

SOME SPECIAL INTEGRALS (XII, R. S. AGGARWAL)

EXERCISE 14 A (Pg.No.: 719)

Evaluate:

1. $\int \frac{1}{(1-9x^2)} dx$

Sol. Let $I = \int \frac{1}{(1-9x^2)} dx \Rightarrow I = \int \frac{1}{9\left(\frac{1}{9}-x^2\right)} dx \Rightarrow I = \frac{1}{9} \int \frac{1}{\left(\frac{1}{3}\right)^2 - (x)^2} dx$

$$\Rightarrow I = \frac{1}{9} \cdot \frac{1}{2\left(\frac{1}{3}\right)} \log \left| \frac{\frac{1}{3}+x}{\frac{1}{3}-x} \right| + c \quad \therefore I = \frac{1}{6} \log \left| \frac{1+3x}{1-3x} \right| + c$$

2. $\int \frac{1}{(25-4x^2)} dx$

Sol. Let $I = \int \frac{1}{(25-4x^2)} dx \Rightarrow I = \int \frac{1}{4\left(\frac{25}{4}-x^2\right)} dx \Rightarrow I = \frac{1}{4} \int \frac{1}{\left(\frac{5}{2}\right)^2 - (x)^2} dx$

$$\Rightarrow I = \frac{1}{4} \cdot \frac{1}{2\left(\frac{5}{2}\right)} \log \left| \frac{\frac{5}{2}+x}{\frac{5}{2}-x} \right| + c \quad \therefore I = \frac{1}{20} \log \left| \frac{5+2x}{5-2x} \right| + c$$

3. $\int \frac{1}{(x^2+16)} dx$

Sol. Let $I = \int \frac{1}{(x^2+16)} dx \Rightarrow I = \int \frac{1}{(x)^2 + (4)^2} dx \quad \therefore I = \frac{1}{4} \tan^{-1} \left(\frac{x}{4} \right) + c$

4. $\int \frac{1}{(4+9x^2)} dx$

Sol. Let $I = \int \frac{1}{(4+9x^2)} dx \Rightarrow I = \int \frac{1}{9\left(\frac{4}{9}+x^2\right)} dx \Rightarrow I = \frac{1}{9} \int \frac{1}{\left(\frac{2}{3}\right)^2 + (x)^2} dx$

$$\Rightarrow I = \frac{1}{9} \cdot \frac{1}{\left(\frac{2}{3}\right)} \tan^{-1} \left(\frac{x}{2/3} \right) + c \quad \therefore I = \frac{1}{6} \tan^{-1} \left(\frac{3x}{2} \right) + c$$

5. $\int \frac{1}{(50+2x^2)} dx$

Sol. Let $I = \int \frac{1}{(50+2x^2)} dx \Rightarrow I = \int \frac{1}{2(25+x^2)} dx \Rightarrow I = \frac{1}{2} \int \frac{1}{(5)^2 + (x)^2} dx$

$$\Rightarrow I = \frac{1}{2} \cdot \frac{1}{5} \tan^{-1} \frac{x}{5} + c \quad \therefore I = \frac{1}{10} \tan^{-1} \frac{x}{5} + c$$

6. $\int \frac{1}{(16x^2 - 25)} dx$

Sol. Let $I = \int \frac{1}{(16x^2 - 25)} dx \Rightarrow I = \int \frac{1}{16 \left(x^2 - \frac{25}{16} \right)} dx \Rightarrow I = \frac{1}{16} \int \frac{1}{(x)^2 - \left(\frac{5}{4} \right)^2} dx$

$$\Rightarrow I = \frac{1}{16} \cdot \frac{1}{2 \left(\frac{5}{4} \right)} \log \left| \frac{x - \frac{5}{4}}{x + \frac{5}{4}} \right| + c \quad \therefore I = \frac{1}{40} \log \left| \frac{4x - 5}{4x + 5} \right| + c$$

7. $\int \frac{(x^2 - 1)}{(x^2 + 4)} dx$

Sol. Let $I = \int \frac{(x^2 - 1)}{(x^2 + 4)} dx = \int \left(\frac{x^2 + 4 - 5}{x^2 + 4} \right) dx = \int \left(\frac{x^2 + 4}{x^2 + 4} - \frac{5}{x^2 + 4} \right) dx \Rightarrow I = \int \left(1 - \frac{5}{x^2 + 4} \right) dx$

$$\Rightarrow I = \int dx - 5 \int \frac{1}{(x^2 + 4)} dx \Rightarrow I = x - 5 \int \frac{1}{(x)^2 + (2)^2} dx \quad \therefore I = x - \frac{5}{2} \tan^{-1} \frac{x}{2} + c$$

