

METHOD OF INTEGRATION (XII, R. S. AGGARWAL)

EXERCISE 13A (Pg.No.: 621)

Evaluate the following integrals:

Very-short Answers Questions

1. $\int (2x+9)^5 dx$

Sol. Let $I = \int (2x+9)^5 dx$, Put $2x+9=t \Rightarrow 2dx=dt \Rightarrow dx=\frac{dt}{2}$

$$I = \int t^5 \cdot \frac{dt}{2} \Rightarrow I = \frac{1}{2} \int t^5 dt \Rightarrow I = \frac{1}{2} \cdot \frac{t^6}{6} + c \therefore I = \frac{(2x+9)^6}{12} + c$$

2. $\int (7-3x)^4 dx$

Sol. Let $I = \int (7-3x)^4 dx$, Put $7-3x=t \Rightarrow dx=-\frac{dt}{3}$

$$I = \int t^4 \left(-\frac{dt}{3}\right) \Rightarrow I = -\frac{1}{3} \int t^4 dt \Rightarrow I = -\frac{1}{3} \cdot \frac{t^5}{5} + c \therefore I = -\frac{1}{15} (7-3x)^5 + c$$

3. $\int \sqrt{3x-5} dx$

Sol. Let $I = \int (3x-5)^{\frac{1}{2}} dx$, Put $3x-5=t \Rightarrow 3=\frac{dt}{dx} \Rightarrow dx=\frac{dt}{3}$

$$I = \int t^{1/2} \cdot \frac{dt}{3} \Rightarrow I = \frac{1}{3} \int t^{1/2} dt \Rightarrow I = \frac{1}{3} \cdot \frac{t^{3/2}}{3/2} + c \therefore I = \frac{2}{9} (3x-5)^{3/2} + c$$

4. $\int \frac{1}{\sqrt{4x+3}} dx$

Sol. Let $I = \int \frac{1}{(4x+3)^{1/2}} dx \Rightarrow I = \int (4x+3)^{-1/2} dx$, Put $4x+3=t \Rightarrow 4=\frac{dt}{dx} \Rightarrow dx=\frac{dt}{4}$

$$I = \int t^{-1/2} \cdot \frac{dt}{4} \Rightarrow I = \frac{1}{4} \int t^{-1/2} dt \Rightarrow I = \frac{1}{4} \cdot \frac{t^{1/2}}{1/2} + c \Rightarrow I = \frac{1}{2} (4x+3)^{1/2} + c \therefore I = \frac{1}{2} \sqrt{4x+3} + c$$

5. $\int \frac{1}{\sqrt{3-4x}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{3-4x}} dx \Rightarrow I = \int (3-4x)^{-1/2} dx$, Put $3-4x=t \Rightarrow dx=-\frac{dt}{4}$

$$I = \int t^{-1/2} \left(-\frac{dt}{4}\right) \Rightarrow I = -\frac{1}{4} \int t^{-1/2} dt \therefore I = -\frac{1}{2} \sqrt{3-4x} + c$$

6. $\int \frac{1}{(2x-3)^{3/2}} dx$

Sol. Let $I = \int \frac{1}{(2x-3)^{3/2}} dx \Rightarrow I = \int (2x-3)^{-3/2} dx$, Put $2x-3=t \Rightarrow 2 = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{2}$

$$I = \frac{1}{2} \int t^{-3/2} dt \Rightarrow I = -\frac{1}{t^{1/2}} + c \quad \therefore I = -\frac{1}{\sqrt{2x-3}} + c$$

7. $\int e^{(2x-1)} dx$

Sol. Let $I = \int e^{(2x-1)} dx$, Put $2x-1=t \Rightarrow 2 = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{2}$

$$I = \int e^t \cdot \frac{dt}{2} \Rightarrow I = \frac{1}{2} \int (e^t) + c \quad \therefore I = \frac{1}{2} e^{2x-1} + c$$

8. $\int e^{(1-3x)} dx$

Sol. Let $I = \int e^{(1-3x)} dx$, Put $1-3x=t \Rightarrow -3 = \frac{dt}{dx} \Rightarrow dx = -\frac{dt}{3}$

$$I = \int e^t \cdot \left(-\frac{dt}{3}\right) \Rightarrow I = -\frac{1}{3} \int e^t dt \Rightarrow I = -\frac{1}{3} e^t + c \quad \therefore I = -\frac{e^{1-3x}}{3} + c$$

9. $\int 3^{2-3x} dx$

Sol. Let $I = \int 3^{2-3x} dx$, Put $2-3x=t \Rightarrow -3 = \frac{dt}{dx} \Rightarrow dx = -\frac{dt}{3}$

$$I = \int 3^t \cdot \left(-\frac{dt}{3}\right) \Rightarrow I = -\frac{1}{3} \int 3^t dt \Rightarrow I = -\frac{1}{3} \int 3^t dt \Rightarrow I = -\frac{1}{3} \cdot 3^t / \log 3$$

$$\therefore I = -\frac{1}{3} \frac{3^{2-3x}}{\log 3} + c$$

10. $\int \sin 3x dx$

Sol. Let $I = \int \sin 3x dx$, Put $3x=t \Rightarrow 3 = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{3}$

$$I = \int \sin t \cdot \frac{dt}{3} \Rightarrow I = \frac{1}{3} \int \sin t dt \Rightarrow I = -\frac{1}{3} \cos t + c \quad \therefore I = -\frac{\cos(3x)}{3} + c$$

Short-Answers Questions

11. $\int \cos(5+6x) dx$

Sol. Let $I = \int \cos(5+6x) dx$, Put $5+6x=t \Rightarrow 6 = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{6}$

$$I = \int \cos t \cdot \frac{dt}{6} \Rightarrow I = \frac{1}{6} \int \cos t dt \Rightarrow I = \frac{1}{6} \sin t + c \quad \therefore I = \frac{\sin(5+6x)}{6} + c$$

12. $\int \sin x \sqrt{1+\cos 2x} dx$

Sol. Let $I = \int \sin x \sqrt{1+\cos 2x} dx \Rightarrow I = \int \sin x \sqrt{2\cos^2 x} dx$

$$\Rightarrow I = \sqrt{2} \int \sin x \cos x dx, \text{ Put } \sin x = t \Rightarrow \cos x = \frac{dt}{dx} \Rightarrow \cos x dx = dt$$

$$\Rightarrow I = \sqrt{2} \int t \cdot dt \Rightarrow I = \sqrt{2} \cdot \frac{t^2}{2} + c \quad \therefore I = \frac{\sin^2 x}{\sqrt{2}} + c$$

13. $\int \operatorname{cosec}^2(2x+5) dx$

Sol. Let $I = \int \operatorname{cosec}^2(2x+5) dx$, Put $2x+5=t \Rightarrow 2 = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{2}$

$$I = \int \operatorname{cosec}^2(t) \cdot \frac{dt}{2} \Rightarrow I = \frac{1}{2} \int \operatorname{cosec}^2 t \cdot dt \Rightarrow I = \frac{1}{2} (-\cot t) + c \quad \therefore I = -\frac{\cot(2x+5)}{2} + c$$

14. $\int \sin x \cos x dx$

Sol. Let $I = \int \sin x \cdot \cos x dx$, Put $\sin x = t \Rightarrow \cos x = \frac{dt}{dx} \Rightarrow \cos x dx = dt$

$$I = \int t \cdot dt \Rightarrow I = \frac{t^2}{2} + c \quad \therefore I = \frac{\sin^2 x}{2} + c$$

15. $\int \sin^3 x \cos x dx$

Sol. Let $I = \int \sin^3 x \cos x dx$, Put $\sin x = t \Rightarrow \cos x = \frac{dt}{dx} \Rightarrow \cos x dx = dt$

$$I = \int t^3 \cdot dt \Rightarrow I = \frac{t^4}{4} + c \quad \therefore I = \frac{\sin^4 x}{4} + c$$

16. $\int \sqrt{\cos x} \cdot \sin x dx$

Sol. Let $I = \int \sqrt{\cos x} \cdot \sin x dx$, Put $\cos x = t \Rightarrow -\sin x = \frac{dt}{dx} \Rightarrow \sin x dx = -dt$

$$I = \int \sqrt{t} (-dt) \Rightarrow I = -\int t^{1/2} dt \Rightarrow I = -\frac{t^{3/2}}{3/2} + c \quad \therefore I = -\frac{2}{3} (\cos x)^{3/2} + c$$

17. $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

Sol. Let $I = \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$, Put $\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} = \frac{dt}{dx} \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$

$$I = \int t \cdot dt \Rightarrow I = \frac{t^2}{2} + c \quad \therefore I = \frac{(\sin^{-1} x)^2}{2} + c$$

18. $\int \frac{\sin(2 \tan^{-1} x)}{1+x^2} dx$

Sol. Let $I = \int \frac{\sin(2 \tan^{-1} x)}{1+x^2} dx$, Put $2 \tan^{-1} x = t \Rightarrow 2 \cdot \frac{1}{1+x^2} = \frac{dt}{dx} \Rightarrow \frac{1}{1+x^2} dx = \frac{dt}{2}$

$$I = \int \sin(t) \cdot \frac{dt}{2} \Rightarrow I = \frac{1}{2} \int \sin t dt \Rightarrow I = -\frac{1}{2} \cos(t) + c \quad \therefore I = -\frac{1}{2} \cos(2 \tan^{-1} x) + c$$

19. $\int \frac{\cos(\log x)}{x} dx$

Sol. Let $I = \int \frac{\cos(\log x)}{x} dx$, Put $\log x = t \Rightarrow \frac{1}{x} = \frac{dt}{dx} \Rightarrow \frac{1}{x} dx = dt$

$$I = \int \cos(t) dt \Rightarrow I = \sin(t) + c \quad \therefore I = \sin(\log x) + c$$

20. $\int \frac{\operatorname{cosec}^2(\log x)}{x} dx$

Sol. Let $I = \int \frac{\operatorname{cosec}^2(\log x)}{x} dx$, Put $\log x = t \Rightarrow \frac{1}{x} = \frac{dt}{dx} \Rightarrow \frac{1}{x} dx = dt$
 $I = \int \operatorname{cosec}^2(t) dt \Rightarrow I = -\cot(t) + c \therefore I = -\cot(\log x) + c$

21. $\int \frac{1}{x \log x} dx$

Sol. Let $I = \int \frac{1}{x \log x} dx$, Put $\log x = t \Rightarrow \frac{1}{x} = \frac{dt}{dx} \Rightarrow \frac{1}{x} dx = dt$
 $I = \int \frac{1}{t} dt \Rightarrow I = \log(t) + c \therefore I = \log(\log x) + c$

22. $\int \frac{(x+1)(x+\log x)^2}{x} dx$

Sol. Let $I = \int \frac{(x+1)(x+\log x)^2}{x} dx$, Put $x+\log x = t \Rightarrow 1 + \frac{1}{x} = \frac{dt}{dx} \Rightarrow \left(\frac{x+1}{x}\right) dx = dt$
 $I = \int t^2 dt \Rightarrow I = \frac{t^3}{3} + c \therefore I = \frac{(x+\log x)^3}{3} + c$

23. $\int \frac{(\log x)^2}{x} dx$

Sol. Let $I = \int \frac{(\log x)^2}{x} dx$, Put $\log x = t \Rightarrow \frac{1}{x} = \frac{dt}{dx} \Rightarrow \frac{1}{x} dx = dt$
 $I = \int t^2 dt \Rightarrow I = \frac{t^3}{3} + c \therefore I = \frac{(\log x)^3}{3} + c$

24. $\int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx$

Sol. $I = \int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx$, Put $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} = \frac{dt}{dx} \Rightarrow \frac{1}{\sqrt{x}} dx = 2dt$
 $I = \int \cos(t) \cdot 2dt \Rightarrow I = 2 \int \cos t dt \Rightarrow I = 2 \sin t + c \therefore I = 2 \sin(\sqrt{x}) + c$

25. $\int e^{\tan x} \sec^2 x dx$

Sol. $I = \int e^{\tan x} \cdot \sec^2 x dx$, Put $\tan x = t \Rightarrow \sec^2 x = \frac{dt}{dx} \Rightarrow \sec^2 x \cdot dx = dt$
 $I = \int e^t dt \Rightarrow I = e^t + c \therefore I = e^{\tan x} + c$

26. $\int e^{\cos^2 x} \sin 2x dx$

Sol. Let $I = \int e^{\cos^2 x} \sin 2x dx$, Put $\cos^2 x = t \Rightarrow 2 \cos x (-\sin x) = \frac{dt}{dx} \Rightarrow \sin 2x dx = -dt$
 $I = \int e^t (-dt) \Rightarrow I = -\int e^t dt \Rightarrow I = -e^t + c \therefore I = -e^{\cos^2 x} + c$

27. $\int \sin(ax+b) \cos(ax+b) dx$

Sol. Let $I = \int \sin(ax+b) \cos(ax+b) dx$

Put $\sin(ax+b) = t \Rightarrow \cos(ax+b) \cdot a = \frac{dt}{dx} \Rightarrow \cos(ax+b) dx = \frac{dt}{a}$

$I = \int t \cdot \frac{dt}{a} \Rightarrow I = \frac{1}{a} \int t dt \Rightarrow I = \frac{1}{a} \cdot \frac{t^2}{2} + c \quad \therefore I = \frac{\sin^2(ax+b)}{2a} + c$

28. $\int \cos^3 x dx$

Sol. Let $I = \int \cos^3 x dx \Rightarrow I = \int \cos^2 x \cos x dx \Rightarrow I = \int (1 - \sin^2 x) \cos x dx$

Put $\sin x = t \Rightarrow \cos x = \frac{dt}{dx} \Rightarrow \cos x dx = dt$

$I = \int (1 - t^2) dt \Rightarrow I = t - \frac{t^3}{3} + c \quad \therefore I = \sin x - \frac{\sin^3 x}{3} + c$

29. $\int \frac{1}{x^2} e^{\frac{1}{x}} dx$

Sol. Let $I = \int \frac{1}{x^2} e^{\frac{1}{x}} dx$, Put $-\frac{1}{x} = t \Rightarrow \frac{1}{x^2} = \frac{dt}{dx} \Rightarrow \frac{1}{x^2} dx = dt$

$I = \int e^t dt \Rightarrow I = e^t + c \quad \therefore I = e^{\frac{1}{x}} + c$

30. $\int \frac{1}{x^2} \cos\left(\frac{1}{x}\right) dx$

Sol. Let $I = \int \frac{1}{x^2} \cos\left(\frac{1}{x}\right) dx$, Put $\frac{1}{x} = t \Rightarrow \frac{1}{x^2} = \frac{dt}{dx} \Rightarrow \frac{1}{x^2} dx = -dt$

$I = \int \cos(t)(-dt) \Rightarrow I = -\int \cos t dt \Rightarrow I = -\sin t + c \quad \therefore I = -\sin\left(\frac{1}{x}\right) + c$

31. $\int \frac{1}{e^x + e^{-x}} dx$

Sol. Let $I = \int \frac{1}{e^x + e^{-x}} dx \Rightarrow I = \int \frac{1}{e^x + \frac{1}{e^x}} dx \Rightarrow I = \int \frac{e^x}{(e^x)^2 + 1} dx$

Put $e^x = t \Rightarrow e^x = \frac{dt}{dx} \Rightarrow e^x dx = dt$

$I = \int \frac{dt}{t^2 + 1} \Rightarrow I = \tan^{-1}(t) + c \quad \therefore I = \tan^{-1}(e^x) + c$

32. $\int \frac{e^{2x}}{e^{2x} - 2} dx$

Sol. Let $I = \int \frac{e^{2x}}{e^{2x} - 2} dx$, Put $e^{2x} - 2 = t \Rightarrow e^{2x} \cdot 2 = \frac{dt}{dx} \Rightarrow e^{2x} dx = \frac{dt}{2}$

$I = \frac{1}{2} \int \frac{dt}{t} \Rightarrow I = \frac{1}{2} \log |t| + c \quad \therefore I = \frac{1}{2} \log |e^{2x} - 2| + c$

33. $\int \cot x \log(\sin x) dx$

Sol. Let $I = \int \cot x \log(\sin x) dx$, Put $\log(\sin x) = t \Rightarrow \frac{1}{\sin x} \cdot \cos x = \frac{dt}{dx} \Rightarrow \cot x dx = dt$

$$I = \int t \cdot dt \Rightarrow I = \frac{t^2}{2} + c \quad \therefore I = \frac{(\log \sin x)^2}{2} + c$$

34. $\int \frac{\cot x}{\log(\sin x)} dx$

Sol. Let $I = \int \frac{\cot x}{\log(\sin x)} dx$, Put $\log(\sin x) = t \Rightarrow \frac{1}{\sin x} \cdot \cos x = \frac{dt}{dx} \Rightarrow \cot x dx = dt$

$$I = \int \frac{1}{t} \cdot dt \Rightarrow I = \log(t) + c \quad \therefore I = \log|\log(\sin x)| + c$$

35. $\int 2x \sin(x^2 + 1) dx$

Sol. Let $I = \int 2x \sin(x^2 + 1) dx$, Put $x^2 + 1 = t \Rightarrow 2x = \frac{dt}{dx} \Rightarrow 2x dx = dt$

$$I = \int \sin(t) \cdot dt \Rightarrow I = -\cos t + c \quad \therefore I = -\cos(x^2 + 1) + c$$

36. $\int \sec x \cdot \log(\sec x + \tan x) dx$

Sol. Let $I = \int \sec x \cdot \log(\sec x + \tan x) dx$

Put $\log(\sec x + \tan x) = t \Rightarrow \frac{1}{\sec x + \tan x} (\sec x \tan x + \sec^2 x) = \frac{dt}{dx}$

$$\Rightarrow \frac{\sec x \cdot \tan x + \sec^2 x}{\sec x + \tan x} = \frac{dt}{dx} \Rightarrow \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} = \frac{dt}{dx} \Rightarrow \sec x dx = dt$$

$$\therefore I = \int t dt \Rightarrow I = \frac{t^2}{2} + c \quad \therefore I = \frac{\{\log(\sec x + \tan x)\}^2}{2} + c$$

37. $\int \frac{\tan \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}} dx$

Sol. Let $I = \int \frac{\tan(\sqrt{x}) \sec^2(\sqrt{x})}{\sqrt{x}} dx$

Put $\tan(\sqrt{x}) = t \Rightarrow \sec^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} = \frac{dt}{dx} \Rightarrow \frac{\sec^2(\sqrt{x})}{\sqrt{x}} dx = 2dt$

$$I = \int t \cdot 2dt \Rightarrow I = 2 \cdot \frac{t^2}{2} + c \Rightarrow I = \tan^2(\sqrt{x}) + c \quad \therefore I = \tan^2(\sqrt{x}) + c$$

38. $\int \frac{x \tan^{-1}(x^2)}{1+x^4} dx$

Sol. Let $I = \int \frac{x \tan^{-1}(x^2)}{1+x^4} dx$, Put $\tan^{-1}(x^2) = t \Rightarrow \frac{1}{1+(x^2)^2} \cdot 2x = \frac{dt}{dx} \Rightarrow \frac{x}{1+x^4} dx = \frac{dt}{2}$

$$I = \int t \cdot \frac{dt}{2} \Rightarrow I = \frac{1}{2} \cdot \frac{t^2}{2} + c \Rightarrow I = \frac{t^2}{4} + c \quad \therefore I = \frac{\{\tan^{-1}(x^2)\}^2}{4} + c$$

39. $\int \frac{x \sin^{-1}(x^2)}{\sqrt{1-x^4}} dx$

Sol. Let $I = \int \frac{x \sin^{-1}(x^2)}{\sqrt{1-x^4}} dx$, Put $\sin^{-1}(x^2) = t \Rightarrow \frac{1}{\sqrt{1-x^4}} \cdot 2x = \frac{dt}{dx} \Rightarrow \frac{x}{\sqrt{1-x^4}} dx = \frac{dt}{2}$

$$I = \int t \cdot \frac{dt}{2} \Rightarrow I = \frac{1}{2} \int t dt \Rightarrow I = \frac{t^2}{4} + c \quad \therefore I = \frac{\{\sin^{-1}(x^2)\}^2}{4} + c$$

40. $\int \frac{1}{\sqrt{1-x^2} \sin^{-1} x} dx$

Sol. Let $I = \int \frac{1}{\sqrt{1-x^2} \sin^{-1} x} dx$, Put $\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} = \frac{dt}{dx} \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$

$$I = \int \frac{1}{t} dt \Rightarrow I = \log |t| + c \quad \therefore I = \log |\sin^{-1} x| + c$$

41. $\int \frac{\sqrt{2+\log x}}{x} dx$

Sol. Let $I = \int \frac{\sqrt{2+\log x}}{x} dx$, Put $2+\log x = t \Rightarrow \frac{1}{x} = \frac{dt}{dx} \Rightarrow \frac{1}{x} dx = dt$

$$I = \int \sqrt{t} dt \Rightarrow I = \frac{t^{3/2}}{3/2} + c \quad \therefore I = \frac{2}{3} (2+\log x)^{3/2} + c$$

42. $\int \frac{\sec^2 x}{1+\tan x} dx$

Sol. Let $I = \int \frac{\sec^2 x}{1+\tan x} dx$, Put $1+\tan x = t \Rightarrow \sec^2 x = \frac{dt}{dx} \Rightarrow \sec^2 x dx = dt$

$$I = \int \frac{dt}{t} \Rightarrow I = \log |t| + c \quad \therefore I = \log |1+\tan x| + c$$

43. $\int \frac{\sin x}{1+\cos x} dx$

Sol. Let $I = \int \frac{\sin x}{1+\cos x} dx$, Put $1+\cos x = t \Rightarrow -\sin x = \frac{dt}{dx} \Rightarrow \sin x dx = -dt$

$$I = -\int \frac{dt}{t} \Rightarrow I = -\log |t| + c \quad \therefore I = -\log |1+\cos x| + c$$

44. $\int \left(\frac{1+\tan x}{1-\tan x} \right) dx$

Sol. Let $I = \int \frac{1+\tan x}{1-\tan x} dx \Rightarrow I = \int \frac{1 + \frac{\sin x}{\cos x}}{1 - \frac{\sin x}{\cos x}} dx \Rightarrow I = \int \frac{\cos x + \sin x}{\cos x - \sin x} dx$

Put $\cos x - \sin x = t \Rightarrow -\sin x - \cos x = \frac{dt}{dx} \Rightarrow (\sin x + \cos x) dx = -dt$

$$I = -\int \frac{dt}{t} \Rightarrow I = -\log |t| + c \quad \therefore I = -\log |\cos x - \sin x| + c$$

45. (i) $\int \frac{1 + \tan x}{x + \log(\sec x)} dx$

(ii) $\int \frac{1 - \sin x}{x + \cos^2 x} dx$

Sol. (i) $\int \frac{1 + \tan x}{x + \log(\sec x)} dx$

Put $x + \log(\sec x) = t \Rightarrow 1 + \frac{1}{\sec x} \cdot \sec x \cdot \tan x = \frac{dt}{dx} \Rightarrow (1 + \tan x) dx = dt$

$I = \int \frac{1}{t} dt \Rightarrow I = \log|t| + c \therefore I = \log|x + \log(\sec x)| + c$

(ii) $I = \int \frac{1 - \sin 2x}{x + \cos^2 x} dx$, Put $x + \cos^2 x = t \Rightarrow 1 + 2 \cos x(-\sin x) = \frac{dt}{dx} \Rightarrow (1 - \sin 2x) dx = dt$

$I = \int \frac{dt}{t} \Rightarrow I = \log|t| + c \therefore I = \log|x + \cos^2 x| + c$

