$

$$y = \sin^{-1}(1 - 2\sin^2 \theta) \Rightarrow y = \sin^{-1}(\cos 2\theta) \Rightarrow y = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\}$$

$$\Rightarrow y = \frac{\pi}{2} - 2\theta, \text{ Putting the value of } \theta, y = \frac{\pi}{2} - 2\sin^{-1} x$$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left(\frac{\pi}{2}\right)}{dx} - 2 \frac{d(\sin^{-1} x)}{dx} \Rightarrow \frac{dy}{dx} = 0 - 2 \cdot \frac{1}{\sqrt{1-x^2}} \therefore \frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

20. $y = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

Sol. $y = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$, Let $x = \sin \theta \therefore \theta = \sin^{-1} x$

$$y = \sec^{-1}\left(\frac{1}{\sqrt{1-\sin^2 \theta}}\right) \Rightarrow y = \sec^{-1}\left(\frac{1}{\sqrt{\cos^2 \theta}}\right)$$

$$\Rightarrow y = \sec^{-1}\left(\frac{1}{\cos \theta}\right) \Rightarrow y = \sec^{-1}(\sec \theta) \Rightarrow y = \theta$$

Putting the value of θ , $y = \sin^{-1} x$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d(\sin^{-1} x)}{dx} \therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

21. $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Sol. $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, Let $x = \sin \theta \therefore \theta = \sin^{-1}(x)$

$$y = \tan^{-1}\left(\frac{\sin \theta}{\sqrt{1-\sin^2 \theta}}\right) \Rightarrow y = \tan^{-1}\left(\frac{\sin \theta}{\cos \theta}\right) \Rightarrow y = \tan^{-1}(\tan \theta) \Rightarrow y = \theta$$

Putting the value of θ , $y = \sin^{-1} x$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d(\sin^{-1} x)}{dx} \therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

22. $y = \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$

Sol. $y = \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$, Let $x = \sin \theta \therefore \theta = \sin^{-1}(x)$

$$y = \tan^{-1}\left(\frac{\sin \theta}{1+\sqrt{1-\sin^2 \theta}}\right) \Rightarrow y = \tan^{-1}\left(\frac{\sin \theta}{1+\cos \theta}\right) \Rightarrow y = \tan^{-1}\left(\frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}}\right)$$

$$\Rightarrow y = \tan^{-1} \left(\frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \right) \Rightarrow y = \tan^{-1} \left(\tan \frac{\theta}{2} \right) \Rightarrow y = \frac{\theta}{2}$$

Putting the value of θ , $y = \frac{1}{2} \sin^{-1} x$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{1}{2} \frac{d(\sin^{-1} x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}} \therefore \frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}$

23. $y = \cot^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right)$

Sol. $y = \cot^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right)$, Let $x = \sin \theta \therefore \theta = \sin^{-1}(x)$

$$y = \cot^{-1} \left(\frac{\sqrt{1-\sin^2 \theta}}{\sin \theta} \right) \Rightarrow y = \cot^{-1} \left(\frac{\cos \theta}{\sin \theta} \right) \Rightarrow y = \cot^{-1}(\cot \theta) \Rightarrow y = \theta$$

Putting the value of θ , $y = \sin^{-1} x$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d(\sin^{-1} x)}{dx} \therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

24. $y = \sec^{-1} \left(\frac{1}{1-2x^2} \right)$

Sol. $y = \sec^{-1} \left(\frac{1}{1-2x^2} \right)$, Let $x = \sin \theta \therefore \theta = \sin^{-1}(x)$

$$y = \sec^{-1} \left(\frac{1}{1-2\sin^2 \theta} \right) \Rightarrow y = \sec^{-1} \left(\frac{1}{\cos 2\theta} \right) \Rightarrow y = \sec^{-1}(\sec 2\theta) \Rightarrow y = 2\theta$$

Putting the value of θ , $y = 2\sin^{-1}(x)$

Differentiating both sides with respect to x , $\frac{dy}{dx} = 2 \frac{d(\sin^{-1} x)}{dx} \Rightarrow \frac{dy}{dx} = 2 \cdot \frac{1}{\sqrt{1-x^2}} \therefore \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

25. $y = \sin^{-1} \left(\frac{1}{\sqrt{1+x^2}} \right)$

Sol. $y = \sin^{-1} \left(\frac{1}{\sqrt{1+x^2}} \right)$, Let $x = \cot \theta \therefore \theta = \cot^{-1}(x)$

$$y = \sin^{-1} \left(\frac{1}{\sqrt{1+\cot^2 \theta}} \right) \Rightarrow y = \sin^{-1} \left(\frac{1}{\sqrt{\operatorname{cosec}^2 \theta}} \right) \Rightarrow y = \sin^{-1} \left(\frac{1}{\operatorname{cosec} \theta} \right)$$

$$\Rightarrow y = \sin^{-1}(\sin \theta) \Rightarrow y = \theta$$

Putting the value of θ , $y = \cot^{-1}(x)$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d\{\cot^{-1}(x)\}}{dx} \therefore \frac{dy}{dx} = -\frac{1}{1+x^2}$

26. $y = \tan^{-1} \left(\frac{1+x}{1-x} \right)$

Sol. $y = \tan^{-1} \left(\frac{1+x}{1-x} \right)$, Let $x = \tan \theta \therefore \theta = \tan^{-1}(x)$

$$y = \tan^{-1} \left(\frac{1+\tan \theta}{1-\tan \theta} \right) \Rightarrow y = \tan^{-1} \left(\frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} \tan \theta} \right) \Rightarrow y = \tan^{-1} \left\{ \tan \left(\frac{\pi}{4} + \theta \right) \right\} \Rightarrow y = \frac{\pi}{4} + \theta$$

Putting the value of θ , $y = \frac{\pi}{4} + \tan^{-1}(x)$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d\left(\frac{\pi}{4}\right)}{dx} + \frac{d(\tan^{-1} x)}{dx}$

$$\Rightarrow \frac{dy}{dx} = 0 + \frac{1}{1+x^2} \therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

27. $y = \cot^{-1} \left(\frac{1+x}{1-x} \right)$

Sol. $y = \cot^{-1} \left(\frac{1+x}{1-x} \right)$, Let $x = \tan \theta \therefore \theta = \tan^{-1}(x)$

$$y = \cot^{-1} \left(\frac{1+\tan \theta}{1-\tan \theta} \right) \Rightarrow y = \cot^{-1} \left(\frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} \tan \theta} \right) \Rightarrow y = \cot^{-1} \left\{ \tan \left(\frac{\pi}{4} + \theta \right) \right\}$$

$$\Rightarrow y = \cot^{-1} \left\{ \cot \left(\frac{\pi}{2} - \left(\frac{\pi}{4} + \theta \right) \right) \right\} \Rightarrow y = \frac{\pi}{2} - \frac{\pi}{4} - \theta \Rightarrow y = \frac{\pi}{4} - \theta$$

Putting the value of ' θ ', $y = \frac{\pi}{4} - \tan^{-1}(x)$

Differentiating both sides with respect to x

$$\Rightarrow \frac{dy}{dx} = \frac{d\left(\frac{\pi}{4}\right)}{dx} - \frac{d(\tan^{-1} x)}{dx} \Rightarrow \frac{dy}{dx} = 0 - \frac{1}{1+x^2} \therefore \frac{dy}{dx} = -\frac{1}{1+x^2}$$

28. $y = \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right)$

Sol. $y = \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right)$, Let $x = \tan \theta \therefore \theta = \tan^{-1}(x)$

$$y = \tan^{-1} \left(\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right) \Rightarrow y = \tan^{-1} (\tan 3\theta) \Rightarrow y = 3\theta$$

Putting the value of ' θ ', $y = 3 \tan^{-1}(x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = 3 \frac{d(\tan^{-1} x)}{dx} \Rightarrow \frac{dy}{dx} = 3 \cdot \frac{1}{1+x^2} \therefore \frac{dy}{dx} = \frac{3}{1+x^2}$$

29. $y = \operatorname{cosec}^{-1} \left(\frac{1+x^2}{2x} \right)$

Sol. $y = \operatorname{cosec}^{-1} \left(\frac{1+x^2}{2x} \right)$, Let $x = \tan \theta \therefore \theta = \tan^{-1}(x)$

$$y = \operatorname{cosec}^{-1} \left(\frac{1+\tan^2 \theta}{2 \tan \theta} \right) \Rightarrow y = \operatorname{cosec}^{-1} \left(\frac{1}{\frac{2 \tan \theta}{1+\tan^2 \theta}} \right)$$

$$\Rightarrow y = \operatorname{cosec}^{-1} \left(\frac{1}{\sin 2\theta} \right) \Rightarrow y = \operatorname{cosec}^{-1} (\operatorname{cosec} 2\theta) \Rightarrow y = 2\theta$$

Putting the value of ' θ ', $y = 2 \tan^{-1}(x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(2 \tan^{-1} x)}{dx} \Rightarrow \frac{dy}{dx} = 2 \cdot \frac{1}{1+x^2} \therefore \frac{dy}{dx} = \frac{2}{1+x^2}$$

30. $y = \sec^{-1} \left(\frac{1+x^2}{1-x^2} \right)$

Sol. $y = \sec^{-1} \left(\frac{1+x^2}{1-x^2} \right)$, Let $x = \tan \theta \therefore \theta = \tan^{-1}(x)$

$$y = \sec^{-1} \left(\frac{1+\tan^2 \theta}{1-\tan^2 \theta} \right) \Rightarrow y = \sec^{-1} \left(\frac{1}{\frac{1-\tan^2 \theta}{1+\tan^2 \theta}} \right)$$

$$y = \sec^{-1} \left(\frac{1}{\cos 2\theta} \right) \Rightarrow y = \sec^{-1} (\sec 2\theta) \Rightarrow y = 2\theta$$

Putting the value of ' θ ', $y = 2 \tan^{-1}(x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = 2 \frac{d(\tan^{-1} x)}{dx} \Rightarrow \frac{dy}{dx} = 2 \cdot \frac{1}{1+x^2} \therefore \frac{dy}{dx} = \frac{2}{1+x^2}$$

31. $y = \sin^{-1} \left(\frac{1}{\sqrt{1+x^2}} \right)$

Sol. $y = \sin^{-1} \left(\frac{1}{\sqrt{1+x^2}} \right)$, Let $x = \tan \theta \therefore \theta = \tan^{-1}(x)$

$$\Rightarrow y = \sin^{-1} \left(\frac{1}{\sqrt{1+\tan^2 \theta}} \right) \Rightarrow y = \sin^{-1} \left(\frac{1}{\sec \theta} \right) \Rightarrow y = \sin^{-1} (\cos \theta)$$

$$\Rightarrow y = \sin^{-1} \left\{ \sin \left(\frac{\pi}{2} - \theta \right) \right\} \Rightarrow y = \frac{\pi}{2} - \theta$$

Putting the value of ' θ ', $y = \frac{\pi}{2} - \tan^{-1}(x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\left(\frac{\pi}{2}\right)}{dx} - \frac{d(\tan^{-1} x)}{dx} \Rightarrow \frac{dy}{dx} = 0 - \frac{1}{1+x^2} \Rightarrow \frac{dy}{dx} = -\frac{1}{1+x^2}$$

32. $y = \sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Sol. $y = \sec^{-1}\left(\frac{x^2+1}{x^2-1}\right) \Rightarrow y = \sec^{-1}\left\{\frac{-(1+x^2)}{(1-x^2)}\right\} \Rightarrow y = \pi - \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

Let $x = \tan \theta \therefore \theta = \tan^{-1}(x)$, $y = \pi - \sec^{-1}\left(\frac{1+\tan^2 \theta}{1-\tan^2 \theta}\right)$

$$\Rightarrow y = \pi - \sec^{-1}\left(\frac{1-\tan^2 \theta}{1+\tan^2 \theta}\right) \Rightarrow y = \pi - \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$

$$\Rightarrow y = \pi - \sec^{-1}(\sec 2\theta) \Rightarrow y = \pi - 2\theta$$

Putting the value of ' θ ', $y = \pi - 2\tan^{-1}(x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\pi)}{dx} - 2 \frac{d(\tan^{-1} x)}{dx} \Rightarrow \frac{dy}{dx} = 0 - 2 \cdot \frac{1}{1+x^2} \Rightarrow \frac{dy}{dx} = -\frac{2}{1+x^2}$$

33. $y = \cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right)$

Sol. $y = \cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right) \Rightarrow y = \cos^{-1}\left\{\frac{1-(x^n)^2}{1+(x^n)^2}\right\}$, Let $\tan \theta = x^n \therefore \theta = \tan^{-1}(x^n)$

$$y = \cos^{-1}\left(\frac{1-\tan^2 \theta}{1+\tan^2 \theta}\right) \Rightarrow y = \cos^{-1}(\cos 2\theta) \Rightarrow y = 2\theta$$

Putting the value of ' θ ', $y = 2\tan^{-1}(x^n)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = 2 \frac{d\{\tan^{-1}(x^n)\}}{d(x^n)} \times \frac{d(x^n)}{dx} \Rightarrow \frac{dy}{dx} = 2 \cdot \frac{1}{1+(x^n)^2} \cdot nx^{n-1} \therefore \frac{dy}{dx} = \frac{2nx^{n-1}}{1+x^{2n}}$$

34. $y = \tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$

Sol. $y = \tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$, Let $x = a \sin \theta \Rightarrow \frac{x}{a} = \sin \theta \therefore \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$$\Rightarrow y = \tan^{-1}\left(\frac{a \sin \theta}{\sqrt{a^2 - a^2 \sin^2 \theta}}\right) \Rightarrow y = \tan^{-1}\left(\frac{a \sin \theta}{\sqrt{a^2(1-\sin^2 \theta)}}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\frac{a \sin \theta}{a \cos \theta}\right) \Rightarrow y = \tan^{-1}(\tan \theta) \Rightarrow y = \theta$$

Putting the value of θ , $y = \sin^{-1}\left(\frac{x}{a}\right)$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d\left\{\sin^{-1}\left(\frac{x}{a}\right)\right\}}{d\left(\frac{x}{a}\right)} \times d\left(\frac{x}{a}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{1}{a} \frac{d(x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \cdot \frac{1}{a} \Rightarrow \frac{dy}{dx} = \frac{a}{\sqrt{a^2-x^2}} \cdot \frac{1}{a} \therefore \frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}}$$

35. $y = \sin^{-1}\left\{2ax\sqrt{1-a^2x^2}\right\}$

Sol. $y = \sin^{-1}\left\{2ax\sqrt{1-a^2x^2}\right\}$, Let $ax = \sin \theta \therefore \theta = \sin^{-1}(ax)$

$$\Rightarrow y = \sin^{-1}\left\{2\sin \theta \sqrt{1-\sin^2 \theta}\right\} \Rightarrow y = \sin^{-1}\{\sin 2\theta\},$$

Putting the value of θ , $y = 2\sin^{-1}ax$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = 2 \frac{d\{\sin^{-1}(ax)\}}{d(ax)} \times \frac{d(ax)}{dx} \Rightarrow \frac{dy}{dx} = 2 \cdot \frac{1}{\sqrt{1-(ax)^2}} \times a \Rightarrow \frac{dy}{dx} = \frac{2a}{\sqrt{1-a^2x^2}}$$

36. $y = \tan^{-1}\left(\frac{\sqrt{1+a^2x^2}-1}{ax}\right)$

Sol. $y = \tan^{-1}\left(\frac{\sqrt{1+(ax)^2}-1}{ax}\right)$, Let $ax = \tan \theta \therefore \theta = \tan^{-1}(ax)$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2 \theta}-1}{\tan \theta}\right) \Rightarrow y = \tan^{-1}\left(\frac{\sec \theta - 1}{\tan \theta}\right) \Rightarrow y = \tan^{-1}\left(\frac{\frac{1}{\cos \theta} - 1}{\frac{\sin \theta}{\cos \theta}}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\frac{1-\cos \theta}{\sin \theta}\right) \Rightarrow y = \tan^{-1}\left(\frac{2\sin^2 \frac{\theta}{2}}{2\sin \frac{\theta}{2} \cos \frac{\theta}{2}}\right) \Rightarrow y = \tan^{-1}\left(\frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\tan \frac{\theta}{2}\right) \Rightarrow y = \frac{\theta}{2}$$

Putting the value of θ , $y = \frac{1}{2}\tan^{-1}(ax)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{1}{2} \frac{d\{\tan^{-1}(ax)\}}{d(ax)} \times \frac{d(ax)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1+(ax)^2} \cdot a \therefore \frac{dy}{dx} = \frac{a}{2(1+a^2x^2)}$$

37. $y = \sin^{-1} \left(\frac{x^2}{\sqrt{x^4 + a^4}} \right)$

Sol. $y = \sin^{-1} \left(\frac{x^2}{\sqrt{x^4 + a^4}} \right)$, Let $x^2 = a^2 \cot \theta \Rightarrow \frac{x^2}{a^2} = \cot \theta \therefore \theta = \cot^{-1} \left(\frac{x^2}{a^2} \right)$

$$y = \sin^{-1} \left(\frac{a^2 \cot \theta}{\sqrt{a^4 \cot^2 \theta + a^4}} \right) \Rightarrow y = \sin^{-1} \left(\frac{a^2 \cot \theta}{\sqrt{a^4 (\cot^2 \theta + 1)}} \right)$$

$$\Rightarrow y = \sin^{-1} \left(\frac{a^2 \cot \theta}{a^2 \operatorname{cosec} \theta} \right) \Rightarrow y = \sin^{-1} \left(\frac{\cos \theta}{\sin \theta} \times \frac{\sin \theta}{1} \right)$$

$$\Rightarrow y = \sin^{-1} (\cos \theta) \Rightarrow y = \sin^{-1} \left\{ \sin \left(\frac{\pi}{2} - \theta \right) \right\} \Rightarrow y = \frac{\pi}{2} - \theta$$

Putting the value of θ , $y = \frac{\pi}{2} - \cot^{-1} \left(\frac{x^2}{a^2} \right)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d \left(\frac{\pi}{2} \right)}{dx} - \frac{d \left\{ \cot^{-1} \left(\frac{x^2}{a^2} \right) \right\}}{d \left(\frac{x^2}{a^2} \right)} \times \frac{d \left(\frac{x^2}{a^2} \right)}{dx} \Rightarrow \frac{dy}{dx} = 0 + \frac{1}{1 + \left(\frac{x^2}{a^2} \right)^2} \cdot \frac{1}{a^2} \cdot 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{a^4}{a^4 + x^4} \cdot \frac{1}{a^2} \cdot 2x \Rightarrow \frac{dy}{dx} = \frac{2a^2 x}{a^4 + x^4}$$

38. $y = \tan^{-1} \left(\frac{e^{2x} + 1}{e^{2x} - 1} \right)$

Sol. $y = \tan^{-1} \left(\frac{e^{2x} + 1}{e^{2x} - 1} \right) \Rightarrow y = \tan^{-1} \left\{ \frac{e^{2x} + 1}{-(1 - e^{2x})} \right\} \Rightarrow y = -\tan^{-1} \left(\frac{1 + e^{2x}}{1 - e^{2x}} \right)$

Let $e^{2x} = \tan \theta \therefore \theta = \tan^{-1} (e^{2x})$

$$y = -\tan^{-1} \left(\frac{1 + \tan \theta}{1 - \tan \theta} \right) \Rightarrow y = -\tan^{-1} \left(\frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} + \tan \theta} \right)$$

$$\Rightarrow y = -\tan^{-1} \left\{ \tan \left(\frac{\pi}{4} + \theta \right) \right\} \Rightarrow y = -\frac{\pi}{4} - \theta$$

Putting the value of θ , $y = -\frac{\pi}{4} - \tan^{-1} (e^{2x})$

Differentiating both sides with respect to x , $\frac{dy}{dx} = -\frac{d \left(\frac{\pi}{4} \right)}{dx} - \frac{d \left\{ \tan^{-1} (e^{2x}) \right\}}{d (e^{2x})} \times \frac{d (e^{2x})}{d (2x)} \times \frac{d (2x)}{dx}$

$$\Rightarrow \frac{dy}{dx} = 0 - \frac{1}{1 + (e^{2x})^2} \cdot e^{2x} \cdot 2 \therefore \frac{dy}{dx} = \frac{-2e^{2x}}{1 + e^{4x}}$$

39. $y = \cos^{-1}(2x) + 2 \cos^{-1}(\sqrt{1-4x^2})$

Sol. $y = \cos^{-1}(2x) + 2 \cos^{-1}(\sqrt{1-4x^2})$, Let $2x = \cos \theta \therefore \theta = \cos^{-1}(2x)$

$$y = \cos^{-1}(\cos \theta) + 2 \cos^{-1}(\sqrt{1-\cos^2 \theta})$$

$$\Rightarrow y = \theta + 2 \cos^{-1}(\sin \theta) \Rightarrow y = \theta + 2 \cos^{-1}\left\{\cos\left(\frac{\pi}{2}-\theta\right)\right\} \Rightarrow y = \theta + 2\left(\frac{\pi}{2}-\theta\right)$$

$$\Rightarrow y = \theta + \pi - 2\theta \Rightarrow y = \pi - \theta$$

Putting the value of θ , $y = \pi - \cos^{-1}(2x)$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d(\pi)}{dx} - \frac{d\{\cos^{-1}(2x)\}}{d(2x)} \times \frac{d(2x)}{dx}$

$$\Rightarrow \frac{dy}{dx} = 0 + \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 \therefore \frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}$$

40. $y = \tan^{-1}\left(\frac{a-x}{1+ax}\right)$

Sol. $y = \tan^{-1}\left(\frac{a-x}{1+ax}\right) = \tan^{-1} a - \tan^{-1} x \Rightarrow \frac{dy}{dx} = 0 - \frac{d}{dx}(\tan^{-1} x) = \frac{-1}{1+x^2}$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{d\{\tan^{-1}(a)\}}{d(a)} \times \frac{d(a)}{dx} - \frac{d(\tan^{-1} x)}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+a^2} \cdot 0 - \frac{1}{1+x^2} \therefore \frac{dy}{dx} = -\frac{1}{1+x^2}$$

41. $y = \tan^{-1}\left(\frac{\sqrt{x}-x}{1+x^{3/2}}\right)$

Sol. $y = \tan^{-1}\left(\frac{\sqrt{x}-x}{1+x^{3/2}}\right)$, $y = \tan^{-1}\left(\frac{\sqrt{x}-x}{1+\sqrt{x}x}\right) = \tan^{-1}\sqrt{x} - \tan^{-1} x$

Differentiating both sides with respect to x , $\frac{dy}{dx} = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} - \frac{1}{1+x^2}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}(1+x)} - \frac{1}{1+x^2}$$

42. $y = \tan^{-1}\left(\frac{\sqrt{a}+\sqrt{x}}{1-\sqrt{ax}}\right)$

Sol. $y = \tan^{-1}\left(\frac{\sqrt{a}+\sqrt{x}}{1-\sqrt{ax}}\right) \Rightarrow y = \tan^{-1}(\sqrt{a}) + \tan^{-1}(\sqrt{x}) \Rightarrow \frac{dy}{dx} = 0 + \frac{d}{dx}(\tan^{-1}\sqrt{x}) = \frac{1}{2\sqrt{x}(1+x)}$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\tan^{-1}(\sqrt{a})\}}{d(\sqrt{a})} \times \frac{d(\sqrt{a})}{dx} + \frac{d\{\tan^{-1}(\sqrt{x})\}}{d(\sqrt{x})} \times \frac{d(\sqrt{x})}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 0 + \frac{1}{2\sqrt{x}(1+x)} \therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}(1+x)}$$

43. $y = \tan^{-1}\left(\frac{3-2x}{1+6x}\right)$

Sol. $y = \tan^{-1}\left(\frac{3-2x}{1+6x}\right) \Rightarrow y = \tan^{-1}(3) - \tan^{-1}(2x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\tan^{-1}(3)\}}{dx} - \frac{d\{\tan^{-1}(2x)\}}{d(2x)} \times \frac{d(2x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 0 - \frac{1}{1+(2x)^2} \cdot 2 \therefore \frac{dy}{dx} = -\frac{2}{1+4x^2}$$

44. $y = \tan^{-1}\left(\frac{5x}{1-6x^2}\right)$

Sol. $y = \tan^{-1}\left(\frac{5x}{1-6x^2}\right) \Rightarrow y = \tan^{-1}\left(\frac{3x+2x}{1-3x \cdot 2x}\right)$

Let $3x = \tan A \therefore A = \tan^{-1}(3x)$ & $2x = \tan B \therefore B = \tan^{-1}(2x)$

$$y = \tan^{-1}\left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) \Rightarrow y = \tan^{-1}\{\tan(A+B)\} \Rightarrow y = A+B$$

Putting the value of A & B , $y = \tan^{-1}(3x) + \tan^{-1}(2x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\tan^{-1}(3x)\}}{d(3x)} \times \frac{d(3x)}{dx} + \frac{d\{\tan^{-1}(2x)\}}{d(2x)} \times \frac{d(2x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+(3x)^2} \cdot 3 + \frac{1}{1+(2x)^2} \therefore \frac{dy}{dx} = \frac{3}{1+9x^2} + \frac{2}{1+4x^2}$$

45. $y = \tan^{-1}\left(\frac{2x}{1+15x^2}\right)$

Sol. $y = \tan^{-1}\left(\frac{2x}{1+15x^2}\right) \Rightarrow y = \tan^{-1}\left(\frac{5x-3x}{1+5x \cdot 3x}\right) \Rightarrow y = \tan^{-1}(5x) - \tan^{-1}(3x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\tan^{-1}(5x)\}}{d(5x)} \times \frac{d(5x)}{dx} - \frac{d\{\tan^{-1}(3x)\}}{d(3x)} \times \frac{d(3x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+(5x)^2} \cdot 5 - \frac{1}{1+(3x)^2} \cdot 3 \therefore \frac{dy}{dx} = \frac{5}{1+25x^2} - \frac{3}{1+9x^2}$$

46. If $y = \tan^{-1}\left(\frac{ax-b}{a+bx}\right)$, prove that $\frac{dy}{dx} = \frac{1}{1+x^2}$

Sol. $y = \tan^{-1}\left(\frac{ax-b}{a+bx}\right)$

Dividing nominator and denominator by a , $y = \tan^{-1} \left(\frac{ax-b}{\frac{a}{a+bx}} \right) \Rightarrow y = \tan^{-1} \left(\frac{\frac{ax-b}{a}}{\frac{a}{a} + \frac{b}{a} \cdot x} \right)$

$$\Rightarrow y = \tan^{-1} \left(\frac{x - \frac{b}{a}}{1 + \frac{b}{a} \cdot x} \right) \Rightarrow y = \tan^{-1}(x) - \tan^{-1}(b/a)$$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d\{\tan^{-1}(x)\}}{dx} - \frac{d\left(\tan^{-1} \frac{b}{a}\right)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2} \therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

47. If $y = \sin^{-1} \left(\frac{2x}{1+x^2} \right) + \sec^{-1} \left(\frac{1+x^2}{1-x^2} \right)$ show that $\frac{dy}{dx} = \frac{4}{1+x^2}$

Sol. $y = \sin^{-1} \left(\frac{2x}{1+x^2} \right) + \sec^{-1} \left(\frac{1+x^2}{1-x^2} \right)$, Let $x = \tan \theta \therefore \theta = \tan^{-1}(x)$

$$y = \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) + \sec^{-1} \left(\frac{1 + \tan^2 \theta}{1 - \tan^2 \theta} \right)$$

$$\Rightarrow y = \sin^{-1}(\sin 2\theta) + \sec^{-1}(\sec 2\theta) \Rightarrow y = 2\theta + 2\theta \Rightarrow y = 4\theta$$

Putting the value of θ , $y = 4 \tan^{-1}(x)$

Differentiating both sides with respect to x $\frac{dy}{dx} = 4 \frac{d(\tan^{-1} x)}{dx} \Rightarrow \frac{dy}{dx} = 4 \cdot \frac{1}{1+x^2} \therefore \frac{dy}{dx} = \frac{4}{1+x^2}$

48. If $y = \sec^{-1} \left(\frac{x+1}{x-1} \right) + \sin^{-1} \left(\frac{x-1}{x+1} \right)$ show that $\frac{dy}{dx} = 0$

Sol. $y = \sec^{-1} \left(\frac{x+1}{x-1} \right) + \sin^{-1} \left(\frac{x-1}{x+1} \right) \Rightarrow y = \cos^{-1} \left(\frac{x-1}{x+1} \right) + \sin^{-1} \left(\frac{x-1}{x+1} \right)$ $\left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \right]$

$$\Rightarrow y = \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \frac{d\left(\frac{\pi}{2}\right)}{dx} \therefore \frac{dy}{dx} = 0$$

49. If $y = \sin \left\{ 2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} \right\}$ show that $\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}$

Sol. $y = \sin \left\{ 2 \tan^{-1} \left(\sqrt{\frac{1-x}{1+x}} \right) \right\}$

Let $x = \cos \theta \therefore \theta = \cos^{-1}(x)$, $y = \sin \left\{ 2 \tan^{-1} \sqrt{\frac{1-\cos \theta}{1+\cos \theta}} \right\} \Rightarrow y = \sin \left\{ 2 \cdot \tan^{-1} \left(\sqrt{\frac{2 \sin^2 \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}}} \right) \right\}$

$$y = \sin \left\{ 2 \tan^{-1} \left(\tan \frac{\theta}{2} \right) \right\} \Rightarrow y = \sin \left\{ 2 \times \frac{\theta}{2} \right\} \Rightarrow y = \sin \theta$$

Putting the value of θ , $y = \sin(\cos^{-1} x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d \sin(\cos^{-1} x)}{d(\cos^{-1} x)} \times \frac{d(\cos^{-1} x)}{dx} \Rightarrow \frac{dy}{dx} = -\cos(\cos^{-1} x) \cdot \frac{1}{\sqrt{1-x^2}} \therefore \frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}$$