8. $\int \frac{x^2}{(9 + 4x^2)} dx$

Sol. Let $I = \int \frac{x^2}{(9 + 4x^2)} dx \Rightarrow I = \frac{1}{4} \int \frac{4x^2}{4x^2 + 9} dx = \frac{1}{4} \int \frac{4x^2 + 9 - 9}{4x^2 + 9} dx$

$$\Rightarrow \frac{1}{4} \int dx - \frac{9}{4} \cdot \frac{1}{4} \int \frac{1}{\left(\frac{9}{4} + x^2 \right)} dx \Rightarrow I = \frac{1}{4} \int dx - \frac{9}{16} \int \frac{1}{(x)^2 + \left(\frac{3}{2} \right)^2} dx$$

$$\Rightarrow I = \frac{1}{4} x - \frac{9}{3/2} \tan^{-1} \left(\frac{x}{3/2} \right) + c = \frac{1}{4} x - \frac{9}{16} \times \frac{2}{3} \tan^{-1} \left(\frac{2x}{3} \right) + c \quad \therefore I = \frac{1}{4} x - \frac{3}{8} \tan^{-1} \left(\frac{2x}{3} \right) + c$$

9. $\int \frac{e^x}{(e^{2x} + 1)} dx$

Sol. Let $I = \int \frac{e^x}{(e^{2x} + 1)} dx$, Put, $e^x = t \Rightarrow e^x dx = dt$

$$\Rightarrow I = \int \frac{1}{(t^2 + 1)} dt \Rightarrow I = \tan^{-1} t + c \quad \therefore I = \tan^{-1} (e^x) + c$$

10. $\int \frac{\sin x}{(1 + \cos^2 x)} dx$

Sol. Let $I = \int \frac{\sin x}{(1 + \cos^2 x)} dx$, Put $\cos x = t \Rightarrow -\sin x dx = dt \Rightarrow \sin x dx = (-dt)$

$$\Rightarrow I = \int \frac{-dt}{(1 + t^2)} \Rightarrow I = -\int \frac{1}{(1 + t^2)} dt \Rightarrow I = -\tan^{-1} t + c \quad \therefore I = -\tan^{-1} (\cos x) + c$$

11. $\int \frac{\cos x}{(1 + \sin^2 x)} dx$

Sol. Let $I = \int \frac{\cos x}{(1 + \sin^2 x)} dx$, Put, $\sin x = t \Rightarrow \cos x dx = dt$

$$\Rightarrow I = \int \frac{1}{(1+t^2)} dt \Rightarrow I = \tan^{-1} t + c \quad \therefore I = \tan^{-1}(\sin x) + c$$

12. $\int \frac{3x^5}{(1+x^{12})} dx$

Sol. Let $I = \int \frac{3x^5}{(1+x^{12})} dx \Rightarrow I = \int \frac{3x^5}{\{1+(x^6)^2\}} dx$, Put $x^6 = t \Rightarrow 6x^5 dx = dt \Rightarrow 3x^5 dx = \frac{dt}{2}$

$$\Rightarrow I = \int \frac{1}{(1+t^2)} \frac{dt}{2} \Rightarrow I = \frac{1}{2} \int \frac{1}{(1+t^2)} dt \Rightarrow I = \frac{1}{2} \tan^{-1} t + c \quad \therefore I = \frac{1}{2} \tan^{-1}(x^6) + c$$

13. $\int \frac{2x^3}{(4+x^8)} dx$

Sol. Let $I = \int \frac{2x^3}{(4+x^8)} dx \Rightarrow I = \int \frac{2x^3}{\{4+(x^4)^2\}} dx$, Put $x^4 = t \Rightarrow 4x^3 dx = dt \Rightarrow 2x^3 dx = \frac{dt}{2}$

$$\Rightarrow I = \int \frac{1}{(4+t^2)} \frac{dt}{2} \Rightarrow I = \frac{1}{2} \int \frac{1}{(4+t^2)} dt \Rightarrow I = \frac{1}{2} \int \frac{1}{(2)^2 + (t)^2} dt$$

$$\Rightarrow I = \frac{1}{2} \cdot \frac{1}{2} \tan^{-1} \left(\frac{t}{2} \right) + c \quad \therefore I = \frac{1}{4} \tan^{-1} \left(\frac{x^4}{2} \right) + c$$

14. $\int \frac{dx}{(e^x + e^{-x})}$

Sol. Let $I = \int \frac{dx}{(e^x + e^{-x})} \Rightarrow I = \int \frac{1}{\left(e^x + \frac{1}{e^x}\right)} dx \Rightarrow I = \int \frac{e^x}{(e^x)^2 + (1)^2} dx$, Put $e^x = t \Rightarrow e^x dx = dt$

$$\Rightarrow I = \int \frac{dt}{t^2 + 1} \quad \therefore I = \tan^{-1}(e^x) + c$$

15. $\int \frac{x}{(1-x^4)} dx$

Sol. Let $I = \int \frac{x}{(1-x^4)} dx \Rightarrow I = \int \frac{x}{\{1-(x^2)^2\}} dx$, Put $x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{dt}{2}$

$$\Rightarrow I = \int \frac{1}{(1-t^2)} \frac{dt}{2} \Rightarrow I = \frac{1}{2} \int \frac{1}{(1-t^2)} dt \Rightarrow I = \frac{1}{2} \int \frac{1}{(1)^2 - (t)^2} dt$$

$$\Rightarrow I = \frac{1}{2} \cdot \frac{1}{2(1)} \log \left| \frac{1+t}{1-t} \right| + c \quad \therefore I = \frac{1}{4} \log \left| \frac{1+x^2}{1-x^2} \right| + c$$