46. $\int \frac{\sin 2x}{a^2 + b^2 \sin^2 x} dx$

Sol. Let $I = \int \frac{\sin 2x}{a^2 + b^2 \sin^2 x} dx$, Put $a^2 + b^2 \sin^2 x = t \Rightarrow b^2 2 \sin x \cos x = \frac{dt}{dx} \Rightarrow \sin 2x dx = \frac{dt}{b^2}$

$I = \frac{1}{b^2} \int \frac{dt}{t} \Rightarrow I = \frac{1}{b^2} \log|t| + c \therefore I = \frac{\log|a^2 + b^2 \sin^2 x|}{b^2} + c$

47. $\int \frac{\sin 2x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$

Sol. Let $I = \int \frac{\sin 2x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$

Put $a^2 \cos^2 x + b^2 \sin^2 x = t \Rightarrow a^2 2 \cos x(-\sin x) + b^2 2 \sin x \cos x = \frac{dt}{dx}$

$\Rightarrow -a^2 \sin 2x + b^2 \sin 2x = \frac{dt}{dx} \Rightarrow \sin 2x(b^2 - a^2) = \frac{dt}{dx} \Rightarrow \sin 2x dx = \frac{dt}{b^2 - a^2}$

$I = \int \frac{1}{t} \cdot \frac{dt}{b^2 - a^2} \therefore I = \frac{1}{b^2 - a^2} \log|a^2 \cos^2 x + b^2 \sin^2 x| + c$

48. $\int \left(\frac{2 \cos x - 3 \sin x}{3 \cos x + 2 \sin x} \right) dx$

Sol. Let $I = \int \left(\frac{2 \cos x - 3 \sin x}{3 \cos x + 2 \sin x} \right) dx$

Put $3 \cos x + 2 \sin x = t \Rightarrow 3(-\sin x) + 2 \cos x = \frac{dt}{dx} \Rightarrow (2 \cos x - 3 \sin x) dx = dt$

$I = \int \frac{dt}{t} \Rightarrow I = \log|t| + c \therefore I = \log|3 \cos x + 2 \sin x| + c$

49. $\int \frac{4x}{2x^2 + 3} dx$

Sol. Let $I = \int \frac{4x}{2x^2 + 3} dx$, Put $2x^2 + 3 = t \Rightarrow 4x = \frac{dt}{dx} \Rightarrow 4x dx = dt$

$I = \int \frac{dt}{t} \Rightarrow I = \log|t| + c \therefore I = \log|2x^2 + 3| + c$

50. $\int \frac{x+1}{x^2+2x-3} dx$

Sol. Let $I = \int \frac{x+1}{x^2+2x-3} dx$

Put $x^2+2x-3=t \Rightarrow (2x+2) = \frac{dt}{dx} \Rightarrow 2(x+1)dx = dt \Rightarrow (x+1)dx = \frac{dt}{2}$

$I = \frac{1}{2} \int \frac{dt}{t} \Rightarrow I = \frac{1}{2} \log|t| + c \therefore I = \frac{1}{2} \log|x^2+2x-3| + c$

51. $\int \frac{4x-5}{2x^2-5x+1} dx$

Sol. Let $I = \int \frac{4x-5}{2x^2-5x+1} dx$, Put $2x^2-5x+1=t \Rightarrow 4x-5 = \frac{dt}{dx} \Rightarrow (4x-5)dx = dt$

$I = \int \frac{1}{t} dt \Rightarrow I = \log|t| + c \therefore I = \log|2x^2-5x+1| + c$

52. $\int \frac{9x^2-4x+5}{3x^3-2x^2+5x+1} dx$

Sol. Let $I = \int \frac{9x^2-4x+5}{3x^3-2x^2+5x+1} dx$

Put $3x^3-2x^2+5x+1=t \Rightarrow (9x^2-4x+5) = \frac{dt}{dx} \Rightarrow (9x^2-4x+5)dx = dt$

$I = \int \frac{1}{t} dt \Rightarrow I = \log|t| + c \therefore I = \log|3x^3-2x^2+5x+1| + c$

53. $\int \frac{\sec x \operatorname{cosec} x}{\log(\tan x)} dx$

Sol. Let $I = \int \frac{\sec x \operatorname{cosec} x}{\log(\tan x)} dx$, Put $\log(\tan x) = t \Rightarrow \frac{1}{\tan x} \sec^2 x = \frac{dt}{dx} \Rightarrow \sec x \operatorname{cosec} x dx = dt$

$I = \int \frac{1}{t} dt \Rightarrow I = \log|t| + c \therefore I = \log|\log(\tan x)| + c$

54. $\int \frac{1+\cos x}{(x+\sin x)^3} dx$

Sol. Let $I = \int \frac{1+\cos x}{(x+\sin x)^3} dx$, Put $x+\sin x = t \Rightarrow 1+\cos x = \frac{dt}{dx} \Rightarrow (1+\cos x)dx = dt$

$I = \int \frac{dt}{t^3} \Rightarrow I = -\frac{1}{2t^2} + c \therefore I = -\frac{1}{2(x+\sin x)^2} + c$

55. $\int \frac{\sin x}{(1+\cos x)^2} dx$

Sol. Let $I = \int \frac{\sin x}{(1+\cos x)^2} dx$, Put $1+\cos x = t \Rightarrow -\sin x = \frac{dt}{dx} \Rightarrow \sin x dx = -dt$

$I = \int -\frac{dt}{t^2} \Rightarrow I = -\int t^{-2} dt \Rightarrow I = -\frac{t^{-1}}{-1} + c \therefore I = \frac{1}{1+\cos x} + c$

56. $\int \frac{2x+3}{\sqrt{x^2+3x-2}} dx$

Sol. Let $I = \int \frac{2x+3}{\sqrt{x^2+3x-2}} dx$, Put $x^2+3x-2=t \Rightarrow (2x+3) = \frac{dt}{dx} \Rightarrow (2x+3)dx = dt$

$$I = \int \frac{dt}{\sqrt{t}} \Rightarrow I = \frac{t^{1/2}}{1/2} + c \quad \therefore I = 2\sqrt{x^2+3x-2} + c$$

57. $\int \frac{2x-1}{\sqrt{x^2-x-1}} dx$

Sol. Let $I = \int \frac{2x-1}{\sqrt{x^2-x-1}} dx$, Put $x^2-x-1=t \Rightarrow (2x-1) = \frac{dt}{dx} \Rightarrow (2x-1)dx = dt$

$$I = \int \frac{dt}{\sqrt{t}} \Rightarrow I = \frac{t^{1/2}}{1/2} + c \quad \therefore I = 2\sqrt{x^2-x-1} + c$$

58. $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} dx \Rightarrow I = \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}} dx$
 $\Rightarrow I = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(\sqrt{x+a})^2 - (\sqrt{x+b})^2} dx \Rightarrow I = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)} dx \Rightarrow I = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{x+a-x-b} dx$
 $\Rightarrow I = \frac{1}{a-b} \left[\int (x+a)^{1/2} dx - \int (x+b)^{1/2} dx \right] \Rightarrow I = \frac{1}{a-b} \left[\frac{(x+a)^{3/2}}{3/2} - \frac{(x+b)^{3/2}}{3/2} \right]$
 $\therefore I = \frac{2}{3(a-b)} \left[(x+a)^{3/2} - (x+b)^{3/2} \right] + c$

59. $\int \frac{1}{\sqrt{1-3x}-\sqrt{5-3x}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{1-3x}-\sqrt{5-3x}} dx \Rightarrow I = \int \frac{1}{\sqrt{1-3x}-\sqrt{5-3x}} \times \frac{\sqrt{1-3x}+\sqrt{5-3x}}{\sqrt{1-3x}+\sqrt{5-3x}} dx$
 $\Rightarrow I = \int \frac{\sqrt{1-3x}+\sqrt{5-3x}}{(\sqrt{1-3x})^2 - (\sqrt{5-3x})^2} dx \Rightarrow I = \int \frac{\sqrt{1-3x}+\sqrt{5-3x}}{1-3x-5+3x} dx$
 $\Rightarrow I = -\frac{1}{4} \left[\int (1-3x)^{1/2} dx + \int (5-3x)^{1/2} dx \right] \Rightarrow I = -\frac{1}{4} \left[\frac{(1-3x)^{3/2}}{3/2(-3)} + \frac{(5-3x)^{3/2}}{3/2(-3)} \right]$
 $\Rightarrow I = \frac{2}{9 \cdot 4} \left[(1-3x)^{3/2} + (5-3x)^{3/2} \right] + c \quad \therefore I = \frac{1}{18} \left[(1-3x)^{3/2} + (5-3x)^{3/2} \right] + c$

60. $\int \frac{x^2}{1+x^6} dx$

Sol. Let $I = \int \frac{x^2}{1+x^6} dx \Rightarrow I = \int \frac{x^2}{1+(x^3)^2} dx$, Put $x^3=t \Rightarrow 3x^2 = \frac{dt}{dx} \Rightarrow x^2 dx = \frac{dt}{3}$

$$I = \frac{1}{3} \int \frac{dt}{1+t^2} \Rightarrow I = \frac{1}{3} \tan^{-1}(t) + c \quad \therefore I = \frac{1}{3} \tan^{-1}(x^3) + c$$

61. $\int \frac{x^3}{1+x^8} dx$

Sol. Let $I = \int \frac{x^3}{1+x^8} dx \Rightarrow I = \int \frac{x^3}{1+(x^4)^2} dx$, Put $x^4 = t \Rightarrow 4x^3 = \frac{dt}{dx} \Rightarrow x^3 dx = \frac{dt}{4}$

$$I = \frac{1}{4} \int \frac{dt}{1+t^2} \Rightarrow I = \frac{1}{4} \tan^{-1}(t) + c \quad \therefore I = \frac{1}{4} \tan^{-1}(x^4) + c$$

62. $\int \frac{x}{1+x^4} dx$

Sol. Let $I = \int \frac{x}{1+x^4} dx \Rightarrow I = \int \frac{x}{1+(x^2)^2} dx$, Put $x^2 = t \Rightarrow 2x = \frac{dt}{dx} \Rightarrow x dx = \frac{dt}{2}$

$$I = \frac{1}{2} \int \frac{dt}{1+t^2} \Rightarrow I = \frac{1}{2} \tan^{-1}(t) + c \quad \therefore I = \frac{1}{2} \tan^{-1}(x^2) + c$$

63. $\int \frac{x^5}{\sqrt{1+x^3}} dx$

Sol. Let $I = \int \frac{x^5}{\sqrt{1+x^3}} dx \Rightarrow I = \int \frac{x^3 \cdot x^2}{\sqrt{1+x^3}} dx$

Put $1+x^3 = t \Rightarrow x^3 = t-1 \Rightarrow 3x^2 = \frac{dt}{dx} \Rightarrow x^2 dx = \frac{dt}{3}$

$$I = \int \frac{t-1}{\sqrt{t}} \frac{dt}{3} \Rightarrow I = \frac{1}{3} \int \frac{t}{\sqrt{t}} dt - \frac{1}{3} \int \frac{1}{\sqrt{t}} dt \Rightarrow I = \frac{1}{3} \left[\int t^{1/2} dt - \int t^{-1/2} dt \right]$$

$$\Rightarrow I = \frac{1}{3} \left[\frac{t^{3/2}}{3/2} - \frac{t^{1/2}}{1/2} \right] + c \Rightarrow I = \frac{2}{3} \left[\frac{(1+x^3)^{3/2}}{3} - (1+x^3)^{1/2} \right] + c \quad \therefore I = \frac{2}{9} (1+x^3)^{3/2} - \frac{2}{3} (1+x^3)^{1/2} + c$$

64. $\int \frac{x}{\sqrt{1+x}} dx$

Sol. Let $I = \int \frac{x}{\sqrt{1+x}} dx$, Put $1+x = t \Rightarrow x = t-1 \Rightarrow dx = dt$

$$\Rightarrow I = \int \frac{t-1}{\sqrt{t}} dt \Rightarrow I = \int \sqrt{t} dt - \int \frac{1}{\sqrt{t}} dt \Rightarrow I = \frac{t^{3/2}}{3/2} - \frac{t^{1/2}}{1/2} + c \quad \therefore I = \frac{2}{3} (1+x)^{3/2} - 2(1+x)^{1/2} + c$$

65. $\int \frac{1}{x\sqrt{x^4-1}} dx$

Sol. Let $I = \int \frac{1}{x\sqrt{x^4-1}} dx \Rightarrow I = \int \frac{x}{x^2\sqrt{(x^2)^2-1}} dx$

Put $x^2 = t \Rightarrow 2x = \frac{dt}{dx} \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{dt}{2}$

$$\Rightarrow I = \frac{1}{2} \int \frac{dt}{t\sqrt{t^2-1}} \Rightarrow I = \frac{1}{2} \sec^{-1}(t) + c \quad \therefore I = \frac{1}{2} \sec^{-1}(x^2) + c$$

66. $I = \int x\sqrt{x-1} dx$

Sol. Let $I = \int x\sqrt{x-1} dx$, Put $x-1 = t \Rightarrow x = t+1 \Rightarrow dx = dt$

$$I = \int (t+1)\sqrt{t} dt \Rightarrow I = \int t\sqrt{t} dt + \int \sqrt{t} dt \Rightarrow I = \int t^{3/2} dt + \int t^{1/2} dt \Rightarrow I = \frac{t^{5/2}}{5/2} + \frac{t^{3/2}}{3/2} + c$$

$$\therefore I = \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + c$$

67. $\int (1-x)\sqrt{1+x} dx$

Sol. Let $I = \int (1-x)\sqrt{1+x} dx$, Put $1+x=t \Rightarrow x=t-1 \Rightarrow dx=dt$

$$I = \int \{1-(t-1)\}\sqrt{t} dt \Rightarrow I = \int (1-t+1)\sqrt{t} dt \Rightarrow I = \int (2-t)\sqrt{t} dt$$

$$\Rightarrow I = 2\int \sqrt{t} dt - \int t\sqrt{t} dt \Rightarrow I = 2\int t^{1/2} dt - \int t^{3/2} dt \Rightarrow I = \frac{2t^{3/2}}{3/2} - \frac{t^{5/2}}{5/2} + c$$

$$\therefore I = \frac{4}{3}(1+x)^{3/2} - \frac{2}{5}(1+x)^{5/2} + c$$

68. $\int x\sqrt{x^2-1} dx$

Sol. Let $I = \int x\sqrt{x^2-1} dx$, Put $x^2-1=t \Rightarrow 2x = \frac{dt}{dx} \Rightarrow x dx = \frac{dt}{2}$

$$I = \int \sqrt{t} \cdot \frac{dt}{2} \Rightarrow I = \frac{1}{2} \int t^{1/2} dt \Rightarrow I = \frac{1}{2} \cdot \frac{t^{3/2}}{3/2} + c \therefore I = \frac{1}{3}(x^2-1)^{3/2} + c$$

69. $\int x\sqrt{3x-2} dx$

Sol. Let $I = \int x\sqrt{3x-2} dx$, Put $3x-2=t \therefore x = \frac{t+2}{3} \Rightarrow 3 = \frac{dt}{dx}$

$$I = \int \left(\frac{t+2}{3}\right)\sqrt{t} \cdot \frac{dt}{3} \Rightarrow I = \frac{1}{9} \left[\int t^{3/2} dt + 2 \int t^{1/2} dt \right] \Rightarrow I = \frac{1}{9} \left[\frac{t^{5/2}}{5/2} + \frac{2t^{3/2}}{3/2} \right] + c$$

$$\Rightarrow I = \frac{1}{9} \left[\frac{2}{5}(3x-2)^{5/2} + \frac{4}{3}(3x-2)^{3/2} \right] + c \therefore I = \frac{2}{45}(3x-2)^{5/2} + \frac{4}{27}(3x-2)^{3/2} + c$$

70. $\int \frac{dx}{x \cos^2(1+\log x)}$

Sol. Let $I = \int \frac{1}{x \cos^2(1+\log x)} dx$, Put $1+\log x=t \Rightarrow \frac{1}{x} = \frac{dt}{dx} \Rightarrow \frac{1}{x} dx = dt$

$$I = \int \frac{1}{\cos^2(t)} dt \Rightarrow I = \int \sec^2(t) dt \Rightarrow I = \tan(t) + c \therefore I = \tan(1+\log x) + c$$

71. $\int x^2 \sin(x^3) dx$

Sol. Let $I = \int x^2 \sin(x^3) dx$, Put $x^3=t \Rightarrow 3x^2 = \frac{dt}{dx} \Rightarrow x^2 dx = \frac{dt}{3}$

$$I = \int \sin(t) \cdot \frac{dt}{3} \Rightarrow I = \frac{1}{3} \int \sin t dt \Rightarrow I = \frac{1}{3} (-\cos t) + c \therefore I = \frac{-\cos(x^3)}{3} + c$$

72. $\int (2x+4)\sqrt{x^2+4x+3} dx$

Sol. Let $I = \int (2x+4)\sqrt{x^2+4x+3} dx$, Put $x^2+4x+3=t \Rightarrow (2x+4) = \frac{dt}{dx} \Rightarrow (2x+4) dx = dt$

$$I = \int \sqrt{t} dt \Rightarrow I = \frac{t^{3/2}}{3/2} + c \therefore I = \frac{2}{3}(x^2+4x+3)^{3/2} + c$$

73. $\int \frac{\sin x}{\sin x - \cos x} dx$

Sol. Let $I = \int \frac{\sin x}{\sin x - \cos x} dx \Rightarrow I = \frac{1}{2} \int \frac{2 \sin x}{\sin x - \cos x} dx$
 $\Rightarrow I = \frac{1}{2} \int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x - \cos x)} dx \Rightarrow I = \frac{1}{2} \int \frac{\sin x + \cos x}{\sin x - \cos x} dx + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x - \cos x} dx$
 Put $\sin x - \cos x = t \Rightarrow \cos x + \sin x = \frac{dt}{dx} \Rightarrow (\cos x + \sin x) dx = dt$
 $I = \frac{1}{2} \int \frac{dt}{t} + \frac{1}{2} \int dx \Rightarrow I = \frac{1}{2} \log |\sin x - \cos x| + \frac{1}{2} x + c \quad \therefore I = \frac{x}{2} + \frac{1}{2} \log |\sin x - \cos x| + c$

74. $\int \frac{1}{1 - \tan x} dx$

Sol. Let $I = \int \frac{1}{1 - \tan x} dx \Rightarrow I = \frac{1}{\frac{1 - \sin x}{\cos x}} \Rightarrow I = \int \frac{\cos x}{\cos x - \sin x} dx \Rightarrow I = \frac{1}{2} \int \frac{2 \cos x}{\cos x - \sin x} dx$
 $\Rightarrow I = \frac{1}{2} \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\cos x - \sin x} dx \Rightarrow \frac{1}{2} \int \frac{\cos x + \sin x}{\cos x - \sin x} dx + \frac{1}{2} \int \frac{\cos x - \sin x}{\cos x - \sin x} dx$
 Put $\cos x - \sin x = t \Rightarrow \sin x - \cos x = \frac{dt}{dx} \Rightarrow (\sin x + \cos x) dx = -dt$
 $I = -\frac{1}{2} \int \frac{dt}{t} + \frac{1}{2} \int dx \Rightarrow I = -\frac{1}{2} \log |t| + \frac{1}{2} x + c \Rightarrow I = -\frac{1}{2} \log |\cos x - \sin x| + \frac{x}{2} + c$
 $\therefore I = \frac{x}{2} - \frac{1}{2} \log |\cos x - \sin x| + c$ and $I = \frac{x}{2} - \frac{1}{2} \log |\sin x - \cos x| + c$

75. $\int \frac{1}{1 - \cot x} dx$

Sol. Let $I = \int \frac{1}{1 - \cot x} dx \Rightarrow I = \int \frac{1}{\frac{1 - \cos x}{\sin x}} dx \Rightarrow I = \int \frac{\sin x}{\sin x - \cos x} dx$
 $\Rightarrow I = \frac{1}{2} \int \frac{2 \sin x}{\sin x - \cos x} dx \Rightarrow I = \frac{1}{2} \int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x - \cos x)} dx$
 $\Rightarrow I = \frac{1}{2} \int \frac{\sin x + \cos x}{\sin x - \cos x} dx + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x - \cos x} dx$
 Put $\sin x - \cos x = t \Rightarrow \cos x + \sin x = \frac{dt}{dx} \Rightarrow (\cos x + \sin x) dx = dt$
 $I = \frac{1}{2} \int \frac{dt}{t} + \frac{1}{2} \int dx \Rightarrow I = \frac{1}{2} \log |\sin x - \cos x| + \frac{1}{2} x + c \quad \therefore I = \frac{x}{2} + \frac{1}{2} \log |\sin x - \cos x| + c$

76. $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$

Sol. Let $I = \int \frac{\cos 2x}{(\sin x + \cos x)^2} dx \Rightarrow I = \int \frac{\cos^2 x - \sin^2 x}{(\sin x + \cos x)^2} dx$
 $\Rightarrow I = \int \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\cos x + \sin x)^2} dx \Rightarrow I = \int \frac{\cos x - \sin x}{(\cos x + \sin x)} dx$

Put $\cos x + \sin x = t \Rightarrow -\sin x + \cos x = \frac{dt}{dx} \Rightarrow (\cos x - \sin x) dx = dt$

$$I = \int \frac{dt}{t} \Rightarrow I = \log|t| + c \quad \therefore I = \log|\cos x + \sin x| + c$$

77. $\int \frac{\cos x - \sin x}{(1 + \sin 2x)} dx$

Sol. Let $I = \int \frac{\cos x - \sin x}{1 + \sin 2x} dx \Rightarrow I = \int \frac{\cos x - \sin x}{(\cos x + \sin x)^2}$

Put $\cos x + \sin x = t \Rightarrow -\sin x + \cos x = \frac{dt}{dx} \Rightarrow (\cos x - \sin x) dx = dt$

$$I = \int \frac{dt}{t^2} \Rightarrow I = -\frac{1}{t} + c \quad \therefore I = -\frac{1}{\cos x + \sin x} + c$$

78. $\int \frac{(x+1)(x+\log x)^2}{x} dx$

Sol. Let $I = \int \frac{(x+1)(x+\log x)^2}{x} dx$, Put $x + \log x = t \Rightarrow 1 + \frac{1}{x} = \frac{dt}{dx} \Rightarrow \frac{x+1}{x} dx = dt$

$$I = \int t^2 dt \Rightarrow I = \frac{t^3}{3} + c \quad \therefore I = \frac{(x+\log x)^3}{3} + c$$

79. $\int x \sin^3(x^2) \cos(x^2) dx$

Sol. Let $I = \int x \sin^3(x^2) \cos(x^2) dx$, Put $\sin(x^2) = t \Rightarrow \cos(x^2) \cdot 2x = \frac{dt}{dx} \Rightarrow x \cos(x^2) dx = \frac{dt}{2}$

$$I = \int t^3 \cdot \frac{dt}{2} \Rightarrow I = \frac{1}{2} \int t^3 dt \Rightarrow I = \frac{1}{2} \cdot \frac{t^4}{4} + c \quad \therefore I = \frac{\sin^4(x^2)}{8} + c$$

80. $\int \frac{\sec^2 x}{\sqrt{1 - \tan^2 x}} dx$

Sol. Let $I = \int \frac{\sec^2 x}{\sqrt{1 - \tan^2 x}} dx$, Put $\tan x = t \Rightarrow \sec^2 x = \frac{dt}{dx} \Rightarrow \sec^2 x dx = dt$

$$I = \int \frac{dt}{\sqrt{1-t^2}} \Rightarrow I = \sin^{-1}(t) + c \quad \therefore I = \sin^{-1}(\tan x) + c$$