50. If $y = \tan^{-1} \left\{ \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right\}$ prove that $\frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}$

Sol. $y = \tan^{-1} \left\{ \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right\}$, Let $x = \cos 2\theta \Rightarrow 2\theta = \cos^{-1}(x) \therefore \theta = \frac{1}{2} \cos^{-1}(x)$

$$\Rightarrow y = \tan^{-1} \left\{ \frac{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}} \right\} \Rightarrow y = \tan^{-1} \left\{ \frac{\sqrt{2\cos^2 \theta} - \sqrt{2\sin^2 \theta}}{\sqrt{2\cos^2 \theta} + \sqrt{2\sin^2 \theta}} \right\}$$

$$\Rightarrow y = \tan^{-1} \left\{ \frac{\sqrt{2} \cos \theta - \sqrt{2} \sin \theta}{\sqrt{2} \cos \theta + \sqrt{2} \sin \theta} \right\} \Rightarrow y = \tan^{-1} \left\{ \frac{\sqrt{2} (\cos \theta - \sin \theta)}{\sqrt{2} (\cos \theta + \sin \theta)} \right\} \Rightarrow y = \tan^{-1} \left(\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \right)$$

Dividing numerator and denominator by $\cos \theta$

$$y = \tan^{-1} \left(\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \right) \Rightarrow y = \tan^{-1} \left(\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \right) \Rightarrow y = \tan^{-1} \left(\frac{1 - \tan \theta}{1 + \tan \theta} \right)$$

$$\Rightarrow y = \tan^{-1} \left(\frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \tan \theta} \right) \Rightarrow y = \tan^{-1} \left\{ \tan \left(\frac{\pi}{4} - \theta \right) \right\} \Rightarrow y = \frac{\pi}{4} - \theta$$

Putting the value of θ , $y = \frac{\pi}{4} - \frac{1}{2} \cos^{-1}(x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d(\pi/4)}{dx} - \frac{1}{2} \frac{d(\cos^{-1} x)}{dx} \Rightarrow \frac{dy}{dx} = 0 + \frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}} \therefore \frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}$$

51. Differentiate $\sin^{-1} \left(\frac{2^{x+1}}{1+4^x} \right)$ w.r.t. x

Sol. $y = \sin^{-1} \left(\frac{2^{x+1}}{1+4^x} \right) \Rightarrow y = \sin^{-1} \left(\frac{2^x \cdot 2^1}{1+(2^2)^x} \right) \Rightarrow y = \sin^{-1} \left(\frac{2 \cdot 2^x}{1+(2^x)^2} \right)$

Let $2^x = \tan \theta \therefore \theta = \tan^{-1}(2^x)$, $y = \sin^{-1} \left(\frac{2 \cdot \tan \theta}{1 + \tan^2 \theta} \right) \Rightarrow y = \sin^{-1}(\sin 2\theta) \Rightarrow y = 2\theta$

Putting the value of θ , $y = 2 \tan^{-1}(2^x)$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = 2 \frac{d\{\tan^{-1}(2^x)\}}{d(2^x)} \times \frac{d(2^x)}{dx} \Rightarrow \frac{dy}{dx} = 2 \cdot \frac{1}{1+(2^x)^2} \cdot 2^x \log 2 \Rightarrow \frac{dy}{dx} = \frac{2^{1+x} \log 2}{1+4^x}$$

Exercise 10 E

Find $\frac{dy}{dx}$, when:

1. $x^2 + y^2 = 4$

Sol. Differentiating both sides with respect to x , $\frac{d(x^2)}{dx} + \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = \frac{d(4)}{dx}$
 $\Rightarrow 2x \cdot 1 + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow 2y \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} \therefore \frac{dy}{dx} = -\frac{x}{y}$

2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol. Differentiating both sides with respect to x , $\frac{1}{a^2} \frac{d(x^2)}{dx} + \frac{1}{b^2} \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = \frac{d(1)}{dx}$
 $\Rightarrow \frac{1}{a^2} \cdot 2x \cdot 1 + \frac{1}{b^2} \cdot 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{2y}{b^2} = \frac{-2x}{a^2} \therefore \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$

3. $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Sol. Differentiating both sides with respect to x

$$\frac{d(\sqrt{x})}{dx} + \frac{d(\sqrt{y})}{dy} \cdot \frac{dy}{dx} = \frac{d(\sqrt{a})}{da} \Rightarrow \frac{1}{2\sqrt{x}} \cdot 1 + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0 \Rightarrow \frac{1}{2\sqrt{y}} \frac{dy}{dx} = -\frac{1}{2\sqrt{x}} \Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} \therefore \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

4. $x^{2/3} + y^{2/3} = a^{2/3}$

Sol. Differentiating both sides with respect to x , $\frac{d(x^{2/3})}{dx} + \frac{d(y^{2/3})}{dy} \cdot \frac{dy}{dx} = \frac{d(a^{2/3})}{da} \cdot \frac{da}{dx}$
 $\Rightarrow \frac{2}{3} x^{\frac{2}{3}-1} \cdot 1 + \frac{2}{3} y^{\frac{2}{3}-1} \frac{dy}{dx} = \frac{2}{3} a^{\frac{2}{3}-1} \cdot 0 \Rightarrow \frac{2}{3} x^{-\frac{1}{3}} + \frac{2}{3} y^{-\frac{1}{3}} \frac{dy}{dx} = 0$
 $\Rightarrow \frac{2}{3} y^{-\frac{1}{3}} \frac{dy}{dx} = -\frac{2}{3} x^{-\frac{1}{3}} \Rightarrow \frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} \therefore \frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

5. $xy = c^2$

Sol. $xy = c^2$... (i)

Differentiating both sides with respect to x

$$x \frac{dy}{dx} + y \frac{dx}{dx} = \frac{d(c^2)}{dc} \cdot \frac{dc}{dx} \Rightarrow x \frac{dy}{dx} + y \cdot 1 = 2c \cdot 0 \Rightarrow x \frac{dy}{dx} + y = 0$$

$$\Rightarrow x \frac{dy}{dx} = -y \Rightarrow \frac{dy}{dx} = -\frac{y}{x} \quad \dots (ii)$$

From equation (i), $xy = c^2 \Rightarrow y = \frac{c^2}{x}$

Putting the value of y in equation (ii), $\frac{dy}{dx} = -\frac{c^2}{x^2}$

6. $x^2 + y^2 - 3xy = 1$

Sol. Differentiating both sides with respect to x , $\frac{d(x^2)}{dx} + \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} - 3 \left[x \frac{dy}{dx} + y \frac{dx}{dx} \right] = \frac{d(1)}{dx}$

$$\Rightarrow 2x \cdot 1 + 2y \frac{dy}{dx} - 3 \left[x \frac{dy}{dx} + y \cdot 1 \right] = 0 \Rightarrow 2x + 2y \frac{dy}{dx} - 3x \frac{dy}{dx} - 3y = 0$$

$$\Rightarrow 2y \frac{dy}{dx} - 3x \frac{dy}{dx} = 3y - 2x \Rightarrow \frac{dy}{dx} (2y - 3x) = 3y - 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{3y - 2x}{2y - 3x} \Rightarrow \frac{dy}{dx} = \frac{-(2x - 3y)}{-(3x - 2y)} \therefore \frac{dy}{dx} = \frac{2x - 3y}{3x - 2y}$$

7. $xy^2 - x^2y - 5 = 0$

Sol. Differentiating both sides with respect to x , $\frac{d(xy^2)}{dx} - \frac{d(x^2y)}{dx} - \frac{d(5)}{dx} = 0$

$$\Rightarrow \left\{ x \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} + y^2 \frac{d(x)}{dx} \right\} - \left\{ x^2 \frac{dy}{dx} + y \frac{d(x^2)}{dx} \right\} - 0 = 0$$

$$\Rightarrow \left(x \cdot 2y \frac{dy}{dx} + y^2 \cdot 1 \right) - \left(x^2 \frac{dy}{dx} + 2xy \right) = 0 \Rightarrow 2xy \frac{dy}{dx} - x^2 \frac{dy}{dx} = 2xy - y^2$$

$$\Rightarrow \frac{dy}{dx} (2xy - x^2) = 2xy - y^2 \Rightarrow \frac{dy}{dx} = \frac{-(y^2 - 2xy)}{-(x^2 - 2xy)} \therefore \frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - 2xy}$$

8. $(x^2 + y^2)^2 = xy$

Sol. Differentiating both sides with respect to x , $\frac{d\{(x^2 + y^2)^2\}}{d(x^2 + y^2)} \times \frac{d(x^2 + y^2)}{dx} = \frac{d(xy)}{dx}$

$$\Rightarrow 2(x^2 + y^2) \times \left\{ \frac{d(x^2)}{dx} + \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} \right\} = x \frac{dy}{dx} + y \frac{dx}{dx}$$

$$\Rightarrow 2(x^2 + y^2) \cdot \left\{ 2x \cdot 1 + 2y \frac{dy}{dx} \right\} = x \frac{dy}{dx} + y \cdot 1 \Rightarrow 2(x^2 + y^2) \left(2x + 2y \frac{dy}{dx} \right) = x \frac{dy}{dx} + y$$

$$\Rightarrow 4x(x^2 + y^2) + 4y(x^2 + y^2) \frac{dy}{dx} = x \frac{dy}{dx} + y \Rightarrow 4y(x^2 + y^2) \frac{dy}{dx} - x \frac{dy}{dx} = y - 4x(x^2 + y^2)$$

$$\Rightarrow \frac{dy}{dx} (4x^2y + 4y^3 - x) = y - 4x^3 - 4xy^2 \therefore \frac{dy}{dx} = \frac{y - 4x^3 - 4xy^2}{4y^3 + 4x^2y - x}$$

9. $x^2 + y^2 = \log(xy)$

Sol. Differentiating both sides with respect to x , $\frac{d(x^2)}{dx} \cdot \frac{dx}{dx} + \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = \frac{d\{\log(xy)\}}{d(xy)} \times \frac{d(xy)}{dx}$

$$\Rightarrow 2x \cdot 1 + 2y \frac{dy}{dx} = \frac{1}{xy} \left\{ x \frac{dy}{dx} + y \frac{dx}{dx} \right\} \Rightarrow 2x + 2y \frac{dy}{dx} = \frac{1}{xy} \left(x \frac{dy}{dx} + y \right)$$

$$\Rightarrow 2x + 2y \frac{dy}{dx} = \frac{1}{xy} \cdot x \frac{dy}{dx} + \frac{1}{xy} \cdot y \Rightarrow 2x + 2y \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx} + \frac{1}{x}$$

$$\Rightarrow 2y \frac{dy}{dx} - \frac{1}{y} \frac{dy}{dx} = \frac{1}{x} - 2x \Rightarrow \frac{dy}{dx} \left(2y - \frac{1}{y} \right) = \left(\frac{1}{x} - 2x \right)$$

$$\Rightarrow \frac{dy}{dx} \left(\frac{2y^2 - 1}{y} \right) = \left(\frac{1 - 2x^2}{x} \right) \Rightarrow \frac{dy}{dx} = \frac{y(1 - 2x^2)}{x(2y^2 - 1)}$$

10. $x^n + y^n = a^n$

Sol. Differentiating both sides with respect to x , $\frac{d(x^n)}{dx} \cdot \frac{dx}{dx} + \frac{d(y^n)}{dy} \cdot \frac{dy}{dx} = \frac{d(a^n)}{da} \cdot \frac{da}{dx}$

$$\Rightarrow nx^{n-1} + ny^{n-1} \frac{dy}{dx} = na^{n-1} \cdot 0 \Rightarrow nx^{n-1} + ny^{n-1} \frac{dy}{dx} = 0$$

$$\Rightarrow ny^{n-1} \frac{dy}{dx} = -nx^{n-1} \Rightarrow \frac{dy}{dx} = -\frac{nx^{n-1}}{ny^{n-1}} \therefore \frac{dy}{dx} = -\frac{x^{n-1}}{y^{n-1}}$$

11. $x \sin 2y = y \cos 2x$

Sol. Differentiating both sides with respect to x

$$x \frac{d(\sin 2y)}{d(2y)} \times \frac{d(2y)}{dx} + \sin 2y \frac{d(x)}{dx} = y \frac{d(\cos 2x)}{d(2x)} \times \frac{d(2x)}{dx} + \cos 2x \frac{d(y)}{dx}$$

$$\Rightarrow x \cos 2y \cdot 2 \frac{dy}{dx} + \sin 2y \cdot 1 = -y \sin 2x \cdot 2 + \cos 2x \frac{dy}{dx}$$

$$\Rightarrow 2x \cos 2y \frac{dy}{dx} - \cos 2x \frac{dy}{dx} = -2y \sin 2x - \sin 2y \Rightarrow \frac{dy}{dx} (2x \cos 2y - \cos 2x) = -(2y \sin 2x + \sin 2y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(2y \sin 2x + \sin 2y)}{-(\cos 2x - 2x \cos 2y)} \therefore \frac{dy}{dx} = \frac{2y \sin 2x + \sin 2y}{\cos 2x - 2x \cos 2y}$$

12. $\sin^2 x + 2 \cos y + xy = 0$

Sol. Differentiating both sides with respect to x , $\frac{d(\sin^2 x)}{d(\sin x)} \times \frac{d(\sin x)}{dx} + 2 \frac{d(\cos y)}{dy} \cdot \frac{dy}{dx} + \frac{d(xy)}{dx} = 0$

$$\Rightarrow 2 \sin x \cos x + 2(-\sin y) \frac{dy}{dx} + \left(x \frac{dy}{dx} + y \frac{dx}{dx} \right) = 0$$

$$\Rightarrow 2 \sin 2x - 2 \sin y \frac{dy}{dx} + x \frac{dy}{dx} + y = 0 \Rightarrow -2 \sin y \frac{dy}{dx} + x \frac{dy}{dx} = -y - \sin 2x$$

$$\Rightarrow \frac{dy}{dx} (-2 \sin y + x) = -(y + \sin 2x) \Rightarrow \frac{dy}{dx} = \frac{-(y + \sin 2x)}{-(2 \sin y - x)} \therefore \frac{dy}{dx} = \frac{y + \sin 2x}{2 \sin y - x}$$

13. $y \sec x + \tan x + x^2 y = 0$

Sol. Differentiating both sides with respect to x , $y \frac{d(\sec x)}{dx} + \sec x \frac{dy}{dx} + \frac{d(\tan x)}{dx} + \left(x^2 \frac{dy}{dx} + y \frac{d(x^2)}{dx} \right) = 0$

$$\Rightarrow y \sec x \cdot \tan x + \sec x \frac{dy}{dx} + \sec^2 x + x^2 \frac{dy}{dx} + y \cdot 2x = 0$$

$$\Rightarrow \sec x \frac{dy}{dx} + x^2 \frac{dy}{dx} = -2xy - \sec^2 x - y \sec x \tan x$$

$$\Rightarrow \frac{dy}{dx} (\sec x + x^2) = -(2xy + \sec^2 x + y \sec x \tan x) \therefore \frac{dy}{dx} = -\frac{(2xy + \sec^2 x + y \sec x \tan x)}{\sec x + x^2}$$

14. $\cot(xy) + xy = y$

Sol. Differentiating both sides with respect to x , $\frac{d\{\cot(xy)\}}{d(xy)} \times \frac{d(xy)}{dx} + \frac{d(xy)}{dx} = \frac{d(y)}{dx}$

$$\Rightarrow -\operatorname{cosec}^2(xy) \left\{ x \frac{dy}{dx} + y \frac{dx}{dx} \right\} + \left\{ x \frac{dy}{dx} + y \frac{dx}{dx} \right\} = \frac{dy}{dx}$$

$$\Rightarrow -\operatorname{cosec}^2(xy) \left\{ x \frac{dy}{dx} + y \right\} + \left\{ x \frac{dy}{dx} + y \right\} = \frac{dy}{dx}$$

$$\Rightarrow -x \operatorname{cosec}^2(xy) \frac{dy}{dx} - y \operatorname{cosec}^2(xy) + x \frac{dy}{dx} + y = \frac{dy}{dx}$$

$$\Rightarrow -x \operatorname{cosec}^2(xy) \frac{dy}{dx} + x \frac{dy}{dx} - \frac{dy}{dx} = -y + y \operatorname{cosec}^2(xy)$$

$$\Rightarrow \frac{dy}{dx} \{-x \operatorname{cosec}^2(xy) + x - 1\} = y \{\operatorname{cosec}^2(xy) - 1\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y \cot^2(xy)}{-1 - x \{\operatorname{cosec}^2(xy) - 1\}} \Rightarrow \frac{dy}{dx} = \frac{-y \cot^2(xy)}{-1 - x \cot^2(xy)} \therefore \frac{dy}{dx} = \frac{-y \cot^2(xy)}{1 + x \cot^2(xy)}$$

15. $y \tan x - y^2 \cos x + 2x = 0$

Sol. Differentiating both sides with respect to x

$$\left\{ y \frac{d(\tan x)}{dx} + \tan x \frac{dy}{dx} \right\} - \left\{ y^2 \frac{d(\cos x)}{dx} + \cos x \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} \right\} + \frac{d(2x)}{dx} = 0$$

$$\Rightarrow y \sec^2 x + \tan x \frac{dy}{dx} + y^2 \sin x - \cos x \cdot 2y \frac{dy}{dx} + 2 = 0$$

$$\Rightarrow \tan x \frac{dy}{dx} - 2y \cos x \frac{dy}{dx} = -2 - y^2 \sin x - y \sec^2 x$$

$$\Rightarrow \frac{dy}{dx} (\tan x - 2y \cos x) = -(2 + y^2 \sin x + y \sec^2 x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(2 + y^2 \sin x + y \sec^2 x)}{-(2y \cos x - \tan x)} \therefore \frac{dy}{dx} = \frac{2 + y^2 \sin x + y \sec x}{2y \cos x - \tan x}$$

16. $e^x \log y = \sin^{-1} x + \sin^{-1} y$

Sol. Differentiating both sides with respect to x

$$e^x \frac{d(\log y)}{dy} \cdot \frac{dy}{dx} + \log y \frac{d(e^x)}{dx} = \frac{d(\sin^{-1} x)}{dx} + \frac{d(\sin^{-1} y)}{dy} \cdot \frac{dy}{dx}$$

$$\Rightarrow e^x \cdot \frac{1}{y} \frac{dy}{dx} + \log y \cdot e^x = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}} \frac{dy}{dx} \Rightarrow \frac{e^x}{y} \frac{dy}{dx} - \frac{1}{\sqrt{1-y^2}} \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \log y \cdot e^x$$

$$\Rightarrow \frac{dy}{dx} \left(\frac{e^x}{y} - \frac{1}{\sqrt{1-y^2}} \right) = \frac{1}{\sqrt{1-x^2}} - \log y \cdot e^x \Rightarrow \frac{dy}{dx} \left(\frac{e^x \sqrt{1-y^2} - y}{y \sqrt{1-y^2}} \right) = \left(\frac{1 - \log y \cdot e^x \sqrt{1-x^2}}{\sqrt{1-x^2}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y \sqrt{1-y^2} (1 - \log y \cdot e^x \sqrt{1-x^2})}{\sqrt{1-x^2} (e^x \sqrt{1-y^2} - y)} \therefore \frac{dy}{dx} = y \sqrt{1-x^2} \left\{ \frac{1 - \log y \cdot e^x \sqrt{1-x^2}}{e^x \sqrt{1-y^2} - y} \right\}$$

17. $xy \log(x+y) = 1$

Sol. $xy \log(x+y) = 1 \quad \dots(1)$

Differentiating both sides with respect to x

$$\begin{aligned} & xy \frac{d\{\log(x+y)\}}{d(x+y)} \times \frac{d(x+y)}{dx} + \log(x+y) \frac{d(xy)}{dx} = \frac{d(1)}{dx} \\ \Rightarrow & xy \cdot \frac{1}{x+y} \left(\frac{dx}{dx} + \frac{dy}{dx} \right) + \log(x+y) \left\{ x \frac{dy}{dx} + y \frac{dx}{dx} \right\} = 0 \\ \Rightarrow & \frac{xy}{x+y} \left(1 + \frac{dy}{dx} \right) + \log(x+y) \left(x \frac{dy}{dx} + y \right) = 0 \\ \Rightarrow & \frac{xy}{x+y} + \frac{xy}{x+y} \frac{dy}{dx} + x \log(x+y) \frac{dy}{dx} + y \log(x+y) = 0 \\ \Rightarrow & \frac{xy}{x+y} \frac{dy}{dx} + x \log(x+y) \frac{dy}{dx} = -y \log(x+y) - \frac{xy}{x+y} \\ \Rightarrow & \frac{dy}{dx} \left(\frac{xy}{x+y} + x \log(x+y) \right) = - \left(y \log(x+y) + \frac{xy}{x+y} \right) \\ \Rightarrow & \frac{dy}{dx} \left\{ \frac{xy + x(x+y) \log(x+y)}{x+y} \right\} = - \left\{ \frac{y(x+y) \log(x+y) + xy}{(x+y)} \right\} \\ \Rightarrow & \frac{dy}{dx} = - \frac{y \{ (x+y) \log(x+y) + x \}}{x \{ y + (x+y) \log(x+y) \}} \quad \dots(ii) \end{aligned}$$

From equation (i), $xy \log(x+y) = 1 \Rightarrow \log(x+y) = \frac{1}{xy}$

Putting the value of $\log(x+y)$ in equation (ii),

$$\begin{aligned} \frac{dy}{dx} &= - \frac{y \left\{ (x+y) \cdot \frac{1}{xy} + x \right\}}{x \{ y + (x+y) \log(x+y) \}} \Rightarrow \frac{dy}{dx} = - \frac{y \left\{ \frac{x+y+x^2y}{xy} \right\}}{x \{ y + (x+y) \log(x+y) \}} \\ \Rightarrow & \frac{dy}{dx} = - \frac{(x+y+x^2y)}{x^2 \{ y + (x+y) \log(x+y) \}} \end{aligned}$$

18. $\tan(x+y) + \tan(x-y) = 1$

Sol. Differentiating both sides with respect to x

$$\begin{aligned} & \frac{d\{\tan(x+y)\}}{d(x+y)} \times \frac{d(x+y)}{dx} + \frac{d\{\tan(x-y)\}}{d(x-y)} \times \frac{d(x-y)}{dx} = \frac{d(1)}{dx} \\ \Rightarrow & \sec^2(x+y) \left\{ \frac{dx}{dx} + \frac{dy}{dx} \right\} + \sec^2(x-y) \left\{ \frac{dx}{dx} - \frac{dy}{dx} \right\} = 0 \\ \Rightarrow & \sec^2(x+y) \left\{ 1 + \frac{dy}{dx} \right\} + \sec^2(x-y) \left\{ 1 - \frac{dy}{dx} \right\} = 0 \\ \Rightarrow & \sec^2(x+y) + \sec^2(x+y) \frac{dy}{dx} + \sec^2(x-y) - \sec^2(x-y) \frac{dy}{dx} = 0 \\ \Rightarrow & \sec^2(x+y) \frac{dy}{dx} - \sec^2(x-y) \frac{dy}{dx} = -\sec^2(x-y) - \sec^2(x+y) \end{aligned}$$

$$\Rightarrow \frac{dy}{dx} \{ \sec^2(x+y) - \sec^2(x-y) \} = - \{ \sec^2(x-y) + \sec^2(x+y) \}$$

$$\Rightarrow \frac{dy}{dx} = \frac{- \{ \sec^2(x-y) + \sec^2(x+y) \}}{- \{ \sec^2(x-y) - \sec^2(x+y) \}} \quad \therefore \frac{dy}{dx} = \frac{\sec^2(x+y) + \sec^2(x-y)}{\sec^2(x-y) - \sec^2(x+y)}$$

19. $\log \sqrt{x^2 + y^2} = \tan^{-1} \left(\frac{y}{x} \right)$

Sol. Given, $\log \sqrt{x^2 + y^2} = \tan^{-1} \left(\frac{y}{x} \right) \Rightarrow \frac{1}{2} \log(x^2 + y^2) = \tan^{-1} \frac{y}{x}$

Differentiating both sides with respect to x

$$\frac{1}{2(x^2 + y^2)} \left\{ \frac{d(x^2)}{dx} + \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} \right\} = \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot \left\{ \frac{x \frac{dy}{dx} - y \frac{dx}{dx}}{(x)^2} \right\}$$

$$\Rightarrow \frac{1}{2(x^2 + y^2)} \cdot \left(2x + 2y \cdot \frac{dy}{dx} \right) = \frac{x^2}{x^2 + y^2} \cdot \frac{(x \frac{dy}{dx} - y)}{x^2} \Rightarrow \frac{2(x + y \frac{dy}{dx})}{2(x^2 + y^2)} = \frac{(x \frac{dy}{dx} - y)}{x^2 + y^2}$$

$$\Rightarrow x + y \frac{dy}{dx} = x \frac{dy}{dx} - y \Rightarrow x + y = x \frac{dy}{dx} - y \frac{dy}{dx} \Rightarrow x + y = \frac{dy}{dx} (x - y) \quad \therefore \frac{dy}{dx} = \frac{x + y}{x - y}$$

20. If $y = x \sin y$, prove that $x \cdot \frac{dy}{dx} = \frac{y}{1 - x \cos y}$

Sol. $y = x \sin y \quad \dots (i)$

Differentiating both sides of (i) with respect to x , $\frac{dy}{dx} = x \cdot \frac{d(\sin y)}{dx} + \sin y \cdot \frac{d(x)}{dx}$

$$\Rightarrow \frac{dy}{dx} = x \cos y \frac{dy}{dx} + \sin y \cdot 1 \Rightarrow \frac{dy}{dx} - x \cos y \frac{dy}{dx} = \sin y$$

$$\Rightarrow \frac{dy}{dx} (1 - x \cos y) = \sin y \Rightarrow \frac{dy}{dx} = \frac{\sin y}{1 - x \cos y} \quad \dots (ii)$$

From equation (i), $y = x \sin y \therefore \sin y = \frac{y}{x}$

Putting the value of $\sin y$ in equation (ii), $\frac{dy}{dx} = \frac{y}{x(1 - x \cos y)} \therefore x \frac{dy}{dx} = \frac{y}{1 - x \cos y}$

21. If $xy = \tan(xy)$ show that $\frac{dy}{dx} = -\frac{y}{x}$

Sol. $xy = \tan(xy) \quad \dots (i)$

Differentiating both sides of (i) with respect to x , $x \frac{dy}{dx} + y \frac{dx}{dx} = \frac{d\{\tan(xy)\}}{d(xy)} \times \frac{d(xy)}{dx}$

$$\Rightarrow x \frac{dy}{dx} + y = \sec^2(xy) \cdot \left\{ x \frac{dy}{dx} + y \frac{dx}{dx} \right\} \Rightarrow x \frac{dy}{dx} + y = \sec^2(xy) \left\{ x \frac{dy}{dx} + y \right\}$$

$$\Rightarrow x \frac{dy}{dx} + y = x \sec^2(xy) \frac{dy}{dx} + y \sec^2(xy) \Rightarrow y - y \sec^2(xy) = x \sec^2(xy) \frac{dy}{dx} - x \frac{dy}{dx}$$

$$\Rightarrow y\{1 - \sec^2(xy)\} = x \frac{dy}{dx} \{\sec^2(xy) - 1\} \Rightarrow \frac{dy}{dx} = -\frac{y\{\sec^2(xy) - 1\}}{x\{\sec^2(xy) - 1\}} \therefore \frac{dy}{dx} = \frac{-y}{x}$$

22. If $y \log x = x - y$ prove that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$

Sol. $y \log x = x - y$... (i)

Differentiating both sides of (i) with respect to x

$$\begin{aligned} y \frac{d(\log x)}{dx} + \log x \frac{dy}{dx} &= \frac{d(x)}{dx} - \frac{d(y)}{dx} \Rightarrow y \cdot \frac{1}{x} + \log x \frac{dy}{dx} = 1 - \frac{dy}{dx} \\ \Rightarrow \log x \frac{dy}{dx} + \frac{dy}{dx} &= 1 - \frac{y}{x} \Rightarrow \frac{dy}{dx} (\log x + 1) = \frac{x - y}{x} \\ \Rightarrow \frac{dy}{dx} &= \frac{x - y}{x(1 + \log x)} \therefore \frac{dy}{dx} = \frac{y \log x}{x(1 + \log x)} \quad \dots (ii) \end{aligned}$$

From equation (i), $y \log x = x - y \Rightarrow y \log x + y = x \Rightarrow y(\log x + 1) = x \Rightarrow \frac{y}{x} = \frac{1}{(1 + \log x)}$

Putting the value of $\frac{y}{x}$ in equation (ii), $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)(1 + \log x)} \therefore \frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$

23. If $\cos y = x \cos(y + a)$ prove that $\frac{dy}{dx} = \frac{\cos^2(y + a)}{\sin a}$

Sol. $\cos y = x \cos(y + a)$... (i)

Differentiating both sides with respect to x , $\frac{\cos y}{\cos(y + a)} = x$

$$\begin{aligned} \Rightarrow \frac{\cos(y + a) \frac{d \cos y}{dx} - \cos y \frac{d \cos(y + a)}{dx}}{\cos^2(y + a)} &= 1 \\ \Rightarrow \frac{\cos(y + a)(-\sin y) \frac{dy}{dx} + \cos y \sin(a + y) \frac{dy}{dx}}{\cos^2(y + a)} &= 1 \end{aligned}$$

$$\Rightarrow \frac{dy}{dx} (\sin(a + y) \cos y - \sin y \cos(a + y)) = \cos^2(y + a)$$

$$\Rightarrow \frac{dy}{dx} (\sin(y + a - y)) = \cos^2(a + y) \therefore \frac{dy}{dx} = \frac{\cos^2(a + y)}{\sin a}$$