16. $\int \frac{x^2}{(a^6 - x^6)} dx$

Sol. Let $I = \int \frac{x^2}{(a^6 - x^6)} dx \Rightarrow I = \int \frac{x^2}{\{a^6 - (x^3)^2\}} dx$, Put $x^3 = t \Rightarrow 3x^2 dx = dt \Rightarrow x^2 dx = \frac{dt}{3}$

$$\Rightarrow I = \int \frac{1}{(a^6 - t^2)} \cdot \frac{dt}{3} \Rightarrow I = \frac{1}{3} \int \frac{1}{(a^3)^2 - (t)^2} dt$$

$$\Rightarrow I = \frac{1}{3} \cdot \frac{1}{2(a^3)} \log \left| \frac{a^3 + t}{a^3 - t} \right| + c \quad \therefore I = \frac{1}{6a^3} \log \left| \frac{a^3 + x^3}{a^3 - x^3} \right| + c$$

17. $\int \frac{1}{(x^2 + 4x + 8)} dx$

Sol. Let $I = \int \frac{1}{(x^2 + 4x + 8)} dx \Rightarrow I = \int \frac{1}{(x)^2 + 2x \cdot 2 + (2)^2 - (2)^2 + 8} dx$

$$\Rightarrow I = \int \frac{1}{(x+2)^2 + 4} dx \Rightarrow I = \int \frac{1}{(x+2)^2 + (2)^2} dx \quad \therefore I = \frac{1}{2} \tan^{-1} \left(\frac{x+2}{2} \right) + c$$

18. $\int \frac{1}{(4x^2 - 4x + 3)} dx$

Sol. Let $I = \int \frac{1}{(4x^2 - 4x + 3)} dx \Rightarrow I = \int \frac{1}{4(x^2 - x + 3/4)} dx \Rightarrow I = \frac{1}{4} \int \frac{1}{(x^2 - x + 3/4)} dx$

$$\Rightarrow I = \frac{1}{4} \int \frac{1}{(x)^2 - 2x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)} dx \Rightarrow I = \frac{1}{4} \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} dx$$

$$\Rightarrow I = \frac{1}{4} \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} dx \Rightarrow I = \frac{1}{4} \cdot \frac{1}{\left(\frac{1}{\sqrt{2}}\right)} \tan^{-1} \left[\frac{\left(x - \frac{1}{2}\right)}{\left(\frac{1}{\sqrt{2}}\right)} \right] + c \quad \therefore I = \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{2x-1}{\sqrt{2}} \right) + c$$

19. $\int \frac{1}{(2x^2 + x + 3)} dx$

Sol. Let $I = \int \frac{1}{(2x^2 + x + 3)} dx \Rightarrow I = \int \frac{1}{2\left(x^2 + \frac{x}{2} + \frac{3}{2}\right)} dx$

$$\Rightarrow I = \frac{1}{2} \int \frac{1}{(x)^2 + 2x \cdot \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + \frac{3}{2}} dx \Rightarrow I = \frac{1}{2} \int \frac{1}{\left(x + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2} dx$$

$$\Rightarrow I = \frac{1}{2} \cdot \frac{1}{\sqrt{23}/4} \tan^{-1} \left[\frac{\left(x + \frac{1}{4}\right)}{\left(\frac{\sqrt{23}}{4}\right)} \right] + c \quad \therefore I = \frac{2}{\sqrt{23}} \tan^{-1} \left(\frac{4x+1}{\sqrt{23}} \right) + c$$

20. $\int \frac{1}{(2x^2 - x - 1)} dx$

Sol. Let $I = \int \frac{1}{(2x^2 - x - 1)} dx \Rightarrow I = \frac{1}{2} \int \frac{1}{\left(x^2 - \frac{x}{2} - \frac{1}{2}\right)} dx$

$$\Rightarrow I = \frac{1}{2} \int \frac{1}{(x)^2 - 2x \cdot \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - \frac{1}{2}} dx \Rightarrow I = \frac{1}{2} \int \frac{1}{\left(x - \frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2} dx$$

$$\Rightarrow I = \frac{1}{2} \cdot \frac{1}{2\left(\frac{3}{4}\right)} \log \left[\frac{x - \frac{1}{4} - \frac{3}{4}}{x - \frac{1}{4} + \frac{3}{4}} \right] + c \Rightarrow I = \frac{1}{3} \log \left(\frac{2x-2}{2x+1} \right) + c \therefore I = \frac{1}{3} \log \left[\frac{2(x-1)}{(2x+1)} \right] + c$$