81. $\int e^{-x} \operatorname{cosec}^2(2e^{-x} + 5) dx$

Sol. Let $I = \int e^{-x} \operatorname{cosec}^2(2e^{-x} + 5) dx$, Put $2e^{-x} + 5 = t \Rightarrow 2e^{-x}(-1) = \frac{dt}{dx} \Rightarrow e^{-x} dx = \frac{dt}{-2}$

$$I = \int \operatorname{cosec}^2(t) \cdot \frac{dt}{-2} \Rightarrow I = -\frac{1}{2} \int \operatorname{cosec}^2(t) dt \Rightarrow I = -\frac{1}{2}(-\cot t) + c$$

$$\therefore I = \frac{1}{2} \cot(2e^{-x} + 5) + c$$

82. $\int 2x \sec^3(x^2 + 3) \tan(x^2 + 3) dx$

Sol. Let $I = \int 2x \sec^2(x^2 + 3) \sec(x^2 + 3) \tan(x^2 + 3) dx$

Put $\sec(x^2 + 3) = t \Rightarrow \sec(x^2 + 3) \tan(x^2 + 3) \cdot 2x = \frac{dt}{dx} \Rightarrow 2x \sec(x^2 + 3) \tan(x^2 + 3) dx = dt$

$$I = \int t^2 dt \Rightarrow I = \frac{t^3}{3} + c \quad \therefore I = \frac{\sec^3(x^2+3)}{3} + c$$

83. $\int \frac{\sin 2x}{(a+b \cos x)^2} dx$

Sol. Let $I = \int \frac{\sin 2x}{(a+b \cos x)^2} dx \Rightarrow I = \int \frac{2 \sin x \cos x}{(a+b \cos x)^2} dx$

Put $a+b \cos x = t \Rightarrow \cos x = \frac{t-a}{b} \Rightarrow \sin x dx = \frac{dt}{-b} \quad \therefore -b \sin x = \frac{dt}{dx}$

$$I = 2 \int \frac{t-a}{b} \cdot \frac{1}{(-b)} \cdot \frac{1}{t^2} dt \Rightarrow I = \frac{2}{-b^2} \int \frac{t-a}{t^2} dt \Rightarrow I = \frac{2}{-b^2} \left[\int \frac{t}{t^2} dt - \int \frac{a}{t^2} dt \right]$$

$$\Rightarrow I = \frac{2}{-b^2} \left[\int \frac{1}{t} dt - a \int \frac{1}{t^2} dt \right] \Rightarrow I = \frac{2}{-b^2} \left[\log |t| - a \left(-\frac{1}{t} \right) + c \right]$$

$$\Rightarrow I = \frac{2}{-b^2} \left[\log |a+b \cos x| + \frac{a}{a+b \cos x} \right] + c \quad \therefore I = -\frac{2}{b^2} \left[\log |a+b \cos x| + \frac{a}{a+b \cos x} \right] + c$$

84. $\int \frac{1}{3-5x} dx$

Sol. Let $I = \int \frac{1}{3-5x} dx$, Put $3-5x = t \Rightarrow -5 = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{-5}$

$$I = \int \frac{1}{t} \cdot \frac{dt}{-5} \Rightarrow I = -\frac{1}{5} \int \frac{1}{t} dt \quad \therefore I = -\frac{1}{5} \log |3-5x| + c$$

85. $\int \sqrt{1+x} dx$

Sol. Let $I = \int \sqrt{1+x} dx$, Put $1+x = t \Rightarrow dx = dt$

$$I = \int \sqrt{t} dt \Rightarrow I = \frac{t^{3/2}}{3/2} + c \quad \therefore I = \frac{2}{3} (1+x)^{3/2} + c$$

86. $\int x^2 e^{x^3} \cos(e^{x^3}) dx$

Sol. Let $I = \int x^2 e^{x^3} \cos(e^{x^3}) dx$, Put $e^{x^3} = t \Rightarrow e^{x^3} \cdot 3x^2 = \frac{dt}{dx} \Rightarrow e^{x^3} \cdot x^2 dx = \frac{dt}{3}$

$$I = \int \cos(t) \cdot \frac{dt}{3} \Rightarrow I = \frac{1}{3} \int \cos(t) dt \Rightarrow I = \frac{1}{3} \sin(t) + c \quad \therefore I = \frac{1}{3} \sin(e^{x^3}) + c$$

87. $\int \frac{e^{m \tan^{-1} x}}{1+x^2} dx$

Sol. Let $I = \int \frac{e^{m \tan^{-1} x}}{1+x^2} dx$, Put $m \tan^{-1} x = t \Rightarrow m \cdot \frac{1}{1+x^2} = \frac{dt}{dx} \Rightarrow \frac{1}{1+x^2} dx = \frac{dt}{m}$

$$I = \int e^t \cdot \frac{dt}{m} \Rightarrow I = \frac{1}{m} e^t + c \quad \therefore I = \frac{1}{m} e^{m \tan^{-1} x} + c$$

88. $\int \frac{(x+1)e^x}{\cos^2(xe^x)} dx$

Sol. Let $I = \int \frac{(x+1)e^x}{\cos^2(xe^x)} dx$, Put $xe^x = t \Rightarrow xe^x + e^x \cdot 1 = \frac{dt}{dx} \Rightarrow e^x(x+1) = \frac{dt}{dx} \Rightarrow (x+1)e^x dx = dt$

$$I = \int \frac{dt}{\cos^2(t)} \Rightarrow I = \int \sec^2(t) dt \Rightarrow I = \tan(t) + c \quad \therefore I = \tan(xe^x) + c$$

89. $\int \frac{e^{\sqrt{x}} \cos(e^{\sqrt{x}})}{\sqrt{x}} dx$

Sol. Let $I = \int \frac{e^{\sqrt{x}} \cos(e^{\sqrt{x}})}{\sqrt{x}} dx$, Put $e^{\sqrt{x}} = t \Rightarrow e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{dt}{dx} \Rightarrow \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 dt$

$$I = \int \cos(t) 2 dt \Rightarrow I = 2 \int \cos(t) dt \Rightarrow I = 2 \sin(t) + c \quad \therefore I = 2 \sin(e^{\sqrt{x}}) + c$$

90. $\int \sqrt{e^x - 1} dx$

Sol. Let $I = \int \sqrt{e^x - 1} dx$, Put $e^x - 1 = t^2 \Rightarrow e^x = t^2 + 1 \Rightarrow e^x = 2t \frac{dt}{dx} \Rightarrow dx = \frac{2t}{e^x} dt \Rightarrow dx = \frac{2t}{t^2 + 1} dt$

$$I = \int \sqrt{t^2} \cdot \frac{2t}{t^2 + 1} dt \Rightarrow I = \int \frac{2t^2}{t^2 + 1} dt = 2 \int \frac{t^2 + 1 - 1}{t^2 + 1} dt = 2 \int \left(1 - \frac{1}{t^2 + 1} \right) dt$$

$$\Rightarrow I = 2 \left[t - \tan^{-1} t \right] + c = 2 \left[\sqrt{e^x - 1} - \tan^{-1} \sqrt{e^x - 1} \right] + c$$

91. $I = \int \frac{1}{x - \sqrt{x}} dx$

Sol. Let $I = \int \frac{1}{x - \sqrt{x}} dx \Rightarrow I = \int \frac{1}{\sqrt{x}(\sqrt{x} - 1)} dx$, Put $\sqrt{x} - 1 = t \Rightarrow \frac{1}{2\sqrt{x}} = \frac{dt}{dx} \Rightarrow \frac{1}{\sqrt{x}} dx = 2 dt$

$$I = \int \frac{1}{t} \cdot \frac{2 dt}{2} \quad \therefore I = \frac{1}{2} \log |t| + c = \frac{1}{2} \log (\sqrt{x} - 1) + c$$

92. $\int \frac{\sec^2(2 \tan^{-1} x)}{1 + x^2} dx$

Sol. Let $I = \int \frac{\sec^2(2 \tan^{-1} x)}{1 + x^2} dx$, Put $2 \tan^{-1} x = t \Rightarrow 2 \cdot \frac{1}{1 + x^2} = \frac{dt}{dx} \Rightarrow \frac{1}{1 + x^2} dx = \frac{dt}{2}$

$$I = \int \sec^2(t) \cdot \frac{dt}{2} \Rightarrow I = \frac{1}{2} \tan(t) + c \quad \therefore I = \frac{1}{2} \tan(2 \tan^{-1} x) + c$$

93. $\int \frac{1 + \sin 2x}{x + \sin^2 x} dx$

Sol. Let $I = \int \frac{1 + \sin 2x}{x + \sin^2 x} dx$, Put $x + \sin^2 x = t \Rightarrow 1 + 2 \sin x \cdot \cos x = \frac{dt}{dx} \Rightarrow (1 + \sin 2x) dx = dt$

$$I = \int \frac{dt}{t} \Rightarrow I = \log |t| + c \quad \therefore I = \log |x + \sin^2 x| + c$$

94. $\int \frac{1 - \tan x}{x + \log(\cos x)} dx$

Sol. Let $I = \int \frac{1 - \tan x}{x + \log(\cos x)} dx$, Put $x + \log(\cos x) = t \Rightarrow 1 + \frac{1}{\cos x}(-\sin x) = \frac{dt}{dx} \Rightarrow (1 - \tan x) dx = dt$

$$I = \int \frac{1}{t} dt \Rightarrow I = \log |t| + c \quad \therefore I = \log |x + \log(\cos x)| + c$$

95. $\int \frac{1 + \cot x}{x + \log \sin x} dx$

Sol. Let $I = \int \frac{1 + \cot x}{x + \log(\sin x)} dx$, Put $x + \log(\sin x) = t \Rightarrow 1 + \frac{1}{\sin x} \cdot \cos x = \frac{dt}{dx} \Rightarrow (1 + \cot x) dx = dt$

$$I = \int \frac{dt}{t} \Rightarrow I = \log|t| + c \quad \therefore I = \log|x + \log(\sin x)| + c$$

96. $\int \frac{\tan x \cdot \sec^2 x}{1 - \tan^2 x} dx$

Sol. Let $I = \int \frac{\tan x \cdot \sec^2 x}{1 - \tan^2 x} dx$, Put $1 - \tan^2 x = t \Rightarrow 0 - 2 \tan x \cdot \sec^2 x = \frac{dt}{dx} \Rightarrow \tan x \cdot \sec^2 x dx = \frac{dt}{-2}$

$$I = \int \frac{1}{t} \cdot \left(\frac{dt}{-2} \right) \Rightarrow I = -\frac{1}{2} \log|t| + c \quad \therefore I = -\frac{1}{2} \log|1 - \tan^2 x| + c$$

97. $\int \frac{\sin(2 \tan^{-1} x)}{1 + x^2} dx$

Sol. Let $I = \int \frac{\sin(2 \tan^{-1} x)}{1 + x^2} dx$, Put $2 \tan^{-1} x = t \Rightarrow 2 \cdot \frac{1}{1 + x^2} = \frac{dt}{dx} \Rightarrow \frac{1}{1 + x^2} dx = \frac{dt}{2}$

$$I = \int \sin(t) \cdot \frac{dt}{2} \Rightarrow I = -\frac{1}{2} \cos(t) + c \quad \therefore I = -\frac{1}{2} \cos(2 \tan^{-1} x) + c$$

98. $\int \frac{1}{x^{1/2} + x^{1/3}} dx$

Sol. Let $I = \int \frac{1}{x^{1/2} + x^{1/3}} dx$

LCM of 2 and 3 is 6. Then put $x = t^6 \Rightarrow dx = 6t^5 dt$

$$I = \int \frac{6t^5}{t^3 + t^2} dt \Rightarrow I = \int \frac{6t^5}{t^2(1+t)} dt \Rightarrow I = 6 \int \frac{t^3}{t+1} dt \Rightarrow I = 6 \int \frac{t^3 + 1 - 1}{t+1} dt$$

$$\Rightarrow I = 6 \left[\int \frac{(t+1)(t^2 - t + 1)}{t+1} dt - \int \frac{1}{t+1} dt \right] \Rightarrow I = 6 \left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1| \right] + c$$

$$\Rightarrow I = \left[2t^3 - 3t^2 + 6t - 6 \log|t+1| \right] + c \quad \therefore I = 2\sqrt{x} - 3x^{1/3} + 6x^{1/6} - 6 \log|1 + x^{1/6}| + c$$

99. $\int (\sin^{-1} x)^2 dx$

Sol. Let $I = \int (\sin^{-1} x)^2 dx$, Put $\sin^{-1} x = t \quad \therefore x = \sin t$

$$\frac{1}{\sqrt{1-x^2}} = \frac{dt}{dx} \Rightarrow dx = \sqrt{1-x^2} dt \Rightarrow dx = \sqrt{1-\sin^2 t} dt \Rightarrow dx = \cos t dt$$

$$\Rightarrow I = \int t^2 \cos t dt \Rightarrow I = t^2 \int \cos t dt - \int \left[\frac{d(t^2)}{dt} \int \cos t dt \right] dt$$

$$\Rightarrow I = t^2 \sin t - \int 2t \cdot \sin t dt \Rightarrow t^2 \sin t - 2 \left\{ t \int \sin t dt - \int \left[\frac{d(t)}{dt} \int \sin t dt \right] dt \right\}$$

$$\Rightarrow I = t^2 \sin t - 2 \left[-t \cos t + \int 1 \cdot \cos t dt \right] \Rightarrow I = t^2 \sin t + 2t \cos t - 2 \sin t + c$$

$$\therefore I = x(\sin^{-1} x)^2 + 2\sqrt{1-x^2} \sin^{-1} x - 2x + c$$

100. $\int \frac{2x \tan^{-1} x^2}{1+x^4} dx$

Sol. Let $I = \int \frac{2x \tan^{-1}(x^2)}{1+x^4} dx$, Put $\tan^{-1}(x^2) = t \Rightarrow \frac{1}{1+(x^2)^2} \cdot 2x = \frac{dt}{dx} \Rightarrow \frac{2x}{1+x^4} dx = dt$

$$I = \int t dt \Rightarrow I = \frac{t^2}{2} + c \Rightarrow I = \frac{t^2}{2} + c \therefore I = \frac{\{\tan^{-1}(x^2)\}^2}{2} + c$$

101. $\int \frac{x^2+1}{x^4+1} dx$

Sol. Let $I = \int \frac{x^2+1}{x^4+1} dx$

Divide Numerator and Denominator by x^2 , $I = \int \frac{\left(1 + \frac{1}{x^2}\right)}{x^2 + \frac{1}{x^2} + 2 - 2} dx \Rightarrow I = \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx$

$$\Rightarrow I = \int \frac{\left(1 + \frac{1}{x^2}\right)}{(x)^2 - 2 \cdot x \cdot \frac{1}{x} + \left(\frac{1}{x}\right)^2 + 2} dx \Rightarrow I = \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + (\sqrt{2})^2} dx$$

Put $t = x - \frac{1}{x} \Rightarrow dt = \left(1 + \frac{1}{x^2}\right) dx$, $I = \int \frac{dt}{(t)^2 + (\sqrt{2})^2} \Rightarrow I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x - \frac{1}{x}}{\sqrt{2}}\right) + c \therefore I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{2}x}\right) + c$$

102. $\int \frac{(\sin x + \cos x)}{\sqrt{\sin 2x}} dx$

Sol. By given integral we get

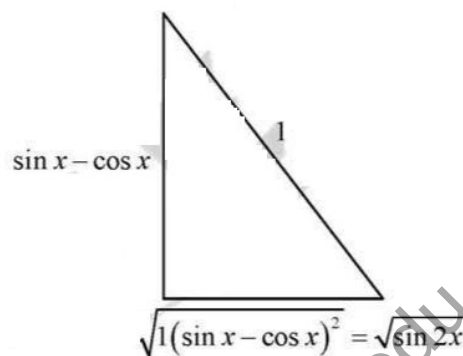
$$\int \frac{dt}{1-t^2} = \sin^{-1} t = \sin^{-1}(\sin x - \cos x)$$

$$= \tan^{-1}\left(\frac{\sin x - \cos x}{\sqrt{\sin 2x}}\right) = \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2} \tan x}\right)$$

Put $\sin x - \cos x = t$... (i)

$$\Rightarrow 1 - \sin 2x = t^2 \Rightarrow \sin 2x = 1 - t^2$$

and (i) $\Rightarrow (\cos x + \sin x) dx = dt$



EXERCISE 13B (Pg. No.: 658)

Evaluate the following integrals:

1. (i) $\int \sin^2 x \, dx$ (ii) $\int \cos^2 x \, dx$

Sol. (i) Let $I = \int \sin^2 x \, dx \Rightarrow I = \int \frac{1 - \cos 2x}{2} \, dx \Rightarrow I = \frac{1}{2} \int (1 - \cos 2x) \, dx$

$$\Rightarrow I = \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right] + c \quad \therefore I = \frac{x}{2} - \frac{\sin 2x}{4} + c$$

(ii) Let $I = \int \cos^2 x \, dx \Rightarrow I = \int \frac{1 + \cos 2x}{2} \, dx \Rightarrow I = \frac{1}{2} \int (1 + \cos 2x) \, dx$

$$\Rightarrow I = \frac{1}{2} \left[x + \frac{\sin 2x}{2} \right] + c \quad \therefore I = \frac{x}{2} + \frac{\sin 2x}{4} + c$$

2. (i) $\int \cos^2 \left(\frac{x}{2} \right) \, dx$ (ii) $\int \cot^2 \left(\frac{x}{2} \right) \, dx$

Sol. (i) Let $I = \int \cos^2 \left(\frac{x}{2} \right) \, dx \Rightarrow I = \int \left(\frac{1 + \cos x}{2} \right) \, dx \Rightarrow I = \frac{1}{2} \int (1 + \cos x) \, dx$

$$\Rightarrow I = \frac{1}{2} (x + \sin x) + c \quad \therefore I = \frac{x}{2} + \frac{\sin x}{2} + c$$

(ii) Let $I = \int \cot^2 \left(\frac{x}{2} \right) \, dx \Rightarrow I = \int \left\{ \operatorname{cosec}^2 \left(\frac{x}{2} \right) - 1 \right\} \, dx \Rightarrow I = \int \operatorname{cosec}^2 \left(\frac{x}{2} \right) \, dx - \int dx$

$$\Rightarrow I = \frac{-\cot \frac{x}{2}}{1/2} - x + c \quad \therefore I = -2 \cot \left(\frac{x}{2} \right) - x + c$$

3. (i) $\int \sin^2 nx \, dx$ (ii) $\int \sin^5 x \, dx$

Sol. (i) Let $I = \int \sin^2 nx \, dx \Rightarrow I = \int \frac{1 - \cos 2nx}{2} \, dx \Rightarrow I = \frac{1}{2} \int (1 - \cos 2nx) \, dx$

$$\Rightarrow I = \frac{1}{2} \left[x - \frac{\sin 2nx}{2n} \right] + c \quad \therefore I = \frac{x}{2} - \frac{\sin 2nx}{4n} + c$$

(ii) Let $I = \int \sin^5 x \, dx \Rightarrow I = \int (\sin^2 x)^2 \cdot \sin x \, dx \Rightarrow I = \int (1 - \cos^2 x)^2 \cdot \sin x \, dx$

Put $\cos x = t \Rightarrow -\sin x = \frac{dt}{dx} \Rightarrow \sin x \, dx = -dt$

$$I = \int (1 - t^2)^2 (-dt) \Rightarrow I = \int (1 - 2t^2 + t^4) (-dt) \Rightarrow I = - \left[t - 2 \frac{t^3}{3} + \frac{t^5}{5} \right] + c$$

$$\Rightarrow I = -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + c \quad \therefore I = \frac{2}{3} \cos^3 x - \cos x - \frac{1}{5} \cos^5 x + c$$

4. $\int \cos^3 (3x+5) \, dx$

Sol. Let $I = \int \cos^3 (3x+5) \, dx \Rightarrow I = \int \cos^2 (3x+5) \cdot \cos (3x+5) \, dx$

$$\Rightarrow I = \int \{1 - \sin^2 (3x+5)\} \cos (3x+5) \, dx$$

Put $\sin (3x+5) = t \Rightarrow \cos (3x+5) \cdot 3 = \frac{dt}{dx} \Rightarrow \cos (3x+5) \, dx = \frac{dt}{3}$

$$I = \int (1-t^2) \frac{dt}{3} \Rightarrow I = \frac{1}{3} \int (1-t^2) dt \Rightarrow I = \frac{1}{3} \left[t - \frac{t^3}{3} \right] + c$$

$$\therefore I = \frac{1}{3} \sin(3x+5) - \frac{\sin^3(3x+5)}{9} + c$$

5. $\int \sin^7(3-2x) dx$

Sol. Let $I = \int \sin^7(3-2x) dx$

$$\Rightarrow I = \int \left\{ \sin^2(3-2x) \right\}^3 \cdot \sin(3-2x) dx \Rightarrow \int \left\{ 1 - \cos^2(3-2x) \right\}^3 \cdot \sin(3-2x) dx$$

$$\text{Put } \cos(3-2x) = t \Rightarrow -\sin(3-2x)(-2) = \frac{dt}{dx} \Rightarrow \sin(3-2x) dx = \frac{dt}{2}$$

$$I = \int (1-t^2)^3 \frac{dt}{2} \Rightarrow I = \frac{1}{2} \int \left\{ (1)^3 - (t^2)^3 - 3(1)^2 t^2 + 3(t^2)^2 \cdot 1 \right\} dt$$

$$\Rightarrow I = \frac{1}{2} \int (1 - t^6 - 3t^2 + 3t^4) dt \Rightarrow I = \frac{1}{2} \left[t - \frac{t^7}{7} - \frac{3t^3}{3} + \frac{3t^5}{5} \right] + c$$

$$\therefore I = \frac{1}{2} \cos(3-2x) - \frac{1}{14} \cos^7(3-2x) - \frac{1}{2} \cos^3(3-2x) + \frac{3}{10} \cos^5(3-2x) + c$$

6. (i) $\int \frac{1-\cos 2x}{1+\cos 2x} dx$ (ii) $I = \int \frac{1+\cos 2x}{1-\cos 2x} dx$

Sol. (i) Let $I = \int \frac{1-\cos 2x}{1+\cos 2x} dx \Rightarrow I = \int \frac{2\sin^2 x}{2\cos^2 x} dx \Rightarrow I = \int \tan^2 x dx$

$$\Rightarrow I = \int (\sec^2 x - 1) dx \therefore I = \tan x - x + c$$

(ii) Let $I = \int \frac{1+\cos 2x}{1-\cos 2x} dx \Rightarrow I = \int \frac{2\cos^2 x}{2\sin^2 x} dx \Rightarrow I = \int \cot^2 x dx$

$$\Rightarrow I = \int (\operatorname{cosec}^2 x - 1) dx \therefore I = -\cot x - x + c$$

7. (i) $\int \frac{1-\cos x}{1+\cos x} dx$ (ii) $\int \frac{1+\cos x}{1-\cos x} dx$

Sol. (i) Let $I = \int \frac{1-\cos x}{1+\cos x} dx \Rightarrow I = \int \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} dx$

$$\Rightarrow I = \int \tan^2 \left(\frac{x}{2} \right) dx \Rightarrow I = \int \left(\sec^2 \frac{x}{2} - 1 \right) dx \therefore I = 2 \tan \frac{x}{2} - x + c$$

(ii) Let $I = \int \frac{1+\cos x}{1-\cos x} dx \Rightarrow I = \int \frac{2\cos^2 \frac{x}{2}}{2\sin^2 \frac{x}{2}} dx \Rightarrow I = \int \cot^2 \frac{x}{2} dx$

$$I = \int \left(\operatorname{cosec}^2 \frac{x}{2} - 1 \right) dx \Rightarrow I = \frac{-\cot \frac{x}{2}}{1/2} - x + c \therefore I = -2 \cot \frac{x}{2} - x + c$$