Putting the value of x in equation (ii),

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos(y + a)}{-\sin y + \frac{\cos y}{\cos(y + a)} \sin(y + a)} \Rightarrow \frac{dy}{dx} = \frac{\cos^2(y + a)}{-\sin y \cos(y + a) + \cos y \sin(y + a)} \\ \Rightarrow \frac{dy}{dx} &= \frac{\cos^2(y + a)}{\sin(y + a - y)} \therefore \frac{dy}{dx} = \frac{\cos^2(y + a)}{\sin a} \quad [\because \sin(A - B) = \sin A \cos B - \cos A \sin B] \end{aligned}$$

24. If $\cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \tan^{-1}(a)$ prove that $\frac{dy}{dx} = \frac{y}{x}$.

Sol. $\cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \tan^{-1}(a) \Rightarrow \frac{x^2 - y^2}{x^2 + y^2} = \cos\{\tan^{-1}(a)\}$

Differentiating both sides with respect to x

$$\begin{aligned} & \frac{(x^2 + y^2) \frac{d(x^2 - y^2)}{dx} - (x^2 - y^2) \frac{d(x^2 + y^2)}{dx}}{(x^2 + y^2)^2} = \frac{d\{\cos(\tan^{-1} a)\}}{dx} \\ & \Rightarrow \frac{(x^2 + y^2) \left(2x - 2y \frac{dy}{dx}\right) - (x^2 - y^2) \left(2x + 2y \frac{dy}{dx}\right)}{(x^2 + y^2)^2} = 0 \\ & \Rightarrow \frac{2(x^2 + y^2) \left(x - y \frac{dy}{dx}\right) - 2(x^2 - y^2) \left(x + y \frac{dy}{dx}\right)}{(x^2 + y^2)^2} = 0 \\ & \Rightarrow 2(x^2 + y^2) \left(x - y \frac{dy}{dx}\right) - 2(x^2 - y^2) \left(x + y \frac{dy}{dx}\right) = 0 \\ & \Rightarrow (x^2 + y^2) \left(x - y \frac{dy}{dx}\right) = (x^2 - y^2) \left(x + y \frac{dy}{dx}\right) \\ & \Rightarrow x^3 - x^2 y \frac{dy}{dx} + y^2 x - y^3 \frac{dy}{dx} = x^3 + x^2 y \frac{dy}{dx} - y^2 x - y^3 \frac{dy}{dx} \\ & \Rightarrow -x^2 y \frac{dy}{dx} - x^2 y \frac{dy}{dx} = -y^2 x - y^2 x \Rightarrow -2x^2 y \frac{dy}{dx} = -2y^2 x \Rightarrow \frac{dy}{dx} = \frac{-2y^2 x}{-2x^2 y} \therefore \frac{dy}{dx} = \frac{y}{x} \end{aligned}$$

EXERCISE 10F (Pg.no.: 425)

Find $\frac{dy}{dx}$, when:

1. $y = x^{\frac{1}{x}}$

Sol. $y = x^{\frac{1}{x}}$, Taking log both sides we get, $\log y = \log \left(x^{\frac{1}{x}} \right) \Rightarrow \log y = \frac{1}{x} \log(x)$

Differentiating both sides with respect to x , $\frac{d(\log y)}{dy} \cdot \frac{dy}{dx} = \frac{1}{x} \cdot \frac{d(\log x)}{dx} + \log x \cdot \frac{d\left(\frac{1}{x}\right)}{dx}$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{x} + \log x \left(-\frac{1}{x^2} \right) \Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{1}{x^2} - \frac{\log x}{x^2} \Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{1 - \log x}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x^2} (1 - \log x) \therefore \frac{dy}{dx} = \frac{x^{\frac{1}{x}} (1 - \log x)}{x^2}$$

2. $y = x^{\sqrt{x}}$

Sol. $y = x^{\sqrt{x}}$, Taking log both sides we get, $\log y = \log \left(x^{\sqrt{x}} \right) \Rightarrow \log y = \sqrt{x} \log x$

Differentiating both sides with respect to x

$$\frac{d(\log y)}{dy} \cdot \frac{dy}{dx} = \sqrt{x} \frac{d(\log x)}{dx} + \log x \cdot \frac{d(\sqrt{x})}{dx} \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \sqrt{x} \cdot \frac{1}{x} + \log x \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{1}{\sqrt{x}} + \frac{\log x}{2\sqrt{x}} \Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{2 + \log x}{2\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{y(2 + \log x)}{2\sqrt{x}} \therefore \frac{dy}{dx} = \frac{x^{\sqrt{x}} (2 + \log x)}{2\sqrt{x}}$$

3. $y = (\log x)^x$

Sol. $y = (\log x)^x$, Taking log both sides we get, $\log y = \log \left\{ (\log x)^x \right\} \Rightarrow \log y = x \log(\log x)$

Differentiating both sides with respect to x

$$\frac{d(\log y)}{dy} \cdot \frac{dy}{dx} = x \frac{d\{\log(\log x)\}}{d(\log x)} \times \frac{d(\log x)}{dx} + \log(\log x) \cdot \frac{d(x)}{dx}$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log(\log x) \cdot 1 \Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{1}{\log x} + \log(\log x) \Rightarrow \frac{dy}{dx} = y \left\{ \frac{1}{\log x} + \log(\log x) \right\}$$

$$\therefore \frac{dy}{dx} = (\log x)^x \left\{ \frac{1}{\log x} + \log(\log x) \right\}$$

4. $y = x^{\sin x}$

Sol. $y = x^{\sin x}$, Taking log both sides we get, $\log y = \log \left(x^{\sin x} \right) \Rightarrow \log y = \sin x \cdot \log x$

Differentiating both sides with respect to x

$$\frac{d(\log y)}{dy} \cdot \frac{dy}{dx} = \sin x \cdot \frac{d(\log x)}{dx} + \log x \cdot \frac{d(\sin x)}{dx} \Rightarrow \frac{1}{y} \frac{dy}{dx} = \sin x \cdot \frac{1}{x} + \log x \cdot \cos x$$

$$\Rightarrow \frac{dy}{dx} = y \left(\frac{\sin x}{x} + \log x \cdot \cos x \right) \therefore \frac{dy}{dx} = x^{\sin x} \left(\frac{\sin x}{x} + \log x \cdot \cos x \right)$$

5. $y = x^{\cos^{-1}(x)}$

Sol. $y = x^{\cos^{-1}(x)}$, Taking log both sides we get, $\log y = \log(x^{\cos^{-1}x}) \Rightarrow \log y = \cos^{-1}x \cdot \log x$

Differentiating both sides with respect to x

$$\frac{d(\log y)}{dy} \cdot \frac{dy}{dx} = \cos^{-1}x \cdot \frac{d(\log x)}{dx} + \log x \cdot \frac{d(\cos^{-1}x)}{dx} \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \cos^{-1}x \cdot \frac{1}{x} + \log x \left(-\frac{1}{\sqrt{1-x^2}} \right)$$

$$\Rightarrow \frac{dy}{dx} = y \left(\frac{\cos^{-1}x}{x} - \frac{\log x}{\sqrt{1-x^2}} \right) \therefore \frac{dy}{dx} = x^{\cos^{-1}x} \left(\frac{\cos^{-1}x}{x} - \frac{\log x}{\sqrt{1-x^2}} \right)$$

6. $y = (\tan x)^{\frac{1}{x}}$

Sol. $y = (\tan x)^{\frac{1}{x}}$, Taking log both sides we get, $\log y = \log \left\{ (\tan x)^{\frac{1}{x}} \right\} \Rightarrow \log y = \frac{1}{x} \log(\tan x)$

Differentiating both sides with respect to x

$$\frac{d(\log y)}{dy} \cdot \frac{dy}{dx} = \frac{1}{x} \cdot \frac{d\{\log(\tan x)\}}{d(\tan x)} \times \frac{d(\tan x)}{dx} + \log(\tan x) \cdot \frac{d\left(\frac{1}{x}\right)}{dx}$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{\tan x} \cdot \sec^2 x + \log(\tan x) \cdot \left(-\frac{1}{x^2} \right) \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} \cdot \frac{2}{2 \sin x \cos x} - \frac{\log(\tan x)}{x^2}$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{2 \operatorname{cosec} 2x}{x} - \frac{\log(\tan x)}{x^2} \Rightarrow \frac{dy}{dx} = y \left\{ \frac{2x(\operatorname{cosec} 2x) - \log(\tan x)}{x^2} \right\}$$

$$\therefore \frac{dy}{dx} = (\tan x)^{\frac{1}{x}} \left\{ \frac{2x \operatorname{cosec} 2x - \log(\tan x)}{x^2} \right\}$$

7. $y = (\sin x)^{\cos x}$

Sol. $y = (\sin x)^{\cos x}$, Taking log both sides we get, $\log y = \log \left\{ (\sin x)^{\cos x} \right\} \Rightarrow \log y = \cos x \cdot \log(\sin x)$

Differentiating both sides with respect to x

$$\frac{d(\log y)}{dy} \cdot \frac{dy}{dx} = \cos x \cdot \frac{d\{\log(\sin x)\}}{d(\sin x)} \times \frac{d(\sin x)}{dx} + \log(\sin x) \cdot \frac{d(\cos x)}{dx}$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \cos x \cdot \frac{1}{\sin x} \cdot \cos x + \log(\sin x) \cdot (-\sin x) \Rightarrow \frac{dy}{dx} = y (\cos x \cot x - \log(\sin x) \cdot \sin x)$$

$$\therefore \frac{dy}{dx} = (\sin x)^{\cos x} (\cot x \cos x - \sin x \cdot \log(\sin x))$$

8. $y = (\log x)^{\sin x}$

Sol. $y = (\log x)^{\sin x}$, Taking log both sides we get, $\log y = \log \left\{ (\log x)^{\sin x} \right\} \Rightarrow \log y = \sin x \cdot \log(\log x)$

Differentiating both sides with respect to x

$$\frac{d(\log y)}{dy} \cdot \frac{dy}{dx} = \sin x \cdot \frac{d\{\log(\log x)\}}{d(\log x)} \times \frac{d(\log x)}{dx} + \log(\log x) \cdot \frac{d(\sin x)}{dx}$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \sin x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log(\log x) \cdot \cos x \Rightarrow \frac{dy}{dx} = y \left\{ \frac{\sin x}{x \log x} + \cos x \cdot \log(\log x) \right\}$$

$$\therefore \frac{dy}{dx} = (\log x)^{\sin x} \left\{ \frac{\sin x}{x \log x} + \cos x \cdot \log(\log x) \right\}$$

9. $y = (\cos x)^{\log x}$

Sol. $y = (\cos x)^{\log x}$, Taking log both sides we get, $\log y = \log \{(\cos x)^{\log x}\} \Rightarrow \log y = \log x \cdot \log(\cos x)$

On differentiating both sides with respect to x

$$\begin{aligned} \frac{d(\log y)}{dy} \cdot \frac{dy}{dx} &= \log x \cdot \frac{d\{\log(\cos x)\}}{d(\cos x)} \times \frac{d(\cos x)}{dx} + \log(\cos x) \cdot \frac{d(\log x)}{dx} \\ \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= \log x \cdot \frac{1}{\cos x} (-\sin x) + \log(\cos x) \cdot \frac{1}{x} \Rightarrow \frac{dy}{dx} = y \left(-\log x \cdot \tan x + \frac{\log(\cos x)}{x} \right) \\ \therefore \frac{dy}{dx} &= (\cos x)^{\log x} \left\{ \frac{\log(\cos x)}{x} - \log x \cdot \tan x \right\} \end{aligned}$$

10. $y = (\tan x)^{\sin x}$

Sol. $y = (\tan x)^{\sin x}$, Taking log both sides we get, $\log y = \log \{(\tan x)^{\sin x}\} \Rightarrow \log y = \sin x \cdot \log(\tan x)$

On differentiating both sides with respect to x

$$\begin{aligned} \frac{d(\log y)}{dy} \cdot \frac{dy}{dx} &= \sin x \cdot \frac{d\{\log(\tan x)\}}{d(\tan x)} \times \frac{d(\tan x)}{dx} + \log(\tan x) \cdot \frac{d(\sin x)}{dx} \\ \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= \sin x \cdot \frac{1}{\tan x} \cdot \sec^2 x + \log(\tan x) \cdot \cos x \\ \Rightarrow \frac{dy}{dx} &= y \{ \sec x + \cos x \cdot \log(\tan x) \} \therefore \frac{dy}{dx} = (\tan x)^{\sin x} \{ \cos x \cdot \log(\tan x) + \sec x \} \end{aligned}$$

11. $y = (\cos x)^{\cos x}$

Sol. $y = (\cos x)^{\cos x}$, Taking log both sides we get, $\log y = \log \{(\cos x)^{\cos x}\} \Rightarrow \log y = \cos x \cdot \log(\cos x)$

On differentiating both sides with respect to x

$$\begin{aligned} \frac{d(\log y)}{dy} \cdot \frac{dy}{dx} &= \cos x \cdot \frac{d(\log(\cos x))}{d(\cos x)} \times \frac{d(\cos x)}{dx} + \log(\cos x) \cdot \frac{d(\cos x)}{dx} \\ \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= \cos x \cdot \frac{1}{\cos x} (-\sin x) + \log(\cos x) (-\sin x) \Rightarrow \frac{dy}{dx} = -y (\sin x + \sin x \cdot \log(\cos x)) \\ \therefore \frac{dy}{dx} &= -(\cos x)^{\cos x} \cdot \sin x (1 + \log \cos x) \end{aligned}$$

12. $y = (\tan x)^{\cot x}$

Sol. $y = (\tan x)^{\cot x}$, Taking log both sides we get, $\log y = \log \{ \tan x^{\cot x} \} \Rightarrow \log y = \cot x \cdot \log(\tan x)$

On differentiating both sides with respect to x

$$\begin{aligned} \frac{d(\log y)}{dy} \times \frac{dy}{dx} &= \cot x \cdot \frac{d\{\log(\tan x)\}}{d(\tan x)} \times \frac{d(\tan x)}{dx} + \log(\tan x) \cdot \frac{d(\cot x)}{dx} \\ \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= \cot x \cdot \frac{1}{\tan x} \cdot \sec^2 x + \log(\tan x) (-\operatorname{cosec}^2 x) \Rightarrow \frac{dy}{dx} = y \{ \operatorname{cosec}^2 x - \log(\tan x) \operatorname{cosec}^2 x \} \end{aligned}$$

$$\Rightarrow \frac{dy}{dx} = (\tan x)^{\cot x} \cdot \operatorname{cosec}^2 x \{1 - \log(\tan x)\}$$

13. $y = x^{\sin 2x}$

Sol. $y = x^{\sin 2x}$, Taking log both sides we get, $\log y = \log(x^{\sin 2x}) \Rightarrow \log y = \sin 2x \cdot \log(x)$

On differentiating both sides with respect to x

$$\begin{aligned} \frac{d(\log y)}{dy} \cdot \frac{dy}{dx} &= \sin 2x \cdot \frac{d(\log x)}{dx} + \log x \cdot \frac{d(\sin 2x)}{d(2x)} \times \frac{d(2x)}{dx} \\ \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= \sin 2x \cdot \frac{1}{x} + \log x \cdot \cos 2x \cdot 2 \Rightarrow \frac{dy}{dx} = y \left(\frac{\sin 2x}{x} + \log x \cdot \cos 2x \cdot 2 \right) \\ \therefore \frac{dy}{dx} &= x^{\sin 2x} \left\{ \frac{\sin 2x}{x} + (2 \cos 2x) \log x \right\} \end{aligned}$$

14. $y = (\sin^{-1} x)^x$

Sol. $y = (\sin^{-1} x)^x$, Taking log both sides we get, $\log y = \log\{(\sin^{-1} x)^x\} \Rightarrow \log y = x \log(\sin^{-1} x)$

On differentiating both sides with respect to x

$$\begin{aligned} \frac{d(\log y)}{dy} \cdot \frac{dy}{dx} &= \frac{x \cdot d\{\log(\sin^{-1} x)\}}{d(\sin^{-1} x)} \times \frac{d(\sin^{-1} x)}{dx} + \log(\sin^{-1} x) \cdot \frac{d(x)}{dx} \\ \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= x \cdot \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} + \log(\sin^{-1} x) \cdot 1 \Rightarrow \frac{dy}{dx} = y \left\{ \frac{x}{\sin^{-1} x \cdot \sqrt{1-x^2}} + \log(\sin^{-1} x) \right\} \\ \therefore \frac{dy}{dx} &= (\sin^{-1} x)^x \left\{ \log(\sin^{-1} x) + \frac{x}{\sin^{-1} x \cdot \sqrt{1-x^2}} \right\} \end{aligned}$$

15. $y = \sin(x^x)$

Sol. $y = \sin(x^x)$, Let $x^x = t \Rightarrow y = \sin(t)$... (i)

On differentiating both sides of (i) with respect to x

$$\frac{dy}{dx} = \frac{d(\sin t)}{dt} \cdot \frac{dt}{dx} \quad \dots (ii)$$

$\Rightarrow t = x^x$, taking log both sides we get, $\log t = \log(x^x) \Rightarrow \log t = x \log x$... (iii)

On differentiating both sides of (iii) with respect to x

$$\begin{aligned} \frac{d(\log t)}{dt} \cdot \frac{dt}{dx} &= x \cdot \frac{d(\log x)}{dx} + \log x \cdot \frac{d(x)}{dx} \Rightarrow \frac{1}{t} \cdot \frac{dt}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1 \\ \Rightarrow \frac{1}{t} \cdot \frac{dt}{dx} &= 1 + \log x \Rightarrow \frac{dt}{dx} = t(1 + \log x) \Rightarrow \frac{dt}{dx} = x^x(1 + \log x) \end{aligned}$$

Putting the value of $\frac{dt}{dx}$ in equation (ii), $\frac{dy}{dx} = \cos(x^x) \cdot x^x(1 + \log x)$

16. $y = (3x+5)^{2x-3}$

Sol. $y = (3x+5)^{2x-3}$

Taking log both sides we get, $\log y = \log\{(3x+5)^{(2x-3)}\} \Rightarrow \log y = (2x-3) \log(3x+5)$

On differentiating both sides with respect to x

$$\begin{aligned}\frac{1}{y} \cdot \frac{dy}{dx} &= (2x-3) \frac{d\{\log(3x+5)\}}{d(3x+5)} \times \frac{d(3x+5)}{dx} + \log(3x+5) \frac{d(2x-3)}{dx} \\ \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= (2x-3) \cdot \frac{1}{3x+5} \cdot 3 + \log(3x+5) \cdot 2 \Rightarrow \frac{dy}{dx} = y \left\{ \frac{3(2x-3)}{3x+5} + 2\log(3x+5) \right\} \\ \therefore \frac{dy}{dx} &= (3x+5)^{2x-3} \left\{ \frac{3(2x-3)}{3x+5} + 2\log(3x+5) \right\}\end{aligned}$$

17. $y = (x+1)^3 (x+2)^4 (x+3)^5$

Sol. $y = (x+1)^3 (x+2)^4 (x+3)^5$, Taking log both sides we get, $\log y = \log \{(x+1)^3 (x+2)^4 (x+3)^5\}$

$$\Rightarrow \log y = \log \{(x+1)^3\} + \log \{(x+2)^4\} + \log \{(x+3)^5\}$$

$$\Rightarrow \log y = 3\log(x+1) + 4\log(x+2) + 5\log(x+3) \quad \dots(i)$$

On differentiating both sides of (i) with respect to x

$$\begin{aligned}\frac{1}{y} \cdot \frac{dy}{dx} &= 3 \cdot \frac{1}{x+1} \cdot 1 + 4 \cdot \frac{1}{x+2} \cdot 1 + 5 \cdot \frac{1}{x+3} \cdot 1 \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{3}{x+1} + \frac{4}{x+2} + \frac{5}{x+3} \\ \Rightarrow \frac{dy}{dx} &= y \left(\frac{3}{x+1} + \frac{4}{x+2} + \frac{5}{x+3} \right) \therefore \frac{dy}{dx} = (x+1)^3 (x+2)^4 (x+3)^5 \left(\frac{3}{x+1} + \frac{4}{x+2} + \frac{5}{x+3} \right)\end{aligned}$$

18. $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Sol. $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \Rightarrow y = \left\{ \frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)} \right\}^{\frac{1}{2}}$

Taking log both sides we get,

$$\begin{aligned}\log y &= \log \left\{ \frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)} \right\}^{\frac{1}{2}} \Rightarrow \log y = \frac{1}{2} \log \left\{ \frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)} \right\} \\ \Rightarrow \log y &= \frac{1}{2} [\log \{(x-1)(x-2)\} - \log \{(x-3)(x-4)(x-5)\}] \\ \Rightarrow \log y &= \frac{1}{2} [\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)] \quad \dots(ii)\end{aligned}$$

On differentiating both sides of (ii) with respect to x

$$\begin{aligned}\frac{1}{y} \cdot \frac{dy}{dx} &= \frac{1}{2} \left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right] \\ \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{1}{2} \left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right] \Rightarrow \frac{dy}{dx} = \frac{y}{2} \left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right] \\ \therefore \frac{dy}{dx} &= \frac{1}{2} \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right]\end{aligned}$$

19. $y = (2-x)^3 (3+2x)^5$

Sol. $y = (2-x)^3 (3+2x)^5$, Taking log both sides we get, $\log y = \log \{(2-x)^3 (3+2x)^5\}$

$$\Rightarrow \log y = \log \{(2-x)^3\} + \log \{(3+2x)^5\} \Rightarrow \log y = 3\log(2-x) + 5\log(3+2x) \quad \dots(i)$$

On differentiating both sides of (i) with respect to x

$$\begin{aligned} \frac{d(\log y)}{dy} \cdot \frac{dy}{dx} &= 3 \frac{d\{\log(2-x)\}}{d(2-x)} \times \frac{d(2-x)}{dx} + 5 \frac{d\{\log(3+2x)\}}{d(3+2x)} \times \frac{d(3+2x)}{dx} \\ \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= 3 \cdot \frac{1}{2-x} \cdot (-1) + 5 \cdot \frac{1}{3+2x} \cdot (2) \Rightarrow \frac{dy}{dx} = y \left(\frac{-3}{2-x} + \frac{10}{3+2x} \right) \\ \therefore \frac{dy}{dx} &= (2-x)^3 (3+2x)^5 \left(\frac{10}{3+2x} - \frac{3}{2-x} \right) \end{aligned}$$

20. $y = \cos x \cdot \cos 2x \cdot \cos 3x$

Sol. $y = \cos x \cdot \cos 2x \cdot \cos 3x$, Taking log both sides we get, $\log y = \log(\cos x \cdot \cos 2x \cdot \cos 3x)$

$$\Rightarrow \log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x) \quad \dots(i)$$

On differentiating both sides of (i) with respect to x

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= \left\{ \frac{1}{\cos x} (-\sin x) + \frac{1}{\cos 2x} (-\sin 2x) \cdot 2 + \frac{1}{\cos 3x} (-\sin 3x) \cdot 3 \right\} \\ \Rightarrow \frac{dy}{dx} &= y(-\tan x - 2\tan 2x - 3\tan 3x) \quad \therefore \frac{dy}{dx} = -\cos x \cos 2x \cos 3x (\tan x + 2\tan 2x + 3\tan 3x) \end{aligned}$$

21. $y = \frac{x^5 \sqrt{x+4}}{(2x+3)^2}$

Sol. $y = \frac{x^5 \sqrt{x+4}}{(2x+3)^2}$, Taking log both sides we get, $\log y = \log \left\{ \frac{x^5 (x+4)^{\frac{1}{2}}}{(2x+3)^2} \right\}$

$$\Rightarrow \log y = \log \left\{ x^5 \cdot (x+4)^{\frac{1}{2}} \right\} - \log \{(2x+3)^2\} \Rightarrow \log y = \log(x^5) + \log \left\{ (x+4)^{\frac{1}{2}} \right\} - \log \{(2x+3)^2\}$$

$$\Rightarrow \log y = 5\log x + \frac{1}{2}\log(x+4) - 2\log(2x+3)$$

On differentiating both sides with respect to x , $\frac{1}{y} \cdot \frac{dy}{dx} = 5 \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{x+4} - 2 \cdot \frac{1}{2x+3} \cdot 2$

$$\Rightarrow \frac{dy}{dx} = y \left(\frac{5}{x} + \frac{1}{2(x+4)} - \frac{4}{2x+3} \right) \quad \therefore \frac{dy}{dx} = \frac{x^5 \sqrt{x+4}}{(2x+3)^2} \left(\frac{5}{x} + \frac{1}{2(x+4)} - \frac{4}{2x+3} \right)$$

22. $y = \frac{(x+1)^2 \sqrt{x-1}}{(x+4)^3 e^x}$

Sol. $y = \frac{(x+1)^2 \sqrt{x-1}}{(x+4)^3 e^x}$, Taking log both sides we get, $\log y = \log \left\{ \frac{(x+1)^2 (x-1)^{\frac{1}{2}}}{(x+4)^3 e^x} \right\}$

$$\Rightarrow \log y = \log \left\{ (x+1)^2 (x-1)^{\frac{1}{2}} \right\} - \log \{(x+4)^3 \cdot e^x\}$$

$$\Rightarrow \log y = \log \{(x+1)^2\} + \log \left\{ (x-1)^{\frac{1}{2}} \right\} - \log \{(x+4)^3\} - \log e^x$$

$$\Rightarrow \log y = 2\log(x+1) + \frac{1}{2}\log(x-1) - 3\log(x+4) - x \log e$$

$$\Rightarrow \log y = 2\log(x+1) + \frac{1}{2}\log(x-1) - 3\log(x+4) - x \quad \dots(i)$$

On differentiating both sides of (i) with respect to x

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \frac{1}{x+1} + \frac{1}{2} \cdot \frac{1}{x-1} - 3 \cdot \frac{1}{x+4} - 1 \Rightarrow \frac{dy}{dx} = y \left(\frac{2}{x+1} + \frac{1}{2(x-1)} - \frac{3}{x+4} - 1 \right)$$

$$\therefore \frac{dy}{dx} = \frac{(x+1)^2 \sqrt{x-1}}{(x+4)^3 e^x} \left(\frac{2}{x+1} + \frac{1}{2(x-1)} - \frac{3}{x+4} - 1 \right)$$

23. $y = \frac{\sqrt{x}(3x+5)^2}{\sqrt{x+1}}$

Sol. $y = \frac{\sqrt{x}(3x+5)^2}{\sqrt{x+1}}$, Taking log both sides we get, $\log y = \log \left(\frac{x^{\frac{1}{2}}(3x+5)^2}{(x+1)^{\frac{1}{2}}} \right)$

$$\Rightarrow \log y = \log \left\{ x^{\frac{1}{2}}(3x+5)^2 \right\} - \log \left\{ (x+1)^{\frac{1}{2}} \right\} \Rightarrow \log y = \log \left(x^{\frac{1}{2}} \right) + \log \left\{ (3x+5)^2 \right\} - \log \left\{ (x+1)^{\frac{1}{2}} \right\}$$

$$\Rightarrow \log y = \frac{1}{2} \log x + 2 \log(3x+5) - \frac{1}{2} \log(x+1) \quad \dots(i)$$

Differentiating both sides of (i) with respect to x ,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{x} + 2 \cdot \frac{1}{3x+5} \cdot 3 - \frac{1}{2} \cdot \frac{1}{x+1} \cdot 1 \Rightarrow \frac{dy}{dx} = y \left(\frac{1}{2x} + \frac{6}{3x+5} - \frac{1}{2(x+1)} \right)$$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{x}(3x+5)^2}{\sqrt{x+1}} \left(\frac{1}{2x} + \frac{6}{3x+5} - \frac{1}{2(x+1)} \right)$$

24. $y = \frac{x^2 \sqrt{1+x}}{(1+x^2)^{\frac{3}{2}}}$

Sol. $y = \frac{x^2 \sqrt{1+x}}{(1+x^2)^{\frac{3}{2}}}$, Taking log both sides we get, $\log y = \log \left(\frac{x^2 (1+x)^{\frac{1}{2}}}{(1+x^2)^{\frac{3}{2}}} \right)$

$$\Rightarrow \log y = \log \left\{ x^2 \cdot (1+x)^{\frac{1}{2}} \right\} - \log \left\{ (1+x^2)^{\frac{3}{2}} \right\} \Rightarrow \log y = \log(x^2) + \log \left\{ (1+x)^{\frac{1}{2}} \right\} - \log \left\{ (1+x^2)^{\frac{3}{2}} \right\}$$

$$\Rightarrow \log y = 2 \log x + \frac{1}{2} \log(1+x) - \frac{3}{2} \log(1+x^2) \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{1+x} - \frac{3}{2} \cdot \frac{1}{1+x^2} \cdot 2x$$

$$\Rightarrow \frac{dy}{dx} = y \left(\frac{2}{x} + \frac{1}{2(1+x)} - \frac{3x}{1+x^2} \right) \therefore \frac{dy}{dx} = \frac{x^2 \sqrt{1+x}}{(1+x^2)^{\frac{3}{2}}} \left(\frac{2}{x} + \frac{1}{2(1+x)} - \frac{3x}{1+x^2} \right)$$

25. $y = \sqrt{(x-2)(2x-3)(3x-4)}$

Sol. $y = \sqrt{(x-2)(2x-3)(3x-4)}$, Taking log both sides we get,

$$\log y = \log \{(x-2)(2x-3)(3x-4)\}^{\frac{1}{2}} \Rightarrow \log y = \frac{1}{2} \log \{(x-2)(2x-3)(3x-4)\}$$

$$\Rightarrow \log y = \frac{1}{2} [\log(x-2) + \log(2x-3) + \log(3x-4)] \quad \dots(i)$$