21. $\int \frac{1}{(3-2x-x^2)} dx$

Sol. Let $I = \int \frac{1}{(3-2x-x^2)} dx \Rightarrow I = \int \frac{1}{-[x^2+2x-3]} dx$

$$\Rightarrow I = \int \frac{1}{-[(x)^2 + 2 \cdot x \cdot 1 + (1)^2 - (1)^2 - 3]} dx \Rightarrow I = \int \frac{1}{-[(x+1)^2 - (2)^2]} dx$$

$$\Rightarrow I = \int \frac{1}{(2)^2 - (x+1)^2} dx \Rightarrow I = \frac{1}{2(2)} \log \left(\frac{2+x+1}{2-x-1} \right) + c \therefore I = \frac{1}{4} \log \left(\frac{3+x}{1-x} \right) + c$$

22. $\int \frac{x}{(x^2+3x+2)} dx$

Sol. For given $I = \int \frac{x}{(x^2+3x+2)} dx$ Let $x = A \cdot \frac{d}{dx}(x^2+3x+2) + B$

$$\Rightarrow x = A(2x+3) + B \Rightarrow x = 2Ax + 3A + B$$

Equating co-effecting we get, $2A = 1 \therefore A = \frac{1}{2}$

And $0 = 3A + B \Rightarrow B = -3A \Rightarrow B = -3\left(\frac{1}{2}\right) \therefore B = \frac{-3}{2}$

$$\Rightarrow I = \int \frac{A(2x+3) + B}{(x^2+3x+2)} dx \Rightarrow I = A \int \frac{(2x+3)}{(x^2+3x+2)} dx + B \int \frac{1}{(x^2+3x+2)} dx$$

$$\Rightarrow I = \frac{1}{2} \cdot I_1 - \frac{3}{2} \cdot I_2 \quad \dots (1)$$

$$\Rightarrow I_1 = \int \frac{(2x+3)}{(x^2+3x+2)} dx, \text{ Put } x^2+3x+2 = t \Rightarrow (2x+3)dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} \Rightarrow I_1 = \log|t| + c \Rightarrow I_1 = \log|x^2+3x+2| + c_1$$

Now, $I_2 = \int \frac{1}{x^2+3x+2} dx \Rightarrow I_2 = \int \frac{1}{(x)^2 + 2x \cdot \frac{3}{2} + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2} dx$

$$\Rightarrow I_2 = \int \frac{1}{\left(x + \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2} dx \Rightarrow I_2 = \frac{1}{2\left(\frac{1}{2}\right)} \log \left[\frac{x + \frac{3}{2} - \frac{1}{2}}{x + \frac{3}{2} + \frac{1}{2}} \right] + c$$

$$\Rightarrow I_2 = \log \left(\frac{2x+3-1}{2x+3+1} \right) + c \Rightarrow I_2 = \log \left(\frac{2x+2}{2x+4} \right) + c \Rightarrow I_2 = \log \left(\frac{x+1}{x+2} \right) + c$$

Putting the value of I_1 & I_2 in equation (1), $I = \frac{1}{2} \log |x^2 + 3x + 2| - \frac{3}{2} \log \left| \frac{x+1}{x+2} \right| + c$

23. $\int \frac{(x-3)}{(x^2+2x-4)} dx$

Sol. Let $I = \int \frac{(x-3)}{(x^2+2x-4)} dx$ and suppose $x-3 = A \frac{d}{dx}(x^2+2x-4) + B$

$$\Rightarrow x-3 = A(2x+2) + B \Rightarrow x-3 = 2Ax + 2A + B$$

Equating co-efficient we get, $1 = 2A \therefore A = \frac{1}{2}$

And $-3 = 2A + B \Rightarrow B = -3 - 2A \Rightarrow B = -3 - 2\left(\frac{1}{2}\right) \therefore B = -4$

$$\Rightarrow I = \int \frac{A(2x+2) + B}{(x^2+2x-4)} dx \Rightarrow I = A \int \frac{(2x+2)}{(x^2+2x-4)} dx + B \int \frac{1}{(x^2+2x-4)} dx$$

$$\Rightarrow I = A \int \frac{(2x+2)}{(x^2+2x-4)} dx + B \int \frac{1}{(x^2+2x-4)} dx \Rightarrow I = \frac{1}{2} I_1 - 4 I_2 \quad \dots (1)$$

$$\Rightarrow I_1 = \int \frac{(2x+2)}{(x^2+2x-4)} dx, \text{ Put, } x^2+2x-4 = t \Rightarrow (2x+2) dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} \Rightarrow I = \log |t| + c_1 \Rightarrow I = \log |x^2+2x-4| + c_1$$

Now, $I_2 = \int \frac{1}{(x^2+2x-4)} dx \Rightarrow I_2 = \int \frac{1}{(x)^2 + 2 \cdot x \cdot 1 + (1)^2 - (1)^2 - 4} dx$

$$\Rightarrow I_2 = \int \frac{1}{(x+1)^2 - (\sqrt{5})^2} dx \Rightarrow I_2 = \frac{1}{2\sqrt{5}} \log \left(\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}} \right) + c$$