8. $\int \sin 3x \cdot \sin 4x dx$

Sol. Let $I = \int \sin 3x \cdot \sin 4x \, dx \Rightarrow I = \frac{1}{2} \int 2 \sin 3x \sin 4x \, dx \Rightarrow I = \frac{1}{2} \int \{\cos(3x-4x) - \cos(3x+4x)\} \, dx$
 $\Rightarrow I = \frac{1}{2} \int (\cos x - \cos 7x) \, dx \Rightarrow I = \frac{1}{2} \left[\sin x - \frac{\sin 7x}{7} \right] + c \therefore I = \frac{1}{2} \sin x - \frac{1}{14} \sin 7x + c$

9. $\int \cos 4x \cos 3x \, dx$

Sol. Let $I = \int \cos 4x \cdot \cos 3x \, dx \Rightarrow I = \frac{1}{2} \int \{\cos(4x+3x) + \cos(4x-3x)\} \, dx$
 $I = \frac{1}{2} \int \{\cos 7x + \cos x\} \, dx \Rightarrow I = \frac{1}{2} \left[\frac{\sin 7x}{7} + \sin x \right] + c$

10. $\int \sin 4x \sin 8x \, dx$

Sol. Let $I = \int \sin 4x \sin 8x \, dx \Rightarrow I = \frac{1}{2} \int 2 \sin 4x \sin 8x \, dx$
 $I = \frac{1}{2} \int \{\cos(4x-8x) - \cos(4x+8x)\} \, dx \Rightarrow I = \frac{1}{2} \int \{\cos(-4x) - \cos(12x)\} \, dx$
 $\Rightarrow I = \frac{1}{2} \int (\cos 4x - \cos 12x) \, dx \Rightarrow I = \frac{1}{2} \left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12} \right] + c \therefore I = \frac{\sin 4x}{8} - \frac{\sin 12x}{24} + c$

11. $\int \sin 6x \cos x \, dx$

Sol. Let $I = \int \sin 6x \cos x \, dx \Rightarrow I = \frac{1}{2} \int 2 \sin 6x \cos x \, dx$
 $\Rightarrow I = \frac{1}{2} \int \{\sin(6x+x) + \sin(6x-x)\} \, dx \Rightarrow I = \frac{1}{2} \int \{\sin 7x + \sin 5x\} \, dx$
 $\Rightarrow I = \frac{1}{2} \left[-\frac{\cos 7x}{7} - \frac{\cos 5x}{5} \right] + c \therefore I = -\frac{\cos 7x}{14} - \frac{\cos 5x}{10} + c$

12. $\int \sin x \sqrt{1+\cos 2x} \, dx$

Sol. Let $I = \int \sin x \sqrt{1+\cos 2x} \, dx \Rightarrow I = \int \sin x \sqrt{2 \cos^2 x} \, dx$
 $\Rightarrow I = \int \sin x \sqrt{2} \cos x \, dx \Rightarrow I = \sqrt{2} \int \sin x \cos x \, dx$
 Put $\sin x = t \Rightarrow \cos x = \frac{dt}{dx} \Rightarrow \cos x \, dx = dt$

$I = \sqrt{2} \int t \cdot dt \Rightarrow I = \sqrt{2} \cdot \frac{t^2}{2} + c \therefore I = \frac{\sin^2 x}{\sqrt{2}} + c$

13. $\int \cos^4 x \, dx$

Sol. Let $I = \int \cos^4 x \, dx \Rightarrow I = \int \cos^2 x \cdot \cos^2 x \, dx \Rightarrow I = \int \left(\frac{1+\cos 2x}{2} \right)^2 \, dx$
 $\Rightarrow \frac{1}{4} \int (1+2\cos 2x+\cos^2 2x) \, dx \Rightarrow I = \frac{1}{4} \left[\int dx + 2 \int \cos 2x \, dx + \int \cos^2 2x \, dx \right]$
 $\Rightarrow I = \frac{1}{4} \left[x + 2 \frac{\sin 2x}{2} + \int \left(\frac{1+\cos 4x}{2} \right) \, dx \right] \Rightarrow I = \frac{1}{4} \left[x + \sin 2x + \frac{1}{2} \left(x + \frac{\sin 4x}{4} \right) \right] + c$
 $\Rightarrow I = \frac{1}{4} \left[\frac{3x}{2} + \sin 2x + \frac{\sin 4x}{8} \right] + c \therefore I = \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + c$

14. $\int \cos 2x \cos 4x \cos 6x \, dx$

Sol. Let $I = \int \cos 2x \cdot \cos 4x \cdot \cos 6x \, dx \Rightarrow I = \frac{1}{2} \int 2 \cos 2x \cdot \cos 4x \cdot \cos 6x \, dx$

$$\Rightarrow I = \frac{1}{2} \int \cos 2x (2 \cos 4x \cos 6x) \, dx \Rightarrow I = \frac{1}{2} \int \cos 2x \{ \cos(4x+6x) + \cos(4x-6x) \} \, dx$$

$$\Rightarrow I = \frac{1}{2} \int \cos 2x (\cos 10x + \cos 2x) \, dx \Rightarrow I = \frac{1}{2} \int \cos 2x \cos 10x \, dx + \frac{1}{2} \int \cos^2 2x \, dx$$

$$\Rightarrow I = \frac{1}{2} \cdot \frac{1}{2} \int 2 \cos 2x \cdot \cos 10x \, dx + \frac{1}{2} \int \frac{1 + \cos 4x}{2} \, dx$$

$$\Rightarrow I = \frac{1}{4} \int \{ \cos(2x+10x) + \cos(2x-10x) \} \, dx + \frac{1}{4} \int (1 + \cos 4x) \, dx$$

$$\Rightarrow I = \frac{1}{4} \int \{ \cos 12x + \cos 8x \} \, dx + \frac{1}{4} \left(x + \frac{\sin 4x}{4} \right)$$

$$\Rightarrow I = \frac{1}{4} \left[\frac{\sin 12x}{12} + \frac{\sin 8x}{8} \right] + \frac{1}{4} x + \frac{\sin 4x}{16} + c \Rightarrow I = \frac{\sin 12x}{48} + \frac{\sin 8x}{32} + \frac{x}{4} + \frac{\sin 4x}{16} + c$$

$$\therefore I = \frac{x}{4} + \frac{\sin 4x}{16} + \frac{\sin 8x}{32} + \frac{\sin 12x}{48} + c$$

15. $\int \sin^3 x \cos x \, dx$

Sol. Let $I = \int \sin^3 x \cdot \cos x \, dx$, Put $\sin x = t \Rightarrow \cos x = \frac{dt}{dx} \Rightarrow \cos x \, dx = dt$

$$I = \int t^3 \, dt \Rightarrow I = \frac{t^4}{4} + c \therefore I = \frac{\sin^4 x}{4} + c$$

16. $\int \sec^4 x \, dx$

Sol. Let $I = \int \sec^4 x \, dx \Rightarrow I = \int \sec^2 x \cdot \sec^2 x \, dx \Rightarrow I = \int (1 + \tan^2 x) \sec^2 x \, dx$

Put $\tan x = t \Rightarrow \sec^2 x = \frac{dt}{dx} \Rightarrow \sec^2 x \, dx = dt$

$$I = \int (1 + t^2) \, dt \Rightarrow I = t + \frac{t^3}{3} + c \therefore I = \tan x + \frac{\tan^3 x}{3} + c$$

17. $\int \cos^3 x \sin^4 x \, dx$

Sol. Let $I = \int \cos^3 x \cdot \sin^4 x \, dx \Rightarrow I = \int \sin^4 x \cdot \cos^2 x \cdot \cos x \, dx \Rightarrow I = \int \sin^4 x (1 - \sin^2 x) \cdot \cos x \, dx$

Put $\sin x = t \Rightarrow \cos x = \frac{dt}{dx} \Rightarrow \cos x \, dx = dt$

$$I = \int t^4 (1 - t^2) \, dt \Rightarrow I = \int t^4 \, dt - \int t^6 \, dt \Rightarrow I = \frac{t^5}{5} - \frac{t^7}{7} + c \therefore I = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$$

18. $\int \cos^4 x \sin^3 x \, dx$

Sol. Let $I = \int \cos^4 x \sin^3 x \, dx \Rightarrow I = \int \cos^4 x \cdot \sin^2 x \cdot \sin x \, dx \Rightarrow I = \int \cos^4 x (1 - \cos^2 x) \sin x \, dx$

Put $\cos x = t \Rightarrow -\sin x = \frac{dt}{dx} \Rightarrow \sin x \, dx = -dt$

$$I = \int t^4 (1 - t^2) (-dt) \Rightarrow I = \int t^4 (t^2 - 1) \, dt \Rightarrow I = \frac{t^7}{7} - \frac{t^5}{5} + c$$

$$\therefore I = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + c$$

19. $\int \sin^{2/3} x \cos^3 x \, dx$

Sol. Let $I = \int \sin^{2/3} x \cos^3 x \, dx \Rightarrow I = \int \sin^{2/3} x \cdot \cos^2 x \cos x \, dx \Rightarrow I = \int \sin^{2/3} x (1 - \sin^2 x) \cos x \, dx$

Put $\sin x = t \Rightarrow \cos x = \frac{dt}{dx} \Rightarrow \cos x \, dx = dt$

$$I = \int t^{2/3} (1 - t^2) dt \Rightarrow I = \int t^{2/3} dt - \int t^{2/3} \cdot t^2 dt \Rightarrow I = \int t^{2/3} dt - \int t^{8/3} dt$$

$$\Rightarrow I = \frac{t^{5/3}}{5/3} - \frac{t^{11/3}}{11/3} + c \therefore I = \frac{3}{5} \sin^{5/3} x - \frac{3}{11} \sin^{11/3} x + c$$

20. $\int \cos^{3/5} x \sin^3 x \, dx$

Sol. Let $I = \int \cos^{3/5} x \sin^3 x \, dx$

$$\Rightarrow I = \int \cos^{3/5} x \sin^2 x \cdot \sin x \, dx \Rightarrow I = \int \cos^{3/5} x (1 - \cos^2 x) \cdot \sin x \, dx$$

Put $\cos x = t \Rightarrow -\sin x = \frac{dt}{dx} \Rightarrow \sin x \, dx = -dt$

$$I = \int t^{3/5} (1 - t^2) (-dt) \Rightarrow \int t^{3/5} (t^2 - 1) dt \Rightarrow I = \int t^{3/5} t^2 dt - \int t^{3/5} dt$$

$$\Rightarrow I = \int t^{13/5} dt - \int t^{3/5} dt \Rightarrow I = \frac{t^{18/5}}{18/5} - \frac{t^{8/5}}{8/5} + c \therefore I = \frac{5}{18} \cos^{18/5} x - \frac{5}{8} \cos^{8/5} x + c$$

21. $\int \operatorname{cosec}^4(2x) \, dx$

Sol. Let $I = \int \operatorname{cosec}^4(2x) \, dx$

$$\Rightarrow I = \int \{\operatorname{cosec}^2(2x)\}^2 dx \Rightarrow I = \int \{1 + \cot^2(2x)\} \operatorname{cosec}^2(2x) dx$$

Put $\cot(2x) = t \Rightarrow -\operatorname{cosec}^2(2x) \cdot 2 = \frac{dt}{dx} \Rightarrow \operatorname{cosec}^2(2x) dx = \frac{dt}{-2}$

$$I = \int (1 + t^2) \left(-\frac{dt}{2}\right) \Rightarrow I = -\frac{1}{2} \int (1 + t^2) dt \Rightarrow I = -\frac{1}{2} \left[t + \frac{t^3}{3}\right] + c$$

$$\therefore I = -\frac{1}{2} \cot(2x) - \frac{1}{6} \cot^3(2x) + c$$

22. $\int \frac{\cos 2x}{\cos x} dx$

Sol. Let $I = \int \frac{\cos 2x}{\cos x} dx \Rightarrow I = \int \frac{2\cos^2 x - 1}{\cos x} dx \Rightarrow I = 2 \int \cos x \, dx - \int \sec x \, dx$

$$\therefore I = 2 \sin x - \log |\sec x + \tan x| + c$$

23. $\int \frac{\cos(x)}{\cos(x+\alpha)} dx$

Sol. Let $I = \int \frac{\cos x}{\cos(x+\alpha)} dx$, Put $x+\alpha = t \Rightarrow x = t - \alpha \Rightarrow 1 = \frac{dt}{dx} \Rightarrow dx = dt$

$$\Rightarrow I = \int \frac{\cos(t-\alpha)}{\cos t} dt \Rightarrow I = \int \frac{\cos t \cdot \cos \alpha + \sin t \cdot \sin \alpha}{\cos t} dt$$

$$\Rightarrow I = \cos \alpha \int dt + \sin \alpha \int \tan t \, dt \Rightarrow I = \cos \alpha \cdot t + \sin \alpha (-\log \cos t) + c$$

$$\therefore I = (x + \alpha) \cos \alpha - \sin \alpha \log \cos |x + \alpha| + c$$

$$I = x \cos \alpha - \sin \alpha \log \cos |x + \alpha| + c$$

24. $\int \cos^3 x \cdot \sin 2x \, dx$

Sol. Let $I = \int \cos^3 x \cdot \sin 2x \, dx \Rightarrow I = \int \cos^3 x \cdot 2 \sin x \cdot \cos x \, dx \Rightarrow I = 2 \int \cos^4 x \sin x \, dx$

Put $\cos x = t \Rightarrow -\sin x = \frac{dt}{dx} \Rightarrow \sin x \, dx = -dt$

$$\Rightarrow I = 2 \int t^4 \cdot (-dt) = -2 \frac{t^5}{5} + c \therefore I = -\frac{2}{5} \cos^5 x + c$$

25. $\int \frac{\cos^9 x}{\sin x} \, dx$

Sol. Let $I = \int \frac{\cos^9 x}{\sin x} \, dx \Rightarrow I = \int \frac{(\cos^2 x)^4 \cdot \cos x}{\sin x} \, dx \Rightarrow I = \int \frac{(1 - \sin^2 x)^4}{\sin x} \cdot \cos x \, dx$

Put $\sin x = t \Rightarrow \cos x = \frac{dt}{dx} \Rightarrow \cos x \, dx = dt$

$$I = \int \frac{(1 - t^2)^4}{t} \, dt \Rightarrow I = \int \left(\frac{1 - 4t^2 + 6t^4 - 4t^6 + t^8}{t} \right) dt$$

$$\Rightarrow I = \int \frac{1}{t} \, dt - 4 \int t \, dt + 6 \int t^3 \, dt - 4 \int t^5 \, dt + \int t^7 \, dt \Rightarrow I = \log |t| - 4 \frac{t^2}{2} + 6 \frac{t^4}{4} - 4 \frac{t^6}{6} + \frac{t^8}{8} + c$$

$$\therefore I = \log |\sin x| - 2 \sin^2 x + \frac{3}{2} \sin^4 x - \frac{2}{3} \sin^6 x + \frac{1}{8} \sin^8 x + c$$

26. $\int \cos^4(2x) \, dx$

Sol. Let $I = \int \cos^4(2x) \, dx \Rightarrow I = \int \cos^2(2x) \cdot \cos^2(2x) \, dx$

$$\Rightarrow I = \int \left(\frac{1 + \cos 4x}{2} \right) \left(\frac{1 + \cos 4x}{2} \right) dx \Rightarrow I = \frac{1}{4} \int (1 + 2 \cos 4x + \cos^2 4x) \, dx$$

$$\Rightarrow I = \frac{1}{4} \left[x + \frac{2 \sin 4x}{4} + \int \left(\frac{1 + \cos 8x}{2} \right) dx \right] \Rightarrow I = \frac{1}{4} \left[x + \frac{\sin 4x}{2} + \frac{1}{2} \left(x + \frac{\sin 8x}{8} \right) \right] + c$$

$$\Rightarrow I = \frac{1}{4} \left[\frac{3x}{2} + \frac{\sin 4x}{2} + \frac{\sin 8x}{16} \right] + c \therefore I = \frac{3}{8} x + \frac{\sin 4x}{8} + \frac{\sin 8x}{64} + c$$

27. $\int \frac{\sin^2 x}{(1 + \cos x)^2} \, dx$

Sol. Let $I = \int \frac{\sin^2 x}{(1 + \cos x)^2} \, dx \Rightarrow I = \int \frac{1 - \cos^2 x}{(1 + \cos x)^2} \, dx \Rightarrow I = \int \frac{(1 + \cos x)(1 - \cos x)}{(1 + \cos x)^2} \, dx$

$$\Rightarrow I = \int \frac{(1 - \cos x)}{(1 + \cos x)} \, dx \Rightarrow I = \int \frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \, dx \Rightarrow I = \int \tan^2 \frac{x}{2} \, dx$$

$$\Rightarrow I = \int \left(\sec^2 \frac{x}{2} - 1 \right) dx \Rightarrow I = \frac{\tan \frac{x}{2}}{1/2} - x + c \quad \therefore I = 2 \tan \frac{x}{2} - x + c$$

28. $\int \frac{dx}{3 \cos x + 4 \sin x}$

Sol. Let $I = \int \frac{1}{3 \cos x + 4 \sin x} dx$, Put $3 = r \sin \theta$... (i) & $4 = r \cos \theta$... (ii)

Square both side and adding equation (i) & (ii)

$$9 + 16 = r^2 \cos^2 \theta + r^2 \sin^2 \theta \Rightarrow 25 = r^2 (\cos^2 \theta + \sin^2 \theta) \Rightarrow r^2 = 25 \quad \therefore r = 5$$

Dividing equation (i) by (ii), $\frac{r \sin \theta}{r \cos \theta} = \frac{3}{4} \Rightarrow \tan \theta = \frac{3}{4} \quad \therefore \theta = \tan^{-1} \left(\frac{3}{4} \right)$

$$I = \int \frac{1}{r \sin \theta \cos x + r \cos \theta \sin x} dx \Rightarrow I = \int \frac{1}{r (\cos x \sin \theta + \sin x \cos \theta)} dx$$

$$\Rightarrow I = \int \frac{1}{r \sin(x + \theta)} dx \Rightarrow I = \frac{1}{r} \int \operatorname{cosec}(x + \theta) dx \Rightarrow I = \frac{1}{5} \log \left| \tan \left(\frac{x + \theta}{2} \right) \right| + c$$

$$\therefore I = \frac{1}{5} \log \left| \tan \left(\frac{x}{2} + \frac{1}{2} \tan^{-1} \frac{3}{4} \right) \right| + c$$

29. $\int \frac{1}{(a \cos x + b \sin x)^2} dx$

Sol. Let $I = \int \frac{1}{(a \cos x + b \sin x)^2} dx$, Put $a = r \sin \theta$... (i) & $b = r \cos \theta$... (ii)

Square both side and adding equation (i) & (ii), $a^2 + b^2 = r^2 \sin^2 \theta + r^2 \cos^2 \theta$

$$\Rightarrow a^2 + b^2 = r^2 (\sin^2 \theta + \cos^2 \theta) \Rightarrow a^2 + b^2 = r^2 \quad \therefore r = \sqrt{a^2 + b^2}$$

Dividing equation (i) by (ii), $\frac{r \sin \theta}{r \cos \theta} = \frac{a}{b} \Rightarrow \tan \theta = \frac{a}{b} \quad \therefore \theta = \tan^{-1} \left(\frac{a}{b} \right)$

$$I = \int \frac{1}{(r \sin \theta \cos x + r \cos \theta \sin x)^2} dx \Rightarrow I = \int \frac{1}{r^2 (\cos x \sin \theta + \sin x \cos \theta)^2} dx$$

$$\Rightarrow I = \int \frac{1}{r^2 \sin^2(x + \theta)} dx \Rightarrow I = \frac{1}{r^2} \int \operatorname{cosec}^2(x + \theta) dx \Rightarrow I = \frac{1}{r^2} \{ -\cot(x + \theta) \} + c$$

$$\therefore I = -\frac{\cot \left(x + \tan^{-1} \left(\frac{a}{b} \right) \right)}{a^2 + b^2} + c$$

30. $\int \frac{1}{(\cos x - \sin x)} dx$

Sol. Let $I = \int \frac{1}{\cos x - \sin x} dx$, Put $1 = r \sin \theta$... (i) & $1 = r \cos \theta$... (ii)

Square both side and adding equation (i) & (ii)

$$1 + 1 = r^2 \sin^2 \theta + r^2 \cos^2 \theta \Rightarrow 2 = r^2 (\sin^2 \theta + \cos^2 \theta) \Rightarrow r^2 = 2 \quad \therefore r = \sqrt{2}$$

Dividing equation (i) by (ii), $\frac{r \sin \theta}{r \cos \theta} = \frac{1}{1} \Rightarrow \tan \theta = 1 \therefore \theta = \frac{\pi}{4}$

$$I = \int \frac{1}{r \sin \theta \cos x - r \cos \theta \sin x} dx \Rightarrow I = \int \frac{1}{r \sin(\theta - x)} dx \Rightarrow I = \frac{1}{r} \int \operatorname{cosec}(\theta - x) dx$$

$$\Rightarrow I = -\frac{1}{r} \log \tan \left(\frac{\theta - x}{2} \right) + c \Rightarrow I = -\frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{1}{2} \cdot \frac{\pi}{4} - \frac{x}{2} \right) \right| + c$$

$$\therefore I = -\frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{\pi}{8} - \frac{x}{2} \right) \right| + c$$

31. $I = \int (2 \tan x - 3 \cot x)^2 dx$

Sol. Let $I = \int (2 \tan x - 3 \cot x)^2 dx \Rightarrow I = \int (4 \tan^2 x - 2 \cdot 2 \tan x \cdot 3 \cot x + 9 \cot^2 x) dx$

$$\Rightarrow I = 4 \int (\sec^2 x - 1) dx - 12 \int dx + 9 \int (\operatorname{cosec}^2 x - 1) dx$$

$$\Rightarrow I = 4(\tan x - x) - 12x + 9(-\cot x - x) + c \therefore I = 4 \tan x - 9 \cot x - 25x + c$$

32. $\int \sin x \cdot \sin 2x \cdot \sin 3x dx$

Sol. Let $I = \int \sin x \cdot \sin 2x \cdot \sin 3x dx \Rightarrow I = \frac{1}{2} \int 2 \sin x \cdot \sin 2x \cdot \sin 3x dx$

$$\Rightarrow I = \frac{1}{2} \int \sin x \cdot (2 \sin 2x \sin 3x) dx \Rightarrow I = \frac{1}{2} \int \sin x \{ \cos(2x - 3x) - \cos(2x + 3x) \} dx$$

$$\Rightarrow I = \frac{1}{2} \int \sin x \{ \cos x - \cos 5x \} dx \Rightarrow I = \frac{1}{2} \int \sin x \cos x dx - \frac{1}{2} \int \sin x \cos 5x dx$$

$$\Rightarrow I = \frac{1}{2} \cdot \frac{1}{2} \int 2 \sin x \cos x dx - \frac{1}{2} \cdot \frac{1}{2} \int 2 \sin x \cos 5x dx$$

$$\Rightarrow I = \frac{1}{4} \int \sin 2x dx - \frac{1}{4} \int \{ \sin(x + 5x) + \sin(x - 5x) \} dx$$

$$\Rightarrow I = \frac{-1}{4} \left(\frac{\cos 2x}{2} \right) - \left[\frac{1}{4} \int (\sin 6x - \sin 4x) dx \right]$$

$$\Rightarrow I = -\frac{1}{8} \cos 2x + \frac{1}{4} \cdot \frac{\cos 6x}{6} - \frac{1}{4} \cdot \frac{\cos 4x}{4} + c \therefore I = \frac{\cos 6x}{24} - \frac{\cos 2x}{8} - \frac{\cos 4x}{16} + c$$