On differentiation of (i) with respect to x ,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2} \left\{ \frac{1}{x-2} \cdot 1 + \frac{1}{2x-3} \cdot 2 + \frac{1}{3x-4} \cdot 3 \right\} \Rightarrow \frac{dy}{dx} = \frac{y}{2} \left[\frac{1}{x-2} + \frac{2}{2x-3} + \frac{3}{3x-4} \right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \sqrt{(x-2)(2x-3)(3x-4)} \left[\frac{1}{x-2} + \frac{2}{2x-3} + \frac{3}{3x-4} \right]$$

26. $y = \sin 2x \sin 3x \sin 4x$

Sol. $y = \sin 2x \sin 3x \sin 4x \quad \dots(i)$

Taking log both sides we get, $\log y = \log(\sin 2x \cdot \sin 3x \cdot \sin 4x)$

$$\Rightarrow \log y = \log(\sin 2x) + \log(\sin 3x) + \log(\sin 4x) \quad \dots(ii)$$

On differentiating both sides of (ii) w.r.t. x ,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{\sin 2x} \cos 2x \cdot 2 + \frac{1}{\sin 3x} \cos 3x \cdot 3 + \frac{1}{\sin 4x} \cos 4x \cdot 4$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 2 \cot 2x + 3 \cot 3x + 4 \cot 4x \Rightarrow \frac{dy}{dx} = y(2 \cot 2x + 3 \cot 3x + 4 \cot 4x)$$

$$\Rightarrow \frac{dy}{dx} = \sin 2x \cdot \sin 3x \cdot \sin 4x (2 \cot 2x + 3 \cot 3x + 4 \cot 4x)$$

27. $y = \frac{x^3 \sin x}{e^x}$

Sol. $y = \frac{x^3 \sin x}{e^x} \quad \dots(i)$

$$\text{Taking log both sides we get, } \log y = \log \left(\frac{x^3 \sin x}{e^x} \right) \Rightarrow \log y = \log(x^3 \cdot \sin x) - \log(e^x)$$

$$\Rightarrow \log y = \log(x^3) + \log(\sin x) - x \log e \Rightarrow \log y = 3 \log x + \log(\sin x) - x \quad \dots(ii)$$

On differentiating both sides of (ii) w.r.t. x ,

$$\frac{1}{y} \cdot \frac{dy}{dx} = 3 \cdot \frac{1}{x} + \frac{1}{\sin x} \cdot \cos x - 1 \Rightarrow \frac{dy}{dx} = y \left(\frac{3}{x} + \cot x - 1 \right) \therefore \frac{dy}{dx} = \frac{x^3 \sin x}{e^x} \left(\frac{3}{x} + \cot x - 1 \right)$$

28. $y = \frac{e^x \log x}{x^2}$

Sol. $y = \frac{e^x \log x}{x^2} \quad \dots(i)$

$$\text{Taking log both sides we get, } \log y = \log \left(\frac{e^x \cdot \log x}{x^2} \right) \Rightarrow \log y = \log(e^x \cdot \log x) - \log(x^2)$$

$$\Rightarrow \log y = x \log e + \log(\log x) - 2 \log x \Rightarrow \log y = x + \log(\log x) - 2 \log x \quad \dots(ii)$$

$$\text{On differentiating both sides of (ii) w.r.t. } x, \frac{1}{y} \cdot \frac{dy}{dx} = 1 + \frac{1}{\log x} \cdot \frac{1}{x} - 2 \cdot \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = y \left(1 + \frac{1}{x \log x} - \frac{2}{x} \right) \Rightarrow \frac{dy}{dx} = \frac{e^x \log x}{x^2} \left(1 + \frac{1}{x \log x} - \frac{2}{x} \right)$$

29. $y = \frac{x \cos^{-1} x}{\sqrt{1-x^2}}$

Sol. $y = \frac{x \cos^{-1} x}{\sqrt{1-x^2}} \dots (i)$

Taking log both sides we get, $\log y = \log \left(\frac{x \cos^{-1} x}{\sqrt{1-x^2}} \right)$

$$\Rightarrow \log y = \log(x \cos^{-1} x) - \log \left\{ (1-x^2)^{\frac{1}{2}} \right\} \Rightarrow \log y = \log x + \log(\cos^{-1} x) - \frac{1}{2} \log(1-x^2) \dots (ii)$$

On differentiating both sides (ii) w.r.t. x , $\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} + \frac{1}{\cos^{-1} x} \left(-\frac{1}{\sqrt{1-x^2}} \right) - \frac{1}{2} \cdot \frac{1}{1-x^2} (-2x)$

$$\Rightarrow \frac{dy}{dx} = y \left(\frac{1}{x} - \frac{1}{\sqrt{1-x^2} \cos^{-1} x} + \frac{x}{1-x^2} \right) \Rightarrow \frac{dy}{dx} = \frac{x \cos^{-1} x}{\sqrt{1-x^2}} \left(\frac{1}{x} - \frac{1}{\cos^{-1} x \sqrt{1-x^2}} + \frac{x}{1-x^2} \right)$$

30. $y = (1+x)(1+x^2)(1+x^4)(1+x^6)$

Sol. $y = (1+x)(1+x^2)(1+x^4)(1+x^6) \dots (i)$

Taking log both sides we get, $\log y = \log \{ (1+x)(1+x^2)(1+x^4)(1+x^6) \}$

$$\Rightarrow \log y = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^6) \dots (ii)$$

On differentiating both sides of (ii) w.r.t. x ,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{1+x} \cdot 1 + \frac{1}{1+x^2} \cdot 2x + \frac{1}{1+x^4} \cdot 4x^3 + \frac{1}{1+x^6} \cdot 6x^5$$

$$\Rightarrow \frac{dy}{dx} = y \left(\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{6x^5}{1+x^6} \right)$$

$$\therefore \frac{dy}{dx} = (1+x)(1+x^2)(1+x^4)(1+x^6) \left(\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{6x^5}{1+x^6} \right)$$

31. $y = x^x - 2^{\sin x}$

Sol. $y = x^x - 2^{\sin x}$, Let $x^x = u$ & $2^{\sin x} = v$

$$y = u - v \dots (i)$$

On differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \dots (ii)$

$$u = x^x$$

Taking log both sides we get, $\log u = \log(x^x) \Rightarrow \log(u) = x \log x$

On differentiating both sides w.r.t. x , $\frac{1}{u} \cdot \frac{du}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1 \Rightarrow \frac{du}{dx} = u(1 + \log x)$

$$\therefore \frac{du}{dx} = x^x (1 + \log x)$$

$$v = 2^{\sin x}$$

On differentiating both sides w.r.t. x , $\frac{dv}{dx} = \frac{d(2^{\sin x})}{d(\sin x)} \cdot \frac{d(\sin x)}{dx} \Rightarrow \frac{dv}{dx} = 2^{\sin x} \cdot \log 2 \cdot \cos x$

Putting the value of $\frac{du}{dx}, \frac{dv}{dx}$ in equation of (ii), $\frac{dy}{dx} = x^x (1 + \log x) - 2^{\sin x} \cdot \log 2 (\cos x)$

32. $y = (\log x)^x + x^{(\log x)}$

Sol. $y = (\log x)^x + x^{(\log x)}$, Let $u = (\log x)^x, v = x^{\log x}$

$$y = u + v \quad \dots (i)$$

On differentiating both sides of (i) w.r.t. x ,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots (ii)$$

$$u = (\log x)^x$$

Taking log both sides we get, $\log u = \log \{(\log x)^x\} \Rightarrow \log u = x \log (\log x)$

On differentiating both sides w.r.t. x , $\frac{1}{u} \cdot \frac{du}{dx} = x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log (\log x) \cdot 1$

$$\Rightarrow \frac{du}{dx} = u \left(\frac{1}{\log x} + \log (\log x) \right) \Rightarrow \frac{dy}{dx} = (\log x)^x \left(\frac{1}{\log x} + \log (\log x) \right)$$

$$v = x^{\log x}$$

Taking log both sides we get, $\log v = \log (x^{\log x}) \Rightarrow \log v = \log x \cdot \log x$

On differentiating both sides w.r.t. x , $\frac{1}{v} \cdot \frac{dv}{dx} = \log x \cdot \frac{1}{x} + \log x \cdot \frac{1}{x}$

$$\Rightarrow \frac{dv}{dx} = v \cdot \frac{2}{x} \log x \Rightarrow \frac{dv}{dx} = v \cdot \frac{2}{x} \log x \Rightarrow \frac{dv}{dx} = x^{\log x} \cdot \frac{2 \log x}{x}$$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii), $\frac{dy}{dx} = (\log x)^x \left\{ \frac{1}{\log x} + \log (\log x) \right\} + x^{\log x} \cdot \frac{2 \log x}{x}$

33. $y = x^{\sin x} + (\sin x)^{\cos x}$

Sol. $y = x^{\sin x} + (\sin x)^{\cos x}$, Let $u = x^{\sin x}, v = (\sin x)^{\cos x}$

$$y = u + v \quad \dots (i)$$

On differentiating both sides of (i) w.r.t. x

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots (ii)$$

$$u = x^{\sin x} \text{ and } v = (\sin x)^{\cos x}$$

$u = x^{\sin x}$, Taking log both sides we get, $\log u = \log (x^{\sin x}) \Rightarrow \log u = \sin x \cdot \log x$

Differentiating both sides with respect to x , $\frac{d(\log u)}{du} \cdot \frac{du}{dx} = \sin x \cdot \frac{d(\log x)}{dx} + \log x \cdot \frac{d(\sin x)}{dx}$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = \sin x \cdot \frac{1}{x} + \log x \cdot \cos x \Rightarrow \frac{du}{dx} = u \left(\frac{\sin x}{x} + \log x \cdot \cos x \right) \Rightarrow \frac{du}{dx} = x^{\sin x} \left(\frac{\sin x}{x} + \log x \cos x \right)$$

and $v = (\sin x)^{\cos x}$, Taking log both sides we get, $\log v = \log \{(\sin x)^{\cos x}\} \Rightarrow \log v = \cos x \cdot \log (\sin x)$

Differentiating both sides with respect to x ,

$$\begin{aligned}\frac{d \log(v)}{dv} \cdot \frac{dv}{dx} &= \cos x \cdot \frac{d\{\log(\sin x)\}}{d(\sin x)} \times \frac{d(\sin x)}{dx} + \log(\sin x) \frac{d(\cos x)}{dx} \\ \Rightarrow \frac{1}{v} \frac{dv}{dx} &= \cos x \cdot \frac{1}{\sin x} \cdot \cos x + \log(\sin x) \cdot (-\sin x) \Rightarrow \frac{dv}{dx} = v(\cos x \cot x - \log(\sin x) \cdot \sin x) \\ \Rightarrow \frac{dv}{dx} &= (\sin x)^{\cos x} (\cot x \cos x - \sin x \cdot \log(\sin x))\end{aligned}$$

Putting the value of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (ii),

$$\frac{dy}{dx} = x^{\sin x} \left(\frac{\sin x}{x} + \log x \cdot \cos x \right) + (\sin x)^{\cos x} (\cot x \cos x - \sin x \cdot \log(\sin x))$$

34. $y = (x \cos x)^x + (x \sin x)^{\frac{1}{x}}$

Sol. $y = (x \cos x)^x + (x \sin x)^{\frac{1}{x}}$, Let $u = (x \cos x)^x$, $v = (x \sin x)^{\frac{1}{x}}$

$$y = u + v \quad \dots (i)$$

On differentiating both sides w.r.t. x in equation (i)

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots (ii)$$

$$u = (x \cos x)^x$$

Taking log both sides we get, $\log u = \log \{(x \cos x)^x\}$

$$\Rightarrow \log u = x \log(x \cos x) \Rightarrow \log u = x \{\log x + \log(\cos x)\} \quad \dots (iii)$$

On differentiating both sides of (iii) w.r.t. x , $\frac{1}{u} \cdot \frac{du}{dx} = x \left\{ \frac{1}{x} + \frac{1}{\cos x} (-\sin x) \right\} + \{\log x + \log \cos x\}$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x (1 - x \tan x + \log x + \log \cos x)$$

$$v = (x \sin x)^{\frac{1}{x}}$$

Taking log both sides we get, $\log v = \log \{(x \sin x)^{\frac{1}{x}}\}$

$$\Rightarrow \log v = \frac{1}{x} \log(x \sin x) \Rightarrow \log v = \frac{1}{x} [\log x + \log \sin x] \quad \dots (iv)$$

On differentiating both sides of (iv) w.r.t. x ,

$$\begin{aligned}\frac{1}{v} \cdot \frac{dv}{dx} &= \frac{1}{x} \left\{ \frac{1}{x} + \frac{1}{\sin x} \cos x \right\} + \left\{ \log x + \log(\sin x) \right\} \left(-\frac{1}{x^2} \right) \\ \Rightarrow \frac{dv}{dx} &= v \left[\frac{1}{x^2} + \frac{1}{x} \cot x - \frac{\log x}{x^2} - \frac{\log(\sin x)}{x^2} \right] \\ \Rightarrow \frac{dv}{dx} &= (x \sin x)^{\frac{1}{x}} \left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log x}{x^2} - \frac{\log(\sin x)}{x^2} \right]\end{aligned}$$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii),

$$\therefore \frac{dy}{dx} = (x \cos x)^x [1 - x \tan x + \log x + \log \cos x] + (x \sin x)^{\frac{1}{x}} \left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log x}{x^2} - \frac{\log(\sin x)}{x^2} \right]$$

35. $y = (\sin x)^x + \sin^{-1}(\sqrt{x})$

Sol. Let $u = (\sin^{-1} x)^x$, $v = \sin^{-1}(\sqrt{x})$

$$y = u + v \quad \dots(i)$$

On differentiating both sides of (i) w.r.t. x

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(ii)$$

$$u = (\sin^{-1} x)^x, \text{ Taking log both sides we get, } \log u = \log \{ (\sin^{-1} x)^x \}$$

$$\Rightarrow \log u = x \log(\sin^{-1} x) \quad \dots(iii)$$

$$\text{On differentiating both sides of (iii) w.r.t. } x, \frac{1}{u} \cdot \frac{du}{dx} = x \cdot \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} + \log(\sin^{-1} x) \cdot 1$$

$$\Rightarrow \frac{du}{dx} = u \left\{ \frac{x}{\sin^{-1} x \cdot \sqrt{1-x^2}} + \log(\sin^{-1} x) \right\}$$

$$v = \sin^{-1}(\sqrt{x}) \quad \dots(iv)$$

$$\text{On differentiating both sides of (iv) w.r.t. } x, \frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} \Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii),

$$\frac{dy}{dx} = (\sin x)^x \left\{ \frac{x}{\sin^{-1} x \sqrt{1-x^2}} + \log(\sin^{-1} x) \right\} + \frac{1}{2\sqrt{x} \cdot \sqrt{1-x}}$$

$$\therefore \frac{dy}{dx} = (\sin x)^x \left(\frac{x}{\sin^{-1} x \sqrt{1-x^2}} + \log(\sin^{-1} x) \right) + \frac{1}{2\sqrt{x-x^2}}$$

36. $y = x^{x \cos x} + \frac{x^2+1}{x^2-1}$

Sol. $y = x^{x \cos x} + \frac{x^2+1}{x^2-1}$, Let $u = x^{x \cos x}$, $v = \frac{x^2+1}{x^2-1}$

$$y = u + v \quad \dots(i)$$

On differentiating both sides of (i) w.r.t. x

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(ii)$$

$$u = x^{x \cos x}, \text{ Taking log both sides we get,}$$

$$\log u = \log(x^{x \cos x}) \Rightarrow \log u = (x \cos x) \cdot \log x \quad \dots(iii)$$

$$\text{On differentiating both sides of w.r.t. } x, \frac{1}{u} \cdot \frac{du}{dx} = (x \cos x) \cdot \frac{1}{x} + \log x \cdot (-x \sin x + \cos x \cdot 1)$$

$$\Rightarrow \frac{du}{dx} = u \{ \cos x - x \log x \sin x + \log x \cos x \}$$

$$\Rightarrow \frac{du}{dx} = x^{\cos x} \{ \cos x - x \log x \sin x + \log x \cdot \cos x \}$$

$$v = \frac{x^2 + 1}{x^2 - 1} \quad \dots (iv)$$

On differentiating both sides of (iv) w.r.t. x ,

$$\begin{aligned} \frac{dv}{dx} &= \frac{(x^2 - 1) \frac{d(x^2 + 1)}{dx} - (x^2 + 1) \frac{d(x^2 - 1)}{dx}}{(x^2 - 1)^2} \Rightarrow \frac{dv}{dx} = \frac{(x^2 - 1)2x - (x^2 + 1)2x}{(x^2 - 1)^2} \\ \Rightarrow \frac{dv}{dx} &= \frac{2x(x^2 - 1 - x^2 - 1)}{(x^2 - 1)^2} \Rightarrow \frac{dv}{dx} = \frac{-4x}{(x^2 - 1)^2} \end{aligned}$$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii), $\frac{dy}{dx} = x^{\cos x} \{ \cos x - x \log x \sin x + \log x \cdot \cos x \} - \frac{4x}{(x^2 - 1)^2}$

37. $y = e^x \sin^3 x \cos^4 x$

Sol. $y = e^x \sin^3 x \cos^4 x \quad \dots (i)$

Taking log both sides, we get, $\log y = \log \{ e^x \cdot \sin^3 x \cdot \cos^4 x \}$

$$\Rightarrow \log y = \log(e^x) + \log(\sin^3 x) + \log(\cos^4 x) \Rightarrow \log y = x \log e + 3 \log(\sin x) + 4 \log(\cos x)$$

$$\Rightarrow \log y = x + 3 \log(\sin x) + 4 \log(\cos x) \quad \dots (ii)$$

On differentiating both sides of (ii) w.r.t. x , $\frac{1}{y} \cdot \frac{dy}{dx} = 1 + 3 \cdot \frac{1}{\sin x} \cdot \cos x + 4 \cdot \frac{1}{\cos x} (-\sin x)$

$$\Rightarrow \frac{dy}{dx} = y(1 + 3 \cot x - 4 \tan x) \therefore \frac{dy}{dx} = e^x \sin^3 x \cos^4 x (1 + 3 \cot x - 4 \tan x)$$

38. $y = 2^x \cdot e^{3x} \sin 4x$

Sol. $y = 2^x \cdot e^{3x} \sin 4x \quad \dots (i)$

Taking log both sides we get, $\log y = \log(2^x \cdot e^{3x} \cdot \sin 4x)$

$$\Rightarrow \log y = \log(2^x) + \log(e^{3x}) + \log(\sin 4x) \Rightarrow \log y = x \log 2 + 3x \log e + \log(\sin 4x)$$

$$\Rightarrow \log y = x \log 2 + 3x \log e + \log(\sin 4x) \Rightarrow \log y = x \log 2 + 3x + \log(\sin 4x) \quad \dots (ii)$$

On differentiating both sides of (ii) w.r.t. x ,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \log 2 \cdot 1 + 3 + \frac{1}{\sin 4x} \cdot \cos 4x \cdot 4 \Rightarrow \frac{dy}{dx} = 2^x \cdot e^{3x} \sin 4x (\log 2 + 3 + 4 \cot 4x)$$

39. $y = x^x \cdot e^{(2x+5)}$

Sol. $y = x^x \cdot e^{(2x+5)} \quad \dots (i)$

Taking log both sides, we get, $\log y = \log \{ x^x \cdot e^{2x+5} \} \Rightarrow \log y = \log(x^x) + \log(e^{2x+5})$

$$\Rightarrow \log y = x \log x + (2x + 5) \log e \Rightarrow \log y = x \log x + (2x + 5) \quad \dots (ii)$$

On differentiating both sides of (ii) w.r.t. x ,

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1 + 2 \Rightarrow \frac{dy}{dx} = y(1 + \log x + 2) \therefore \frac{dy}{dx} = x^x \cdot e^{2x+5} (3 + \log x)$$

40. $y = (2x+3)^5 (3x-5)^7 (5x-1)^3$

Sol. $y = (2x+3)^5 (3x-5)^7 (5x-1)^3$... (i)

Taking log both sides, we get, $\log y = \log \{(2x+3)^5 (3x-5)^7 (5x-1)^3\}$

$$\Rightarrow \log y = \log \{(2x+3)^5 + \log(3x-5)^7\} + \log \{(5x-1)^3\}$$

$$\Rightarrow \log y = 5 \log(2x+3) + 7 \log(3x-5) + 3 \log(5x-1) \quad \dots (ii)$$

On differentiating both sides of (ii) w.r.t. x , $\frac{1}{y} \cdot \frac{dy}{dx} = 5 \cdot \frac{1}{2x+3} \cdot 2 + 7 \cdot \frac{1}{3x-5} \cdot 3 + 3 \cdot \frac{1}{5x-1} \cdot 5$

$$\Rightarrow \frac{dy}{dx} = y \left(\frac{10}{2x+3} + \frac{21}{3x-5} + \frac{15}{5x-1} \right) \quad \therefore \frac{dy}{dx} = (2x+3)^5 (3x-5)^7 (5x-1)^3 \left\{ \frac{10}{2x+3} + \frac{21}{3x-5} + \frac{15}{5x-1} \right\}$$

41. $(\cos x)^y = (\cos y)^x$

Sol. $(\cos x)^y = (\cos y)^x$, Taking log both sides, we get, $\log \{(\cos x)^y\} = \log \{(\cos y)^x\}$

$$\Rightarrow y \log(\cos x) = x \log(\cos y) \quad \dots (i)$$

On differentiating both sides of (i) w.r.t. x ,

$$y \cdot \frac{1}{\cos x} (-\sin x) + \log(\cos x) \cdot \frac{dy}{dx} = x \cdot \frac{1}{\cos y} (-\sin y) \frac{dy}{dx} + \log(\cos y) \cdot 1$$

$$\Rightarrow -y \tan x + \log(\cos x) \frac{dy}{dx} = -x \tan y \frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow \log(\cos x) \frac{dy}{dx} + x \tan y \frac{dy}{dx} = \log(\cos y) + y \tan x$$

$$\Rightarrow \frac{dy}{dx} \{ \log(\cos x) + x \tan y \} = \{ \log(\cos y) + y \tan x \} \quad \therefore \frac{dy}{dx} = \frac{\log(\cos y) + y \tan x}{\log \cos x + x \tan y}$$

42. $(\tan x)^y = (\tan y)^x$

Sol. $(\tan x)^y = (\tan y)^x$... (i)

Taking both sides, we get, $\log \{(\tan x)^y\} = \log \{(\tan y)^x\}$

$$\Rightarrow y \log(\tan x) = x \log(\tan y) \quad \dots (ii)$$

On differentiating both sides of (ii) w.r.t. x ,

$$y \cdot \frac{1}{\tan x} \cdot \sec^2 x + \log(\tan x) \frac{dy}{dx} = x \cdot \frac{1}{\tan y} \sec^2 y \frac{dy}{dx} + \log(\tan y) \cdot 1$$

$$\Rightarrow y \cdot \frac{2}{\sin 2x} + \log(\tan x) \frac{dy}{dx} = x \cdot \frac{2}{\sin 2y} \frac{dy}{dx} + \log(\tan y)$$

$$\Rightarrow 2y \operatorname{cosec} 2x + \log(\tan x) \frac{dy}{dx} = 2x \operatorname{cosec} 2y \frac{dy}{dx} + \log(\tan y)$$

$$\Rightarrow \log(\tan x) \frac{dy}{dx} - 2x \operatorname{cosec} 2y \frac{dy}{dx} = \log(\tan y) - 2y \operatorname{cosec} 2x$$

$$\Rightarrow \frac{dy}{dx} (\log(\tan x) - 2x \operatorname{cosec} 2y) = \log(\tan y) - 2y \operatorname{cosec} 2x \quad \therefore \frac{dy}{dx} = \frac{\log(\tan y) - 2y \operatorname{cosec} 2x}{\log(\tan x) - 2x \operatorname{cosec} 2y}$$

43. $y = (\log x)^x + x^{(\log x)}$

Sol. $y = (\log x)^x + x^{(\log x)}$, Let $u = (\log x)^x$, $v = x^{\log x}$

$$y = u + v \quad \dots(i)$$

$$\text{On differentiating both sides of (ii) w.r.t. } x, \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(ii)$$

$$u = (\log x)^x$$

$$\text{Taking log both sides, we get, } \log u = \log \{(\log x)^x\} \Rightarrow \log u = x \log(\log x) \quad \dots(iii)$$

$$\text{On differentiating both sides of (iii) w.r.t. } x, \frac{1}{u} \cdot \frac{du}{dx} = x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log(\log x) \cdot 1$$

$$\Rightarrow \frac{du}{dx} = u \left\{ \frac{1}{\log x} + \log(\log x) \right\} \Rightarrow \frac{du}{dx} = (\log x)^x \left\{ \frac{1}{\log x} + \log(\log x) \right\}$$

$$\Rightarrow v = x^{\log x}$$

$$\text{Taking log both sides we get, } \log v = \log(x^{\log x}) \Rightarrow \log v = \log x \cdot \log x \quad \dots(iv)$$

On differentiating both sides of (iv) w.r.t. x ,

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log x \cdot \frac{1}{x} + \log x \cdot \frac{1}{x} \Rightarrow \frac{dv}{dx} = v \left(\frac{2 \log x}{x} \right) \Rightarrow \frac{dv}{dx} = x^{\log x} \left(\frac{2 \log x}{x} \right)$$

Putting the value differentiation $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii),

$$\frac{dy}{dx} = (\log x)^x \left\{ \frac{1}{\log x} + \log(\log x) \right\} + x^{\log x} \cdot \frac{2}{x} (\log x)$$

44. If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ prove that $(1-x^2) \frac{dy}{dx} = xy + 1$

Sol. $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}} \quad \dots(i)$

On differentiating both sides w.r.t. x ,

$$\frac{dy}{dx} = \frac{(\sqrt{1-x^2}) \frac{d(\sin^{-1} x)}{dx} - \sin^{-1} x \frac{d(\sqrt{1-x^2})}{dx}}{(\sqrt{1-x^2})^2} \times \frac{d(1-x^2)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(\sqrt{1-x^2}) \cdot \frac{1}{\sqrt{1-x^2}} - \sin^{-1} x \cdot \frac{1}{2\sqrt{1-x^2}} \cdot (-2x)}{(1-x^2)} \Rightarrow (1-x^2) \frac{dy}{dx} = 1 + \frac{x \sin^{-1} x}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2) \frac{dy}{dx} = (1+xy) \text{ Proved.}$$

45. If $y = \sqrt{x+y}$ prove that $\frac{dy}{dx} = \frac{1}{2y-1}$

Sol. $y = \sqrt{x+y} \quad \dots(i)$

$$\text{Squaring both sides we get, } (y)^2 = (\sqrt{x+y})^2, y^2 = x+y \quad \dots(ii)$$

On differentiation of both sides at (ii) w.r.t. x ,

$$2y \frac{dy}{dx} = 1 + \frac{dy}{dx} \Rightarrow 2y \frac{dy}{dx} - \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} (2y-1) = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y-1}$$

46. If $x^a y^b = (x+y)^{a+b}$ prove that $\frac{dy}{dx} = \frac{y}{x}$

Sol. $x^a y^b = (x+y)^{a+b}$... (i)

Taking log both sides of (i) w.r.t. x , $\log(x^a y^b) = \log\{(x+y)^{a+b}\}$

$$\Rightarrow \log(x^a) + \log(y^b) = (a+b)\log(x+y)$$

$$\Rightarrow a \log x + b \log y = (a+b)\log(x+y) \quad \dots (ii)$$

On differentiating both sides of (ii) w.r.t. x , $a \cdot \frac{1}{x} + b \cdot \frac{1}{y} \cdot \frac{dy}{dx} = (a+b) \cdot \frac{1}{x+y} \left(1 + \frac{dy}{dx}\right)$

$$\Rightarrow \frac{a}{x} + \frac{b}{y} \frac{dy}{dx} = \frac{a+b}{x+y} + \frac{a+b}{x+y} \frac{dy}{dx} \Rightarrow \frac{b}{y} \frac{dy}{dx} - \left(\frac{a+b}{x+y}\right) \frac{dy}{dx} = \frac{a+b}{x+y} - \frac{a}{x}$$

$$\Rightarrow \frac{dy}{dx} \left(\frac{b}{y} - \frac{a+b}{x+y}\right) = \frac{a+b}{x+y} - \frac{a}{x} \Rightarrow \frac{dy}{dx} \left(\frac{b(x+y) - y(a+b)}{y(x+y)}\right) = \frac{(a+b)x - a(x+y)}{(x+y)x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(ax + bx - ax - ay)}{x(bx + by - ay - by)} \Rightarrow \frac{dy}{dx} = \frac{y(bx - ay)}{x(bx - ay)} \therefore \frac{dy}{dx} = \frac{y}{x}$$

47. If $x^x + y^y = 1$ show that $\frac{dy}{dx} = -\left\{\frac{x^x(1+\log x) + y^y \log y}{xy^{x-1}}\right\}$

Sol. $x^x + y^y = 1$

Let $u = x^x$, $v = y^y$

$$u + v = 1 \quad \dots (i)$$

On differentiating both sides w.r.t. x , $\frac{du}{dx} + \frac{dv}{dx} = \frac{d(1)}{dx}$, $\frac{du}{dx} + \frac{dv}{dx} = 0 \quad \dots (ii)$

$u = x^x$, Taking log both sides we get, $\log u = \log(x^x) \Rightarrow \log u = x \log x \quad \dots (iii)$

On differentiating both sides of (iii) w.r.t. x ,

$$\frac{1}{u} \frac{du}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1 \Rightarrow \frac{du}{dx} = u(1 + \log x) \Rightarrow \frac{du}{dx} = x^x(1 + \log x)$$

$v = y^y$, Taking log both sides we get, $\log v = \log(y^y) \Rightarrow \log v = y \log y \quad \dots (iv)$

On differentiating both sides of (iv) w.r.t. x ,

$$\frac{1}{v} \frac{dv}{dx} = y \cdot \frac{1}{y} \frac{dy}{dx} + \log y \cdot 1 \Rightarrow \frac{dv}{dx} = v \left(\frac{y}{y} \frac{dy}{dx} + \log y\right) \Rightarrow \frac{dv}{dx} = y^y \left(\frac{y}{y} \frac{dy}{dx} + \log y\right)$$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii),

$$x^x(1 + \log x) + y^y \left(\frac{y}{y} \frac{dy}{dx} + \log y\right) = 0 \Rightarrow x^x(1 + \log x) + \frac{y^y \cdot y}{y} \frac{dy}{dx} + y^y \log y = 0$$

$$\Rightarrow xy^{x-1} \frac{dy}{dx} = -\{y^y \log y + x^x(1 + \log x)\} \therefore \frac{dy}{dx} = -\frac{\{x^x + (1 + \log x) + y^y \log y\}}{xy^{x-1}}$$

48. If $y = e^{\sin x} + (\tan x)^x$ prove that $\frac{dy}{dx} = e^{\sin x} \cos x + (\tan x)^2 [2x \operatorname{cosec} 2x + \log \tan x]$

Sol. $y = e^{\sin x} + (\tan x)^x$

Let $u = e^{\sin x}$, $v = (\tan x)^x$, $y = u + v$... (i)

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$... (ii)

$u = e^{\sin x}$... (iii)

Differentiating both sides of (iii) w.r.t. x , $\frac{du}{dx} = e^{\sin x} \cdot \cos x$

$v = (\tan x)^x$, Taking log both sides we get,

$\log v = \log \{(\tan x)^x\} \Rightarrow \log v = x \log(\tan x)$... (iv)

On differentiating both sides of (iv) w.r.t. x , $\frac{1}{v} \cdot \frac{dv}{dx} = x \cdot \frac{1}{\tan x} \cdot \sec^2 x + \log(\tan x) \cdot 1$

$\Rightarrow \frac{dv}{dx} = v \{2x \operatorname{cosec} 2x + \log(\tan x)\} \Rightarrow \frac{dv}{dx} = (\tan x)^x (2x \operatorname{cosec} 2x + \log(\tan x))$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii), $\frac{dy}{dx} = e^{\sin x} \cdot \cos x + (\tan x)^x [2x \operatorname{cosec} 2x + \log(\tan x)]$

49. If $y = \log(x + \sqrt{1+x^2})$ prove that $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$

Sol. $y = \log(x + \sqrt{1+x^2})$... (i)

On differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = \frac{1}{x + \sqrt{1+x^2}} \left\{ 1 + \frac{1}{2\sqrt{1+x^2}} \cdot 2x \right\}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{1+x^2}} \cdot \left\{ \frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}} \right\} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$ Proved.