Putting the value of I_1 & I_2 in equation (1), $I = \frac{1}{2} \log |x^2+2x-4| - \frac{2}{\sqrt{5}} \log \left| \frac{x+1-\sqrt{5}}{x+1+\sqrt{5}} \right| + c$

24. $\int \frac{(2x-3)}{(x^2+3x-18)} dx$

Sol. Let $I = \int \frac{(2x-3)}{(x^2+3x-18)} dx$ and suppose $(2x-3) = A \frac{d}{dx}(x^2+3x-18) + B$

$$\Rightarrow (2x-3) = A(2x+3) + B \Rightarrow 2x-3 = 2Ax + 3A + B$$

Equating co-efficient we get, $2 = 2A \therefore A = 1$

And $-3 = 3A + B \Rightarrow B = -3 - 3A = -3 - 3(1) = -6 \therefore B = -6$

$$\Rightarrow I = \int \frac{A(2x+3)+B}{(x^2+3x-18)} dx \Rightarrow I = A \int \frac{(2x+3)}{(x^2+3x-18)} dx + B \int \frac{1}{(x^2+3x-18)} dx$$

$$\Rightarrow I = I_1 - 6 I_2 \quad \dots (1)$$

$$\Rightarrow I_1 = \int \frac{(2x+3)}{(x^2+3x-18)} dx, \text{ Put } x^2+3x-18=t \Rightarrow (2x+3)dx=dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} \Rightarrow I_1 = \log|t| + c_1 \Rightarrow I_1 = \log|x^2+3x-18| + c_1$$

$$\text{Now, } I_2 = \int \frac{1}{(x^2+3x-18)} dx \Rightarrow I_2 = \int \frac{1}{(x)^2 + 2x \cdot \frac{3}{2} + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 18} dx$$

$$\Rightarrow I_2 = \int \frac{1}{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2} dx \Rightarrow I_2 = \frac{1}{2\left(\frac{9}{2}\right)} \log \left[\frac{x + \frac{3}{2} - \frac{9}{2}}{x + \frac{3}{2} + \frac{9}{2}} \right] + c_2$$

$$\Rightarrow I_2 = \frac{1}{9} \log \left(\frac{2x+3-9}{2x+3+9} \right) + c_2 \Rightarrow I_2 = \frac{1}{9} \log \left(\frac{2x-6}{2x+12} \right) + c_2 \Rightarrow I_2 = \frac{1}{9} \log \left| \frac{x-3}{x+6} \right| + c_2$$

$$\text{Putting the value of } I_1 \text{ \& } I_2 \text{ in equation (1), } I = \log|x^2+3x-18| - \frac{2}{3} \log \left| \frac{x-3}{x+6} \right| + c$$

25. $\int \frac{x^2}{(x^2+6x-3)} dx$

Sol. Let $I = \int \frac{x^2}{(x^2+6x-3)} dx \Rightarrow I = \int \left(1 - \frac{6x-3}{x^2+6x-3} \right) dx$

$$\Rightarrow I = \int dx - \int \frac{6x-3}{x^2+6x-3} dx \Rightarrow I = x - \int \frac{6x-3}{x^2+6x-3} dx$$

$$\text{Now let } 6x-3 = A(2x+6)+B \Rightarrow 6x-3 = 2Ax+6A+B$$

$$\text{Equating co-efficient, we get, } 6 = 2A \therefore A = 3$$

$$\text{And } -3 = 6A+B \Rightarrow B = -3-6A = -3-6(3) = -3-18 = -21 \therefore B = -21$$

$$\Rightarrow I = x - \int \frac{A(2x+6)+B}{(x^2+6x-3)} dx \Rightarrow I = x - \left\{ A \int \frac{2x+6}{x^2+6x-3} dx + B \int \frac{1}{x^2+6x-3} dx \right\}$$

$$\Rightarrow I = x - (3I_1 - 21I_2) \quad \dots (1)$$

$$\Rightarrow I_1 = \int \frac{2x+6}{x^2+6x-3} dx, \text{ Put } x^2+6x-3=t \Rightarrow (2x+6)dx=dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} \Rightarrow I_1 = \log|t| + c_1 \therefore I_1 = \log|x^2+6x-3| + c$$

$$\text{Now, } I_2 = \int \frac{1}{(x^2+6x-3)} dx \Rightarrow I_2 = \int \frac{1}{(x)^2 + 2x \cdot 3 + (3)^2 - (3)^2 - 3} dx$$

$$\Rightarrow I_2 = \int \frac{1}{(x+3)^2 - (2\sqrt{3})^2} dx \Rightarrow I_2 = \frac{1}{2(2\sqrt{3})} \log \left| \frac{x+3-2\sqrt{3}}{x+3+2\sqrt{3}} \right| + c_2$$

$$\Rightarrow I_1 = \frac{1}{4\sqrt{3}} \log \left(\frac{x+3-2\sqrt{3}}{x+3+2\sqrt{3}} \right) + c$$