33. $\int \frac{1 - \cot x}{1 + \cot x} dx$

Sol. Let $I = \int \frac{1 - \cot x}{1 + \cot x} dx \Rightarrow I = \int \frac{1 - \frac{\cos x}{\sin x}}{1 + \frac{\cos x}{\sin x}} dx \Rightarrow I = \int \frac{\sin x - \cos x}{\sin x + \cos x} dx$

Put $\sin x + \cos x = t \Rightarrow \cos x - \sin x = \frac{dt}{dx} \Rightarrow -(\sin x - \cos x) dx = dt \Rightarrow (\sin x - \cos x) dx = -dt$

$$I = -\int \frac{dt}{t} \Rightarrow I = -\log |t| + c \therefore I = -\log |\sin x + \cos x| + c$$

34. $\int \frac{dx}{2 \sin x + \cos x + 3}$

Sol. Let $I = \int \frac{1}{2 \sin x + \cos x + 3} dx$, Put $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$, $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$I = \int \frac{1}{2 \left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right) + \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right) + 3} dx \Rightarrow I = \int \frac{1 + \tan^2 \frac{x}{2}}{4 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2} + 3 \left(1 + \tan^2 \frac{x}{2} \right)} dx$$

$$\Rightarrow I = \int \frac{\sec^2 \frac{x}{2}}{4 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2} + 3 + 3 \tan^2 \frac{x}{2}} dx \Rightarrow I = \int \frac{\sec^2 \frac{x}{2}}{2 \tan^2 \frac{x}{2} + 4 \tan \frac{x}{2} + 4} dx$$

Put $\tan \frac{x}{2} = t \Rightarrow \sec^2 \frac{x}{2} \cdot \frac{1}{2} = \frac{dt}{dx} \Rightarrow \sec^2 \frac{x}{2} dx = 2 dt$

$$I = 2 \int \frac{dt}{2t^2 + 4t + 4} \Rightarrow I = \frac{2}{2} \int \frac{1}{t^2 + 2t + 2} dt \Rightarrow I = \int \frac{1}{(t)^2 + 2t \cdot 1 + (1)^2 + (1)^2 - (1)^2 + 2}$$

$$\Rightarrow I = \int \frac{1}{(t+1)^2 + (1)^2} dt \Rightarrow I = \tan^{-1} \left(\frac{t+1}{1} \right) + c \therefore I = \tan^{-1} \left(\tan \frac{x}{2} + 1 \right) + c$$

EXERCISE 13C (Pg.no.: 678)

Evaluate the following integrals :

1. $\int x e^x dx$

Sol. Let $I = \int x e^x dx \Rightarrow I = x \int e^x dx - \int \left[\frac{d(x)}{dx} \int e^x dx \right] dx \Rightarrow I = x e^x - \int 1 \cdot e^x dx$
 $\Rightarrow I = x e^x - \int e^x dx \Rightarrow I = x e^x - e^x + c \quad \therefore I = e^x (x-1) + c$

2. $\int x \cos x dx$

Sol. Let $I = \int x \cos x dx \Rightarrow I = x \int \cos x dx - \int \left[\frac{d(x)}{dx} \int \cos x dx \right] dx$
 $\Rightarrow I = x \sin x - \int 1 \cdot \sin x dx \Rightarrow I = x \sin x - \int \sin x dx \quad \therefore I = x \sin x + \cos x + c$

3. $\int x e^{2x} dx$

Sol. Let $I = \int x e^{2x} dx \Rightarrow I = x \int e^{2x} dx - \int \left[\frac{d(x)}{dx} \int e^{2x} dx \right] dx \Rightarrow I = x \cdot \frac{e^{2x}}{2} - \int 1 \cdot \frac{e^{2x}}{2} dx$
 $\Rightarrow I = \frac{x}{2} e^{2x} - \frac{1}{2} \int e^{2x} dx \Rightarrow I = \frac{x}{2} e^{2x} - \frac{1}{2} \cdot \frac{e^{2x}}{2} + c \quad \therefore I = \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} + c$

4. $\int x \sin 3x dx$

Sol. Let $I = \int x \sin 3x dx \Rightarrow I = x \int \sin 3x dx - \int \left[\frac{d(x)}{dx} \int \sin 3x dx \right] dx$
 $\Rightarrow I = x \left(-\frac{\cos 3x}{3} \right) - \int 1 \cdot \left(-\frac{\cos 3x}{3} \right) dx \Rightarrow I = -\frac{x}{3} \cos 3x + \frac{1}{3} \int \cos 3x dx$
 $\Rightarrow I = -\frac{x}{3} \cos 3x + \frac{1}{3} \cdot \frac{\sin 3x}{3} + c \quad \therefore I = -\frac{x \cos 3x}{3} + \frac{\sin 3x}{9} + c$

5. $\int x \cos 2x dx$

Sol. Let $I = \int x \cos 2x dx \Rightarrow I = x \int \cos 2x dx - \int \left[\frac{d(x)}{dx} \int \cos 2x dx \right] dx$
 $\Rightarrow I = \frac{x \sin 2x}{2} - \int 1 \cdot \frac{\sin 2x}{2} dx \Rightarrow I = \frac{x}{2} \sin 2x - \frac{1}{2} \left(-\frac{\cos 2x}{2} \right) + c$
 $\therefore I = \frac{x}{2} \sin 2x + \frac{\cos 2x}{4} + c$

6. $I = \int x \log 2x dx$

Sol. Let $I = \log 2x \int x dx - \int \left[\frac{d(\log 2x)}{dx} \int x dx \right] dx$
 $\Rightarrow I = \log 2x \cdot \frac{x^2}{2} - \int \frac{1}{2x} \cdot 2 \cdot \frac{x^2}{2} dx \Rightarrow I = \frac{x^2}{2} \log 2x - \frac{1}{2} \int x dx$
 $\Rightarrow I = \frac{x^2}{2} \log 2x - \frac{1}{2} \cdot \frac{x^2}{2} + c \quad \therefore I = \frac{x^2}{2} \log 2x - \frac{x^2}{4} + c$

7. $\int x \operatorname{cosec}^2 x \, dx$

Sol. Let $I = \int x \operatorname{cosec}^2 x \, dx \Rightarrow I = x \int \operatorname{cosec}^2 x \, dx - \int \left[\frac{d(x)}{dx} \int \operatorname{cosec}^2 x \, dx \right] dx$

$$\Rightarrow I = x(-\cot x) - \int 1(-\cot x) \, dx \Rightarrow I = -x \cot x + \int \cot x \, dx$$

$$\therefore I = -x \cot x + \log |\sin x| + c$$

8. $\int x^2 \cos x \, dx$

Sol. Let $I = \int x^2 \cos x \, dx \Rightarrow I = x^2 \int \cos x \, dx - \int \left[\frac{d(x^2)}{dx} \int \cos x \, dx \right] dx$

$$\Rightarrow I = x^2 \sin x - \int 2x \sin x \, dx \Rightarrow I = x^2 \sin x - 2 \left[x \int \sin x \, dx - \int \left[\frac{d(x)}{dx} \int \sin x \, dx \right] dx \right]$$

$$\Rightarrow I = x^2 \sin x - 2 \left[-x \cos x - \int 1(-\cos x) \, dx \right] \Rightarrow I = x^2 \sin x - 2 \left[-x \cos x + \int \cos x \, dx \right]$$

$$\Rightarrow I = x^2 \sin x - 2 \left[-x \cos x + \sin x \right] + c \quad \therefore I = x^2 \sin x + 2x \cos x - 2 \sin x + c$$

9. $\int x \sin^2 x \, dx$

Sol. Let $I = \int x \sin^2 x \, dx \Rightarrow I = \int x \left(\frac{1 - \cos 2x}{2} \right) dx = \frac{1}{2} \int (x - x \cos 2x) \, dx$

$$\Rightarrow I = \frac{1}{2} \int x \, dx - \frac{1}{2} \int x \cos 2x \, dx = \frac{1}{2} \times \frac{x^2}{2} - \frac{1}{2} \left[x \int \cos 2x \, dx - \int \left(\frac{dx}{dx} \int \cos 2x \, dx \right) dx \right]$$

$$\Rightarrow I = \frac{x^2}{4} - \frac{1}{2} \left[x \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \, dx \right] = \frac{x^2}{4} - \frac{1}{2} \left[\frac{x}{2} \sin 2x + \frac{\cos 2x}{4} \right] + c = \frac{x^2}{4} - \frac{x}{4} \sin 2x - \frac{1}{8} \cos 2x + c$$

10. $\int x \tan^2 x \, dx$

Sol. Let $I = \int x \tan^2 x \, dx \Rightarrow I = x \int \tan^2 x \, dx - \int \left[\frac{d(x)}{dx} \int \tan^2 x \, dx \right] dx$

$$\Rightarrow I = x \int (\sec^2 x - 1) \, dx - \int \left[1 \cdot \int (\sec^2 x - 1) \, dx \right] dx$$

$$\Rightarrow I = x(\tan x - x) - \int (\tan x - x) \, dx \Rightarrow I = x \tan x - x^2 - \int \tan x \, dx + \int x \, dx$$

$$\Rightarrow I = x \tan x - x^2 + \log |\cos x| + \frac{x^2}{2} + c \quad \therefore I = x \tan x - \frac{x^2}{2} + \log |\cos x| + c$$

11. $\int x^2 e^x \, dx$

Sol. Let $I = \int x^2 e^x \, dx \Rightarrow I = x^2 \int e^x \, dx - \int \left[\frac{d(x^2)}{dx} \int e^x \, dx \right] dx$

$$\Rightarrow I = x^2 e^x - \int 2x e^x \, dx \Rightarrow I = x^2 e^x - 2 \left[x \int e^x \, dx - \int \left[\frac{d(x)}{dx} \int e^x \, dx \right] dx \right]$$

$$\Rightarrow I = x^2 e^x - 2 \left[x e^x - e^x \right] + c \Rightarrow I = x^2 e^x - 2x e^x + 2e^x + c \quad \therefore I = e^x (x^2 - 2x + 2) + c$$

12. $\int x^2 \cos^3 x \, dx$

Sol. Let $I = \int x^2 \cos x \, dx \Rightarrow I = x^2 \int \cos x \, dx - \int \left[\frac{d(x^2)}{dx} \int \cos x \, dx \right] dx$

$$\Rightarrow I = x^2 \sin x - \int 2x(\sin x) \, dx \Rightarrow I = x^2 \sin x - 2 \left[x \int \sin x \, dx - \int \left[\frac{d(x)}{dx} \int \sin x \, dx \right] dx \right]$$

$$\Rightarrow I = x^2 \sin x - 2 \left[-x \cos x - \int 1 \cdot (-\cos x) \, dx \right] \Rightarrow I = x^2 \sin x - 2(-x \cos x + \sin x) + c$$

$$\therefore I = x^2 \sin x + 2x \cos x - 2 \sin x + c \Rightarrow \frac{2x}{9} \cos 3x + \frac{\sin 3x}{36} (4 + 3x^2)$$

13. $\int x^2 e^{3x} \, dx$

Sol. Let $I = \int x^2 e^{3x} \, dx \Rightarrow I = x^2 \int e^{3x} \, dx - \int \left[\frac{d(x^2)}{dx} \int e^{3x} \, dx \right] dx \Rightarrow I = x^2 \cdot \frac{e^{3x}}{3} - \int \left(2x \cdot \frac{e^{3x}}{3} \right) dx$

$$\Rightarrow I = \frac{x^2}{3} e^{3x} - \frac{2}{3} \int x e^{3x} \, dx \Rightarrow I = \frac{x^2}{3} e^{3x} - \frac{2}{3} \left[x \int e^{3x} \, dx - \int \left[\frac{d(x)}{dx} \int e^{3x} \, dx \right] dx \right]$$

$$\Rightarrow I = \frac{x^2}{3} e^{3x} - \frac{2}{3} \left[x \cdot \frac{e^{3x}}{3} - \int 1 \cdot \frac{e^{3x}}{3} \, dx \right] \Rightarrow I = \frac{x^2}{3} e^{3x} - \frac{2}{3} \left[\frac{x e^{3x}}{3} - \frac{1}{3} \cdot \frac{e^{3x}}{3} \right] + c$$

$$\Rightarrow I = \frac{x^2}{3} e^{3x} - \frac{2x}{9} e^{3x} + \frac{2}{27} e^{3x} + c \therefore I = e^{3x} \left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27} \right) + c$$

14. $\int x^2 \sin^2 x \, dx$

Sol. Let $I = \int x^2 \sin^2 x \, dx \Rightarrow I = x^2 \int \sin^2 x \, dx - \int \left[\frac{d(x^2)}{dx} \int \sin^2 x \, dx \right] dx$

$$\Rightarrow I = x^2 \int \frac{1 - \cos 2x}{2} \, dx - \int \left[2x \cdot \int \frac{1 - \cos 2x}{2} \, dx \right] dx$$

$$\Rightarrow I = \frac{x^2}{2} \left(x - \frac{\sin 2x}{2} \right) - \int x \left(x - \frac{\sin 2x}{2} \right) dx \Rightarrow I = \frac{x^3}{2} - \frac{x^2 \sin 2x}{4} - \int x^2 \, dx + \frac{1}{2} \int x \sin 2x \, dx$$

$$\Rightarrow I = \frac{x^3}{2} - \frac{x^2 \sin 2x}{4} - \frac{x^3}{3} + \frac{1}{2} \left[x \int \sin 2x \, dx - \int \left[\frac{d(x)}{dx} \int \sin 2x \, dx \right] dx \right]$$

$$\Rightarrow I = \frac{x^3}{2} - \frac{x^2 \sin 2x}{4} - \frac{x^3}{3} + \frac{1}{2} \left[-\frac{x \cos 2x}{2} - \int 1 \cdot \left(-\frac{\cos 2x}{2} \right) dx \right]$$

$$\Rightarrow I = \frac{x^3}{6} - \frac{x^2 \sin 2x}{4} + \frac{1}{2} \left[-\frac{x \cos 2x}{2} + \frac{1}{2} \cdot \frac{\sin 2x}{2} \right] + c$$

$$\therefore I = \frac{1}{6} x^3 - \frac{1}{4} x^2 \sin 2x - \frac{1}{4} x \cos 2x + \frac{1}{8} \sin 2x + c$$

15. $\int x^3 \log 2x \, dx$

Sol. Let $I = \int x^3 \log 2x \, dx \Rightarrow I = \log 2x \int x^3 \, dx - \int \left[\frac{d(\log 2x)}{dx} \int x^3 \, dx \right] dx$

$$\Rightarrow I = \log 2x \cdot \frac{x^4}{4} - \int \frac{1}{2x} \cdot 2 \cdot \frac{x^4}{4} dx \Rightarrow I = \frac{x^4}{4} \log 2x - \frac{1}{4} \int x^3 dx$$

$$\Rightarrow I = \frac{x^4}{4} \log 2x - \frac{1}{4} \cdot \frac{x^4}{4} + c \quad \therefore I = \frac{x^4}{4} \log 2x - \frac{x^4}{16} + c$$

16. $\int (1+x) \log x \, dx$

Sol. Let $I = \int (1+x) \log x \, dx \Rightarrow I = \log x \int (1+x) dx - \int \left[\frac{d(\log x)}{dx} \int (1+x) dx \right] dx$

$$\Rightarrow I = \log x \left(x + \frac{x^2}{2} \right) - \int \frac{1}{x} \left(x + \frac{x^2}{2} \right) dx \Rightarrow I = \left(\frac{x^2}{2} + x \right) \log x - \int dx - \frac{1}{2} \int x dx$$

$$\Rightarrow I = \left(\frac{x^2}{2} + x \right) \log x - x - \frac{1}{2} \cdot \frac{x^2}{2} + c \quad \therefore I = \left(\frac{x^2}{2} + x \right) \log x - \left(x + \frac{x^2}{4} \right) + c$$

17. $\int \frac{\log x}{x^n} dx$

Sol. Let $I = \int \frac{\log x}{x^n} dx \Rightarrow I = \int x^{-n} \log x \, dx \Rightarrow I = \log x \int x^{-n} dx - \int \left[\left(\frac{d(\log x)}{dx} \right) \left(\int x^{-n} dx \right) \right] dx$

$$\Rightarrow I = \log x \cdot \frac{x^{-n+1}}{-n+1} - \int \frac{1}{x} \cdot \frac{x^{-n+1}}{(-n+1)} dx \Rightarrow I = \frac{x^{-n+1} \cdot \log x}{-n+1} - \int \frac{x^{-n} \cdot x}{x \cdot (-n+1)} dx$$

$$\Rightarrow I = \frac{x^{-n+1} \log x}{(-n+1)} - \frac{1}{(-n+1)} \cdot \frac{x^{-n+1}}{(-n+1)} + c \quad \therefore I = \frac{x^{1-n} \cdot \log x}{(1-n)} - \frac{x^{1-n}}{(1-n)^2} + c$$

18. $\int 2x^3 e^{x^2} dx$

Sol. Let $I = \int 2x^3 e^{x^2} dx \Rightarrow I = \int 2x \cdot x^2 e^{x^2} dx$, Put $x^2 = t \Rightarrow 2x = \frac{dt}{dx} \Rightarrow 2x \, dx = dt$

$$I = \int t \cdot e^t \, dt \Rightarrow I = \int t \cdot e^t \, dt \Rightarrow I = t \int e^t \, dt - \int \left[\frac{d(t)}{dt} \int e^t \, dt \right] dt$$

$$\Rightarrow I = t \cdot e^t - \int 1 \cdot e^t \, dt \Rightarrow I = t \cdot e^t - e^t + c \Rightarrow I = e^t (t-1) + c \quad \therefore I = e^{x^2} (x^2 - 1) + c$$

19. $\int x \sin^3 x \, dx$

Sol. $I = \int x \sin^3 x \, dx \Rightarrow I = x \int \sin^3 x \, dx - \int \left[\frac{d(x)}{dx} \int \sin^3 x \, dx \right] dx$

$$\Rightarrow I = x \int \left(\frac{3 \sin x - \sin 3x}{4} \right) dx - \int \left[1 \cdot \int \left(\frac{3 \sin x - \sin 3x}{4} \right) dx \right] dx$$

$$\Rightarrow I = \frac{x}{4} \int (3 \sin x - \sin 3x) dx - \frac{1}{4} \int \left(-3 \cos x + \frac{\cos 3x}{3} \right) dx$$

$$\Rightarrow I = \frac{x}{4} \left(-3 \cos x + \frac{\cos 3x}{3} \right) - \frac{1}{4} \left(-3 \sin x + \frac{\sin 3x}{3 \cdot 3} \right) + c$$

$$\therefore I = -\frac{3x}{4} \cos x + \frac{x \cos 3x}{12} + \frac{3}{4} \sin x - \frac{\sin 3x}{36} + c$$

20. $\int x \cos^3 x \, dx$

Sol. Let $I = \int x \cos^3 x \, dx \Rightarrow I = x \int \cos^3 x \, dx - \int \left[\frac{d(x)}{dx} \int \cos^3 x \, dx \right] dx$

$$\Rightarrow I = x \int \left(\frac{\cos 3x + 3 \cos x}{4} \right) dx - \int \left[1 \cdot \int \left(\frac{\cos 3x + 3 \cos x}{4} \right) dx \right] dx$$

$$\Rightarrow I = \frac{x}{4} \int (\cos 3x + 3 \cos x) dx - \frac{1}{4} \int \left(\frac{\sin 3x}{3} + 3 \sin x \right) dx$$

$$\Rightarrow I = \frac{x}{4} \left(\frac{\sin 3x}{3} + 3 \sin x \right) - \frac{1}{4} \left(-\frac{\cos 3x}{9} - 3 \cos x \right) + c \quad \therefore I = \frac{x \sin 3x}{12} + \frac{3x}{4} \sin x + \frac{\cos 3x}{36} + \frac{3 \cos x}{4} + c$$

21. $\int x^3 \cos(x^2) \, dx$

Sol. Let $I = \int x^2 \cdot x \cos(x^2) \, dx$, Put $x^2 = t \Rightarrow 2x = \frac{dt}{dx} \Rightarrow x \, dx = \frac{dt}{2}$

$$I = \int t \cdot \cos(t) \cdot \frac{dt}{2} \Rightarrow I = \frac{1}{2} \int t \cos(t) \, dt \Rightarrow I = \frac{1}{2} \left[t \int \cos t \, dt - \int \left[\frac{d(t)}{dt} \int \cos t \, dt \right] dt \right]$$

$$\Rightarrow I = \frac{1}{2} \left[t \sin t - \int 1 \cdot \sin t \, dt \right] \Rightarrow I = \frac{1}{2} [t \sin t + \cos t] + c \quad \therefore I = \frac{x^2 \sin(x^2)}{2} + \frac{\cos(x^2)}{2} + c$$

22. $\int \sin x \log(\cos x) \, dx$

Sol. Let $I = \int \sin x \log(\cos x) \, dx$, Put $\cos x = t \Rightarrow -\sin x = \frac{dt}{dx} \Rightarrow \sin x \, dx = -dt$

$$I = \int \log(t) (-dt) \Rightarrow I = -\int \log(t) \, dt \Rightarrow I = -\left[\log t \int dt - \int \left(\frac{d(\log t)}{dt} \int dt \right) dt \right]$$

$$\Rightarrow I = -\left[\log t \cdot t - \int \frac{1}{t} \cdot t \, dt \right] \Rightarrow I = -\left[t \log t - \int dt \right] \Rightarrow I = -[t \log t - t] + c$$

$$\therefore I = -\cos x \log(\cos x) + \cos x + c$$

23. $\int x \sin x \cos x \, dx$

Sol. Let $I = \int x \sin x \cos x \, dx \Rightarrow I = \frac{1}{2} \int x \cdot 2 \sin x \cos x \, dx \Rightarrow I = \frac{1}{2} \int x \sin 2x \, dx$

$$\Rightarrow I = \frac{1}{2} \left[x \int \sin 2x \, dx - \int \left(\frac{d(x)}{dx} \int \sin 2x \, dx \right) dx \right]$$

$$\Rightarrow I = \frac{1}{2} \left[-\frac{x \cos 2x}{2} - \int 1 \cdot \left(-\frac{\cos 2x}{2} \right) dx \right] \Rightarrow I = \frac{1}{2} \left[-\frac{x \cos 2x}{2} + \int \frac{\cos 2x}{2} dx \right]$$

$$\Rightarrow I = \frac{1}{2} \left[-\frac{x \cos 2x}{2} + \frac{1}{2} \cdot \frac{\sin 2x}{2} \right] + c \quad \therefore I = -\frac{x \cos 2x}{4} + \frac{\sin 2x}{8} + c$$

24. $\int \cos(\sqrt{x}) \, dx$

Sol. Let $I = \int \cos(\sqrt{x}) \, dx$, Put $(\sqrt{x}) = t \Rightarrow \frac{1}{2\sqrt{x}} = \frac{dt}{dx} \Rightarrow dx = 2\sqrt{x} \, dt \Rightarrow dx = 2t \, dt$

$$I = \int \cos(t) \cdot 2t \, dt \Rightarrow I = 2 \int t \cdot \cos t \, dt \Rightarrow I = 2 \left[t \int \cos t \, dt - \int \left(\frac{d(t)}{dt} \int \cos t \, dt \right) dt \right]$$

$$\Rightarrow I = 2 \left[t \sin t - \int 1 \cdot \sin t \, dt \right] \Rightarrow I = 2 \left[t \sin t - \int \sin t \, dt \right] \Rightarrow I = 2 \left[t \sin t + \cos t \right] + c$$

$$\therefore I = 2 \left[\sqrt{x} \cdot \sin(\sqrt{x}) + \cos(\sqrt{x}) \right] + c$$