50. If $y = \log \sin(\sqrt{x^2+1})$ Proved that $\frac{dy}{dx} = \frac{x \cot \sqrt{x^2+1}}{\sqrt{x^2+1}}$

Sol. $y = \log \left\{ \sin(\sqrt{x^2+1}) \right\}$... (i)

On differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = \frac{1}{\sin \sqrt{x^2+1}} \cdot \cos(\sqrt{x^2+1}) \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x$

$\Rightarrow \frac{dy}{dx} = \cot(\sqrt{x^2+1}) \cdot \frac{x}{\sqrt{x^2+1}}$ $\therefore \frac{dy}{dx} = \frac{x \cot(\sqrt{x^2+1})}{\sqrt{x^2+1}}$

51. If $y = \log \left(\frac{1 - \cos x}{1 + \cos x} \right)$ show that $\frac{dy}{dx} = \operatorname{cosec} x$

Sol. $y = \log \left(\frac{1 - \cos x}{1 + \cos x} \right) \Rightarrow y = \log \left(\frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right) \Rightarrow y = \log \left(\tan^2 \frac{x}{2} \right)$... (i)

On differentiating both sides of (i) w.r.t. x ,

$\frac{dy}{dx} = \frac{1}{\tan^2 \frac{x}{2}} \cdot \sec^2 \frac{x}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{\sin x} \Rightarrow \frac{dy}{dx} = \operatorname{cosec} x$

52. If $y = \log \left\{ \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right\}$ show that $\frac{dy}{dx} = \sec x$.

Sol. $y = \log \left\{ \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right\}$... (i)

On differentiating both sides of (i) w.r.t. x ,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\tan \left(\frac{\pi}{4} + \frac{x}{2} \right)} \cdot \sec^2 \left(\frac{\pi}{4} + \frac{x}{2} \right) \cdot \frac{1}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2 \sin \left(\frac{\pi}{4} + \frac{x}{2} \right) \cos \left(\frac{\pi}{4} + \frac{x}{2} \right)} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\sin \left\{ 2 \left(\frac{\pi}{4} + \frac{x}{2} \right) \right\}} \Rightarrow \frac{dy}{dx} = \frac{1}{\sin \left(\frac{\pi}{2} + x \right)} \Rightarrow \frac{dy}{dx} = \frac{1}{\cos x} \Rightarrow \frac{dy}{dx} = \sec x \end{aligned}$$

53. If $y = \sqrt{\frac{1 - \sin 2x}{1 + \sin 2x}}$ show that $\frac{dy}{dx} + \sec^2 \left(\frac{\pi}{4} - x \right) = 0$

Sol. $y = \sqrt{\frac{1 - \sin 2x}{1 + \sin 2x}}$... (i)

$$y = \sqrt{\frac{(\cos x - \sin x)^2}{(\cos x + \sin x)^2}} \Rightarrow y = \frac{\cos x - \sin x}{\cos x + \sin x}$$

Dividing nominator and denominator $\cos x$, $y = \frac{\cos x - \sin x}{\cos x + \sin x} \Rightarrow y = \frac{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}}$

$$\Rightarrow y = \frac{1 - \tan x}{1 + \tan x} \Rightarrow y = \frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \tan x} \Rightarrow y = \tan \left(\frac{\pi}{4} - x \right) \quad \dots (ii)$$

On differentiating both sides of (ii) w.r.t. x , $\frac{dy}{dx} = \sec^2 \left(\frac{\pi}{4} - x \right) (-1) \therefore \frac{dy}{dx} = -\sec^2 \left(\frac{\pi}{4} - x \right)$

$$\text{L.H.S.} = \frac{dy}{dx} + \sec^2 \left(\frac{\pi}{4} - x \right) = -\sec^2 \left(\frac{\pi}{4} - x \right) + \sec^2 \left(\frac{\pi}{4} - x \right) = 0$$

54. If $y = \log \sqrt{\frac{1 + \cos^2 x}{1 - e^{2x}}}$ show that $\frac{dy}{dx} = \frac{e^{2x}}{1 - e^{2x}} - \frac{\sin x \cos x}{1 + \cos^2 x}$

Sol. $y = \log \left(\sqrt{\frac{1 + \cos^2 x}{1 - e^{2x}}} \right)$... (i)

On differentiating both sides of (i) w.r.t. x ,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sqrt{\frac{1 + \cos^2 x}{1 - e^{2x}}}} \cdot \frac{1}{2 \sqrt{\frac{1 + \cos^2 x}{1 - e^{2x}}}} \left[\frac{(1 - e^{2x}) \frac{d(1 + \cos^2 x)}{dx} - (1 + \cos^2 x) \frac{d(1 - e^{2x})}{dx}}{(1 - e^{2x})^2} \right] \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{2 \left(\frac{1 + \cos^2 x}{1 - e^{2x}} \right)} \left[\frac{(1 - e^{2x})(-2) \sin x \cos x - (1 + \cos^2 x) e^{2x} (-2)}{(1 - e^{2x})^2} \right] \end{aligned}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-e^{2x}}{1+\cos^2 x} \cdot \left\{ \frac{(1-e^{2x})\sin x \cos x + e^{2x}(1+\cos^2 x)}{(1-e^{2x})^2} \right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(1-e^{2x})\sin x \cos x + e^{2x}(1+\cos^2 x)}{(1+\cos^2 x)(1-e^{2x})}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sin x \cos x}{1+\cos^2 x} + \frac{e^{2x}}{1-e^{2x}} \quad \therefore \frac{dy}{dx} = \frac{e^{2x}}{1-e^{2x}} - \frac{\sin x \cos x}{1+\cos^2 x}$$

55. If $y = x^{\cos x} + (\sin x)^{\tan x}$ prove that $\frac{dy}{dx} = x \cos x \left\{ \frac{\cos x}{x} - (\sin x) \log x \right\} + (\sin x)^{\tan x} \{1 + \log(\sin x) \cdot \sec^2 x\}$

Sol. $y = x^{\cos x} + (\sin x)^{\tan x}$, Let $u = x^{\cos x}$, $v = (\sin x)^{\tan x}$

$$y = u + v \quad \dots (i)$$

On differentiating both sides of (i) w.r.t. x

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots (ii)$$

$$u = x^{\cos x}, \text{ Taking log both sides, we get, } \log u = \log(x^{\cos x}) \Rightarrow \log u = \cos x \cdot \log x \quad \dots (iii)$$

On differentiating both sides of (iii) w.r.t. x , $\frac{1}{u} \cdot \frac{du}{dx} = \cos x \cdot \frac{1}{x} + \log x (-\sin x)$

$$\Rightarrow \frac{du}{dx} = u \left(\frac{\cos x}{x} - \log x \sin x \right) \Rightarrow \frac{du}{dx} = x^{\cos x} \left(\frac{\cos x}{x} - \log x \sin x \right)$$

$$v = (\sin x)^{\tan x}, \text{ Taking log both sides we get,}$$

$$\log v = \log \{ \sin^{\tan x} \} \Rightarrow \log v = \tan x \cdot \log(\sin x) \quad \dots (iv)$$

On differentiating both sides of (iv) w.r.t. x , $\frac{1}{v} \cdot \frac{dv}{dx} = \tan x \cdot \frac{1}{\sin x} \cdot \cos x + \log(\sin x) \cdot \sec^2 x$

$$\Rightarrow \frac{dv}{dx} = v \{1 + \log(\sin x) \cdot \sec^2 x\} \Rightarrow \frac{dv}{dx} = (\sin x)^{\tan x} (1 + \log(\sin x) \sec^2 x)$$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii),

$$\frac{dy}{dx} = x^{\cos x} \left(\frac{\cos x}{x} - \log x \sin x \right) + (\sin x)^{\tan x} (1 + \log(\sin x) \cdot \sec^2 x)$$

56. If $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$ prove that

$$\frac{dy}{dx} = (\sin x)^{\cos x} [\cot x \cos x - \sin x \log(\sin x)] + (\cos x)^{\sin x} [\cos x \log(\cos x) - \sin x \tan x]$$

Sol. $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$, Let $u = (\sin x)^{\cos x}$, $v = (\cos x)^{\sin x}$

$$y = u + v \quad \dots (i)$$

On differentiating both sides of (i) w.r.t. x

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots (ii)$$

$$u = (\sin x)^{\cos x}$$

Taking log both sides, we get, $\log u = \log \{ (\sin x)^{\cos x} \} \Rightarrow \log u = \cos x \cdot \log(\sin x) \quad \dots (iii)$

On differentiating both sides of (iii) w.r.t. x , $\frac{1}{v} \cdot \frac{du}{dx} = \cos x \cdot \frac{1}{\sin x} \cdot \cos x + \log(\sin x)(-\sin x)$

$$\Rightarrow \frac{dv}{dx} = v(\cot x \cos x - \log(\sin x) \cdot \sin x) \Rightarrow \frac{du}{dx} = (\sin x)^{\cos x} \{ \cot x \cos x - \sin x \cdot \log(\sin x) \}$$

$$v = (\cos x)^{\sin x}$$

Taking log both sides, we get, $\log v = \log \{ (\cos x)^{\sin x} \} \Rightarrow \log v = \sin x \cdot \log(\cos x) \dots (iv)$

On differentiating both sides of (iv) w.r.t. x , $\frac{1}{v} \cdot \frac{dv}{dx} = \sin x \cdot \frac{1}{\cos x} \cdot (-\sin x) + \log(\cos x) \cos x$

$$\Rightarrow \frac{dv}{dx} = v(-\sin x \tan x + \cos x \cdot \log(\cos x)) \Rightarrow \frac{dv}{dx} = (\cos x)^{\sin x} (\cos x \cdot \log(\cos x) - \sin x \cdot \tan x)$$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (i),

$$\frac{dy}{dx} = (\sin x)^{\cos x} [\cot x \cos x - \sin x \log(\sin x)] + (\cos x)^{\sin x} [\cos x \cdot \log(\cos x) - \sin x \tan x]$$

57. If $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$ prove that

$$\frac{dy}{dx} = (\tan x)^{\cot x} \operatorname{cosec}^2 x (1 - \log(\tan x) + (\cot x)^{\tan x} \sec^2 x) [\log(\cot x) - 1]$$

Sol. $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$, Let $u = (\tan x)^{\cot x}$, $v = (\cot x)^{\tan x}$

$$y = u + v \dots (i)$$

On differentiating both sides of (ii) w.r.t. x

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \dots (ii)$$

$$u = (\tan x)^{\cot x}$$

Taking log both sides, we get, $\log u = \log \{ (\tan x)^{\cot x} \} \Rightarrow \log u = \cot x \cdot \log(\tan x) \dots (iii)$

On differentiating both sides of (iii) w.r.t. x ,

$$\frac{1}{u} \cdot \frac{du}{dx} = \cot x \cdot \frac{1}{\tan x} \cdot \sec^2 x + \log(\tan x) \cdot (-\operatorname{cosec}^2 x)$$

$$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = \operatorname{cosec}^2 x - \log(\tan x) \cdot \operatorname{cosec}^2 x \Rightarrow \frac{dv}{dx} = u \operatorname{cosec}^2 x (1 - \log(\tan x))$$

$$\Rightarrow \frac{du}{dx} = (\tan x)^{\cot x} \cdot \operatorname{cosec}^2 x (1 - \log(\tan x)), \quad v = (\cot x)^{\tan x}$$

Taking log both sides we get, $\log v = \log \{ (\cot x)^{\tan x} \} \Rightarrow \log v = \tan x \cdot \log(\cot x) \dots (iv)$

On differentiating both sides of (iv) w.r.t. x ,

$$\frac{1}{v} \cdot \frac{dv}{dx} = \tan x \cdot \frac{1}{\cot x} \cdot (-\operatorname{cosec}^2 x) + \log(\cot x) \sec^2 x$$

$$\Rightarrow \frac{dv}{dx} = v(-\sec^2 x + \log(\cot x) \sec^2 x) \Rightarrow \frac{dv}{dx} = (\cot x)^{\tan x} \sec^2 x \{ \log(\cot x) - 1 \}$$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii),

$$\frac{dy}{dx} = (\tan x)^{\cot x} \operatorname{cosec}^2 x (1 - \log \tan x) + (\cot x)^{\tan x} \cdot \sec^2 x \{ \log(\cot x) - 1 \}$$

58. If $y = x^{\cos x} + (\cos x)^x$ prove that

$$\frac{dy}{dx} = x^{\cos x} \left\{ \frac{\cos x}{x} - (\sin x) \cdot \log x \right\} + (\cos x)^x [\log(\cos x) - x \tan x]$$

Sol. $y = x^{\cos x} + (\cos x)^x$, Let $u = x^{\cos x}$, $v = (\cos x)^x$

$$y = u + v \quad \dots(i)$$

On differentiating both sides of (i) w.r.t. x

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(ii)$$

$$u = x^{\cos x}$$

$$\text{Taking log both sides, we get, } \log u = \log(x^{\cos x}) \Rightarrow \log u = \cos x \cdot \log x \quad \dots(iii)$$

$$\text{On differentiating both sides of (iii) w.r.t. } x, \frac{1}{u} \cdot \frac{du}{dx} = \cos x \cdot \frac{1}{x} + \log x \cdot (-\sin x)$$

$$\Rightarrow \frac{du}{dx} = u \left(\frac{\cos x}{x} - \log x \cdot \sin x \right) \Rightarrow \frac{du}{dx} = x^{\cos x} \left(\frac{\cos x}{x} - \sin x \cdot \log x \right)$$

$$v = (\cos x)^x$$

$$\text{Taking log both sides, we get, } \log v = \log(\cos x)^x \Rightarrow \log v = x \log(\cos x) \quad \dots(iv)$$

$$\text{On differentiating both sides of (iv) w.r.t. } x, \frac{1}{v} \cdot \frac{dv}{dx} = x \cdot \frac{1}{\cos x} \cdot (-\sin x) + \log(\cos x) \cdot 1$$

$$\Rightarrow \frac{dv}{dx} = (\cos x)^x [\log(\cos x) - x \tan x]$$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii),

$$\frac{dy}{dx} = x^{\cos x} \left[\frac{\cos x}{x} - \sin x \cdot \log x \right] + (\cos x)^x [\log \cos x - x \tan x]$$

59. If $y = x^{\log x} + (\log x)^x$ prove that $\frac{dy}{dx} = x^{\log x} \left\{ \frac{2 \log x}{x} \right\} + (\log x)^x \left\{ \frac{1}{\log x} + \log(\log x) \right\}$

Sol. Let $u = (\log x)^x$, $v = x^{\log x}$

$$y = u + v \quad \dots(i)$$

On differentiating both sides of (i) w.r.t. x ,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(ii)$$

$$u = (\log x)^x$$

$$\text{Taking log both sides we get, } \log u = \log\{(\log x)^x\} \Rightarrow \log u = x \log(\log x)$$

$$\text{On differentiating both sides w.r.t. } x, \frac{1}{u} \cdot \frac{du}{dx} = x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log(\log x) \cdot 1$$

$$\Rightarrow \frac{du}{dx} = u \left(\frac{1}{\log x} + \log(\log x) \right) \Rightarrow \frac{du}{dx} = (\log x)^x \left(\frac{1}{\log x} + \log(\log x) \right)$$

$$v = x^{\log x}, \text{ Taking log both sides we get, } \log v = \log(x^{\log x}) \Rightarrow \log v = \log x \cdot \log x$$

On differentiating both sides w.r.t. x , $\frac{1}{v} \cdot \frac{dv}{dx} = \log x \cdot \frac{1}{x} + \log x \cdot \frac{1}{x}$

$$\Rightarrow \frac{dv}{dx} = v \cdot \frac{2}{x} \log x \Rightarrow \frac{dv}{dx} = v \cdot \frac{2}{x} \log x \Rightarrow \frac{dv}{dx} = x^{\log x} \cdot \frac{2 \log x}{x}$$

Putting the value of $\frac{du}{dv}$ & $\frac{dv}{dx}$ in equation (ii), $\frac{dy}{dx} = (\log x)^x \left\{ \frac{1}{\log x} + \log(\log x) \right\} + x^{\log x} \cdot \frac{2 \log x}{x}$

60. If $y = x^{(x^2-3)} + (x-3)^{x^2}$, find $\frac{dy}{dx}$

Sol. $y = x^{(x^2-3)} + (x-3)^{x^2}$, Let $u = x^{(x^2-3)}$, $v = (x-3)^{x^2}$

$$y = u + v \quad \dots(i)$$

On differentiating both sides of (i) w.r.t. x

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(ii)$$

$$u = x^{(x^2-3)}, \text{ taking log both sides, we get, } \log u = \log \left\{ x^{(x^2-3)} \right\} \Rightarrow \log u = (x^2-3) \log x \quad \dots(iii)$$

On differentiating both sides of (iii) w.r.t. x , $\frac{1}{u} \cdot \frac{du}{dx} = (x^2-3) \cdot \frac{1}{x} + \log x \cdot 2x$

$$\Rightarrow \frac{du}{dx} = u \left(\frac{x^2-3}{x} + 2x \log x \right) \Rightarrow \frac{du}{dx} = x^{(x^2-3)} \left\{ \frac{x^2-3}{x} + 2x \log x \right\}$$

$$v = (x-3)^{x^2}$$

Taking log both sides we get, $\log v = \log \left\{ (x-3)^{x^2} \right\} \Rightarrow \log v = x^2 \log(x-3) \quad \dots(iv)$

On differentiating both sides of (iv) w.r.t. x , $\frac{1}{v} \cdot \frac{dv}{dx} = x^2 \cdot \frac{1}{x-3} + \log(x-3) \cdot 2x$

$$\Rightarrow \frac{dv}{dx} = v \left\{ \frac{x^2}{x-3} + 2x \log(x-3) \right\} \Rightarrow \frac{dv}{dx} = (x-3)^{x^2} \left\{ \frac{x^2}{x-3} + 2x \log(x-3) \right\}$$

Putting the value of $\frac{du}{dx}$ & $\frac{dv}{dx}$ in equation (ii),

$$\frac{dy}{dx} = x^{(x^2-3)} \left\{ \frac{x^2-3}{x} + 2x \log x \right\} + (x-3)^{x^2} \left\{ \frac{x^2}{x-3} + 2x \log(x-3) \right\}$$

61. If $f(x) = \left(\frac{3+x}{1+x} \right)^{2+3x}$ find $f'(0)$.

Sol. $f(x) = \left(\frac{3+x}{1+x} \right)^{2+3x} \quad \dots(i)$

Taking log both sides we get, $\log f(x) = \log \left\{ \left(\frac{3+x}{1+x} \right)^{2+3x} \right\}$

$$\Rightarrow \log f(x) = (2+3x) \log \left(\frac{3+x}{1+x} \right) \Rightarrow \log f(x) = (2+3x) \{ \log(3+x) - \log(1+x) \} \quad \dots(ii)$$

On differentiating both sides of w.r.t. x ,

$$\begin{aligned}\frac{1}{f(x)} f'(x) &= (2+3x) \cdot \left(\frac{1}{3+x} - \frac{1}{1+x} \right) + (\log(3+x) - \log(1+x)) \cdot 3 \\ \Rightarrow f'(x) &= f(x) \cdot \left[(2+3x) \left\{ \frac{1+x-3-x}{(1+x)(3+x)} \right\} + 3 \{ \log(3+x) - \log(1+x) \} \right] \\ \Rightarrow f'(x) &= \left(\frac{3+x}{1+x} \right)^{2+3x} \left[(-2) \frac{(2+3x)}{(1+x)(3+x)} + 3 \{ \log(3+x) - \log(1+x) \} \right] \\ \text{At } x=0, f'(0) &= \left(\frac{3+0}{1+0} \right)^{2+3 \cdot 0} \left\{ (-2) \frac{(2+3 \cdot 0)}{(1+0)(3+0)} + 3 \{ \log(3+0) - \log(1+0) \} \right\} \\ \Rightarrow f'(0) &= 3^2 \left\{ \frac{(-4)}{3} 3 \log 3 \right\} \Rightarrow f'(0) = -12 + 27 \log 3 \quad \therefore f'(0) = 27 \log 3 - 12\end{aligned}$$

EXERCISE 10G (Pg.No.: 435)

1. If $y = (\sin x)^{(\sin x)^{\sin x} \dots \infty}$ prove that $\frac{dy}{dx} = \frac{y^2 \cot x}{1 - y \log \tan x}$

Sol. $y = (\sin x)^{(\sin x)^{\sin x} \dots \infty} \Rightarrow y = (\sin x)^y$

Taking log both sides, we get, $\log y = \log \{ (\sin x)^y \} \Rightarrow \log y = y \log (\sin x) \quad \dots (i)$

On differentiating both sides of (i) w.r.t. x ,

$$\begin{aligned}\frac{d(\log y)}{dy} \cdot \frac{dy}{dx} &= y \cdot \frac{d\{\log(\sin x)\}}{d(\sin x)} \times \frac{d(\sin x)}{dx} + \log(\sin x) \frac{dy}{dx} \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= y \cdot \frac{1}{\sin x} \cos x + \log(\sin x) \frac{dy}{dx} \Rightarrow \frac{1}{y} \frac{dy}{dx} - \log(\sin x) \frac{dy}{dx} = y \cot x \\ \Rightarrow \frac{dy}{dx} \left\{ \frac{1}{y} - \log(\sin x) \right\} &= y \cot x \Rightarrow \frac{dy}{dx} \left\{ \frac{1 - y \log(\sin x)}{y} \right\} = y \cot x \quad \therefore \frac{dy}{dx} = \frac{y^2 \cot x}{1 - y \log(\sin x)}\end{aligned}$$

2. If $y = (\cos x)^{(\cos x)^{\cos x} \dots \infty}$, prove that $\frac{dy}{dx} = \frac{-y^2 \tan x}{1 - y \log \cos x}$

Sol. $y = (\cos x)^{(\cos x)^{\cos x} \dots \infty} \Rightarrow y = (\cos x)^y$

Taking log both sides, we get, $\log y = \log \{ (\cos x)^y \} \Rightarrow \log y = y \log (\cos x) \quad \dots (i)$

On differentiating both sides of (i) w.r.t. x ,

$$\begin{aligned}\frac{d(\log y)}{dy} \cdot \frac{dy}{dx} &= y \cdot \frac{d\{\log(\cos x)\}}{d(\cos x)} \times \frac{d(\cos x)}{dx} + \log(\cos x) \frac{dy}{dx} \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= y \cdot \frac{1}{\cos x} (-\sin x) + \log(\cos x) \frac{dy}{dx} \Rightarrow \frac{1}{y} \frac{dy}{dx} - \log(\cos x) \frac{dy}{dx} = -y \tan x \\ \Rightarrow \frac{dy}{dx} \left\{ \frac{1}{y} - \log(\cos x) \right\} &= -y \tan x \Rightarrow \frac{dy}{dx} \left\{ \frac{1 - y \log(\cos x)}{y} \right\} = -y \tan x \quad \therefore \frac{dy}{dx} = \frac{-y^2 \tan x}{1 - y \log(\cos x)}\end{aligned}$$

3. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \dots \infty}}}$, prove that $\frac{dy}{dx} = \frac{1}{2y-1}$.

Sol. $y = \sqrt{x + \sqrt{x + \sqrt{x + \dots \infty}}} \Rightarrow y = \sqrt{x + y}$

Squaring both sides, we get, $(y)^2 = (\sqrt{x + y})^2 \Rightarrow y^2 = x + y$

On differentiating both sides of (i) w.r.t. x , $\frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = \frac{d(x)}{dx} + \frac{dy}{dx}$

$\Rightarrow 2y \cdot \frac{dy}{dx} = 1 + \frac{dy}{dx} \Rightarrow 2y \frac{dy}{dx} - \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} (2y - 1) = 1 \therefore \frac{dy}{dx} = \frac{1}{2y-1}$

4. If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \dots \infty}}}$, prove that $\frac{dy}{dx} = \frac{\sin x}{1-2y}$.

Sol. $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \dots \infty}}} \Rightarrow y = \sqrt{\cos x + y}$

Squaring both sides, we get, $(y)^2 = (\sqrt{\cos x + y})^2 \Rightarrow y^2 = \cos x + y \dots (i)$

On differentiating both sides of (i) w.r.t. x , $\frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = \frac{d(\cos x)}{dx} + \frac{dy}{dx}$

$\Rightarrow 2y \frac{dy}{dx} - \frac{dy}{dx} = -\sin x \Rightarrow \frac{dy}{dx} (2y - 1) = -\sin x \Rightarrow \frac{dy}{dx} = \frac{-\sin x}{-(1-2y)} \therefore \frac{dy}{dx} = \frac{\sin x}{1-2y}$

5. If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \infty}}}$, prove that $\frac{dy}{dx} = \frac{\sec^2 x}{2y-1}$.