Putting the values of I_1 & I_2 in equation (i), we get,

$$I = x - \left\{ 3 \log |x^2 + 6x - 3| + (-21) \frac{1}{4\sqrt{3}} \log \left(\frac{x+3-2\sqrt{3}}{x+3+2\sqrt{3}} \right) \right\} + c$$

$$\therefore I = x - 3 \log |x^2 + 6x - 3| + \frac{7\sqrt{3}}{4} \log \left| \frac{x+3-2\sqrt{3}}{x+3+2\sqrt{3}} \right| + c$$

26. $\int \frac{(2x-1)}{(2x^2+2x+1)} dx$

Sol. Let $I = \int \frac{(2x-1)}{(2x^2+2x+1)} dx$ and suppose $2x-1 = A \cdot \frac{d}{dx}(2x^2+2x+1) + B$

$$\Rightarrow 2x-1 = A(4x+2) + B \Rightarrow 2x-1 = 4Ax + 2A + B$$

Equating co-efficient both side, we get, $2 = 4A \quad \therefore A = \frac{1}{2}$

And $-1 = 2A + B \Rightarrow B = -1 - 2A = -1 - 2\left(\frac{1}{2}\right) = -2 \quad \therefore B = -2$

$$\Rightarrow I = \int \frac{A(4x+2) + B}{(2x^2+2x+1)} dx \Rightarrow I = A \int \frac{(4x+2)}{(2x^2+2x+1)} dx + B \int \frac{1}{(2x^2+2x+1)} dx$$

$$\Rightarrow I = \frac{1}{2} I_1 - 2 I_2 \quad \dots (1)$$

$$\Rightarrow I_1 = \int \frac{4x+2}{2x^2+2x+1} dx, \text{ Put } 2x^2+2x+1 = t \Rightarrow (4x+2) dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} \Rightarrow I_1 = \log |t| + c_1 \Rightarrow I_1 = \log |2x^2+2x+1| + c_1$$

$$\text{Now, } I_2 = \int \frac{1}{(2x^2+2x+1)} dx \Rightarrow I_2 = \frac{1}{2} \int \frac{1}{\left(x^2+x+\frac{1}{2}\right)} dx$$

$$\Rightarrow I_2 = \frac{1}{2} \int \frac{1}{(x)^2 + 2 \cdot x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)} dx \Rightarrow I_2 = \frac{1}{2} \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} dx$$

$$\Rightarrow I_2 = \frac{1}{2} \cdot \frac{1}{1/2} \tan^{-1} \left[\frac{(x+1/2)}{1/2} \right] + c \Rightarrow I_2 = \tan^{-1} (2x+1) + c$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$I = \frac{1}{2} \log |2x^2+2x+1| + (-2) \tan^{-1} (2x+1) + c$$

$$\therefore I = \frac{1}{2} \log |2x^2+2x+1| - 2 \tan^{-1} (2x+1) + c$$

27. $\int \frac{(1-3x)}{(3x^2+4x+2)} dx$

Sol. Let $I = \int \frac{(1-3x)}{(3x^2+4x+2)} dx$ and suppose $(1-3x) = A \cdot \frac{d}{dx}(3x^2+4x+2) + B$

$$\Rightarrow 1-3x = A(6x+4) + B \Rightarrow 1-3x = 6Ax + 4A + B$$

Equating co-efficient both side, we get, $-3 = 6A \quad \therefore A = -\frac{1}{2}$

And $1 = 4A + B \Rightarrow B = 1 - 4A = 1 - 4\left(-\frac{1}{2}\right) = 3$

$$\Rightarrow I = \int \frac{A(6x+4)+B}{(3x^2+4x+2)} dx + B \int \frac{1}{(3x^2+4x+2)} dx \Rightarrow I = -\frac{1}{2}I_1 + 3I_2 \quad \dots(1)$$

$$\Rightarrow I_1 = \int \frac{6x+4}{3x^2+4x+2} dx, \text{ Put } 3x^2+4x+2 = t \Rightarrow (6x+4)dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} \Rightarrow I_1 = \log|t| + c_1 \Rightarrow I_1 = \log|3x^2+4x+2| + c_1$$

Now, $I_2 = \int \frac{1}{(3x^2+4x+2)} dx \Rightarrow I_2 = \frac{1}{3} \int \frac{1}{\left(x^2 + \frac{4x}{3} + \frac{2}{3}\right)} dx$

$$\Rightarrow I_2 = \frac{1}{3} \int \frac{1}{\left(x\right)^2 + 2 \cdot x \cdot \frac{2}{3} + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} dx \Rightarrow I_2 = \frac{1}{3} \int \frac{1}{\left(x + \frac{2}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2}$$

$$\Rightarrow I_2 = \frac{1}{3} \cdot \frac{1}{\frac{\sqrt{2}}{3}} \tan^{-1} \left[\frac{\left(x + \frac{2}{3}\right)}{\left(\frac{\sqrt{2}}{3}\right)} \right] + c \Rightarrow I_2 = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{3x+2}{\sqrt{2}} \right) + c$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$I = -\frac{1}{2} \log|3x^2+4x+2| + 3 \left(\frac{1}{\sqrt{2}} \right) \tan^{-1} \left(\frac{3x+2}{\sqrt{2}} \right) + c$$

$$\therefore I = -\frac{1}{2} \log|3x^2+4x+2| + \frac{3}{\sqrt{2}} \tan^{-1} \left(\frac{3x+2}{\sqrt{2}} \right) + c$$