25. $\int \operatorname{cosec}^3 x \, dx$

Sol. Let $I = \int \operatorname{cosec}^3 x \, dx \Rightarrow I = \int \operatorname{cosec}^2 x \cdot \operatorname{cosec} x \, dx$

$$\Rightarrow I = \operatorname{cosec} x \int \operatorname{cosec}^2 x \, dx - \int \left[\frac{d(\operatorname{cosec} x)}{dx} \int \operatorname{cosec}^2 x \, dx \right] dx$$

$$\Rightarrow I = \operatorname{cosec} x (-\cot x) - \int (-\operatorname{cosec} x \cot x)(-\cot x) \, dx$$

$$\Rightarrow I = -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x \cot^2 x \, dx \Rightarrow I = -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x (\operatorname{cosec}^2 x - 1) \, dx$$

$$\Rightarrow I = -\operatorname{cosec} x \cot x - \int \operatorname{cosec}^3 x \, dx + \int \operatorname{cosec} x \, dx \Rightarrow I = -\operatorname{cosec} x \cot x - I + \log \left| \tan \frac{x}{2} \right| + c$$

$$\Rightarrow 2I = -\operatorname{cosec} x \cot x + \log \left| \tan \frac{x}{2} \right| + c \therefore I = -\frac{1}{2} \operatorname{cosec} x \cot x + \frac{1}{2} \log \left| \tan \frac{x}{2} \right| + c$$

26. $\int x \sin^3 x \cos x \, dx$

Sol. Let $I = \int x \sin^3 x \cos x \, dx \Rightarrow I = \frac{x \sin^4 x}{4} - \frac{1}{4} \int \sin^4 x \, dx$

$$\Rightarrow I = \frac{x}{4} \sin^4 x - \frac{1}{4} \int \sin^2 x \cdot \sin^2 x \, dx \Rightarrow I = \frac{x}{4} \sin^4 x - \frac{1}{4} \int \left(\frac{1 - \cos 2x}{2} \right) \cdot \left(\frac{1 - \cos 2x}{2} \right) dx$$

$$\Rightarrow I = \frac{x}{4} \sin^4 x - \frac{1}{16} \int (1 - 2 \cos 2x + \cos^2 2x) \, dx$$

$$\Rightarrow I = \frac{x}{4} \sin^4 x - \frac{1}{16} \int dx + \frac{1}{16} \cdot 2 \int \cos 2x \, dx - \frac{1}{16} \int \cos^2 2x \, dx$$

$$\Rightarrow I = \frac{x}{4} \sin^4 x - \frac{1}{16} x + \frac{1}{8} \cdot \frac{\sin 2x}{2} - \frac{1}{16} \int \left(\frac{1 + \cos 4x}{2} \right) dx$$

$$\Rightarrow I = \frac{x}{4} \sin^4 x - \frac{x}{16} + \frac{\sin 2x}{16} - \frac{1}{32} \left(x + \frac{\sin 4x}{4} \right) + c \therefore I = \frac{x}{4} \sin^4 x - \frac{3x}{32} + \frac{\sin 2x}{16} - \frac{\sin 4x}{128} + c$$

27. $\int \sin x \cdot \log(\cos x) \, dx$

Sol. Let $I = \int \sin x \log(\cos x) \, dx$, Put $\cos x = t \Rightarrow -\sin x = \frac{dt}{dx} \Rightarrow \sin x \, dx = -dt$

$$I = \int \log(t)(-dt) \Rightarrow I = -\int \log(t) \, dt \Rightarrow I = -\left[\log t \int dt - \int \left(\frac{d(\log t)}{dt} \int dt \right) dt \right]$$

$$\Rightarrow I = -\left[\log t \cdot t - \int \frac{1}{t} \cdot t \, dt \right] \Rightarrow I = -\left[t \log t - \int dt \right] \Rightarrow I = -[t \log t - t] + c,$$

$$\Rightarrow I = -t \log t + t + c, \Rightarrow I = -\cos x \log \cos x + \cos x + c, \text{ where } t = \cos x$$

28. $\int \frac{\log(\log x)}{x} dx$

Sol. Let $I = \int \frac{\log(\log x)}{x} dx$, Put $\log x = t \Rightarrow \frac{1}{x} = \frac{dt}{dx} \Rightarrow \frac{1}{x} dx = dt$

$$I = \int \log(t) dt \Rightarrow I = \log t \int dt - \int \left[\frac{d(\log t)}{dt} \int dt \right] dt \Rightarrow I = \log t \cdot t - \int \frac{1}{t} dt$$

$$\Rightarrow I = t \log t - \int \frac{1}{t} dt \Rightarrow I = t \log t - t + c \Rightarrow I = \log x \log(\log x) - \log(x) + c$$

$$\therefore I = \log x [\log(\log x) - 1] + c$$

29. $\int \log(2+x^2) dx$

Sol. Let $I = \int \log(2+x^2) dx \Rightarrow I = \log(2+x^2) \int dx - \int \left[\frac{d\{\log(2+x^2)\}}{dx} \int dx \right] dx$

$$\Rightarrow I = \log(2+x^2) \cdot x - \int \frac{1}{2+x^2} (2x) x dx \Rightarrow I = x \log(2+x^2) - 2 \int \frac{x^2}{x^2+2} dx$$

$$\Rightarrow I = x \log(2+x^2) - 2 \int \frac{(x^2+2)-2}{x^2+2} dx \Rightarrow I = x \log(2+x^2) - 2 \int \frac{x^2+2}{x^2+2} dx + 4 \int \frac{1}{x^2+2} dx$$

$$\Rightarrow I = x \log(2+x^2) - 2 \int dx + 4 \int \frac{1}{(x)^2 + (\sqrt{2})^2} dx$$

$$\Rightarrow I = x \log(2+x^2) - 2x + 4 \cdot \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + c \therefore I = x \log(2+x^2) - 2x + 2\sqrt{2} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + c$$

30. $\int \frac{x}{1+\sin x} dx$

Sol. Let $I = \int \frac{x}{1+\sin x} dx \Rightarrow I = \int \frac{x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx \Rightarrow I = \int \frac{x(1-\sin x)}{(1-\sin^2 x)} dx$

$$\Rightarrow I = \int \frac{x(1-\sin x)}{\cos^2 x} dx \Rightarrow I = \int x(\sec^2 x - \tan x \cdot \sec x) dx$$

$$\Rightarrow I = x \int (\sec^2 x - \tan x \cdot \sec x) dx - \int \left[\frac{d(x)}{dx} \int (\sec^2 x - \sec x \cdot \tan x) dx \right]$$

$$\Rightarrow I = x(\tan x - \sec x) - \int 1 \cdot [(\tan x - \sec x) dx]$$

$$\Rightarrow I = x(\tan x - \sec x) - \int \tan x dx + \int \sec x dx$$

$$\therefore I = x(\tan x - \sec x) + \log |\cos x| + \log |\sec x + \tan x| + c$$

31. $I = \int \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx$

Sol. Let $I = \int \left(\frac{1}{(\log x)} - \frac{1}{(\log x)^2} \right) dx$. Put $t = \log x$, $e^t = e^{\log x} = x \Rightarrow \frac{dx}{dt} = e^t \Rightarrow dx = e^t dt$

$$I = \int \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt = e^t \cdot \frac{1}{t} - \frac{x}{\log x} + c \quad \therefore \int e^x (f(x) + f'(x)) dx = e^x f(x)$$

32. $I = \int e^{-x} \cos 2x \cos 4x dx$

Sol. Let $I = \frac{1}{2} \int e^{-x} 2 \cos 2x \cos 4x dx \Rightarrow I = \frac{1}{2} \int e^{-x} \{ \cos(2x+4x) + \cos(2x-4x) \} dx$

$$\Rightarrow I = \frac{1}{2} \int e^{-x} (\cos 6x + \cos 2x) dx \Rightarrow I = \frac{1}{2} \int e^{-x} \cos 6x dx + \frac{1}{2} \int e^{-x} \cos 2x dx$$

$$\Rightarrow I = \frac{1}{2} I_1 + \frac{1}{2} I_2 \quad \dots (1)$$

$$I_1 = \int e^{-x} \cos 6x dx \Rightarrow I_1 = e^{-x} \int \cos 6x dx - \int \left[\frac{d(e^{-x})}{dx} \int \cos 6x dx \right] dx$$

$$\Rightarrow I_1 = e^{-x} \frac{\sin 6x}{6} - \int e^{-x} (-1) \frac{\sin 6x}{6} dx \Rightarrow I_1 = \frac{1}{6} e^{-x} \sin 6x + \frac{1}{6} \int e^{-x} \sin 6x dx$$

$$\Rightarrow I_1 = \frac{1}{6} e^{-x} \sin 6x + \frac{1}{6} \left[e^{-x} \int \sin 6x dx - \int \left[\frac{d(e^{-x})}{dx} \int \sin 6x dx \right] dx \right]$$

$$\Rightarrow I_1 = \frac{1}{6} e^{-x} \sin 6x + \frac{1}{6} \left[e^{-x} \left(-\frac{\cos 6x}{6} \right) - \int (-e^{-x}) \left(-\frac{\cos 6x}{6} \right) dx \right]$$

$$\Rightarrow I_1 = \frac{1}{6} e^{-x} \sin 6x - \frac{e^{-x} \cos 6x}{36} - \frac{1}{36} \int e^{-x} \cos 6x dx$$

$$\Rightarrow I_1 = \frac{1}{6} e^{-x} \sin 6x - \frac{1}{36} e^{-x} \cos 6x - \frac{1}{36} I_1 \Rightarrow \frac{37}{36} I_1 = \frac{1}{6} e^{-x} \sin 6x - \frac{1}{36} e^{-x} \cos 6x + C_1$$

$$\Rightarrow I_1 = \frac{6}{37} e^{-x} \sin 6x - \frac{1}{37} e^{-x} \cos 6x + C_1 \quad \therefore I_1 = \frac{e^{-x}}{37} (6 \sin 6x - \cos 6x) + c_1$$

$$I_2 = \int e^{-x} \cos 2x dx \Rightarrow I_2 = e^{-x} \int \cos 2x dx - \int \left[\frac{d(e^{-x})}{dx} \int \cos 2x dx \right] dx$$

$$\Rightarrow I_2 = e^{-x} \frac{\sin 2x}{2} + \int e^{-x} \frac{\sin 2x}{2} dx \Rightarrow I_2 = \frac{1}{2} e^{-x} \sin 2x + \frac{1}{2} \int e^{-x} \sin 2x dx$$

$$\Rightarrow I_2 = \frac{1}{2} e^{-x} \sin 2x + \frac{1}{2} \left[e^{-x} \int \sin 2x dx - \int \left[\frac{d(e^{-x})}{dx} \int \sin 2x dx \right] dx \right]$$

$$\Rightarrow I_2 = \frac{e^{-x} \sin 2x}{2} + \frac{1}{2} \left[-e^{-x} \frac{\cos 2x}{2} - \int e^{-x} \frac{\cos 2x}{2} dx \right]$$

$$\Rightarrow I_2 = \frac{1}{2} e^{-x} \sin 2x - \frac{1}{4} e^{-x} \cos 2x - \frac{1}{4} \int e^{-x} \cos 2x dx \Rightarrow I_2 = \frac{1}{2} e^{-x} \sin 2x - \frac{1}{4} e^{-x} \cos 2x - \frac{1}{4} I_2$$

$$\Rightarrow I_2 + \frac{1}{4} I_2 = \frac{1}{2} e^{-x} \sin 2x - \frac{1}{4} e^{-x} \cos 2x + c_2 \Rightarrow \frac{5}{4} I_2 = \frac{1}{4} e^{-x} (2 \sin 2x - \cos 2x) + c_2$$

$$\therefore I_2 = \frac{e^{-x}}{5} (2 \sin 2x - \cos 2x) + c_2$$

Putting the value of I_1 & I_2 equation (1), $I = e^{-x} \left[\frac{1}{74} (6 \sin 6x - \cos 6x) + \frac{1}{10} (2 \sin 2x - \cos 2x) \right] + c$

33. $\int e^{\sqrt{x}} dx$

Sol. Let $I = \int e^{\sqrt{x}} dx$, Put $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} = \frac{dt}{dx} \Rightarrow dx = 2\sqrt{x} dt \Rightarrow dx = 2t dt$

$$I = 2 \int t \cdot e^t dt \Rightarrow I = 2 \left[t \int e^t dt - \int \left[\frac{d(t)}{dt} \int e^t dt \right] dt \right] \Rightarrow I = 2 \left[t \cdot e^t - \int 1 \cdot e^t dt \right]$$

$$\Rightarrow I = 2 [t \cdot e^t - e^t] + c \Rightarrow I = e^t \cdot 2(t-1) + c \quad \therefore I = 2 e^{\sqrt{x}} (\sqrt{x} - 1) + c$$

34. $\int e^{\sin x} \sin 2x dx$

Sol. Let $I = \int e^{\sin x} \sin 2x dx$, Put $\sin x = t \Rightarrow \cos x = \frac{dt}{dx} \Rightarrow \cos x dx = dt$

$$I = \int e^{\sin x} 2 \sin x \cos x dx \Rightarrow I = 2 \int e^t t dt$$

$$\Rightarrow I = 2 \left[t \int e^t dt - \int \left[\frac{d(t)}{dt} \int e^t dt \right] dt \right] \Rightarrow I = 2 \left[t \cdot e^t - \int 1 \cdot e^t dt \right]$$

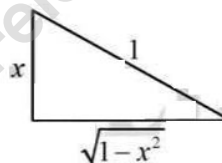
$$\Rightarrow I = 2 [t \cdot e^t - e^t] + c \Rightarrow I = e^t \cdot 2(t-1) + c \quad \therefore I = 2 e^{\sin x} (\sin x - 1) + c$$

35. $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

Sol. Let $I = \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

Put $\sin^{-1} x = t \Rightarrow x = \sin t$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}} = \frac{dt}{dx} \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$$



$$I = \int \sin t \cdot t dt \Rightarrow I = t \int \sin t dt - \int \left[\frac{d(t)}{dt} \int \sin t dt \right] dt$$

$$\Rightarrow I = -t \cos t + \int 1 \cdot \cos t dt \Rightarrow I = -t \cos t + \sin t + c$$

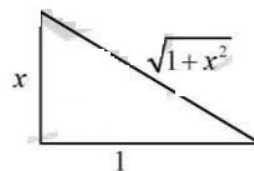
$$\Rightarrow I = -\sin^{-1} x \cos(\sin^{-1} x) + \sin(\sin^{-1} x) + c \quad \therefore I = -\sqrt{1-x^2} \sin^{-1} x + x + c$$

36. $\int \frac{x^2 \tan^{-1} x}{(1+x^2)} dx$

Sol. Let $I = \int \frac{x^2 \tan^{-1} x}{1+x^2} dx$

Put $\tan^{-1} x = t \Rightarrow x = \tan t$

$$\frac{1}{1+x^2} = \frac{dt}{dx} \Rightarrow \frac{1}{1+x^2} dx = dt$$



$$I = \int \tan^2 t \cdot t dt \Rightarrow I = t \int \tan^2 t dt - \int \left[\frac{d(t)}{dt} \int \tan^2 t dt \right] dt$$

$$\Rightarrow I = t \int (\sec^2 t - 1) dt - \int [1 \cdot \int (\sec^2 t - 1) dt] dt$$

$$\Rightarrow I = t(\tan t - t) - \int (\tan t - t) dt \Rightarrow I = t \tan t - t^2 + \log(\cos t) + \frac{t^2}{2} + c$$

$$x + \tan^{-1} x - \frac{1}{x} \log(1+x^2) - \frac{1}{2}(\tan^{-1} x)^2 + c$$

$$\therefore I = x \tan^{-1} x^2 - \frac{1}{2} \log(1+x^2) - \frac{(\tan^{-1} x)^2}{2} + c$$

37. $\int \frac{\log(x+2)}{(x+2)^2} dx$

Sol. Let $I = \int \frac{\log(x+2)}{(x+2)^2} dx$, Put $x+2 = t \Rightarrow dx = dt$

$$I = \int \frac{\log t}{t^2} dt \Rightarrow I = \int \log t \cdot t^{-2} dt \Rightarrow I = \log t \int t^{-2} dt - \int \left[\frac{d(\log t)}{dt} \int t^{-2} dt \right] dt$$

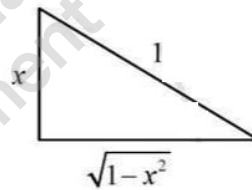
$$I = \log t \cdot \frac{t^{-1}}{(-1)} - \int \frac{1}{t} \cdot \frac{t^{-1}}{(-1)} dt \Rightarrow I = -\frac{\log t}{t} - \frac{1}{t} + c \therefore I = -\frac{\log(x+2) + 1}{(x+2)} + c$$

38. $\int x \sin^{-1} x dx$

Sol. Let $I = \int x \sin^{-1} x dx$

Put $t = \sin^{-1} x$, $\frac{x}{1} = \sin t$

$$\Rightarrow \frac{dt}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dt}{dx} = \frac{1}{\sqrt{1-\sin^2 t}} = \cos t dt = dx$$



$$I = \int \sin t \cdot t \cdot \cos t dt = \frac{1}{2} \int t \cdot 2 \sin t \cdot \cos t dt$$

$$= \frac{1}{2} \int t \cdot \sin 2t dt = \frac{1}{2} \left[t \int \sin 2t dt - \int \left(\frac{dt}{dt} \int \sin 2t dt \right) dt \right]$$

$$= \frac{1}{2} \left[t \left(-\frac{\cos 2t}{2} \right) + \int \frac{\cos 2t}{2} dt \right] = \frac{1}{2} \left[-\frac{t}{2} \cos 2t + \frac{\sin 2t}{4} \right] + c$$

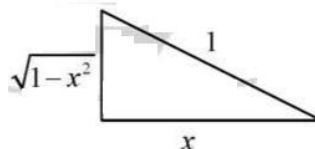
$$= \frac{1}{4} \left[-\left(t \cdot \cos 2t + \frac{2 \sin t \cdot \cos t}{2} \right) \right] + c = \frac{1}{4} \left[-\sin^{-1} x (1 - 2 \sin^2 t) + x \sqrt{1-x^2} \right] + c$$

$$= \frac{1}{4} \left[-\sin^{-1} x (1 - 2x^2) + x \sqrt{1-x^2} \right] + c$$

39. $\int x \cos^{-1} x dx$

Sol. Let $I = \int x \cos^{-1} x dx$

Put $t = \cos^{-1} x \Rightarrow \frac{x}{1} = \cos t$



$$\Rightarrow dt = \frac{-1}{\sqrt{1-x^2}} dx \Rightarrow dt = \frac{-1}{\sqrt{1-\cos^2 t}} dx \Rightarrow -\sin t dt = dx$$

$$I = \int \cos t \cdot t \cdot (-\sin t dt) = -\frac{1}{2} \int t \cdot 2 \sin t \cdot \cos t dt = -\frac{1}{2} \int t \sin 2t dt$$

$$\begin{aligned}
 &= \frac{-1}{2} \left[t \int \sin 2t \cdot dt - \left(\frac{dt}{dt} \int \sin 2t \, dt \right) dt \right] = \frac{-1}{2} \left[t \cdot \left(\frac{-\cos 2t}{2} \right) + \int \frac{\cos 2t}{2} dt \right] \\
 &= \frac{1}{4} \left[-t \cos 2t + \frac{\sin 2t}{2} \right] + c = \frac{-1}{4} \left[-\cos^{-1} x (2 \cos^2 t - 1) + \frac{2 \sin t \cdot \cos t}{2} \right] + c \\
 &= \frac{-1}{4} \left[-\cos^{-1} x (2x^2 - 1) + \sqrt{1-x^2} \cdot x \right] + c = \frac{-1}{4} \left[-(2x^2 - 1) \cos^{-1} x + x \sqrt{1-x^2} \right] + c
 \end{aligned}$$

40. $\int \cot^{-1} x \, dx$

Sol. Let $I = \int \cot^{-1} x \, dx \Rightarrow I = \cot^{-1} x \int dx - \int \left[\frac{d(\cot^{-1} x)}{dx} \int dx \right] dx$

$$\Rightarrow I = \cot^{-1} x \cdot x + \int \frac{1}{1+x^2} \cdot x \, dx \Rightarrow I = x \cot^{-1} x + \int \frac{x}{1+x^2} dx$$

Put $1+x^2 = t \Rightarrow 2x = \frac{dt}{dx} \Rightarrow x \, dx = \frac{dt}{2}$

$$I = x \cot^{-1} x + \frac{1}{2} \int \frac{dt}{t} \therefore I = x \cot^{-1} x + \frac{1}{2} \log(1+x^2) + c$$

41. $\int x \cot^{-1} x \, dx$

Sol. Let $I = \int x \cot^{-1}(x) \, dx \Rightarrow I = \cot^{-1} x \int x \, dx - \int \left[\frac{d(\cot^{-1} x)}{dx} \int x \, dx \right] dx$

$$\Rightarrow I = \cot^{-1} x \cdot \frac{x^2}{2} + \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} dx \Rightarrow I = \frac{x^2}{2} \cot^{-1} x + \frac{1}{2} \int \frac{x^2}{x^2+1} dx$$

$$\Rightarrow I = \frac{x^2}{2} \cot^{-1} x + \frac{1}{2} \int \frac{(x^2+1)-1}{x^2+1} dx \Rightarrow I = \frac{x^2}{2} \cot^{-1} x + \frac{1}{2} \int dx - \frac{1}{2} \int \frac{1}{1+x^2} dx$$

$$\therefore I = \frac{x^2}{2} \cot^{-1} x + \frac{1}{2} \cdot x - \frac{1}{2} \tan^{-1} x + c$$

42. $\int x^2 \cot^{-1} x \, dx$

Sol. Let $I = \int x^2 \cot^{-1} x \, dx \Rightarrow I = \cot^{-1} x \int x^2 \, dx - \int \left[\frac{d(\cot^{-1} x)}{dx} \int x^2 \, dx \right] dx$

$$\Rightarrow I = \cot^{-1} x \cdot \frac{x^3}{3} + \int \frac{1}{1+x^2} \cdot \frac{x^3}{3} dx \Rightarrow I = \frac{x^3}{3} \cot^{-1} x + \frac{1}{3} \int \frac{x^3}{x^2+1} dx$$

$$\Rightarrow I = \frac{x^3}{3} \cot^{-1} x + \frac{1}{3} \int \left(x - \frac{x}{x^2+1} \right) dx \Rightarrow I = \frac{x^3}{3} \cot^{-1} x + \frac{1}{3} \int x \, dx - \frac{1}{3} \int \frac{x}{x^2+1} dx$$

Put $x^2+1 = t \Rightarrow 2x = \frac{dt}{dx} \Rightarrow x \, dx = \frac{dt}{2}$

$$I = \frac{x^3}{3} \cot^{-1} x + \frac{1}{3} \cdot \frac{x^2}{2} - \frac{1}{3} \cdot \frac{1}{2} \int \frac{1}{t} dt \Rightarrow I = \frac{x^3}{3} \cot^{-1} x + \frac{x^2}{6} - \frac{1}{6} \log|t| + c$$

$$\therefore I = \frac{x^3}{3} \cot^{-1} x + \frac{x^2}{6} - \frac{1}{6} \log|x^2+1| + c$$

43. $\int \sin^{-1}(\sqrt{x}) dx$

Sol. Let $I = \int \sin^{-1}(\sqrt{x}) dx$

Put $t = \sin^{-1} \sqrt{x} \Rightarrow \sqrt{x} = \sin t \Rightarrow x = \sin^2 t$

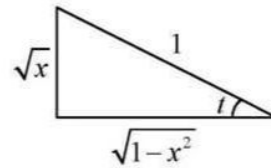
$\Rightarrow \frac{dt}{dx} = \frac{d \sin^{-1} \sqrt{x}}{d \sqrt{x}} \times \frac{d \sqrt{x}}{dx} \Rightarrow \frac{dt}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$