Sol. $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \infty}}} \Rightarrow y = \sqrt{\tan x + y}$

Squaring both sides we get, $(y)^2 = (\sqrt{\tan x + y})^2 \Rightarrow y^2 = (\tan x + y) \dots (i)$

On differentiating both sides of (i) w.r.t. x ,

$\frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = \frac{d(\tan x)}{dx} + \frac{dy}{dx} \Rightarrow 2y \cdot \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \Rightarrow 2y \frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$

$\Rightarrow \frac{dy}{dx} (2y - 1) = \sec^2 x \therefore \frac{dy}{dx} = \frac{\sec^2 x}{2y-1}$

6. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \dots \infty}}}$ show that $(2y-1) \frac{dy}{dx} = \frac{1}{x}$

Sol. $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \dots \infty}}} \Rightarrow y = \sqrt{\log x + y}$

Squaring both sides we get, $(y)^2 = (\sqrt{\log x + y})^2 \Rightarrow y^2 = \log x + y$

On differentiating both sides of (i) w.r.t. x , $\frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = \frac{d(\log x)}{dx} + \frac{dy}{dx}$

$\Rightarrow 2y \frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx} \Rightarrow 2y \frac{dy}{dx} - \frac{dy}{dx} = \frac{1}{x} \Rightarrow \frac{dy}{dx} (2y - 1) = \frac{1}{x} \therefore (2y - 1) \frac{dy}{dx} = \frac{1}{x}$

7. If $y = a^{x^y}$ Prove that of $\frac{dy}{dx} = \frac{y^2 \log y}{x(1 - y \log x \log y)}$

Sol. $y = a^{x^y} \Rightarrow y = a^{x^y}$

Taking both sides, we get, $\log y = \log(a^{x^y}) \Rightarrow \log y = x^y \log(a)$

Again taking log both sides, we get, $\log(\log y) = \log(x^y \cdot \log a)$

$\Rightarrow \log(\log y) = \log(x^y) + \log(\log a) \Rightarrow \log(\log y) = y \log x + \log(\log a)$

$\Rightarrow \log(\log y) = y \log x + \log(\log a) \dots (i)$

On differentiating both sides of (i) w.r.t. x,

$$\begin{aligned} \frac{d\{\log(\log y)\}}{d(\log y)} \times \frac{d(\log y)}{dy} \cdot \frac{dy}{dx} &= y \frac{d(\log x)}{dx} + \log a \frac{dy}{dx} + \frac{d\{\log(\log a)\}}{d(\log a)} \times \frac{d(\log a)}{da} \cdot \frac{d(a)}{dx} \\ \Rightarrow \frac{1}{\log y} \cdot \frac{1}{y} \frac{dy}{dx} &= y \frac{1}{x} + \log a \frac{dy}{dx} + \frac{1}{\log a} \cdot \frac{1}{a} \cdot 0 \Rightarrow \frac{1}{y \log y} \frac{dy}{dx} - \log x \frac{dy}{dx} = \frac{y}{x} \\ \Rightarrow \frac{dy}{dx} \left(\frac{1}{y \log y} - \log x \right) &= \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y^2 \log y}{x(1 - y \log x \log y)} \end{aligned}$$

8. If $y = x + \frac{1}{x + \frac{1}{x + \frac{1}{x + \dots}}}$ prove that $\frac{dy}{dx} = \frac{y}{2y - x}$

Sol. $y = x + \frac{1}{y} \Rightarrow y = \frac{xy + 1}{y} \Rightarrow y^2 = xy + 1 \dots (i)$

On differentiating both sides of (i) w.r.t. x, $\frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = x \frac{dy}{dx} + y \frac{dx}{dx} + \frac{d(1)}{dx}$

$$\Rightarrow 2y \frac{dy}{dx} = x \frac{dy}{dx} + y \cdot 1 + 0 \Rightarrow 2y \frac{dy}{dx} = x \frac{dy}{dx} + y \Rightarrow 2y \frac{dy}{dx} - x \frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{dx} (2y - x) = y \therefore \frac{dy}{dx} = \frac{y}{2y - x}$$

EXERCISE 10H (Pg.No.: 438)

1. Differentiation x^6 with respect to $\left(\frac{1}{\sqrt{x}}\right)$

Sol. Let $u = x^6$, $v = \frac{1}{\sqrt{x}}$, find $\frac{du}{dv}$

$$u = x^6 \dots (i)$$

On differentiating both sides of (i) w.r.t. x, $\frac{du}{dx} = \frac{d(x^6)}{dx} \Rightarrow \frac{du}{dx} = 6x^{6-1} \therefore \frac{du}{dx} = 6x^5$

$$v = \frac{1}{\sqrt{x}} \dots (ii)$$

On differentiating both sides of (ii) w.r.t. x, $\frac{dv}{dx} = \frac{d\left(x^{-\frac{1}{2}}\right)}{dx} \Rightarrow \frac{dv}{dx} = -\frac{1}{2} x^{-\frac{1}{2}-1}$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{6x^5}{-\frac{1}{2}x^{-3/2}} \Rightarrow \frac{du}{dv} = -12x^{\frac{13}{2}}$$

2. Differentiation $(\log x)$ with respect to $\cot x$.

Sol. Let $u = \log x$, $v = \cot x$, find $\frac{du}{dv}$

$$u = \log x \quad \dots (i)$$

On differentiating both sides of (i) w.r.t. x , $\frac{du}{dx} = \frac{1}{x}$

$$v = \cot x \quad \dots (ii)$$

On differentiating both sides of (ii) w.r.t. x , $\frac{dv}{dx} = -\operatorname{cosec}^2 x$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x} \Rightarrow \frac{du}{dx} = -\frac{1}{x} \cdot \frac{1}{x \operatorname{cosec}^2 x} \Rightarrow \frac{du}{dv} = \frac{-\operatorname{cosec}^2 x}{x}$$

3. Differentiation $e^{\sin x}$ with respect to $\cos x$.

Sol. Let $u = e^{\sin x}$, $v = \cos x$ find $\frac{du}{dv}$

$$u = e^{\sin x} \quad \dots (i)$$

On differentiating both sides of (i) w.r.t. x , $\frac{du}{dx} = \frac{d(e^{\sin x})}{d(\sin x)} \times \frac{d(\sin x)}{dx} \Rightarrow \frac{du}{dx} = e^{\sin x} \cdot \cos x$

$$v = \cos x \quad \dots (ii)$$

On differentiating both sides of (ii) w.r.t. x , $\frac{dv}{dx} = \frac{d(\cos x)}{dx} \Rightarrow \frac{dv}{dx} = -\sin x$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{e^{\sin x} \cdot \cos x}{-\sin x} \Rightarrow \frac{du}{dv} = -e^{\sin x} \cot x$$

4. Differentiation $\tan^{-1} \sqrt{\frac{1-x^2}{1+x^2}}$ with respect to $\cos^{-1}(x^2)$.

Sol. Let $u = \tan^{-1} \sqrt{\frac{1-x^2}{1+x^2}}$, $v = \cos^{-1}(x^2)$ find $\frac{du}{dv}$

$$u = \tan^{-1} \left(\sqrt{\frac{1-x^2}{1+x^2}} \right), \quad \text{Let } x^2 = \cos 2\theta \Rightarrow 2\theta = \cos^{-1}(x^2) \Rightarrow \theta = \frac{1}{2} \cos^{-1}(x^2)$$

$$u = \tan^{-1} \left(\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \right) \Rightarrow u = \tan^{-1} \left(\sqrt{\frac{2\sin^2 \theta}{2\cos^2 \theta}} \right) \Rightarrow u = \tan^{-1} (\tan \theta)$$

$$\Rightarrow u = \tan^{-1} (\tan \theta) \Rightarrow u = \theta$$

Putting the value of θ , $u = \frac{1}{2} \cos^{-1}(x^2)$

Differentiating both sides with respect to x ,

$$\frac{du}{dx} = \frac{1}{2} \frac{d\{\cos^{-1}(x^2)\}}{d(x^2)} \times \frac{d(x^2)}{dx} \Rightarrow \frac{du}{dx} = -\frac{1}{2} \cdot \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x \Rightarrow \frac{du}{dx} = -\frac{x}{\sqrt{1-x^4}}$$

$$v = \cos^{-1}(x^2)$$

Differentiating both sides with respect to x , $\frac{dv}{dx} = \frac{d\{\cos^{-1}(x^2)\}}{d(x^2)} \times \frac{d(x^2)}{dx}$

$$\Rightarrow \frac{dv}{dx} = -\frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x \Rightarrow \frac{dv}{dx} = -\frac{2x}{\sqrt{1-x^4}} \Rightarrow \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{-x}{\sqrt{1-x^4}} \cdot \frac{\sqrt{1-x^4}}{-2x} \therefore \frac{du}{dv} = \frac{1}{2}$$

5. Differentiation $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Sol. Let $u = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ find $\frac{du}{dv}$

$$v = \sin^{-1}\left(\frac{2x}{1+x^2}\right); \text{ Let } x = \tan \theta \therefore \theta = \tan^{-1}(x)$$

$$u = \tan^{-1}\left(\frac{2 \tan \theta}{1 - \tan^2 \theta}\right) \Rightarrow u = \tan^{-1}(\tan 2\theta) \Rightarrow u = 2\theta$$

Putting the value of θ , $u = 2 \tan^{-1}(x)$

Differentiating both sides with respect to x , $\frac{du}{dx} = 2 \frac{d\{\tan^{-1}(x)\}}{dx} \Rightarrow \frac{du}{dx} = 2 \cdot \frac{1}{1+x^2} \therefore \frac{du}{dv} = \frac{2}{1+x^2}$

$$v = \sin^{-1}\left(\frac{2x}{1+x^2}\right); \text{ Let } x = \tan \theta \therefore \theta = \tan^{-1}(x)$$

$$\Rightarrow v = \sin^{-1}\left(\frac{2 \tan \theta}{1 + \tan^2 \theta}\right) \Rightarrow v = \sin^{-1}(\sin 2\theta) \Rightarrow v = 2\theta$$

Putting the value of θ , $v = 2 \tan^{-1}(x)$

Differentiating both sides with respect to x , $\frac{dv}{dx} = 2 \frac{d\{\tan^{-1}(x)\}}{dx} \Rightarrow \frac{dv}{dx} = 2 \cdot \frac{1}{1+x^2}$

$$\Rightarrow \frac{dv}{dx} = \frac{2}{1+x^2} \therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}} = 1$$

6. Differentiation $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $\cos^{-1}(2x^2-1)$.

Sol. Let $u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, $v = \cos^{-1}(2x^2-1)$ find $\frac{du}{dv}$

$$u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right), \text{ Let } x = \cos \theta \therefore \theta = \cos^{-1}(x)$$

$$u = \tan^{-1} \left(\frac{\cos \theta}{\sqrt{1 - \cos^2 \theta}} \right) \Rightarrow u = \tan^{-1} \left(\frac{\cos \theta}{\sin \theta} \right) \Rightarrow u = \tan^{-1} (\cot \theta)$$

$$\Rightarrow u = \tan^{-1} \left\{ \tan \left(\frac{\pi}{2} - \theta \right) \right\} \Rightarrow u = \frac{\pi}{2} - \theta$$

Putting the value of θ , $u = \frac{\pi}{2} - \cos^{-1}(x)$

Differentiating both sides with respect to x , $\frac{du}{dx} = \frac{d\left(\frac{\pi}{2}\right)}{dx} - \frac{d(\cos^{-1} x)}{dx}$

$$\Rightarrow \frac{du}{dx} = 0 + \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$v = \cos^{-1}(2x^2 - 1); \text{ Let } x = \cos \theta \therefore \theta = \cos^{-1}(x)$$

$$\Rightarrow v = \cos^{-1}(2\cos^2 \theta - 1) \Rightarrow v = \cos^{-1}(\cos 2\theta) \Rightarrow v = 2\theta$$

Putting the value of θ , $v = 2\cos^{-1}(x)$

Differentiating both sides with respect to x ,

$$\frac{dv}{dx} = -2 \cdot \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dv}{dx} = \frac{-2}{\sqrt{1-x^2}} \therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{-2}{\sqrt{1-x^2}}} = -\frac{1}{2}$$

7. Differentiation $\sin^3 x$ with respect to $\cos^3 x$.

Sol. Let $u = \sin^3 x$, $v = \cos^3 x$, find $\frac{du}{dv}$

$u = \sin^3 x$, Differentiating both sides with respect to x ,

$$\frac{du}{dx} = \frac{d(\sin^3 x)}{d(\sin x)} \cdot \frac{d(\sin x)}{dx} \Rightarrow \frac{du}{dx} = 3\sin^2 x \cos x$$

$$v = \cos^3 x$$

Differentiating both sides with respect to x , $\frac{dv}{dx} = \frac{d(\cos^3 x)}{d(\cos x)} \times \frac{d(\cos x)}{dx} \Rightarrow \frac{dv}{dx} = -3\cos^2 x \sin x$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{3\sin^2 x \cos x}{-3\cos^2 x \sin x} = -\frac{\sin x}{\cos x} \Rightarrow \frac{du}{dv} = -\tan x$$

8. Differentiation $\cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$ with respect to $\tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right)$.

Sol. Let $u = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$, $v = \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right)$ find $\frac{du}{dv}$

$$u = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right), \text{ Let } x = \tan \theta \therefore \theta = \tan^{-1}(x)$$

$$u = \cos^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right) \Rightarrow u = \cos^{-1}(\cos 2\theta) \Rightarrow u = 2\theta$$

Putting the value of θ , $u = 2 \tan^{-1}(x)$

Differentiating both sides with respect to x , $\frac{du}{dx} = 2 \frac{d(\tan^{-1} x)}{dx} \Rightarrow \frac{du}{dx} = \frac{2}{1+x^2}$

$$v = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \text{ Let } x = \tan \theta \therefore \theta = \tan^{-1}(x)$$

$$\Rightarrow v = \tan^{-1}\left(\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}\right) \Rightarrow v = \tan^{-1}(\tan 3\theta) \Rightarrow v = 3\theta$$

Putting the value of θ , $v = 3 \tan^{-1} x$

Differentiating both sides with respect to x ,

$$\frac{dv}{dx} = 3 \frac{d\{\tan^{-1}(x)\}}{dx} \Rightarrow \frac{dv}{dx} = 3 \cdot \frac{1}{1+x^2} \Rightarrow \frac{dv}{dx} = \frac{3}{1+x^2}$$

$$\therefore \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\frac{2}{1+x^2}}{\frac{3}{1+x^2}} = \frac{2}{3} \Rightarrow \frac{du}{dv} = \frac{2}{3}$$

9. Differentiation $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Sol. Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ find $\frac{du}{dv}$

$$u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right), \text{ Let } x = \tan \theta \therefore \theta = \tan^{-1}(x)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2 \theta}-1}{\tan \theta}\right) \Rightarrow u = \tan^{-1}\left(\frac{\sec \theta - 1}{\tan \theta}\right) \Rightarrow u = \tan^{-1}\left(\frac{\frac{1}{\cos \theta} - 1}{\frac{\sin \theta}{\cos \theta}}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{1 - \cos \theta}{\sin \theta}\right) \Rightarrow u = \tan^{-1}\left(\frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}\right) \Rightarrow u = \tan^{-1}\left(\frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}\right) \Rightarrow u = \tan^{-1}\left(\tan \frac{\theta}{2}\right) \Rightarrow u = \frac{\theta}{2}$$

Putting the value of θ , $u = \frac{1}{2} \tan^{-1}(x)$

Differentiating both sides with respect to x ,

$$\frac{du}{dx} = \frac{1}{2} \frac{d\{\tan^{-1}(x)\}}{dx} \Rightarrow \frac{du}{dx} = \frac{1}{2} \cdot \frac{1}{1+x^2} \Rightarrow \frac{du}{dx} = \frac{1}{2(1+x^2)}$$

$$v = \sin^{-1}\left(\frac{2x}{1+x^2}\right), \text{ Let } x = \tan \theta \therefore \theta = \tan^{-1}(x)$$

$$v = \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) \Rightarrow v = \sin^{-1} (\sin 2\theta) \Rightarrow v = 2\theta$$

Putting the value of θ , $v = 2 \tan^{-1}(x)$

Differentiating both sides with respect to x , $\frac{dv}{dx} = 2 \frac{d(\tan^{-1} x)}{dx}$

$$\Rightarrow \frac{dv}{dx} = 2 \cdot \frac{1}{1+x^2} \Rightarrow \frac{dv}{dx} = \frac{2}{1+x^2} \Rightarrow \frac{du}{dx} = 2 \frac{1}{(1+x^2)}$$

$$\therefore \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\frac{2(1+x^2)}{2}}{\frac{2}{1+x^2}} = \frac{1}{4} \Rightarrow \frac{du}{dv} = \frac{1}{4}$$

10. Differentiate $\tan^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right)$ w.r.t. $\cos^{-1} (2x\sqrt{1-x^2})$, when $x \neq 0$

Sol. Put $x = \cos \theta$ so that $\theta = \cos^{-1} x$

$$\text{Let } u = \tan^{-1} \left[\frac{\sqrt{1-x^2}}{x} \right] = \tan^{-1} \left[\frac{\sqrt{1-\cos^2 \theta}}{\cos \theta} \right] = \tan^{-1} \left[\frac{\sin \theta}{\cos \theta} \right] = \tan^{-1} (\tan \theta) = \theta = \cos^{-1} x$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx} \cos^{-1} x = \left(\frac{-1}{\sqrt{1-x^2}} \right)$$

$$\text{and let } v = \cos^{-1} (2x\sqrt{1-x^2}) = \cos^{-1} \{ 2 \cos \theta \sqrt{1-\cos^2 \theta} \}$$

$$= \cos^{-1} \{ 2 \cos \theta \sin \theta \} = \cos^{-1} \{ \sin 2\theta \} = \cos^{-1} \left\{ \cos \left(\frac{\pi}{2} - 2\theta \right) \right\} = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2 \cos^{-1} x$$

$$\Rightarrow \frac{dv}{dx} = 0 - 2 \left(\frac{-1}{\sqrt{1-x^2}} \right) = \frac{2}{\sqrt{1-x^2}} \therefore \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\left(\frac{-1}{\sqrt{1-x^2}} \right)}{\left(\frac{2}{\sqrt{1-x^2}} \right)} = \left(\frac{-1}{2} \right)$$

EXERCISE 10 I (Pg.No.: 442)

Find $\frac{dy}{dx}$, when:

1. $x = at^2$, $y = 2at$

Sol. $x = at^2$... (i)

Differentiating both sides of (i) w.r.t. t , $\frac{dx}{dt} = a \frac{d(t^2)}{dt} \Rightarrow \frac{dx}{dt} = a \cdot 2t \Rightarrow \frac{dx}{dt} = 2at$

$y = 2at$... (ii)

Differentiating both sides of (ii) w.r.t. t , $\frac{dy}{dt} = 2a \frac{d(t)}{dt} \Rightarrow \frac{dy}{dt} = 2a \cdot 1 \Rightarrow \frac{dy}{dt} = 2a$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

2. $x = a \cos \theta, y = b \sin \theta$

Sol. $x = a \cos \theta$... (i)

Differentiating both sides of (i) w.r.t. θ , $\frac{dx}{d\theta} = a \frac{d(\cos \theta)}{d\theta} \Rightarrow \frac{dx}{d\theta} = -a \sin \theta$

$y = b \sin \theta$... (ii)

Differentiating both sides of (ii) w.r.t. θ , $\frac{dy}{d\theta} = b \frac{d(\sin \theta)}{d\theta} \Rightarrow \frac{dy}{d\theta} = b \cos \theta$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b \cos \theta}{-a \sin \theta} \Rightarrow \frac{dy}{dx} = -\frac{b}{a} \cot \theta$$

3. $x = a \cos^2 \theta, y = b \sin^2 \theta$ $\frac{dy}{dx}$

Sol. $x = a \cos^2 \theta$... (i)

Differentiating both sides of (i) w.r.t. θ , $\frac{dx}{d\theta} = a 2 \cos \theta (-\sin \theta) \Rightarrow \frac{dx}{d\theta} = -a \sin 2\theta$

$y = b \sin^2 \theta$... (ii)

Differentiating both sides of (ii) w.r.t. θ , $\frac{dy}{d\theta} = b 2 \sin \theta \cos \theta$

$$\Rightarrow \frac{dy}{d\theta} = b \sin 2\theta \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b \sin 2\theta}{-a \sin^2 \theta} \therefore \frac{dy}{dx} = -\frac{b}{a}$$

4. $x = a \cos^3 \theta, y = a \sin^3 \theta$

Sol. $x = a \cos^3 \theta$... (i)

Differentiating both sides of (i) w.r.t. θ , $\frac{dx}{d\theta} = a 3 \cos^2 \theta (-\sin \theta) \Rightarrow \frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta$

$y = a \sin^3 \theta$... (ii)

Differentiating both sides of (ii) w.r.t. θ , $\frac{dy}{d\theta} = a \sin^2 \theta \cos \theta$

$$\Rightarrow \frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta \therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta} \Rightarrow \frac{dy}{dx} = -\frac{\sin \theta}{\cos \theta} \Rightarrow \frac{dy}{dx} = -\tan \theta$$

5. $x = a(1 - \cos \theta), y = a(\theta + \sin \theta)$

Sol. $x = a(1 - \cos \theta)$... (i)

Differentiating both sides of (i) w.r.t. θ

$$\frac{dx}{d\theta} = a \left\{ \frac{d(1)}{d\theta} - \frac{d(\cos \theta)}{d\theta} \right\} \Rightarrow \frac{dx}{d\theta} = a \{0 + \sin \theta\} \Rightarrow \frac{dx}{d\theta} = a \sin \theta$$

$y = a(\theta + \sin \theta)$... (ii)

Differentiating both sides of (ii) w.r.t. θ , $\frac{dy}{d\theta} = a \left\{ \frac{d(\theta)}{d\theta} + \frac{d(\sin \theta)}{d\theta} \right\} \Rightarrow \frac{dy}{d\theta} = a \{1 + \cos \theta\}$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a(1+\cos\theta)}{a\sin\theta} \Rightarrow \frac{dy}{dx} = \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \Rightarrow \frac{dy}{dx} = \cot\frac{\theta}{2}$$

6. $x = a \log t, y = b \sin t$

Sol. $x = a \log t$... (i)

Differentiating both sides of (i) w.r.t t , $\frac{dx}{dt} = a \cdot \frac{1}{t} \Rightarrow \frac{dx}{dt} = \frac{a}{t}$

$y = b \sin t$... (ii)

Differentiating both sides of (ii) w.r.t t , $\frac{dy}{dt} = b \cos t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b \cos t}{\frac{a}{t}} \Rightarrow \frac{dy}{dx} = \frac{bt \cos t}{a}$$

7. $x = (\log t + \cos t), y = e^t + \sin t$ find $\frac{dy}{dx}$

Sol. $x = \log t + \cos t$... (i)

Differentiating both sides of (i) w.r.t t , $\frac{dx}{dt} = \frac{1}{t} + (-\sin t) \Rightarrow \frac{dx}{dt} = \frac{1}{t} - \sin t \Rightarrow \frac{dx}{dt} = \frac{1-t \sin t}{t}$

$y = e^t + \sin t$... (ii)

Differentiating both sides of (ii) w.r.t t , $\frac{dy}{dt} = e^t + \cos t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t + \cos t}{\frac{1-t \sin t}{t}} \Rightarrow \frac{dy}{dx} = \frac{t(e^t + \cos t)}{1-t \sin t}$$

8. $x = \cos \theta + \cos 2\theta, y = \sin \theta + \sin 2\theta$

Sol. $x = \cos \theta + \cos 2\theta$... (i)

Differentiating both sides of (i) w.r.t θ , $\frac{dx}{d\theta} = -\sin \theta + (-\sin 2\theta) \cdot 2 \Rightarrow \frac{dx}{d\theta} = -(\sin \theta + 2 \sin 2\theta)$

$y = \sin \theta + \sin 2\theta$... (ii)

Differentiating both sides of (ii) w.r.t θ , $\frac{dy}{d\theta} = \cos \theta + \cos 2\theta \cdot 2$

$$\Rightarrow \frac{dy}{d\theta} = \cos \theta + 2 \cos 2\theta \Rightarrow \frac{dy}{d\theta} = \cos \theta + 2 \cos 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos \theta + 2 \cos 2\theta}{-(\sin \theta + 2 \sin 2\theta)} \Rightarrow \frac{dy}{dx} = -\frac{(\cos \theta + 2 \cos 2\theta)}{(\sin \theta + 2 \sin 2\theta)}$$

9. $x = \sqrt{\sin 2\theta}, y = \sqrt{\cos 2\theta}$

Sol. $x = \sqrt{\sin 2\theta}$... (i)

Differentiating both sides of (i) w.r.t θ , $\frac{dx}{d\theta} = \frac{1}{2\sqrt{\sin 2\theta}} \cdot \cos 2\theta \cdot 2 \Rightarrow \frac{dx}{d\theta} = \frac{\cos 2\theta}{\sqrt{\sin 2\theta}}$

$y = \sqrt{\cos 2\theta}$... (ii)

Differentiating both sides of (ii) w.r.t. θ , $\frac{dy}{d\theta} = \frac{1}{2\sqrt{\cos 2\theta}} \{-(\sin 2\theta) \cdot 2\} \Rightarrow \frac{dy}{d\theta} = -\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}}{\frac{\cos 2\theta}{\sqrt{\sin 2\theta}}} \Rightarrow \frac{dy}{dx} = -(\tan 2\theta)^{\frac{3}{2}}$$

10. $x = e^{\theta}(\sin \theta + \cos \theta), y = e^{\theta}(\sin \theta - \cos \theta)$

Sol. $x = e^{\theta}(\sin \theta + \cos \theta) \dots (i)$

Differentiating both sides of (i) w.r.t. θ , $\frac{dx}{d\theta} = e^{\theta} \left\{ \frac{d(\sin \theta + \cos \theta)}{d\theta} \right\} + \left\{ (\sin \theta + \cos \theta) \frac{d(e^{\theta})}{d\theta} \right\}$

$$\Rightarrow \frac{dx}{d\theta} = e^{\theta} \{(\cos \theta - \sin \theta) + (\sin \theta + \cos \theta)\} \Rightarrow \frac{dx}{d\theta} = 2e^{\theta} \cos \theta$$

$y = e^{\theta}(\sin \theta - \cos \theta) \dots (ii)$

Differentiating both sides of (ii) w.r.t. θ , $\frac{dy}{d\theta} = e^{\theta} \frac{d(\sin \theta - \cos \theta)}{d\theta} + (\sin \theta - \cos \theta) \frac{d(e^{\theta})}{d\theta}$

$$\Rightarrow \frac{dy}{d\theta} = e^{\theta}(\cos \theta + \sin \theta) + (\sin \theta - \cos \theta)e^{\theta} \Rightarrow \frac{dy}{d\theta} = e^{\theta}(\cos \theta + \sin \theta + \sin \theta - \cos \theta)$$

$$\Rightarrow \frac{dy}{d\theta} = 2e^{\theta} \sin \theta \Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2e^{\theta} \sin \theta}{2e^{\theta} \cos \theta} \therefore \frac{dy}{dx} = \tan \theta$$

11. $x = a(\cos \theta + \theta \sin \theta), y = a(\sin \theta - \theta \cos \theta)$

Sol. $x = a(\cos \theta + \theta \sin \theta) \dots (i)$

Differentiating both sides of (i) w.r.t. θ , $\frac{dx}{d\theta} = a \left\{ \frac{d(\cos \theta)}{d\theta} + \frac{d(\theta \sin \theta)}{d\theta} \right\}$

$$\Rightarrow \frac{dx}{d\theta} = a \{-\sin \theta + \theta \cos \theta + \sin \theta \cdot 1\} \Rightarrow \frac{dx}{d\theta} = a\theta \cos \theta$$

$y = a(\sin \theta - \theta \cos \theta) \dots (ii)$

Differentiating both sides of (ii) w.r.t. θ , $\frac{dy}{d\theta} = a \left\{ \frac{d(\sin \theta)}{d\theta} - \frac{d(\theta \cos \theta)}{d\theta} \right\}$

$$\Rightarrow \frac{dy}{d\theta} = a \left[\cos \theta - \left\{ \theta \frac{d(\cos \theta)}{d\theta} + \cos \theta \frac{d\theta}{d\theta} \right\} \right] \Rightarrow \frac{dy}{d\theta} = a \{ \cos \theta + \theta \sin \theta - \cos \theta \} \Rightarrow \frac{dy}{d\theta} = a\theta \sin \theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta \sin \theta}{a\theta \cos \theta} \Rightarrow \frac{dy}{dx} = \tan \theta$$

12. $x = \frac{3at}{1+t^2}, y = \frac{3at^2}{1+t^2}$

Sol. $x = \frac{3at}{1+t^2} \dots (i)$

Differentiating both sides of (i) w.r.t t , $\frac{dx}{dt} = \left\{ \frac{(1+t^2) \frac{d(3at)}{dt} - 3at \frac{d(1+t^2)}{dt}}{(1+t^2)^2} \right\}$

$$\Rightarrow \frac{dx}{dt} = \left\{ \frac{(1+t^2)3a - 3at \cdot 2t}{(1+t^2)^2} \right\} \Rightarrow \frac{dx}{dt} = \left\{ \frac{3a + 3t^2a - 6at^2}{(1+t^2)^2} \right\}$$

$$\Rightarrow \frac{dx}{dt} = \left\{ \frac{3a - 3at^2}{(1+t^2)^2} \right\} \Rightarrow \frac{dx}{dt} = \frac{3a(1-t^2)}{(1+t^2)^2}$$

$$y = \frac{3at^2}{1+t^2} \quad \dots(ii)$$

Differentiating both sides of (ii) w.r.t t , $\frac{dy}{dt} = \left\{ \frac{(1+t^2) \frac{d(3at^2)}{dt} - 3at^2 \frac{d(1+t^2)}{dt}}{(1+t^2)^2} \right\}$

$$\Rightarrow \frac{dy}{dt} = \left\{ \frac{(1+t^2)2t3a - 3at^2 \cdot 2t}{(1+t^2)^2} \right\} \Rightarrow \frac{dy}{dt} = \left\{ \frac{6at + 6t^3a - 6t^3a}{(1+t^2)^2} \right\} \Rightarrow \frac{dy}{dt} = \frac{6at}{(1+t^2)^2}$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{6at}{(1+t^2)^2}}{\frac{3a(1-t^2)}{(1+t^2)^2}} \Rightarrow \frac{dy}{dx} = \frac{2t}{1-t^2}$$

13. $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$

Sol. $x = \frac{1-t^2}{1+t^2} \quad \dots(i)$

Differentiating both sides of (i) w.r.t t , $\frac{dx}{dt} = \left\{ \frac{(1+t^2) \frac{d(1-t^2)}{dt} - (1-t^2) \frac{d(1+t^2)}{dt}}{(1+t^2)^2} \right\}$

$$\Rightarrow \frac{dx}{dt} = \left\{ \frac{(1+t^2)(-2t) - (1-t^2)2t}{(1+t^2)^2} \right\} \Rightarrow \frac{dx}{dt} = \frac{2t\{-1-t^2-1+t^2\}}{(1+t^2)^2} \Rightarrow \frac{dx}{dt} = -\frac{4t}{(1+t^2)^2}$$

$$y = \frac{2t}{1+t^2} \quad \dots(ii)$$

Differentiating both sides of (ii) w.r.t t , $\frac{dy}{dt} = \left\{ \frac{(1+t^2) \frac{d(2t)}{dt} - 2t \frac{d(1+t^2)}{dt}}{(1+t^2)^2} \right\}$

$$\Rightarrow \frac{dy}{dt} = \left\{ \frac{(1+t^2) \cdot 2 - 2t \cdot 2t}{(1+t^2)^2} \right\} \Rightarrow \frac{dy}{dt} = \left\{ \frac{2+2t^2-4t^2}{(1+t^2)^2} \right\} \Rightarrow \frac{dy}{dt} = \frac{2-2t^2}{(1+t^2)^2} \Rightarrow \frac{dy}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{2(1-t^2)}{(1+t^2)^2}}{\frac{-4t}{(1+t^2)^2}} = \frac{2(1-t^2)}{-4t} \Rightarrow \frac{dy}{dx} = \frac{t^2-1}{2t}$$