28. $\int \frac{2x}{(2+x-x^2)} dx$

Sol. Let $I = \int \frac{2x}{(2+x-x^2)} dx$ and suppose $2x = A \cdot \frac{d}{dx}(2+x-x^2) + B$

$$\Rightarrow 2x = A(1-2x) + B \Rightarrow 2x = A - 2Ax + B$$

Equating co-efficient both side, we get, $2 = -2A \quad \therefore A = -1$

And $0 = A + B \Rightarrow B = -A = 1$

$$\Rightarrow I = \int \frac{A(1-2x)+B}{(2+x-x^2)} dx \Rightarrow I = A \int \frac{(1-2x)}{(2+x-x^2)} dx + B \int \frac{1}{(2+x-x^2)} dx$$

$$\Rightarrow I = -I_1 + I_2 \quad \dots(1)$$

$$\Rightarrow I_1 = \int \frac{1-2x}{2+x-x^2} dx, \quad \text{Put } 2+x-x^2 = t \Rightarrow (1-2x)dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} \Rightarrow I_1 = \log|t| + c_1 \Rightarrow I_1 = \log|2+x-x^2| + c_1$$

$$\text{Now, } I_2 = \int \frac{1}{(2+x-x^2)} dx \Rightarrow I_2 = \int \frac{1}{-(x^2-x-2)} dx$$

$$\Rightarrow I_2 = \int \frac{1}{-\left[\left(x^2\right)-2 \cdot x \cdot \frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2-2\right]} dx \Rightarrow I_2 = \int \frac{1}{-\left[\left(x-\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2\right]} dx$$

$$\Rightarrow I_2 = \int \frac{1}{\left(\frac{3}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} dx \Rightarrow I_2 = \frac{1}{2\left(\frac{3}{2}\right)} \log \left| \frac{\frac{3}{2}+x-\frac{1}{2}}{\frac{3}{2}-x+\frac{1}{2}} \right| + c_2$$

$$\Rightarrow I_2 = \frac{1}{3} \log \left| \frac{3+2x-1}{3-2x+1} \right| + c_2 \Rightarrow I_2 = \frac{1}{3} \log \left| \frac{2+2x}{4-2x} \right| + c_2 \Rightarrow I_2 = \frac{1}{3} \log \left| \frac{1+x}{2-x} \right| + c_2$$

$$\text{Putting the value of } I_1 \text{ \& } I_2 \text{ in equation (1), we get, } I = -\log|2+x-x^2| + \frac{1}{3} \log \left| \frac{1+x}{2-x} \right| + c$$

29. $\int \frac{1}{(1+\cos^2 x)} dx$

Sol. Let $I = \int \frac{1}{(1+\cos^2 x)} dx$

$$\Rightarrow I = \int \frac{\frac{1}{\cos^2 x}}{\frac{1+\cos^2 x}{\cos^2 x}} dx \Rightarrow I = \int \frac{\sec^2 x}{(\sec^2 x + 1)} dx \Rightarrow I = \int \frac{\sec^2 x}{(1+\tan^2 x)+1} dx$$

$$\Rightarrow I = \int \frac{\sec^2 x}{(2+\tan^2 x)} dx, \text{ Put } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\Rightarrow I = \int \frac{dt}{2+t^2} \Rightarrow I = \int \frac{dt}{(\sqrt{2})^2 + (t)^2} \Rightarrow I = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t}{\sqrt{2}} \right) + c \quad \therefore I = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x}{\sqrt{2}} \right) + c$$

30. $\int \frac{dx}{(2+\sin^2 x)}$

Sol. Let $I = \int \frac{dx}{(2+\sin^2 x)}$

$$\Rightarrow I = \int \frac{\frac{1}{\cos^2 x}}{\frac{2+\sin^2 x}{\cos^2 x}} dx \Rightarrow I = \int \frac{\sec^2 x}{2\sec^2 x + \tan^2 x} dx \Rightarrow I = \int \frac{\sec^2 x}{2(1+\tan^2 x) + \tan^2 x} dx$$

$$\Rightarrow I = \int \frac{\sec^2 x}{2+2\tan^2 x + \tan^2 x} dx \Rightarrow I = \int \frac{\sec^2 x}{2+3\tan^2 x} dx, \text{ Put } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\Rightarrow I = \int \frac{dt}{(2+3t^2)} \Rightarrow I = \frac{1}{3} \int \frac{1}{\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2 + (t)^2} dt$$

$$\Rightarrow I = \frac{1}{3} \cdot \frac{1}{\left(\sqrt{2}/\sqrt{3}\right)} \tan^{-1} \left(\frac{t}{\sqrt{2}/\sqrt{3}} \right) + c \quad \therefore I = \frac{1}{\sqrt{6}} \tan^{-1} \left(\frac{\sqrt{3} \tan x}{\sqrt{2}} \right) + c$$