$\Rightarrow \frac{dt}{dx} = \frac{1}{\sqrt{1-\sin^2 t}} \cdot \frac{1}{2 \sin t} \Rightarrow \frac{dt}{dx} = \frac{1}{2 \sin t \cos t} \Rightarrow \sin 2t dt = dx$

$\Rightarrow \int t \cdot \sin 2t dt = t \int \sin 2t dt - \int \left(\frac{dt}{dt} \right) \sin 2t dt = -t \frac{\cos 2t}{2} + \int \frac{\cos 2t}{2} dt = \frac{-t}{2} \cos 2t + \frac{\sin 2t}{4} + c$

$= \frac{-1}{2} \sin^{-1} \sqrt{x} (1 - 2 \sin^2 t) + \frac{2 \sin t \cdot \cos t}{4} + c = \frac{-1}{2} (1 - 2x) \sin^{-1} \sqrt{x} + \frac{1}{2} \sqrt{x} \sqrt{1-x} + c$

$= \frac{-1}{2} (1 - 2x) \sin^{-1} \sqrt{x} + \frac{1}{2} \sqrt{x(1-x)} + c$

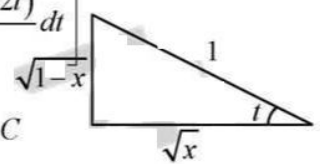


44. $\int \cos^{-1}(\sqrt{x}) dx$

Sol. $I = -\int t \sin 2t dt = -\left[t \int \sin 2t dt - \int \left(\frac{dt}{dt} \right) \sin 2t dt \right] = -\left[\frac{t(-\cos 2t)}{2} - \int \frac{(-\cos 2t)}{2} dt \right]$

$= -\left[\frac{-t \cos 2t}{2} + \frac{\sin 2t}{4} \right] + C = \frac{t \cos 2t}{2} - \frac{\sin 2t}{4} + C = \frac{t(2 \cos^2 t - 1)}{2} - \frac{\sin t \cos t}{2} + C$

$= \frac{\cos^{-1} \sqrt{x} (2(\sqrt{x})^2 - 1)}{2} - \frac{\sqrt{1-x} \times \sqrt{x}}{2} + C = \frac{1}{2} (2x - 1) \cos^{-1} \sqrt{x} - \frac{1}{2} \sqrt{x(1-x)} + C$



45. $\int \cos^{-1}(4x^3 - 3x) dx$

Sol. Let $I = \int \cos^{-1}(4x^3 - 3x) dx$

Put $x = \cos \theta \therefore \theta = \cos^{-1} x \Rightarrow dx = -\sin \theta d\theta$

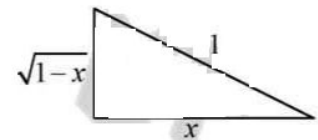
$I = \int \cos^{-1}(4 \cos^3 \theta - 3 \cos \theta) (-\sin \theta) d\theta$

$\Rightarrow I = \int \cos^{-1}(\cos 3\theta) \sin \theta d\theta \Rightarrow I = -\int 3\theta \sin \theta d\theta$

$\Rightarrow I = -3 \left[\theta \int \sin \theta d\theta - \int \left[\frac{d(\theta)}{d\theta} \right] \sin \theta d\theta \right]$

$\Rightarrow I = -3 \left[-\theta \cos \theta + \int 1 \cdot \cos \theta d\theta \right] \Rightarrow I = 3\theta \cos \theta - 3 \sin \theta + c$

$\Rightarrow I = 3(\cos^{-1} x)x - 3\sqrt{1-x^2} + c \therefore I = 3x \cos^{-1} x - 3\sqrt{1-x^2} + c$



46. $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx$

Sol. Let $I = \int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx$

Put $x = \tan \theta \therefore \theta = \tan^{-1}(x) \Rightarrow x = \sec^2 \theta \cdot d\theta$

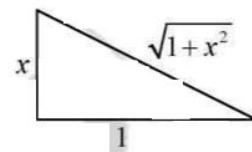


$$\begin{aligned}
 I &= \int \cos^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right) \cdot \sec^2 \theta \, d\theta \Rightarrow I = \int \cos^{-1} (\cos 2\theta) \cdot \sec^2 \theta \, d\theta \\
 &\Rightarrow I = \int 2\theta \sec^2 \theta \, d\theta \Rightarrow I = 2 \int \theta \sec^2 \theta \, d\theta \\
 &\Rightarrow I = 2 \left[\theta \int \sec^2 \theta \, d\theta - \int \left[\frac{d\theta}{d\theta} \int \sec^2 \theta \, d\theta \right] d\theta \right] \Rightarrow I = 2 \left[\theta \tan \theta - \int 1 \cdot \tan \theta \, d\theta \right] \\
 &\Rightarrow I = 2 \left[\theta \tan \theta + \log(\cos \theta) \right] + c \Rightarrow I = 2\theta \tan \theta + 2 \log(\cos \theta) + c \\
 &\Rightarrow I = 2x \tan^{-1} x - \frac{1}{2} \cdot 2 \log(1 + x^2) + c \quad \therefore I = 2x \tan^{-1} x - \log(1 + x^2) + c
 \end{aligned}$$

47. $\int \tan^{-1} \left(\frac{2x}{1-x^2} \right) dx$

Sol. Let $I = \int \tan^{-1} \left(\frac{2x}{1-x^2} \right) dx$

Put $x = \tan \theta \quad \therefore \theta = \tan^{-1}(x) \Rightarrow dx = \sec^2 \theta \, d\theta$



$$\begin{aligned}
 I &= \int \tan^{-1} \left(\frac{2 \tan \theta}{1 - \tan^2 \theta} \right) \sec^2 \theta \, d\theta \Rightarrow I = \int \tan^{-1} (\tan 2\theta) \sec^2 \theta \, d\theta \\
 &\Rightarrow I = 2 \int \theta \sec^2 \theta \, d\theta \Rightarrow I = 2 \left[\theta \tan \theta - \int 1 \cdot \tan \theta \, d\theta \right] \Rightarrow I = 2 \left[\theta \tan \theta + \log(\cos \theta) \right] + c \\
 &\Rightarrow I = 2\theta \tan \theta + 2 \log(\cos \theta) + c \Rightarrow I = 2x \tan^{-1} x - \frac{1}{2} \cdot 2 \log(1 + x^2) + c \\
 \therefore I &= 2x \tan^{-1} x - \log(1 + x^2) + c
 \end{aligned}$$

48. $\int \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right) dx$

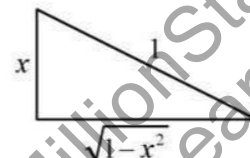
Sol. Let $I = \int \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right) dx$, Put $x = \tan \theta \quad \therefore \theta = \tan^{-1}(x) \Rightarrow dx = \sec^2 \theta \, d\theta$

$$\begin{aligned}
 I &= \int \tan^{-1} \left(\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right) \sec^2 \theta \, d\theta \Rightarrow I = \int 3\theta \sec^2 \theta \, d\theta \Rightarrow I = 3 \int \theta \sec^2 \theta \, d\theta \\
 &\Rightarrow I = 3 \left[\theta \int \sec^2 \theta \, d\theta - \int \left[\frac{d\theta}{d\theta} \int \sec^2 \theta \, d\theta \right] d\theta \right] \Rightarrow I = 3 \left[\theta \tan \theta - \int 1 \cdot \tan \theta \, d\theta \right] \\
 &\Rightarrow I = 3 \left[\theta \tan \theta + \log(\cos \theta) \right] + C \Rightarrow I = 3\theta \tan \theta + 3 \log(\cos \theta) + C \\
 &\Rightarrow I = 3 \tan^{-1} x \tan(\tan^{-1} x) + 3 \log \left| \cos(\tan^{-1} x) \right| + C \\
 &\Rightarrow I = 3x \tan^{-1} x - \frac{1}{2} \cdot 3 \log(1 + x^2) + C \quad \therefore I = 3x \tan^{-1} x - \frac{3}{2} \log(1 + x^2) + C
 \end{aligned}$$

49. $\int \frac{\sin^{-1} x}{x^2} dx$

Sol. Let $I = \int \frac{\sin^{-1} x}{x^2} dx$

Put $x = \sin \theta \quad \therefore \theta = \sin^{-1} x \Rightarrow dx = \cos \theta \, d\theta$



$$I = \int \frac{\sin^{-1}(\sin \theta)}{\sin^2 \theta} \cos \theta \, d\theta \Rightarrow I = \int \theta \cdot \operatorname{cosec}^2 \theta \cos \theta \, d\theta$$

$$\Rightarrow I = \int \theta \cdot \cot \theta \operatorname{cosec} \theta \, d\theta \Rightarrow I = \theta \int \operatorname{cosec} \theta \cot \theta \, d\theta - \int \left[\frac{d(\theta)}{d\theta} \int \operatorname{cosec} \theta \cot \theta \, d\theta \right] d\theta$$

$$\Rightarrow I = -\theta \operatorname{cosec} \theta + \int 1 \cdot \operatorname{cosec} \theta \, d\theta \Rightarrow I = -\theta \operatorname{cosec} \theta + \log |\operatorname{cosec} \theta - \cot \theta| + c$$

$$\therefore I = -\frac{\sin^{-1} x}{x} + \log \left| \frac{1}{x} - \frac{\sqrt{1-x^2}}{x} \right| + c$$

50. $\int \frac{\tan x \cdot \sec^2 x}{1 - \tan^2 x} \, dx$

Sol. Let $I = \int \frac{\tan x \cdot \sec^2 x}{1 - \tan^2 x} \, dx$, Put $\tan x = t \Rightarrow \sec^2 x = \frac{dt}{dx} \Rightarrow \sec^2 x \, dx = dt$

$$I = \int \frac{t}{1-t^2} \, dt, \text{ Put } 1-t^2 = z \Rightarrow -2t = \frac{dz}{dt} \Rightarrow t \, dt = \frac{dz}{-2}$$

$$I = -\frac{1}{2} \int \frac{1}{z} \, dz \Rightarrow I = -\frac{1}{2} \log |z| + c \Rightarrow I = -\frac{1}{2} \log |1-t^2| + c$$

$$\therefore I = -\frac{1}{2} \log |1 - \tan^2 x| + c$$

51. $\int e^{3x} \sin 4x \, dx$

Sol. Let $I = \int e^{3x} \sin 4x \, dx \Rightarrow I = \sin 4x \int e^{3x} \, dx - \int \left[\frac{d(\sin 4x)}{dx} \int e^{3x} \, dx \right] dx$

$$\Rightarrow I = \sin 4x \cdot \frac{e^{3x}}{3} - \int \cos 4x \cdot 4 \cdot \frac{e^{3x}}{3} \, dx \Rightarrow I = \frac{1}{3} \sin 4x \cdot e^{3x} - \frac{4}{3} \int e^{3x} \cos 4x \, dx$$

$$\Rightarrow I = \frac{1}{3} e^{3x} \sin 4x - \frac{4}{3} \left[\cos 4x \int e^{3x} \, dx - \int \frac{d(\cos 4x)}{dx} \int e^{3x} \, dx \right]$$

$$\Rightarrow I = \frac{1}{3} e^{3x} \sin 4x - \frac{4}{3} \left[\cos 4x \cdot \frac{e^{3x}}{3} + \int \sin 4x \cdot 4 \cdot \frac{e^{3x}}{3} \, dx \right]$$

$$\Rightarrow I = \frac{1}{3} e^{3x} \sin 4x - \frac{4}{9} e^{3x} \cos 4x - \frac{16}{9} \int e^{3x} \sin 4x \, dx \Rightarrow \frac{25}{9} I = \frac{1}{3} e^{3x} \sin 4x - \frac{4}{9} e^{3x} \cos 4x + C$$

$$\Rightarrow I = \frac{3}{25} e^{3x} \sin 4x - \frac{4}{25} e^{3x} \cos 4x + c \therefore I = \frac{e^{3x}}{25} (3 \sin 4x - 4 \cos 4x) + c$$

52. $\int e^{2x} \sin x \, dx$

Sol. Let $I = \int e^{2x} \sin x \, dx \Rightarrow I = \sin x \int e^{2x} \, dx - \int \left[\frac{d(\sin x)}{dx} \int e^{2x} \, dx \right] dx$

$$\Rightarrow I = \sin x \cdot \frac{e^{2x}}{2} - \int \cos x \cdot \frac{e^{2x}}{2} \, dx \Rightarrow I = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x \, dx$$

$$\Rightarrow I = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \left[\cos x \int e^{2x} \, dx - \int \frac{d(\cos x)}{dx} \int e^{2x} \, dx \right] dx$$

$$\Rightarrow I = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \left[\cos x \cdot \frac{e^{2x}}{2} + \int \sin x \cdot \frac{e^{2x}}{2} dx \right] \Rightarrow I = \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x - \frac{1}{4} \int e^{2x} \sin x dx$$

$$\Rightarrow I = \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x - \frac{1}{4} I \Rightarrow \frac{5I}{4} = \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x + C$$

$$\therefore I = \frac{e^{2x}}{5} (2 \sin x - \cos x) + c$$

53. $\int e^{2x} \sin x \cos x dx$

Sol. Let $I = \int e^{2x} \sin x \cos x dx \Rightarrow I = \frac{1}{2} \int e^{2x} (2 \sin x \cos x) dx$

$$\Rightarrow I = \frac{1}{2} \int e^{2x} \sin 2x dx \Rightarrow I = \frac{1}{2} \left[\sin 2x \int e^{2x} dx - \int \left[\frac{d(\sin 2x)}{dx} \int e^{2x} dx \right] dx \right]$$

$$\Rightarrow I = \frac{1}{2} \left[\sin 2x \cdot \frac{e^{2x}}{2} - \int \cos 2x \cdot 2 \cdot \frac{e^{2x}}{2} dx \right] \Rightarrow I = \frac{1}{2} \left[\frac{e^{2x}}{2} \sin 2x - \int e^{2x} \cos 2x dx \right]$$

$$\Rightarrow I = \frac{e^{2x}}{4} \sin 2x - \frac{1}{2} \int e^{2x} \cos 2x dx$$

$$\Rightarrow I = \frac{e^{2x}}{4} \sin 2x - \frac{1}{2} \left[\cos 2x \int e^{2x} dx - \int \left[\frac{d(\cos 2x)}{dx} \int e^{2x} dx \right] dx \right]$$

$$\Rightarrow I = \frac{e^{2x}}{4} \sin 2x - \frac{1}{2} \left[\cos 2x \cdot \frac{e^{2x}}{2} + \int \sin 2x \cdot 2 \cdot \frac{e^{2x}}{2} dx \right]$$

$$\Rightarrow I = \frac{e^{2x}}{4} \sin 2x - \frac{e^{2x} \cos 2x}{4} - \frac{1}{2} \int e^{2x} \sin 2x dx = \frac{e^{2x} \sin 2x}{4} - \frac{e^{2x} \cos 2x}{4} = I$$

$$\Rightarrow 2I = \frac{e^{2x}}{4} \sin 2x - \frac{e^{2x}}{4} \cos 2x + c \quad \therefore I = \frac{e^{2x}}{8} (\sin 2x - \cos 2x) + c$$

54. $\int e^{2x} \cos(3x+4) dx$

Sol. Let $I = \int e^{2x} \cos(3x+4) dx \Rightarrow I = \cos(3x+4) \int e^{2x} dx - \int \left[\frac{d\{\cos(3x+4)\}}{dx} \int e^{2x} dx \right] dx$

$$\Rightarrow I = \cos(3x+4) \cdot \frac{e^{2x}}{2} + \int \sin(3x+4) \cdot 3 \cdot \frac{e^{2x}}{2} dx$$

$$\Rightarrow I = \frac{e^{2x}}{2} \cos(3x+4) + \frac{3}{2} \int e^{2x} \sin(3x+4) dx$$

$$\Rightarrow I = \frac{e^{2x}}{2} \cos(3x+4) + \frac{3}{2} \left[\sin(3x+4) \int e^{2x} dx - \int \left[\frac{d \sin(3x+4)}{dx} \int e^{2x} dx \right] dx \right]$$

$$\Rightarrow I = \frac{e^{2x}}{2} \cos(3x+4) + \frac{3}{2} \sin(3x+4) \cdot \frac{e^{2x}}{2} - \int \cos(3x+4) \cdot 3 \cdot \frac{e^{2x}}{2} dx$$

$$\Rightarrow I = \frac{e^{2x}}{2} \cos(3x+4) + \frac{3}{4} e^{2x} \sin(3x+4) - \frac{9}{4} \int e^{2x} \cos(3x+4) dx$$

$$\Rightarrow \frac{13}{4} I = \frac{e^{2x}}{4} \{2 \cos(3x+4) + 3 \sin(3x+4)\} \quad \therefore I = \frac{e^{2x}}{13} \{2 \cos(3x+4) + 3 \sin(3x+4)\}$$

55. $\int e^{-x} \cos x \, dx$

Sol. Let $I = \int e^{-x} \cos x \, dx \Rightarrow I = \cos x \int e^{-x} dx - \int \left[\frac{d(\cos x)}{dx} \int e^{-x} dx \right] dx$

$$\Rightarrow I = \cos x \cdot \frac{e^{-x}}{-1} + \int \sin x \cdot \frac{e^{-x}}{-1} dx \Rightarrow I = -e^{-x} \cos x - \int e^{-x} \sin x \, dx$$

$$\Rightarrow I = -e^{-x} \cos x - \left[\sin x \int e^{-x} dx - \int \left[\frac{d(\sin x)}{dx} \int e^{-x} dx \right] dx \right]$$

$$\Rightarrow I = -e^{-x} \cos x - \left[\sin x \frac{e^{-x}}{-1} - \int \cos x \cdot \frac{e^{-x}}{-1} dx \right] \Rightarrow I = -e^{-x} \cos x + e^{-x} \sin x - \int e^{-x} \cos x \, dx$$

$$\Rightarrow 2I = -e^{-x} \cos x + e^{-x} \sin x + c \quad \therefore I = \frac{e^{-x}}{2} (\sin x - \cos x) + c$$

56. $\int e^x (\sin x + \cos x) dx$

Sol. Let $I = \int e^x (\sin x + \cos x) dx$, where $f(x) = \sin x$, $f'(x) = \cos x$

$$\Rightarrow I = \int e^x \{f(x) + f'(x)\} dx \Rightarrow I = e^x \cdot f(x) + c \quad \therefore I = e^x \sin x + c$$

57. $\int e^x (\cot x - \operatorname{cosec}^2 x) dx$

Sol. Let $I = \int e^x (\cot x - \operatorname{cosec}^2 x) dx$, where $f(x) = \cot x$, $f'(x) = -\operatorname{cosec}^2 x$

$$\Rightarrow I = \int e^x \{f(x) + f'(x)\} dx \Rightarrow I = e^x \cdot f(x) + c \quad \therefore I = e^x \cot x + c$$

58. $\int e^x (\sec x + \sec x \tan x) dx$

Sol. Let $I = \int e^x (\sec x + \sec x \tan x) dx$, where $f(x) = \sec x$, $f'(x) = \sec x \tan x$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x f(x) + c \quad \therefore I = e^x \sec x + c$$

59. $\int e^x \left(\tan^{-1} x + \frac{1}{1+x^2} \right) dx$

Sol. Let $I = \int e^x \left(\tan^{-1} x + \frac{1}{1+x^2} \right) dx$, where $f(x) = \tan^{-1} x$, $f'(x) = \frac{1}{1+x^2}$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x \cdot f(x) + c \quad \therefore I = e^x \tan^{-1} x + c$$

60. $\int e^x \{ \cot x + \log(\sin x) \} dx$

Sol. Let $I = \int e^x \{ \cot x + \log(\sin x) \} dx$, where $f(x) = \log(\sin x)$, $f'(x) = \cot x$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x f(x) + c \quad \therefore I = e^x \log(\sin x) + c$$

61. $\int e^x (\tan x - \log(\cos x)) dx$

Sol. Let $I = \int e^x (\tan x - \log(\cos x)) dx$, where $f(x) = -\log(\cos x)$, $f'(x) = \tan x$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x f(x) + C \quad \therefore I = -e^x \{ \log(\cos x) \} + C$$

62. $\int e^x [\sec x + \log(\sec x + \tan x)] dx$

Sol. Let $I = \int e^x [\sec x + \log(\sec x + \tan x)] dx$, where $f(x) = \log(\sec x + \tan x)$, $f'(x) = \sec x$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x f(x) + c \quad \therefore e^x \log(\sec x + \tan x) + c$$

63. $\int e^x \left(\frac{1 + \sin x \cos x}{\cos^2 x} \right) dx$

Sol. Let $I = \int e^x \left(\frac{1 + \sin x \cos x}{\cos^2 x} \right) dx \Rightarrow I = \int e^x \left(\frac{1}{\cos^2 x} + \frac{\sin x \cos x}{\cos^2 x} \right) dx$

$$\Rightarrow I = \int e^x (\sec^2 x + \tan x) dx, \text{ where } f(x) = \tan x, f'(x) = \sec^2 x$$

$$\Rightarrow I = \int e^x \{f(x) + f'(x)\} dx \Rightarrow I = e^x f(x) + c \quad \therefore I = e^x \tan x + c$$

64. $\int e^x \left(\frac{\sin x \cos x - 1}{\sin^2 x} \right) dx$

Sol. Let $I = \int e^x \left(\frac{\sin x \cos x - 1}{\sin^2 x} \right) dx \Rightarrow I = \int e^x \left(\frac{\sin x \cos x}{\sin^2 x} - \frac{1}{\sin^2 x} \right) dx$

$$\Rightarrow I = \int e^x (\cot x - \operatorname{cosec}^2 x) dx, \text{ where } f(x) = \cot x, f'(x) = -\operatorname{cosec}^2 x$$

$$\Rightarrow I = \int e^x \{f(x) + f'(x)\} dx \Rightarrow I = e^x f(x) + c \quad \therefore I = e^x \cot x + c$$

65. $\int e^x \left(\frac{\cos x + \sin x}{\cos^2 x} \right) dx$

Sol. Let $I = \int e^x \left(\frac{\cos x + \sin x}{\cos^2 x} \right) dx \Rightarrow I = \int e^x \left(\frac{\cos x}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \right) dx$

$$\Rightarrow I = \int e^x (\sec x + \tan x \sec x) dx, \text{ where } f(x) = \sec x, f'(x) = \sec x \tan x$$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x f(x) + c \quad \therefore I = e^x \sec x + c$$

66. $\int e^x \left(\frac{2 - \sin 2x}{1 - \cos 2x} \right) dx$

Sol. Let $I = \int e^x \left(\frac{2 - \sin 2x}{1 - \cos 2x} \right) dx \Rightarrow I = \int e^x \left(\frac{2 - \sin 2x}{2 \sin^2 x} \right) dx \Rightarrow I = \int e^x \left(\frac{2}{2 \sin^2 x} - \frac{2 \sin x \cos x}{2 \sin^2 x} \right) dx$

$$\Rightarrow I = \int e^x (\operatorname{cosec}^2 x - \cot x) dx, \text{ where } f(x) = -\cot x, f'(x) = \operatorname{cosec}^2 x$$

$$\Rightarrow I = \int e^x \{f(x) + f'(x)\} dx \Rightarrow I = e^x f(x) + c \quad \therefore I = -e^x \cot x + c$$

67. $\int e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx$

Sol. Let $I = \int e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx$

$$\Rightarrow I = \int e^x \left(\frac{1 + \sin x}{2 \cos^2 \frac{x}{2}} \right) dx \Rightarrow I = \int e^x \left(\frac{1}{2 \cos^2 \frac{x}{2}} + \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right) dx$$

$$\Rightarrow I = \int e^x \left(\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right) dx, \text{ where } f(x) = \tan \frac{x}{2}, f'(x) = \sec^2 \frac{x}{2} \cdot \frac{1}{2}$$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x f(x) + c \quad \therefore I = e^x \tan \frac{x}{2} + c$$