14. $x = \cos^{-1}\left(\frac{1}{\sqrt{1+t^2}}\right), y = \sin^{-1}\left(\frac{t}{\sqrt{1+t^2}}\right)$

Sol. $x = \cos^{-1}\left(\frac{1}{\sqrt{1+t^2}}\right)$, Let $t = \tan \theta \therefore \theta = \tan^{-1}(t)$

$$x = \cos^{-1}\left(\frac{1}{\sqrt{1+\tan^2 \theta}}\right) \Rightarrow x = \cos^{-1}\left(\frac{1}{\sec \theta}\right) \Rightarrow x = \cos^{-1}(\cos \theta) \Rightarrow x = \theta$$

Putting the value of θ , $x = \tan^{-1}(t)$... (i)

On differentiating both sides of (i) w.r.t. t , $\frac{dx}{dt} = \frac{1}{1+t^2}$

$$y = \sin^{-1}\left(\frac{t}{\sqrt{1+t^2}}\right), \text{ Let } t = \tan \theta \therefore \theta = \tan^{-1}(t)$$

$$\Rightarrow y = \sin^{-1}\left(\frac{\tan \theta}{\sqrt{1+\tan^2 \theta}}\right) \Rightarrow y = \sin^{-1}\left(\frac{\tan \theta}{\sec \theta}\right) \Rightarrow y = \sin^{-1}(\sin \theta) \Rightarrow y = \theta$$

Putting the value of θ , $y = \tan^{-1}(t)$... (ii)

On differentiating both sides of (i) w.r.t. t , $\frac{dy}{dt} = \frac{1}{1+t^2} \therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{1+t^2}}{\frac{1}{1+t^2}} = 1$

15. If $x = 3 \cos t - 2 \cos^3 t$, $y = 3 \sin t - 2 \sin^3 t$ show that $\frac{dy}{dx} = \cot t$

Sol. $x = 3 \cos t - 2 \cos^3 t$... (i)

Differentiating both sides of (i) w.r.t. t , $\frac{dx}{dt} = -3 \sin t - 2 \cdot 3 \cos^2 t (-\sin t)$

$$\Rightarrow \frac{dx}{dt} = -3 \sin t + 6 \cos^2 t \sin t \Rightarrow \frac{dx}{dt} = -3 \sin t (1 - 2 \cos^2 t)$$

$y = 3 \sin t - 2 \sin^3 t$... (ii)

Differentiating both sides of (ii) w.r.t. t , $\frac{dy}{dt} = 3 \cos t - 2 \cdot 3 \sin^2 t \cos t$

$$\Rightarrow \frac{dy}{dt} = 3 \cos t \{1 - 2 \sin^2 t\} \Rightarrow \frac{dy}{dx} = 3 \cos t \{1 - 2(1 - \cos^2 t)\} \Rightarrow \frac{dy}{dx} = 3 \cos t \{1 - 2 + 2 \cos^2 t\}$$

$$\Rightarrow \frac{dy}{dx} = 3 \cos t \{-1 + 2 \cos^2 t\} \Rightarrow \frac{dy}{dx} = -3 \cos t \{1 - 2 \cos^2 t\}$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3 \cos t \{1 - 2 \cos^2 t\}}{-3 \sin t (1 - 2 \cos^2 t)} \Rightarrow \frac{dy}{dx} = \cot t$$

16. If $x = \frac{1 + \log t}{t^2}$ and $y = \frac{3 + 2 \log t}{t}$ show that $\frac{dy}{dx} = t$

Sol. $x = \frac{1 + \log t}{t^2} \dots (i)$

Differentiating both sides of (i) w.r.t. t , $\frac{dx}{dt} = \left\{ \frac{t^2 \frac{d(1 + \log t)}{dt} - (1 + \log t) \frac{d(t^2)}{dt}}{(t^2)^2} \right\}$

$$\Rightarrow \frac{dx}{dt} = \left\{ \frac{t^2 \cdot \frac{1}{t} - (1 + \log t) \cdot 2t}{t^4} \right\} \Rightarrow \frac{dx}{dt} = \left\{ \frac{t - 2t - 2t \log t}{t^4} \right\} \Rightarrow \frac{dx}{dt} = \frac{-t - 2t \log t}{t^4}$$

$$\Rightarrow \frac{dx}{dt} = \frac{-t(1 + 2 \log t)}{t^4} \Rightarrow \frac{dx}{dt} = -\frac{t(1 + 2 \log t)}{t^4} \Rightarrow \frac{dx}{dt} = \frac{-(1 + 2 \log t)}{t^3}$$

$y = \frac{3 + 2 \log t}{t} \dots (ii)$

Differentiating both sides of (ii) w.r.t. t , $\frac{dy}{dx} = \left\{ \frac{t \frac{d(3 + 2 \log t)}{dt} - (3 + 2 \log t) \frac{d(t)}{dt}}{t^2} \right\}$

$$\Rightarrow \frac{dy}{dt} = \left\{ \frac{t \cdot 2 \cdot \frac{1}{t} - (3 + 2 \log t) \cdot 1}{t^2} \right\} \Rightarrow \frac{dy}{dt} = \left\{ \frac{2 - 3 - 2 \log t}{t^2} \right\} \Rightarrow \frac{dy}{dt} = \frac{-1 - 2 \log t}{t^2}$$

$$\Rightarrow \frac{dy}{dt} = \frac{-(1 + 2 \log t)}{t^2} \therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-(1 + 2 \log t)}{t^2}}{\frac{-(1 + 2 \log t)}{t^3}} \Rightarrow \frac{dy}{dx} = t$$

17. If $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{2}$

Sol. $x = a(\theta - \sin \theta) \dots (i)$

Differentiating both sides of (i) w.r.t. θ , $\frac{dx}{d\theta} = a \left\{ \frac{d(\theta)}{d\theta} - \frac{d(\sin \theta)}{d\theta} \right\} \Rightarrow \frac{dx}{d\theta} = a \{1 - \cos \theta\}$

$y = a(1 - \cos \theta) \dots (ii)$

Differentiating both sides of (ii) w.r.t. θ , $\frac{dy}{d\theta} = a \left\{ \frac{d(1)}{d\theta} - \frac{d(\cos \theta)}{d\theta} \right\}$

$$\Rightarrow \frac{dy}{d\theta} = a \{0 + \sin \theta\} \Rightarrow \frac{dy}{d\theta} = a \sin \theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a \sin \theta}{a(1 - \cos \theta)} \Rightarrow \frac{dy}{dx} = \frac{\sin \theta}{1 - \cos \theta} \Rightarrow \frac{dy}{dx} \bigg|_{\theta = \frac{\pi}{2}} = \frac{\sin \frac{\pi}{2}}{1 - \cos \frac{\pi}{2}} = \frac{1}{1 - 0} \Rightarrow \frac{dy}{dx} = 1$$

18. If $x = 2 \cos \theta - \cos 2\theta$ and $y = 2 \sin \theta - \sin 2\theta$ show that $\frac{dy}{dx} = -1$ at $\theta = \frac{\pi}{2}$

Sol. $x = 2 \cos \theta - \cos 2\theta$... (i)

Differentiating both sides of (i) w.r.t. θ , $\frac{dx}{d\theta} = 2 \frac{d(\cos \theta)}{d\theta} - \frac{d(\cos 2\theta)}{d\theta}$

$$\Rightarrow \frac{dx}{d\theta} = -2 \sin \theta + \sin 2\theta \cdot 2 \Rightarrow \frac{dx}{d\theta} = -2 \sin \theta + 2 \sin 2\theta \Rightarrow \frac{dx}{d\theta} = 2(\sin 2\theta - \sin \theta)$$

$y = 2 \sin \theta - \sin 2\theta$... (ii)

Differentiating both sides of (ii) w.r.t. θ , $\frac{dy}{d\theta} = 2 \frac{d(\sin \theta)}{d\theta} - \frac{d(\sin 2\theta)}{d\theta}$

$$\Rightarrow \frac{dy}{d\theta} = 2 \cos \theta - \cos 2\theta \cdot 2 \Rightarrow \frac{dy}{d\theta} = 2(\cos \theta - \cos 2\theta)$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2(\cos \theta - \cos 2\theta)}{2(\sin 2\theta - \sin \theta)} \Rightarrow \frac{dy}{dx} = \frac{\cos \theta - \cos 2\theta}{\sin 2\theta - \sin \theta}$$

$$\Rightarrow \frac{dy}{dx} \text{ at } \left(\theta = \frac{\pi}{2} \right) = \frac{\cos \frac{\pi}{2} - \cos 2 \cdot \frac{\pi}{2}}{\sin \left(2 \cdot \frac{\pi}{2} \right) - \sin \frac{\pi}{2}} = \frac{0 - (-1)}{0 - 1} = \frac{1}{-1} \Rightarrow \frac{dy}{dx} = -1$$

19. If $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$, $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$ find $\frac{dy}{dx}$

Sol. $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$... (i)

Taking log both sides we get, $\log x = \log \left(\frac{\sin^3 t}{\sqrt{\cos 2t}} \right)$

$$\Rightarrow \log x = \log(\sin^3 t) - \log(\sqrt{\cos 2t}) \Rightarrow \log x = 3 \log(\sin t) - \frac{1}{2} \log(\cos 2t) \quad \dots (ii)$$

On differentiating both sides (ii) w.r.t. t , $\frac{1}{x} \frac{dx}{dt} = 3 \cdot \frac{1}{\sin t} \cdot \cos t - \frac{1}{2} \cdot \frac{1}{\cos 2t} \cdot (-\sin 2t) \cdot 2$

$$\Rightarrow \frac{dx}{dt} = x(3 \cot t + \tan 2t) \Rightarrow \frac{dx}{dt} = \frac{\sin^3 t}{\sqrt{\cos 2t}} (3 \cot t + \tan 2t)$$

$y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$... (iii)

Taking log both sides we get, $\log y = \log \left(\frac{\cos^3 t}{\sqrt{\cos 2t}} \right)$

$$\Rightarrow \log y = \log(\cos^3 t) - \log(\sqrt{\cos 2t}) \Rightarrow \log y = 3 \log(\cos t) - \frac{1}{2} \log(\cos 2t) \quad \dots (iv)$$

On differentiating both sides (iv) w.r.t. t , $\frac{1}{y} \frac{dy}{dt} = 3 \cdot \frac{1}{\cos t} \cdot (-\sin t) - \frac{1}{2} \cdot \frac{1}{\cos 2t} \cdot (-\sin 2t) \cdot 2$

$$\Rightarrow \frac{dy}{dt} = y(-3 \tan t + \tan 2t) \Rightarrow \frac{dy}{dt} = \frac{\cos^3 t}{\sqrt{\cos 2t}}(-3 \tan t + \tan 2t)$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{\cos^3 t}{\sqrt{\cos 2t}}(-3 \tan t + \tan 2t)}{\frac{\sin^3 t}{\sqrt{\cos 2t}}(3 \cot(t) + \tan 2t)} = \frac{\cot^3 t \left(-3 \tan t + \frac{2 \tan t}{1 - \tan^2 t} \right)}{\left(\frac{3}{\tan t} + \frac{2 \tan t}{1 - \tan^2 t} \right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(1 - 3 \tan^2 t)}{3 \tan t - \tan^3 t} = -\frac{1}{\tan 3t} \Rightarrow \frac{dy}{dx} = -\cot 3t$$

20. If $x = 2 \cos \theta - \cos 2\theta$ and $y = 2 \sin \theta - \sin 2\theta$ find $\frac{d^2 y}{dx^2}$, $\theta = \frac{\pi}{2}$.

Sol. $x = 2 \cos \theta - \cos 2\theta$... (i)

On differentiating both sides of (i) w.r.t. θ , $\frac{dx}{d\theta} = -2 \sin \theta + \sin 2\theta$ $\Rightarrow \frac{dx}{d\theta} = 2(\sin 2\theta - \sin \theta)$

$y = 2 \sin \theta - \sin 2\theta$... (ii)

On differentiating both sides of (ii) w.r.t. θ , $\frac{dy}{d\theta} = 2 \cos \theta - \cos 2\theta$ $\Rightarrow \frac{dy}{d\theta} = 2(\cos \theta - \cos 2\theta)$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2(\cos \theta - \cos 2\theta)}{2(\sin 2\theta - \sin \theta)} \therefore \frac{dy}{dx} = \frac{\cos \theta - \cos 2\theta}{\sin 2\theta - \sin \theta} \dots (iii)$$

On differentiating both sides of (iii) w.r.t. θ

$$\frac{d^2 y}{dx^2} = \left\{ \frac{(\sin 2\theta - \sin \theta) \frac{d(\cos \theta - \cos 2\theta)}{d\theta} - (\cos \theta - \cos 2\theta) \frac{d(\sin 2\theta - \sin \theta)}{d\theta}}{(\sin 2\theta - \sin \theta)^2} \right\} \times \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \left\{ \frac{(\sin 2\theta - \sin \theta)(-\sin \theta + \sin 2\theta) - (\cos \theta - \cos 2\theta)(\cos 2\theta - \cos \theta)}{(\sin 2\theta - \sin \theta)^2} \right\} \times \frac{1}{2(\sin 2\theta - \sin \theta)}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{(\sin 2\theta - \sin \theta)(2 \sin 2\theta - \sin \theta) - (\cos \theta - \cos 2\theta)(2 \cos 2\theta - \cos \theta)}{2(\sin 2\theta - \sin \theta)^2}$$

$$\Rightarrow \frac{d^2 y}{dx^2} \left(\theta = \frac{\pi}{2} \right) = -\frac{3}{2} \therefore \frac{d^2 y}{dx^2} = -\frac{3}{2}$$

21. If $x = a(\theta - \sin \theta)$, $y = a(1 + \cos \theta)$, find $\frac{d^2 y}{dx^2}$

Sol. $x = a(\theta - \sin \theta)$

Diff. both side w.r.t. $\frac{dx}{d\theta} = a(1 - \cos \theta)$

Now, $y = a(1 + \cos \theta)$

Diff both side w.r.t. θ , we get $\frac{dy}{d\theta} = a(0 + (-\sin \theta))$

$$\frac{dy}{d\theta} = -a \sin \theta$$

$$\therefore \frac{dy}{dx} = \frac{-a \sin \theta}{a(1 - \cos \theta)} = \frac{-2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}}$$

$$\frac{dy}{dx} = -\cot \frac{\theta}{2}$$

Again differentiating w.r.t x we get $\frac{d^2 y}{dx^2} = -\frac{d}{dx} \left(\cot \frac{\theta}{2} \right)$

$$\frac{d^2 y}{dx^2} = \operatorname{cosec}^2 \frac{\theta}{2} \times \frac{1}{2} \times \frac{d\theta}{dx}$$

$$\frac{d^2 y}{dx^2} = \frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2} \times \frac{1}{2a \sin^2 \frac{\theta}{2}}$$

$$\frac{d^2 y}{dx^2} = \frac{1}{4a} \cdot \operatorname{cosec}^4 \frac{\theta}{2}$$

EXERCISE 10 J (Pg.No.: 449)

1. Find the second derivative of

(i) x^{11}

(ii) 5^x

(iii) $y = \tan x$

(iv) $y = \cos^{-1} x$

Sol. (i) $y = x^{11}$... (i)

Differentiating both sides of (i), w.r.t. x , $\frac{dy}{dx} = \frac{d(x^{11})}{dx} \Rightarrow \frac{dy}{dx} = 11x^{10}$... (ii)

Again Differentiating both sides of (ii) w.r.t. x , $\frac{d^2 y}{dx^2} = 11 \frac{d(x^{10})}{dx} \therefore \frac{d^2 y}{dx^2} = 110x^9$

(ii) $y = 5^x$... (i)

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = \frac{d(5^x)}{dx} \Rightarrow \frac{dy}{dx} = 5^x \cdot \log 5$... (ii)

Again Differentiating both sides of (ii) w.r.t. x , $\frac{d^2 y}{dx^2} = \log 5 \cdot 5^x \log 5 \therefore \frac{d^2 y}{dx^2} = 5^x (\log 5)^2$

(iii) $y = \tan x$... (i)

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = \frac{d(\tan x)}{dx} \Rightarrow \frac{dy}{dx} = \sec^2 x$... (ii)

Again Differentiating both sides of (ii) w.r.t. x ,

$$\frac{d^2 y}{dx^2} = 2 \sec x (\tan x \cdot \sec x) \therefore \frac{d^2 y}{dx^2} = 2 \sec^3 x \tan x$$

(iv) $y = \cos^{-1}(x)$... (i)

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$... (ii)

Again Differentiating both sides of (ii) w.r.t. x , $\frac{d^2 y}{dx^2} = -\left[\frac{\sqrt{1-x^2} \frac{d(1)}{dx} - \frac{d(\sqrt{1-x^2})}{d(1-x^2)} \cdot \frac{d(1-x^2)}{dx}}{(\sqrt{1-x^2})^2} \right]$

$$\Rightarrow \frac{d^2 y}{dx^2} = - \left[\frac{\sqrt{1-x^2} \cdot 0 - \frac{1}{2\sqrt{1-x^2}} (-2x)}{(1-x^2)} \right] \Rightarrow \frac{d^2 y}{dx^2} = - \frac{x}{(\sqrt{1-x^2})(1-x^2)} \Rightarrow \frac{d^2 y}{dx^2} = \frac{-x}{(1-x^2)^{3/2}}$$

2. Find the second derivative of

(i) $x \sin x$ (ii) $e^{2x} \cos 3x$ (iii) $x^3 \log x$

Sol. (i) $y = x \sin x$... (i)

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = x \frac{d(\sin x)}{dx} + \sin x \frac{d(x)}{dx} \Rightarrow \frac{dy}{dx} = x \cos x + \sin x \cdot 1$

$\Rightarrow \frac{dy}{dx} = x \cos x + \sin x$... (ii)

Again Differentiating both sides of (ii) w.r.t. x , $\frac{d^2 y}{dx^2} = x \frac{d(\cos x)}{dx} + \cos x \cdot \frac{d(x)}{dx} + \frac{d(\sin x)}{dx}$

$\Rightarrow \frac{d^2 y}{dx^2} = -x \sin x + \cos x \cdot 1 + \cos x \quad \therefore \frac{d^2 y}{dx^2} = -x \sin x + 2 \cos x$

(ii) $y = e^{2x} \cos 3x$... (i)

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = e^{2x} (-\sin 3x) \cdot 3 + \cos 3x \cdot e^{2x} \cdot 2$

$\Rightarrow \frac{dy}{dx} = e^{2x} (-3 \sin 3x + 2 \cos 3x)$... (ii)

Again Differentiating both sides of (ii) w.r.t. x ,

$\frac{d^2 y}{dx^2} = e^{2x} \frac{d(-3 \sin 3x + 2 \cos 3x)}{dx} + (-3 \sin 3x + 2 \cos 3x) \cdot \frac{d(e^{2x})}{dx}$
 $\Rightarrow \frac{d^2 y}{dx^2} = e^{2x} \{(-3 \cos 3x \cdot 3 + 2(-\sin 3x) \cdot 3)\} + (-3 \sin 3x + 2 \cos 3x) \cdot e^{2x} \cdot 2$

$\Rightarrow \frac{d^2 y}{dx^2} = e^{2x} (-9 \cos 3x - 6 \sin 3x - 6 \sin 3x + 4 \cos 3x) \Rightarrow \frac{d^2 y}{dx^2} = e^{2x} (-5 \cos 3x - 12 \sin 3x)$

$\therefore \frac{d^2 y}{dx^2} = -e^{2x} (5 \cos 3x + 12 \sin 3x)$

(iii) $y = x^3 \log x$... (i)

Differentiating both sides of (i) w.r.t. x ,

$\frac{dy}{dx} = x^3 \cdot \frac{1}{x} + \log x \cdot 3x^2 \Rightarrow \frac{dy}{dx} = x^2 + 3 \log x \cdot x^2 \Rightarrow \frac{dy}{dx} = x^2 (1 + 3 \log x)$... (ii)

Again Differentiating both sides of (ii) w.r.t. x , $\frac{d^2 y}{dx^2} = x^2 \frac{d(1 + 3 \log x)}{dx} + (1 + 3 \log x) \frac{d(x^2)}{dx}$

$\Rightarrow \frac{d^2 y}{dx^2} = x^2 \left(3 \cdot \frac{1}{x} \right) + (1 + 3 \log x) \cdot 2x \Rightarrow \frac{d^2 y}{dx^2} = 3x + 2x + 6x \log x \quad \therefore \frac{d^2 y}{dx^2} = 5x + 6x \log x$

3. If $y = x + \tan x$ show that $\cos^2 x \cdot \frac{d^2 y}{dx^2} - 2y + 2x = 0$.

Sol. $y = x + \tan x$... (i)

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = 1 + \sec^2 x \Rightarrow \frac{dy}{dx} = 1 + \frac{1}{\cos^2 x}$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2 x + 1}{\cos^2 x} \Rightarrow \cos^2 x \frac{dy}{dx} = \cos^2 x + 1 \quad \dots (ii)$$

Again Differentiating both sides of (ii) w.r.t.x, $\cos^2 x \cdot \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot 2 \cos x (-\sin x) = 2 \cos x (-\sin x)$

$$\Rightarrow \cos^2 x \cdot \frac{d^2 y}{dx^2} = -\sin 2x + \sin 2x \frac{dy}{dx} \Rightarrow \cos^2 x \cdot \frac{d^2 y}{dx^2} = \sin 2x \left(-1 + \frac{dy}{dx} \right)$$

$$\Rightarrow \cos^2 x \cdot \frac{d^2 y}{dx^2} = \sin 2x \{ -1 + (1 + \sec^2 x) \} \Rightarrow \cos^2 x \cdot \frac{d^2 y}{dx^2} = \sin 2x \cdot \sec^2 x$$

$$\Rightarrow \cos^2 x \cdot \frac{d^2 y}{dx^2} = 2 \tan x \Rightarrow \cos^2 x \cdot \frac{d^2 y}{dx^2} - 2 \tan x - 2x + 2x = 0$$

$$\Rightarrow \cos^2 x \cdot \frac{d^2 y}{dx^2} - 2(\tan x + x) + 2x = 0 \therefore \cos^2 x \cdot \frac{d^2 y}{dx^2} - 2y + 2x = 0$$

4. If $y = 2 \sin x + 3 \cos x$ show that $y + \frac{d^2 y}{dx^2} = 0$.

Sol. $y = 2 \sin x + 3 \cos x \quad \dots (i)$

Differentiating both sides of (i) w.r.t.x, $\frac{dy}{dx} = 2 \cos x - 3 \sin x \quad \dots (ii)$

Again Differentiating both sides of (ii) w.r.t.x, $\frac{d^2 y}{dx^2} = -2 \sin x - 3 \cos x$

$$\Rightarrow \frac{d^2 y}{dx^2} = -(2 \sin x + 3 \cos x) \Rightarrow \frac{d^2 y}{dx^2} = -y \quad [\text{From (i)}] \therefore \frac{d^2 y}{dx^2} + y = 0$$

5. If $y = 3 \cos(\log x) + 4 \sin(\log x)$ prove that $x^2 y_2 + xy_1 + y = 0$

Sol. $y = 3 \cos(\log x) + 4 \sin(\log x) \quad \dots (1)$

Differentiating both sides of (i) w.r.t. x,

$$\frac{dy}{dx} = -3 \sin(\log x) \cdot \frac{1}{x} + 4 \cos(\log x) \cdot \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{1}{x} \{ -3 \sin(\log x) + 4 \cos(\log x) \}$$

$$\Rightarrow x \frac{dy}{dx} = -3 \sin(\log x) + 4 \cos(\log x) \quad \dots (ii)$$

Again Differentiating both sides of (ii) w.r.t.x, $x \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot 1 = -3 \cos(\log x) \cdot \frac{1}{x} - 4 \sin(\log x) \cdot \frac{1}{x}$

$$\Rightarrow x \frac{d^2 y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x} \{ 3 \cos(\log x) + 4 \sin(\log x) \} \Rightarrow x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = -y \quad [\text{From (i)}]$$

$$\Rightarrow x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0 \therefore x^2 y_2 + xy_1 + y = 0$$

6. If $y = e^{-x} \cos x$ show that $\frac{d^2 y}{dx^2} = 2e^{-x} \sin x$

Sol. $y = e^{-x} \cos x \quad \dots (i)$

Differentiating both sides of (i) w.r.t. x, $\frac{dy}{dx} = e^{-x} (-\sin x) + \cos x \cdot (e^{-x}) (-1)$

$$\Rightarrow \frac{dy}{dx} = -e^{-x} (\sin x + \cos x) \quad \dots (ii)$$

Again Differentiating both sides of (ii) w.r.t. x , $\frac{d^2y}{dx^2} = -[e^{-x}(\cos x - \sin x) + (\sin x + \cos x)e^{-x}(-1)]$

$$\Rightarrow \frac{d^2y}{dx^2} = -e^{-x}[\cos x - \sin x - \sin x - \cos x] \Rightarrow \frac{d^2y}{dx^2} = 2e^{-x} \sin x$$

7. If $y = \sec x - \tan x$ show that $\cos x \cdot \frac{d^2y}{dx^2} = y^2$

Sol. $y = \sec x - \tan x$... (i)

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = \sec x \tan x - \sec^2 x \Rightarrow \frac{dy}{dx} = \sec x(\tan x - \sec x)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos x} \{-(\sec x - \tan x)\} \Rightarrow \cos x \cdot \frac{dy}{dx} = -y \quad [\text{From (i)}] \quad \dots (ii)$$

Again Differentiating both sides of (ii) w.r.t. x , $\cos x \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx}(-\sin x) = -\frac{dy}{dx}$

$$\Rightarrow \cos x \cdot \frac{d^2y}{dx^2} = -\frac{dy}{dx} + \sin x \frac{dy}{dx} \Rightarrow \cos x \cdot \frac{d^2y}{dx^2} = \frac{dy}{dx}(-1 + \sin x)$$

$$\Rightarrow \cos x \cdot \frac{d^2y}{dx^2} = \{-(1 - \sin x)\} \cdot \left\{ -\frac{(\sec x - \tan x)}{\cos x} \right\} \Rightarrow \cos x \cdot \frac{d^2y}{dx^2} = \left(\frac{1 - \sin x}{\cos x} \right) (\sec x - \tan x)$$

$$\Rightarrow \cos x \cdot \frac{d^2y}{dx^2} = (\sec x - \tan x)(\sec x - \tan x) \therefore \cos x \cdot \frac{d^2y}{dx^2} = y^2$$

8. If $y = \operatorname{cosec} x + \cot x$ prove that $\sin x \cdot \frac{d^2y}{dx^2} - y^2 = 0$

Sol. $y = \operatorname{cosec} x + \cot x$... (i)

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = -\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x$

$$\Rightarrow \frac{dy}{dx} = -\operatorname{cosec} x(\cot x + \operatorname{cosec} x) \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin x}(\cot x + \operatorname{cosec} x)$$

$$\Rightarrow \sin x \frac{dy}{dx} = -(\cot x + \operatorname{cosec} x) \Rightarrow \sin x \cdot \frac{dy}{dx} = -y \quad [\text{From (i)}] \quad \dots (ii)$$

Again Differentiating both sides of (ii) w.r.t. x , $\sin x \frac{d^2y}{dx^2} + \frac{dy}{dx} \cos x = -\frac{dy}{dx}$

$$\Rightarrow \sin x \cdot \frac{d^2y}{dx^2} = -\frac{dy}{dx} - \cos x \cdot \frac{dy}{dx} \Rightarrow \sin x \cdot \frac{d^2y}{dx^2} = -\frac{dy}{dx}(1 + \cos x)$$

$$\Rightarrow \sin x \cdot \frac{d^2y}{dx^2} = +\frac{1}{\sin x}(\operatorname{cosec} x + \cot x)(1 + \cos x)$$

$$\Rightarrow \sin x \cdot \frac{d^2y}{dx^2} = (\operatorname{cosec} x + \cot x)(\operatorname{cosec} x + \cot x) \Rightarrow \sin x \cdot \frac{d^2y}{dx^2} = y^2 \Rightarrow \sin x \cdot \frac{d^2y}{dx^2} - y^2 = 0$$

9. If $y = \tan^{-1} x$ show that $(1+x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 0$.

Sol. $y = \tan^{-1} x$... (i)

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = \frac{1}{1+x^2} \Rightarrow (1+x^2) \frac{dy}{dx} = 1$... (ii)

Again Differentiating both sides of (ii) w.r.t. x , $(1+x^2) \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 2x = 0 \therefore (1+x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 0$

10. If $y = \sin(\sin x)$ prove that $\frac{d^2y}{dx^2} + (\tan x) \frac{dy}{dx} + y \cos^2 x = 0$.

Sol. $y = \sin(\sin x)$... (i)

Differentiation both sides of (i) w.r.t. x , $\frac{dy}{dx} = \cos(\sin x) \cdot \cos x$... (ii)

Again Differentiating both sides of (ii) w.r.t. x , $\frac{d^2y}{dx^2} = \cos(\sin x)(-\sin x) + \cos x \{-\sin(\sin x) \cos x\}$

$$\Rightarrow \frac{d^2y}{dx^2} = -\{\sin x \cos(\sin x) + \cos^2 x \sin(\sin x)\} \Rightarrow \frac{d^2y}{dx^2} = -\left\{\sin x \cdot \frac{1}{\cos x} \frac{dy}{dx} + \cos^2 x \cdot y\right\}$$

$$\therefore \frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y \cos^2 x = 0 \text{ . Hence proved.}$$

11. If $y = a \cos(\log x) + b \sin(\log x)$ prove that $x^2 y_2 + x y_1 + y = 0$.

Sol. $y = a \cos(\log x) + b \sin(\log x)$... (i)