68. $\int e^x \left(\frac{\sin 4x - 4}{1 - \cos 4x} \right) dx$

Sol. Let $I = \int e^x \left(\frac{\sin 4x - 4}{1 - \cos 4x} \right) dx \Rightarrow I = \int e^x \left(\frac{\sin 4x - 4}{2 \sin^2 2x} \right) dx$
 $\Rightarrow I = \int e^x \left(\frac{\sin 4x}{2 \sin^2 2x} - \frac{4}{2 \sin^2 2x} \right) dx \Rightarrow I = \int e^x \left(\frac{2 \sin 2x \cos 2x}{2 \sin^2 2x} - \frac{2}{\sin^2 2x} \right) dx$
 $\Rightarrow I = \int e^x (\cot 2x - 2 \operatorname{cosec}^2 2x) dx$, where $f(x) = \cot 2x$, $f'(x) = -2 \operatorname{cosec}^2 2x$
 $\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x \cdot f(x) + c \quad \therefore I = e^x \cot 2x + c$

69. $\int \frac{e^x [\sqrt{1-x^2} \sin^{-1} x + 1]}{\sqrt{1-x^2}} dx$

Sol. Let $I = \int e^x \left[\frac{\sqrt{1-x^2} \sin^{-1} x + 1}{\sqrt{1-x^2}} \right] dx \Rightarrow I = \int e^x \left[\frac{\sqrt{1-x^2} \sin^{-1} x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} \right] dx$
 $\Rightarrow I = \int e^x \left[\sin^{-1} x + \frac{1}{\sqrt{1-x^2}} \right] dx$, where $f(x) = \sin^{-1} x$, $f'(x) = \frac{1}{\sqrt{1-x^2}}$
 $\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x f(x) + c \quad \therefore I = e^x \sin^{-1} x + c$

70. $\int e^x \left(\frac{1+x \log x}{x} \right) dx$

Sol. Let $I = \int e^x \left(\frac{1+x \log x}{x} \right) dx \Rightarrow I = \int e^x \left(\frac{1}{x} + \frac{x \log x}{x} \right) dx$
 $\Rightarrow I = \int e^x \left(\frac{1}{x} + \log x \right) dx$, where $f(x) = \log x$, $f'(x) = \frac{1}{x}$
 $\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x f(x) + c \quad \therefore I = e^x \log x + c$

71. $\int e^x \left\{ \frac{x}{(1+x)^2} \right\} dx$

Sol. Let $I = \int e^x \left\{ \frac{x}{(1+x)^2} \right\} dx \Rightarrow I = \int e^x \left\{ \frac{(x+1)-1}{(x+1)^2} \right\} dx \Rightarrow I = \int e^x \left\{ \frac{1}{x+1} - \frac{1}{(x+1)^2} \right\} dx$
 $\Rightarrow I = \int e^x \left\{ \frac{1}{x+1} - \frac{1}{(x+1)^2} \right\} dx$, where $f(x) = \frac{1}{x+1}$, $f'(x) = -\frac{1}{(x+1)^2}$
 $\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x f(x) + c \Rightarrow I = e^x \cdot \frac{1}{x+1} + c \quad \therefore I = \frac{e^x}{x+1} + c$

72. $\int e^x \left\{ \frac{x-1}{(x+1)^3} \right\} dx$

Sol. Let $I = \int e^x \left\{ \frac{x-1}{(x+1)^3} \right\} dx \Rightarrow I = \int e^x \left\{ \frac{(x+1)-2}{(x+1)^3} \right\} dx \Rightarrow I = \int e^x \left\{ \frac{x+1}{(x+1)^3} - \frac{2}{(x+1)^3} \right\} dx$

$$\Rightarrow I = \int e^x \left\{ \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3} \right\} dx, \text{ where } f(x) = \frac{1}{(x+1)^2}, f'(x) = -\frac{2}{(x+1)^3}$$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x \cdot f(x) + c \quad \therefore I = e^x \cdot \frac{1}{(x+1)^2} + c$$

73. $\int e^x \left\{ \frac{2-x}{(1-x)^2} \right\} dx$

Sol. Let $I = \int e^x \left\{ \frac{2-x}{(1-x)^2} \right\} dx \Rightarrow I = \int e^x \left\{ \frac{(1-x)+1}{(1-x)^2} \right\} dx \Rightarrow I = \int e^x \left\{ \frac{(1-x)}{(1-x)^2} + \frac{1}{(1-x)^2} \right\} dx$

$$\Rightarrow I = \int e^x \left\{ \frac{1}{1-x} + \frac{1}{(1-x)^2} \right\} dx, \text{ where } f(x) = \frac{1}{(1-x)}, f'(x) = \frac{1}{(1-x)^2}$$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x f(x) + c \quad \therefore I = e^x \cdot \frac{1}{1-x} + c$$

74. $\int e^x \left\{ \frac{x-3}{(x-1)^3} \right\} dx$

Sol. Let $I = \int e^x \left\{ \frac{x-3}{(x-1)^3} \right\} dx \Rightarrow I = \int e^x \left\{ \frac{x-1-2}{(x-1)^3} \right\} dx \Rightarrow I = \int e^x \left\{ \frac{x-1}{(x-1)^3} - \frac{2}{(x-1)^3} \right\} dx$

$$\Rightarrow I = \int e^x \left\{ \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right\} dx, \text{ where } f(x) = \frac{1}{(x-1)^2}, f'(x) = -\frac{2}{(x-1)^3}$$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x \cdot f(x) + c \quad \therefore I = e^x \cdot \frac{1}{(x-1)^2} + c$$

75. $\int e^{3x} \left(\frac{3x-1}{9x^2} \right) dx$

Sol. Let $I = \int e^{3x} \left(\frac{3x-1}{9x^2} \right) dx$, Put $3x = t \Rightarrow dx = \frac{dt}{3} \Rightarrow 3 = \frac{dt}{dx}$

$$I = \int e^t \left(\frac{t-1}{t^2} \right) \frac{dt}{3} \Rightarrow I = \frac{1}{3} \int e^t \left(\frac{t-1}{t^2} \right) dt, \text{ where } f(t) = \frac{1}{t}, f'(t) = -\frac{1}{t^2}$$

$$\Rightarrow I = \int e^t [f(t) + f'(t)] dx$$

$$\Rightarrow I = \frac{1}{3} e^t \cdot f(t) + c \Rightarrow I = \frac{1}{3} e^t \cdot \frac{1}{t} + c \Rightarrow I = \frac{1}{3} \cdot \frac{e^t}{3x} + c$$

$$\therefore I = \frac{e^{3x}}{9x} + c$$

76. $\int e^x \left\{ \frac{x+1}{(x+2)^2} \right\} dx$

Sol. Let $I = \int e^x \left\{ \frac{x+1}{(x+2)^2} \right\} dx \Rightarrow I = \int e^x \left\{ \frac{x+2-1}{(x+2)^2} \right\} dx \Rightarrow I = \int e^x \left\{ \frac{x+2}{(x+2)^2} - \frac{1}{(x+2)^2} \right\} dx$

$$\Rightarrow I = \int e^x \left\{ \frac{1}{x+2} - \frac{1}{(x+2)^2} \right\} dx, \text{ where } f(x) = \frac{1}{x+2}, f'(x) = -\frac{1}{(x+2)^2}$$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x \cdot f(x) + c \quad \therefore I = e^x \cdot \frac{1}{x+2} + c$$

77. $\int \frac{x e^{2x}}{(1+2x)^2} dx$

Sol. Let $I = \int e^{2x} \cdot \frac{x}{(1+2x)^2} dx$, Put $2x = t \Rightarrow 2 = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{2}$

$$\Rightarrow I = \int e^t \cdot \frac{t}{2(1+t)^2} \cdot \frac{dt}{2} \Rightarrow I = \frac{1}{4} \int e^t \left\{ \frac{t}{(t+1)^2} \right\} dt$$

$$\Rightarrow I = \frac{1}{4} \int e^t \left\{ \frac{(t+1)-1}{(t+1)^2} \right\} dt \Rightarrow I = \frac{1}{4} \int e^t \left\{ \frac{t+1}{(t+1)^2} - \frac{1}{(t+1)^2} \right\} dt$$

$$\Rightarrow I = \frac{1}{4} \int e^t \left\{ \frac{1}{t+1} - \frac{1}{(t+1)^2} \right\} dt, \text{ where } f(x) = \frac{1}{t+1}, f'(x) = -\frac{1}{(t+1)^2}$$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x \cdot f(x) + c \Rightarrow I = \frac{1}{4} \cdot e^t \cdot \frac{1}{t+1} + c \quad \therefore I = \frac{e^{2x}}{4(2x+1)} + c$$

78. $\int e^{2x} \left(\frac{2x-1}{4x^2} \right) dx$

Sol. Let $I = \int e^{2x} \left(\frac{2x-1}{4x^2} \right) dx$, Put $2x = t \Rightarrow 2 = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{2}$

$$I = \frac{1}{2} \int e^t \left(\frac{t-1}{t^2} \right) dt \Rightarrow I = \frac{1}{2} \int e^t \left(\frac{t}{t^2} - \frac{1}{t^2} \right) dt$$

$$\Rightarrow I = \frac{1}{2} \int e^t \left(\frac{1}{t} - \frac{1}{t^2} \right) dt, \text{ where } f(x) = \frac{1}{t}, f'(x) = -\frac{1}{t^2}$$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x \cdot f(x) + c \Rightarrow I = \frac{1}{2} \cdot e^t \cdot \frac{1}{t} + c \Rightarrow I = \frac{e^{2x}}{2 \cdot 2x} + c$$

$$\therefore I = \frac{e^{2x}}{4x} + c$$

79. $\int e^x \left(\log x + \frac{1}{x^2} \right) dx$

Sol. Let $I = \int e^x \left(\log x + \frac{1}{x^2} \right) dx \Rightarrow I = \int e^x \left[\left(\log x + \frac{1}{x} \right) + \left\{ -\frac{1}{x} + \frac{1}{x^2} \right\} \right] dx$

$$\Rightarrow I = \int e^x \left(\log x + \frac{1}{x} \right) dx - \int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx,$$

$$\Rightarrow I = \int e^x [f_1(x) + f_1'(x)] dx - \int e^x [f_2(x) + f_2'(x)] dx$$

$$\text{where } f_1(x) = \log x, f_1'(x) = \frac{1}{x}, f_2(x) = \frac{1}{x}, f_2'(x) = -\frac{1}{x^2}$$

$$\Rightarrow I = e^x f_1(x) - e^x f_2(x) + c \Rightarrow I = e^x \cdot \log x - e^x \cdot \frac{1}{x} + c \quad \therefore I = e^x \left(\log x - \frac{1}{x} \right) + c$$

80. $\int \frac{\log x}{(1+\log x)^2} dx$

Sol. Let $I = \int \frac{\log x}{(1+\log x)^2} dx$, Put $\log x = t \quad \therefore x = e^t \Rightarrow \frac{1}{x} = \frac{dt}{dx} \Rightarrow dx = x dt \Rightarrow dx = e^t dt$

$$I = \int e^t \cdot \frac{t}{(1+t)^2} dt \Rightarrow I = \int e^t \left\{ \frac{t+1-1}{(t+1)^2} \right\} dt \Rightarrow I = \int e^t \left\{ \frac{t+1}{(t+1)^2} - \frac{1}{(t+1)^2} \right\} dt$$

$$\Rightarrow I = \int e^t \left\{ \frac{1}{t+1} - \frac{1}{(t+1)^2} \right\} dt \Rightarrow I = e^t \cdot \frac{1}{t+1} + c \quad \therefore I = \frac{x}{\log x + 1} + c$$

81. $\int \{ \sin(\log x) + \cos(\log x) \} dx$

Sol. Let $I = \int \{ \sin(\log x) + \cos(\log x) \} dx$

Put $\log x = t \quad \therefore x = e^t \Rightarrow \frac{1}{x} = \frac{dt}{dx} \Rightarrow dx = x dt \Rightarrow dx = e^t dt$

$$I = \int e^t (\sin t + \cos t) dt, \text{ where } f(x) = \sin t, f'(x) = \cos t$$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x \cdot f(x) + c \Rightarrow I = e^t \sin t + c \quad \therefore I = x \sin(\log x) + c$$

82. $\int \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx$

Sol. Let $I = \int \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx$, Put $\log x = t \quad \therefore x = e^t$

$$\frac{1}{x} = \frac{dt}{dx} \Rightarrow \frac{1}{x} dx = dt \Rightarrow dx = x dt \Rightarrow dx = e^t dt$$

$$I = \int \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt, \text{ where } f(x) = \frac{1}{t}, f'(x) = -\frac{1}{t^2}$$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x \cdot f(x) + c \Rightarrow I = e^t \cdot \frac{1}{t} + c \quad \therefore I = \frac{x}{\log x} + c$$

83. $\int \left\{ \log(\log x) + \frac{1}{(\log x)^2} \right\} dx$

Sol. Let $I = \int \left\{ \log(\log x) + \frac{1}{(\log x)^2} \right\} dx$

Put $\log x = t \quad \therefore x = e^t \Rightarrow \frac{1}{x} = \frac{dt}{dx} \Rightarrow dx = x dt \Rightarrow dx = e^t dt$

$$I = \int e^t \left(\log t + \frac{1}{t^2} \right) dt \Rightarrow I = \int e^t \left[\left(\log t + \frac{1}{t} \right) + \left\{ -\frac{1}{t} + \frac{1}{t^2} \right\} \right] dt$$

$$\Rightarrow I = \int e^t \left(\log t + \frac{1}{t} \right) dt - \int e^t \left(\frac{1}{t} - \frac{1}{t^2} \right) dt,$$

$$\Rightarrow I = \int e^t [f_1(t) + f_1'(t)] dt - \int e^t [f_2(t) + f_2'(t)] dt \Rightarrow I = e^t f_1(t) - e^t f_2(t) + c$$

$$\text{where } f_1(t) = \log t, f_1'(t) = \frac{1}{t}, f_2(t) = \frac{1}{t}, f_2'(t) = -\frac{1}{t^2}$$

$$\Rightarrow I = e^t \cdot \log t - e^t \cdot \frac{1}{t} + c \Rightarrow I = e^t \left(\log t - \frac{1}{t} \right) + c \therefore I = x \left(\log(\log x) - \frac{1}{\log x} \right) + c$$

84. $\int \frac{\sin^{-1}(\sqrt{x}) - \cos^{-1}(\sqrt{x})}{\sin^{-1}(\sqrt{x}) + \cos^{-1}(\sqrt{x})} dx$

Sol. Let $I = \int \frac{\sin^{-1}(\sqrt{x}) - \cos^{-1}(\sqrt{x})}{\sin^{-1}(\sqrt{x}) + \cos^{-1}(\sqrt{x})} dx$ $[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}]$

$$\Rightarrow I = \int \frac{\sin^{-1}(\sqrt{x}) - \cos^{-1}(\sqrt{x})}{\pi/2} dx \Rightarrow I = \frac{2}{\pi} \int \left\{ \sin^{-1}(\sqrt{x}) - \left[\frac{\pi}{2} - \sin^{-1}(\sqrt{x}) \right] \right\} dx$$

$$\Rightarrow I = \frac{2}{\pi} \int \left\{ \sin^{-1}(\sqrt{x}) - \left[\frac{\pi}{2} - \sin^{-1}(\sqrt{x}) \right] \right\} dx \Rightarrow I = \frac{2}{\pi} \int \left\{ 2 \sin^{-1}(\sqrt{x}) - \frac{\pi}{2} \right\} dx$$

$$\Rightarrow I = \frac{2}{\pi} \int \left\{ 2 \sin^{-1}(\sqrt{x}) - \frac{\pi}{2} \right\} dx \Rightarrow I = \frac{4}{\pi} \int \sin^{-1}(\sqrt{x}) dx - \int dx \Rightarrow I = \frac{4}{\pi} I_1 - x \dots (1)$$

$$I_1 = \int \sin^{-1}(\sqrt{x}) dx, \text{ Put } \sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} \cdot \frac{dt}{dx} \Rightarrow dx = 2\sqrt{x} dt \Rightarrow dx = 2t dt$$

$$\Rightarrow I_1 = \int \sin^{-1}(t) \cdot 2t dt \Rightarrow I_1 = 2 \int t \sin^{-1}(t) dt$$

$$\Rightarrow I_1 = 2 \sin^{-1} t \int t dt - \int \left[\frac{d(\sin^{-1} t)}{dt} \int t dt \right] dt \Rightarrow I_1 = 2 \left[\sin^{-1} t \cdot \frac{t^2}{2} - \int \frac{1}{\sqrt{1-t^2}} \cdot \frac{t^2}{2} dt \right]$$

$$\Rightarrow I_1 = 2 \left[\frac{t^2}{2} \sin^{-1} t + \frac{1}{2} \int -\frac{t^2}{\sqrt{1-t^2}} dt \right] \Rightarrow I_1 = 2 \left[\frac{t^2}{2} \sin^{-1} t + \frac{1}{2} \int \frac{(1-t^2)-1}{\sqrt{1-t^2}} dt \right]$$

$$\Rightarrow I_1 = 2 \left[\frac{t^2}{2} \sin^{-1} t + \frac{1}{2} \int \frac{1-t^2}{\sqrt{1-t^2}} dt - \frac{1}{2} \int \frac{1}{\sqrt{1-t^2}} dt \right]$$

$$\Rightarrow I_1 = 2 \left[\frac{t^2}{2} \sin^{-1} t + \frac{1}{2} \int \sqrt{1-t^2} dt - \frac{1}{2} \sin^{-1} t \right] + c$$

$$\Rightarrow I_1 = 2 \left[\frac{t^2}{2} \sin^{-1} t + \frac{t}{4} \sqrt{1-t^2} + \frac{1}{4} \sin^{-1} t - \frac{1}{2} \sin^{-1} t \right] + c$$

$$\Rightarrow I_1 = 2 \left[\frac{t^2}{2} \sin^{-1} t + \frac{t}{4} \sqrt{1-t^2} - \frac{1}{4} \sin^{-1} t \right] + c \Rightarrow I_1 = \left[x \sin^{-1}(\sqrt{x}) + \frac{\sqrt{x}}{2} \sqrt{1-x} - \frac{1}{2} \sin^{-1} \sqrt{x} \right] + c$$

Put the value of I_1 in equations (1), $\therefore I = \frac{4}{\pi} \left[x \sin^{-1}(\sqrt{x}) + \frac{\sqrt{x}}{2} \sqrt{1-x} - \frac{1}{2} \sin^{-1} \sqrt{x} \right] - x + c$

85. $\int 5^{5^{5^x}} \cdot 5^{5^x} \cdot 5^x dx$

Sol. Let $I = \int 5^{5^{5^x}} \cdot 5^{5^x} \cdot 5^x dx$, Put $t = 5^{5^x} \Rightarrow \log t = 5^x \log 5 \Rightarrow \frac{1}{t} dt = 5^x \log 5 \cdot \log 5 dx$

$$\Rightarrow dt = t \cdot 5^x (\log 5)^2 dx \Rightarrow dt = 5^{5x} \cdot 5^x (\log 5)^2 dx \Rightarrow \frac{dt}{(\log 5)^2} = 5^{5x} \cdot 5^x dx$$

$$\therefore I = \int 5^t \cdot \frac{dt}{(\log 5)^2} = \frac{1}{(\log 5)^2} \cdot \frac{5^t}{\log 5} + c = \frac{5^t}{(\log 5)^3} + c = \frac{5^{5x}}{(\log 5)^3} + c$$

86. $\int e^{2x} \left(\frac{1 + \sin 2x}{1 + \cos 2x} \right) dx$

Sol. Let $I = \int e^{2x} \left(\frac{1 + \sin 2x}{1 + \cos 2x} \right) dx$

Put $t = 2x \Rightarrow \frac{dt}{2} = dx$, $\int e^t \left(\frac{1 + \sin t}{1 + \cos t} \right) \frac{dt}{2} = \frac{1}{2} \int e^t \left(\frac{1}{1 + \cos t} + \frac{\sin t}{1 + \cos t} \right) dt$

$$= \frac{1}{2} \int e^t \left(\frac{1}{2 \cos^2 \frac{t}{2}} + \frac{2 \sin \frac{t}{2} \cos \frac{t}{2}}{2 \cos^2 \frac{t}{2}} \right) dt = \frac{1}{2} \int e^t \left(\frac{1}{2} \sec^2 \frac{t}{2} + \tan \frac{t}{2} \right) dt$$

Here, $f(t) = \tan \frac{t}{2} \Rightarrow f'(t) = \frac{1}{2} \sec^2 \frac{t}{2}$

$$\begin{aligned} \int e^t \left(\frac{1 + \sin t}{1 + \cos t} \right) \frac{dt}{2} &= \frac{1}{2} \int e^t \{f(t) + f'(t)\} dt = \frac{1}{2} e^t \cdot f(t) + c = \frac{1}{2} e^{2x} \cdot \tan \frac{t}{2} + c \\ &= \frac{1}{2} e^{2x} \tan \left(\frac{2x}{2} \right) + c = \frac{1}{2} e^{2x} \tan x + c \end{aligned}$$

87. $\int e^{2x} \left(\frac{1 - \sin 2x}{1 - \cos 2x} \right) dx$

Sol. Let $I = \int e^{2x} \left(\frac{1 - \sin 2x}{1 - \cos 2x} \right) dx$

Again Let $2x = t$

Diff on the both side w.r. to x , we get $\frac{d}{dx}(2x) = \frac{d}{dx}(t)$

$$2 = \frac{dt}{dx}$$

$$dx = \frac{dt}{2}$$

$$I = \int e^t \left(\frac{1 - \sin t}{1 - \cos t} \right) \frac{dt}{2}$$

$$I = \frac{1}{2} \int e^t \left(\frac{1 - \sin t}{1 - \cos t} \right) dt$$

$$I = \frac{1}{2} \int e^t \left(\frac{1 - 2 \sin \left(\frac{t}{2} \right) \cos \left(\frac{t}{2} \right)}{2 \sin^2 \left(\frac{t}{2} \right)} \right) dt$$

$$I = \frac{1}{2} \int e^t \left(\frac{1}{2 \sin^2(t/2)} - \frac{2 \sin(t/2) \cos(t/2)}{2 \sin^2(t/2)} \right) dt$$

$$I = \frac{1}{2} \int e^t \left(\frac{1}{2 \sin^2(t/2)} - \frac{\cos(t/2)}{\sin(t/2)} \right) dt$$

$$I = \frac{1}{2} \int e^t \left(\frac{1}{2} \operatorname{cosec}^2(t/2) - \cot(t/2) \right) dt$$

$$I = -\frac{1}{2} \int e^t \left[\cot(t/2) + \left(-\frac{1}{2} \operatorname{cosec}^2(t/2) \right) \right] dt$$

$$\text{Let } F(t) = \cot(t/2)$$

Differentiation on the both side w.r. to t we get

$$\frac{d}{dt} [F(t)] = \frac{d}{dt} [\cot(t/2)]$$

$$F'(t) = -\operatorname{cosec}^2(t/2) \times \frac{1}{2}$$

$$F'(t) = -\frac{1}{2} \operatorname{cosec}^2(t/2)$$

$$\text{Using } \int e^t [F(t) + F'(t)] dt = e^t F(t) + c$$

$$I = -\frac{1}{2} F(t) + c$$

$$I = -\frac{1}{2} e^t \cot(t/2) + c$$

$$I = -\frac{1}{2} e^{2x} \cot\left(\frac{2x}{2}\right) + c$$

$$I = -\frac{1}{2} e^{2x} \cot x + c$$