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = -a \sin(\log x) \cdot \frac{1}{x} + b \cos(\log x) \cdot \frac{1}{x}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \{-a \sin(\log x) + b \cos(\log x)\} \Rightarrow x \frac{dy}{dx} = -a \sin(\log x) + b \cos(\log x) \text{ ... (ii)}$$

Again differentiating both sides of eq. (ii) w.r.t. x , we get,

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 1 = a \cos(\log x) \cdot \frac{1}{x} - b \sin(\log x) \cdot \frac{1}{x}$$

$$\Rightarrow x \frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{x} \{a \cos(\log x) + b \sin(\log x)\} \Rightarrow x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = -y \text{ [From equation (i)]}$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0 \therefore x^2 y_2 + x y_1 + y = 0$$

12. Find the second derivative of $e^{3x} \sin 4x$.

Sol. $y = e^{3x} \sin 4x$... (i)

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = e^{3x} \cos 4x \cdot 4 + \sin 4x \cdot e^{3x} \cdot 3$

$$\Rightarrow \frac{dy}{dx} = e^{3x} (4 \cos 4x + 3 \sin 4x) \text{ ... (ii)}$$

Again Differentiating both sides of (ii) w.r.t. x

$$\frac{d^2y}{dx^2} = e^{3x} (-4 \sin 4x \cdot 4 + 3 \cos 4x \cdot 4) + (4 \cos 4x + 3 \sin 4x) e^{3x} \cdot 3$$

$$\Rightarrow \frac{d^2y}{dx^2} = 4e^{3x} (-4 \sin 4x + 3 \cos 4x) + 3e^{3x} (4 \cos 4x + 3 \sin 4x)$$

$$\Rightarrow \frac{d^2 y}{dx^2} = e^{3x} (-16 \sin 4x + 12 \cos 4x + 12 \cos 4x + 9 \sin 4x) \Rightarrow \frac{d^2 y}{dx^2} = e^{3x} (24 \cos 4x - 7 \sin 4x)$$

13. Find the second derivative of $\sin 3x \cdot \cos 5x$

Sol. $y = \sin 3x \cdot \cos 5x$... (i)

$$\Rightarrow y = \frac{1}{2} (2 \sin 3x \cdot \cos 5x) \Rightarrow y = \frac{1}{2} [\sin (3x + 5x) + \sin (3x - 5x)] \Rightarrow y = \frac{1}{2} [\sin 8x + \sin (-2x)]$$

$$y = \frac{1}{2} [\sin 8x - \sin 2x] \quad \dots (ii)$$

Differentiating both sides of (ii) w.r.t. x , $\frac{dy}{dx} = \frac{1}{2} [\cos 8x \cdot 8 - \cos 2x \cdot 2]$

$$\Rightarrow \frac{dy}{dx} = 4 \cos 8x - \cos 2x \quad \dots (ii)$$

Again Differentiating both sides of (ii) w.r.t. x , $\frac{d^2 y}{dx^2} = -4 \sin 8x \cdot 8 + \sin 2x \cdot 2$

$$\Rightarrow \frac{d^2 y}{dx^2} = -32 \sin 8x + 2 \sin 2x \quad \therefore \frac{d^2 y}{dx^2} = (2 \sin 2x - 32 \sin 8x)$$

14. If $y = e^{\tan x}$ prove that $(\cos^2 x) \frac{d^2 y}{dx^2} - (1 + \sin 2x) \frac{dy}{dx} = 0$

Sol. $y = e^{\tan x}$... (i)

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = e^{\tan x} \cdot \sec^2 x \Rightarrow \frac{dy}{dx} = \frac{e^{\tan x}}{\cos^2 x}$

$$\Rightarrow \cos^2 x \frac{dy}{dx} = e^{\tan x} \Rightarrow \cos^2 x \frac{dy}{dx} = y \quad [\text{From equation (i)}] \quad \dots (ii)$$

Again Differentiating both sides of (ii) w.r.t. x , $\cos^2 x \cdot \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot 2 \cos x (-\sin x) = \frac{dy}{dx}$

$$\Rightarrow \cos^2 x \cdot \frac{d^2 y}{dx^2} - \sin 2x \frac{dy}{dx} = \frac{dy}{dx} \Rightarrow \cos^2 x \frac{d^2 y}{dx^2} - (1 + \sin 2x) \frac{dy}{dx} = 0$$

15. If $y = \frac{\log x}{x}$ show that $\frac{d^2 y}{dx^2} = \frac{2 \log x - 3}{x^3}$

Sol. $y = \frac{\log x}{x}$... (i)

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = \left\{ \frac{x \cdot \frac{1}{x} - \log x}{(x)^2} \right\} \Rightarrow \frac{dy}{dx} = \frac{1 - \log x}{x^2} \quad \dots (ii)$

Again Differentiating both sides of (ii) w.r.t. x , $\frac{d^2 y}{dx^2} = \left\{ \frac{x^2 \left(0 - \frac{1}{x} \right) - (1 - \log x) \cdot 2x}{(x^2)^2} \right\}$

$$\Rightarrow \frac{d^2 y}{dx^2} = \left\{ \frac{-x - 2x(1 - \log x)}{x^4} \right\} \Rightarrow \frac{d^2 y}{dx^2} = \frac{-(1 + 2 - 2 \log x)}{x^3} \Rightarrow \frac{d^2 y}{dx^2} = \frac{2 \log x - 3}{x^3}$$

16. If $y = e^{ax} \cos bx$ show that $\frac{d^2 y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0$

Sol. $y = e^{ax} \cos bx \quad \dots(i)$

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = -e^{ax} \sin bx \cdot b + \cos bx \cdot e^{ax} \cdot a$

$$\Rightarrow \frac{dy}{dx} = e^{ax} (-b \sin bx + a \cos bx) \quad \dots(ii)$$

Again Differentiating both sides of (ii) w.r.t. x ,

$$\begin{aligned} \frac{d^2y}{dx^2} &= e^{ax} (-b \cos bx \cdot b - a \sin bx \cdot b) + (-b \sin bx + a \cos bx) e^{ax} \cdot a \\ \Rightarrow \frac{d^2y}{dx^2} &= e^{ax} \{-b^2 \cos bx - ab \sin bx - ab \sin bx + a^2 \cos bx\} \\ \Rightarrow \frac{d^2y}{dx^2} &= e^{ax} \{\cos bx (a^2 - b^2) - 2ab \sin bx\} \Rightarrow \frac{d^2y}{dx^2} = (a^2 - b^2) e^{ax} \cos bx - 2ab e^{ax} \sin bx \\ \Rightarrow \frac{d^2y}{dx^2} &= (a^2 - b^2) y - 2ab e^{ax} \sin bx \quad \dots(iii) \end{aligned}$$

From equation (ii), $\frac{dy}{dx} = -be^{ax} \sin bx + ae^{ax} \cos bx \Rightarrow be^{ax} \sin bx = \left(ae^{ax} \cos bx - \frac{dy}{dx} \right)$

From equation (iii), $\frac{d^2y}{dx^2} = (a^2 - b^2) y - 2a \left(ae^{ax} \cos bx - \frac{dy}{dx} \right)$

$$\begin{aligned} \Rightarrow \frac{d^2y}{dx^2} &= (a^2 - b^2) y - 2a^2 e^{ax} \cos bx + 2a \frac{dy}{dx} \Rightarrow \frac{d^2y}{dx^2} = (a^2 - b^2) y - 2a^2 y + 2a \frac{dy}{dx} \\ \Rightarrow \frac{d^2y}{dx^2} &= -(a^2 + b^2) y + 2a \frac{dy}{dx} \quad \therefore \frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2) y = 0 \text{ . Hence proved.} \end{aligned}$$

17. If $y = e^{a \cos^{-1} x}$, $-1 \leq x \leq 1$ show that $(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$

Sol. $y = e^{a \cos^{-1} x} \quad \dots(i)$

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = e^{a \cos^{-1} x} \cdot a \left(-\frac{1}{\sqrt{1-x^2}} \right)$

$$\Rightarrow \sqrt{1-x^2} \frac{dy}{dx} = -ay \quad [\text{From equation (i)}] \quad \dots(ii)$$

Squaring both sides we get, $\left(\sqrt{1-x^2} \frac{dy}{dx} \right)^2 = (-ay)^2 \Rightarrow (1-x^2) \left(\frac{dy}{dx} \right)^2 = a^2 y^2 \quad \dots(iii)$

Again Differentiating both sides of (iii) w.r.t. x ,

$$\begin{aligned} (1-x^2) 2 \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 (-2x) &= a^2 \cdot 2y \frac{dy}{dx} \Rightarrow 2 \frac{dy}{dx} \left\{ (1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} \right\} = a^2 2y \frac{dy}{dx} \\ \Rightarrow (1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} &= a^2 y \quad \therefore (1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0 \end{aligned}$$

18. If $x = at^2$, $y = 2at$ find $\frac{d^2y}{dx^2}$ at $t = 2$

Sol. $x = at^2 \dots(i)$, $\frac{dx}{dt} = a \cdot 2t$; $y = 2at \dots(ii)$, $\frac{dy}{dt} = 2a$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} \Rightarrow \frac{dy}{dx} = \frac{1}{t} \quad \dots(iii)$$

$$\begin{aligned} \text{Again differentiation w.r.t. } x, \quad \frac{d^2y}{dx^2} &= \frac{d\left(\frac{1}{t}\right)}{dt} \cdot \frac{dt}{dx} \Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{t^2} \cdot \frac{1}{2at} \Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{2at^3} \\ \Rightarrow \frac{d^2y}{dx^2} \Big|_{(t=2)} &= -\frac{1}{2a(2)^3} \quad \therefore \frac{d^2y}{dx^2} = -\frac{1}{16a} \end{aligned}$$

19. If $x = a(\theta - \sin \theta)$, and $y = a(1 - \cos \theta)$ find $\frac{d^2y}{dx^2}$ at $\theta = \pi$

Sol. $x = a(\theta - \sin \theta) \quad \dots(i)$

Differentiating both sides of (i) w.r.t. x , $\frac{dx}{d\theta} = a(1 - \cos \theta)$

$y = a(1 - \cos \theta) \quad \dots(ii)$

Differentiating both sides of (ii) w.r.t. x , $\frac{dy}{d\theta} = a(0 + \sin \theta) \Rightarrow \frac{dy}{d\theta} = a \sin \theta$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a \sin \theta}{a(1 - \cos \theta)} \Rightarrow \frac{dy}{dx} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} \Rightarrow \frac{dy}{dx} = \cot \frac{\theta}{2} \quad \dots(ii)$$

$$\begin{aligned} \text{Again Differentiating both sides of (ii) w.r.t. } x, \quad \frac{d^2y}{dx^2} &= -\operatorname{cosec}^2 \frac{\theta}{2} \cdot \frac{1}{2} \cdot \frac{d\theta}{dx} \\ \Rightarrow \frac{d^2y}{dx^2} &= -\frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2} \cdot \frac{1}{a(1 - \cos \theta)} \Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2} \cdot \frac{1}{a \cdot 2 \sin^2 \frac{\theta}{2}} \end{aligned}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{4a} \operatorname{cosec}^4 \left(\frac{\theta}{2} \right) \Rightarrow \frac{d^2y}{dx^2} \Big|_{\left(\theta = \frac{\pi}{2} \right)} = -\frac{1}{4a} \operatorname{cosec}^4 \left(\frac{\pi}{2} \right) = -\frac{1}{4a}$$

20. If $y = \sin(\log x)$ Prove that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$

Sol. $y = \sin(\log x) \quad \dots(i)$

Differentiating both sides of (i) w.r.t. x , $\frac{dy}{dx} = \cos(\log x) \cdot \frac{1}{x}$

$x \frac{dy}{dx} = \cos(\log x) \quad \dots(ii)$

Again Differentiating both sides of (ii) w.r.t. x , $x \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 1 = -\sin(\log x) \cdot \frac{1}{x}$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = -y \quad \therefore x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$$

21. If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, show that $(1-x^2) \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} - y = 0$

Sol. Differentiate the function 'y' w.r.t. x

$$\frac{dy}{dx} = \sqrt{1-x^2} \times \frac{d}{dx} \sin^{-1} x - \sin^{-1} x \times \frac{d}{dx} \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{1-x^2} \times \frac{1}{\sqrt{1-x^2}} - \sin^{-1} x \times \frac{1}{2\sqrt{1-x^2}} \times -2x}{(1-x^2)}$$

$$\frac{dy}{dx} = \frac{1 + \frac{x + \sin^{-1} x}{\sqrt{1-x^2}}}{(1-x^2)} = \frac{1}{(1-x^2)} + \frac{x \sin^{-1} x}{(1-x^2)^{3/2}}$$

$$\text{Again Differentiate } \frac{d^2 y}{dx^2} = \frac{(1-x^2) \frac{d}{dx}[1] - 1 \frac{d}{dx}(1-x^2)}{(1-x^2)^2} + \frac{(1-x^2)^{3/2} \frac{d}{dx}(x \sin^{-1} x) - x \sin^{-1} x \frac{d}{dx}(1-x^2)^{3/2}}{\{(1-x^2)^{3/2}\}^2}$$

$$\frac{d^2 y}{dx^2} = \frac{0 - (-2x)}{(1-x^2)^2} + \frac{(1-x^2)^{3/2} \times \left[\frac{x}{\sqrt{1-x^2}} + \sin^{-1} x \right] - x \sin^{-1} x \times \frac{3}{2} (1-x^2)^{1/2} \times -2x}{\{(1-x^2)^{3/2}\}^2}$$

$$\frac{d^2 y}{dx^2} = \frac{2x}{(1-x^2)^2} + \frac{(1-x^2)\sqrt{1-x^2} \left[\frac{x + \sqrt{1-x^2} \sin^{-1} x}{\sqrt{1-x^2}} \right] + 3x^2 \sin^{-1} x \sqrt{1-x^2}}{(1-x^2)^3}$$

$$\frac{d^2 y}{dx^2} = \frac{2x}{(1-x^2)^2} + \frac{(1-x^2) \left[x + \sqrt{1-x^2} \sin^{-1} x \right] + 3x^2 \sin^{-1} x \sqrt{1-x^2}}{(1-x^2)^3}$$

$$\frac{d^2 y}{dx^2} = \frac{2x}{(1-x^2)^2} + \frac{(1-x^2) \left[x + \sqrt{1-x^2} \sin^{-1} x \right]}{(1-x^2)^3} + \frac{3x^2 \sin^{-1} x \sqrt{1-x^2}}{(1-x^2)^3}$$

$$\frac{d^2 y}{dx^2} = \left[\frac{2x}{(1-x^2)^2} + \frac{x}{(1-x^2)^2} + \frac{\sqrt{1-x^2} \sin^{-1} x}{(1-x^2)^2} + \frac{3x^2 \sin^{-1} x \sqrt{1-x^2}}{(1-x^2)^3} \right]$$

$$\text{Now, L.H.S. } (1-x^2) \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} - y = 0$$

$$\Rightarrow (1-x^2) \left[\frac{2x}{(1-x^2)^2} + \frac{x}{(1-x^2)^2} + \frac{\sqrt{1-x^2} \sin^{-1} x}{(1-x^2)^2} + \frac{3x^2 \sin^{-1} x \sqrt{1-x^2}}{(1-x^2)^3} \right] - 3x \left[\frac{1}{(1-x^2)} + \frac{x \sin^{-1} x}{(1-x^2)^{3/2}} \right] - \frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{3x}{(1-x^2)} + \frac{\sqrt{1-x^2} \sin^{-1} x}{(1-x^2)} + \frac{3x^2 \sin^{-1} x \sqrt{1-x^2}}{(1-x^2)^2} - \frac{3x}{(1-x^2)} - \frac{3x^2 \sin^{-1} x}{\sqrt{1-x^2}(1-x^2)} - \frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{\sin^{-1} x}{\sqrt{1-x^2}} - \frac{\sin^{-1} x}{\sqrt{1-x^2}} + \frac{3x^2 \sin^{-1} x \sqrt{1-x^2}}{(1-x^2)\sqrt{1-x^2}} - \frac{3x^2 \sin^{-1} x}{(1-x^2)\sqrt{1-x^2}}$$

$$\Rightarrow 0 = R.H.S$$

22. If $y = e^x \sin x$, prove that $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$

Sol. $\frac{dy}{dx} = e^x \times \frac{d}{dx}(\sin x) + \sin x \times \frac{d}{dx}(e^x)$

Again diff. the function $\frac{d^2 y}{dx^2} = e^x \frac{d}{dx}(\cos x) + \cos x \times \frac{d}{dx}(e^x) + \frac{d}{dx}(\sin x \cdot e^x)$

$$\frac{d^2 y}{dx^2} = e^x - \sin x + \cos x \times e^x + \sin x \cdot e^x + e^x \cos x$$

$$\frac{d^2 y}{dx^2} = -e^x \sin x + \cos x \cdot e^x + e^x \sin x + \cos x e^x$$

$$\frac{d^2 y}{dx^2} = 2e^x \cos x$$

Now, L.H.S

$$\Rightarrow 2e^x \cos x - 2(e^x \cos x + \sin e^x) + 2(e^x \sin x)$$

$$\Rightarrow 2e^x \cos x - 2e^x \cos x + 2e^x \sin x - 2e^x \sin x \Rightarrow O = R.H.S$$

23. If $x = a \left(\cos \theta + \log \tan \frac{\theta}{2} \right)$ and $y = a \sin \theta$, show that the value of $\frac{d^2 y}{dx^2}$ at $\theta = \frac{\pi}{4}$ is $\frac{4}{a}$

Sol. Differentiate the function 'x' w.r.t. θ

$$\frac{dx}{d\theta} = a \left[-\sin \theta + \frac{1}{\tan \frac{\theta}{2}} \sec^2 \frac{\theta}{2} \times \frac{1}{2} \right]$$

$$\frac{dx}{d\theta} = a \left[-\sin \theta + \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \cdot \frac{1}{\cos^2 \frac{\theta}{2}} \cdot \frac{1}{2} \right]$$

$$\frac{dx}{d\theta} = a \left[-\sin \theta + \frac{1}{\sin \theta} \right] = a \left[\frac{-\sin^2 \theta + 1}{\sin \theta} \right] = a \left[\frac{\cos^2 \theta}{\sin \theta} \right] = a \cos \theta \cdot \cot \theta$$

$$\frac{dy}{d\theta} = a \cos \theta$$

$\therefore$ Now diff the function y w.r.t. x

$$\frac{dy}{dx} = \frac{a \cos \theta}{a \cos \theta \cdot \cot \theta} = \tan \theta$$

$$\frac{dy}{dx} = \sec^2 \theta \cdot \frac{d\theta}{dx} = \sec^2 \theta \times \frac{1}{a \cos \theta \cdot \cot \theta} = \frac{1}{a \cos^2 \theta \cdot \cos \theta \cdot \frac{\cos \theta}{\sin \theta}}$$

$$\frac{d^2 y}{dx^2} = \frac{\sin \theta}{a \cos^4 \theta}$$

$$\therefore \frac{d^2 y}{dx^2} \Big|_{\theta=\frac{\pi}{4}} = \frac{\sin \frac{\pi}{4}}{a \left(\cos \frac{\pi}{4} \right)^4} = \frac{\frac{1}{\sqrt{2}}}{a \left(\frac{1}{\sqrt{2}} \right)^4} = \frac{\frac{1}{\sqrt{2}}}{a \times \frac{1}{4}} = \frac{4}{\sqrt{2}a} \text{ proved}$$

24. If $x = \cot + \log \tan \frac{t}{2}$, $y = \sin t$ then find the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$

Sol. Diff the function 'x' w.r.t. t

$$\frac{dx}{dt} = -\sin t + \frac{1}{\tan \frac{t}{2}} \times \sec^2 \frac{t}{2} \times \frac{t}{2}$$

Again, diff the function 'y' w.r.t. t we get $\frac{dy}{dt} = \cos t$

Again differentiate w.r.t. to t we get $\frac{d^2y}{dt^2} = -\sin t$

$$\left. \frac{d^2y}{dt^2} \right|_{t=\frac{\pi}{4}} = -\sin \frac{\pi}{4} = \frac{-1}{\sqrt{2}}$$

$$\text{Now, } \frac{dx}{dt} = -\sin t + \frac{1}{\tan \frac{t}{2}} \times \sec^2 \frac{t}{2} \times \frac{1}{2}$$

$$\frac{dx}{dt} = -\sin t + \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} \times \frac{1}{\cos^2 \frac{t}{2}} \times \frac{1}{2}$$

$$\frac{dx}{dt} = -\sin t + \frac{1}{\sin t} = \frac{-\sin^2 t + 1}{\sin t} = \frac{\cos^2 t}{\sin t} = \cos t \cdot \cot t$$

$$\therefore \frac{dy}{dx} = \frac{\cos t}{\cot \cdot \cot t} = \frac{1}{\cot t} = \tan t$$

$$\frac{d^2y}{dx^2} = \sec^2 t \frac{dt}{dx} = \sec^2 t \times \frac{1}{\cos t \cdot \cot t} = \frac{\sin t}{\cos^4 t}$$

$$\text{Now } \left. \frac{d^2y}{dx^2} \right|_{t=\frac{\pi}{4}} = \frac{\sin \frac{\pi}{4}}{\left(\cos \frac{\pi}{4} \right)^4} = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{4}} = \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = 2\sqrt{2}$$

25. if $y = x^x$ prove that $\frac{d^2y}{dx^2} - \frac{1}{y} \left(\frac{dy}{dx} \right)^2 - \frac{y}{x} = 0$

Sol. Diff both side w.r.t. x

$$\frac{dy}{dx} = x^x (1 + \log x)$$

$$\text{Again differentiate } \frac{d^2y}{dx^2} = x^x \left(\frac{1}{x} \right) + (1 + \log x) \{ x^x (1 + \log x) \}$$

$$\text{L.H.S } \frac{d^2y}{dx^2} - \frac{1}{y} \left(\frac{dy}{dx} \right)^2 - \frac{y}{x} = 0$$

$$\Rightarrow \frac{x^x}{x} + x^x (1 + \log x)^2 - \frac{1}{x^x} (1 + \log x)^2 - \frac{x^x}{x}$$

$$\Rightarrow x^x (1 + \log x)^2 - x^x (1 + \log x)^2 \Rightarrow 0$$

$$\therefore \text{L.H.S} = \text{R.H.S}$$

26. if $y = (\cot^{-1} x)^2$, then show that $(x^2 + 1)^2 \frac{d^2 y}{dx^2} + 2x(x^2 + 1) \frac{dy}{dx} = 2$

Sol. Diff. both side w.r.t. x

$$\frac{dy}{dx} = 2 \cot^{-1} x \left(\frac{-1}{1+x^2} \right) = \frac{-2 \cot^{-1} x}{1+x^2}$$

$$\frac{d^2 y}{dx^2} = -2 \left[\frac{(1+x^2) - \frac{1}{1+x^2} - \cot^{-1} x (2x)}{(1+x^2)^2} \right]$$

$$\frac{d^2 y}{dx^2} = -2 \left[\frac{-1 - 2x \cot^{-1} x}{(1+x^2)^2} \right] = \frac{2 + 4x \cot^{-1} x}{(1+x^2)^2}$$

$$\text{L.H.S.} = (x^2 + 1)^2 \left[\frac{2 + 4x \cot^{-1} x}{(1+x^2)^2} \right] + 2x(x^2 + 1) \left[\frac{-2 \cot^{-1} x}{(1+x^2)} \right]$$

$$= 2 + 4x \cot^{-1} x - 4x \cot^{-1} x = 2 = R.H.S$$

27. If $y = \left\{ x + \sqrt{x^2 + 1} \right\}^m$, then show that $(x^2 + 1) \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - m^2 y = 0$

Sol. Diff the function w.r.t. x

$$\frac{dy}{dx} = m \left\{ x + \sqrt{x^2 + 1} \right\}^{m-1} \left[1 + \frac{1}{2\sqrt{x^2 + 1}} 2x \right]$$

$$\frac{dy}{dx} = m \frac{\left\{ x + \sqrt{x^2 + 1} \right\}^m}{\left\{ x + \sqrt{x^2 + 1} \right\}} \times \left[1 + \frac{x}{\sqrt{x^2 + 1}} \right]$$

$$\frac{dy}{dx} = m \frac{\left\{ x + \sqrt{x^2 + 1} \right\}^m}{\left\{ x + \sqrt{x^2 + 1} \right\}} \times \left[\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \right]$$

$$\frac{dy}{dx} = m \frac{\left\{ x + \sqrt{x^2 + 1} \right\}^m}{\sqrt{x^2 + 1}}$$

$$\frac{d^2 y}{dx^2} = m \left[\frac{\sqrt{x^2 + 1} \left\{ m \frac{\left\{ x + \sqrt{x^2 + 1} \right\}^m}{\sqrt{x^2 + 1}} \right\} - \left\{ x + \sqrt{x^2 + 1} \right\}^m \frac{1}{2\sqrt{x^2 + 1}} 2x}{(\sqrt{x^2 + 1})^2} \right]$$

$$\frac{d^2 y}{dx^2} = \frac{m^2 \left\{ x + \sqrt{x^2 + 1} \right\}^m - \left\{ x + \sqrt{x^2 + 1} \right\}^m \frac{x}{\sqrt{x^2 + 1}}}{x^2 + 1}$$

$$\begin{aligned} \text{L.H.S } (x^2+1) & \left[\frac{m^2 \{x+\sqrt{x^2+1}\}^m - m(x+\sqrt{x^2+1})^m \frac{x}{\sqrt{x^2+1}}}{x^2+1} \right] + x \\ & + \frac{xm \{x+\sqrt{x^2+1}\}^m}{\sqrt{x^2+1}} - m^2 \{x+\sqrt{x^2+1}\}^m \\ & \Rightarrow 0 \end{aligned}$$

28. if $y = \log [x + \sqrt{x^2 + a^2}]$, then prove that $(x^2 + a^2) \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = 0$

Sol. Diff both side w.r.t x

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x + \sqrt{x^2 + a^2}} \times \left[1 + \frac{1}{2\sqrt{x^2 + a^2}} 2x \right] \\ \frac{dy}{dx} &= \frac{1}{x + \sqrt{x^2 + a^2}} \left[\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}} \right] = \frac{1}{\sqrt{x^2 + a^2}} \end{aligned}$$

Again differentiate w.r.t x

$$\frac{d^2 y}{dx^2} = \frac{\sqrt{x^2 + a^2} \frac{d}{dx} [1] - 1 \frac{d}{dx} [\sqrt{x^2 + a^2}]}{(\sqrt{x^2 + a^2})^2}$$

$$\frac{d^2 y}{dx^2} = \frac{-\frac{1}{2\sqrt{x^2 + a^2}} \times 2x}{(x^2 + a^2)} = \frac{-x}{(x^2 + a^2)\sqrt{x^2 + a^2}}$$

$$\text{L.H.S } \Rightarrow (x^2 + a^2) \frac{(x)}{(x^2 + a^2)\sqrt{x^2 + a^2}} + x \frac{1}{\sqrt{x^2 + a^2}} = 0 = R.H.S$$

29. If $x = a(\cos \theta + \theta \sin \theta)$ and $y = a(\sin \theta - \theta \cos \theta)$, show that $\frac{d^2 y}{dx^2} = \frac{1}{a} \left(\frac{\sec^3 \theta}{0} \right)$

Sol. Differentiate the function 'x' w.r.t θ

$$\frac{dx}{d\theta} = a[-a \sin \theta + \theta \cos \theta + \sin \theta] = (a\theta \cos \theta)$$

Again diff the function 'y' w.r.t θ

$$\frac{dy}{d\theta} = a[\cos \theta - (\theta(-\sin \theta) + \cos \theta)]$$

$$\frac{dy}{d\theta} = a[\cos \theta + \theta \sin \theta - \cos \theta]$$

$$\frac{dy}{d\theta} = a\theta \sin \theta$$

$$\therefore \frac{dy}{dx} = \frac{a\theta \sin \theta}{a\theta \cos \theta} = \tan \theta$$

$$\therefore \frac{d^2 y}{dx^2} = \sec^2 \theta \frac{d\theta}{dx} = \sec^2 \theta \times \frac{1}{a\theta \cos \theta} = \frac{1}{a\theta} \sec^2 \theta \cdot \sec \theta = \left(\frac{\sec^3 \theta}{a\theta} \right)$$

30. If $x = a \cos \theta + b \sin \theta$ and $y = a \sin \theta - b \cos \theta$ show that $y^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = 0$

Sol. Diff the 'x' w.r.t. θ

$$\frac{dx}{d\theta} = -a \sin \theta + b \cos \theta$$

$$\frac{dx}{d\theta} = -[a \sin \theta - b \cos \theta]$$

$$\frac{dx}{d\theta} = -y$$

Again diff 'y' w.r.t. θ

$$\frac{dy}{d\theta} = a \cos \theta + b \sin \theta$$

$$\frac{dy}{d\theta} = x$$

$$\therefore \frac{dy}{dx} = \frac{x}{-y} = -\frac{x}{y}$$

$$\text{Again diff } \frac{d^2 y}{dx^2} = \left[\frac{y \cdot 1 - x \frac{dy}{dx}}{y^2} \right] \Rightarrow - \left[\frac{y - x \left(-\frac{x}{y} \right)}{y^2} \right]$$

$$\frac{d^2 y}{dx^2} = - \left[\frac{y + \frac{x^2}{y}}{y^2} \right]$$

$$\text{L.H.S } y^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = 0$$

$$\Rightarrow -y^2 \left[\frac{y + \frac{x^2}{y}}{y^2} \right] - x \left(-\frac{x}{y} \right) + y \Rightarrow -y - \frac{x^2}{y} + \frac{x^2}{y} + y = 0 \text{ proved}$$