31. $\int \frac{1}{(a^2 \cos^2 x + b^2 \sin^2 x)} dx$

Sol. Let $I = \int \frac{1}{(a^2 \cos^2 x + b^2 \sin^2 x)} dx$,

$$\Rightarrow I = \int \frac{\frac{1}{\cos^2 x}}{\left(\frac{a^2 \cos^2 x + b^2 \sin^2 x}{\cos^2 x} \right)} dx \Rightarrow I = \int \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx \Rightarrow I = \frac{1}{b^2} \int \frac{\sec^2 x}{\frac{a^2}{b^2} + \tan^2 x} dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$, we get, $I = \frac{1}{b^2} \int \frac{dt}{\left(\frac{a}{b}\right)^2 + (t)^2}$

$$\Rightarrow I = \frac{1}{b} \cdot \frac{1}{(a/b)} \tan^{-1} \left(\frac{t}{a/b} \right) + c \quad \therefore I = \frac{1}{ab} \tan^{-1} \left(\frac{b \tan x}{a} \right) + c$$

32. $\int \frac{dx}{(\cos^2 x - 3 \sin^2 x)}$

Sol. Let $I = \int \frac{1}{(\cos^2 x - 3 \sin^2 x)} dx$,

$$\Rightarrow I = \int \frac{\frac{1}{\cos^2 x}}{\frac{\cos^2 x - 3 \sin^2 x}{\cos^2 x}} dx \Rightarrow I = \int \frac{\sec^2 x}{1 - 3 \tan^2 x} dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$; we get, $I = \int \frac{dt}{1 - 3t^2} \Rightarrow I = \frac{1}{3} \int \frac{1}{\frac{1}{3} - t^2} dt$

$$\Rightarrow I = \frac{1}{3} \int \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2 - (t)^2} dt \Rightarrow I = \frac{1}{3} \cdot \frac{1}{2 \cdot \left(\frac{1}{\sqrt{3}}\right)} \log \left| \frac{\frac{1}{\sqrt{3}} + t}{\frac{1}{\sqrt{3}} - t} \right| + c \quad \therefore I = \frac{1}{2\sqrt{3}} \log \left| \frac{1 + \sqrt{3} \tan x}{1 - \sqrt{3} \tan x} \right| + c$$

33. $\int \frac{1}{(\sin^2 x - 4 \cos^2 x)} dx$

Sol. Let $I = \int \frac{1}{(\sin^2 x - 4 \cos^2 x)} dx$, Dividing numerator and denominator by $\cos^2 x$.

$$\Rightarrow I = \int \frac{\frac{1}{\cos^2 x}}{\left(\frac{\sin^2 x - 4 \cos^2 x}{\cos^2 x} \right)} dx \Rightarrow I = \int \frac{\sec^2 x}{\tan^2 x - 4} dx \Rightarrow I = \int \frac{dt}{(t)^2 - (2)^2}$$

Put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$, we get, $I = \frac{1}{2(2)} \cdot \log \left| \frac{t-2}{t+2} \right| + c \therefore I = \frac{1}{4} \log \left| \frac{\tan x - 2}{\tan x + 2} \right| + c$

34. $\int \frac{1}{(\sin x \cos x + 2 \cos^2 x)} dx$

Sol. Let $I = \int \frac{1}{(\sin x \cos x + 2 \cos^2 x)} dx$, Dividing numerator and denominator by $\cos^2 x$.

$$\Rightarrow I = \int \frac{\frac{1}{\cos^2 x}}{\left(\frac{\sin x \cos x + 2 \cos^2 x}{\cos^2 x} \right)} dx \Rightarrow I = \int \frac{\sec^2 x}{\left(\frac{\sin x \cos x}{\cos^2 x} + \frac{2 \cos^2 x}{\cos^2 x} \right)} dx \Rightarrow I = \int \frac{\sec^2 x}{(\tan x + 2)} dx$$

Put $\tan x + 2 = t \Rightarrow \sec^2 x \, dx = dt$, we get, $I = \int \frac{dt}{t} \Rightarrow I = \log |t| + c \therefore I = \log |\tan x + 2| + c$

35. $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$

Sol. Let $I = \int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$, Dividing numerator and denominator by $\cos^4 x$.

$$\Rightarrow I = \int \frac{\frac{\sin 2x}{\cos^4 x}}{\frac{\sin^4 x + \cos^4 x}{\cos^4 x}} dx = \int \frac{\frac{2 \sin x \cdot \cos x}{\cos^4 x}}{\frac{\sin^4 x + \cos^4 x}{\cos^4 x}} dx$$

$$\Rightarrow I = \int \frac{2 \tan x \cdot \sec^2 x}{1 + (\tan^2 x)^2} dx, \text{ Put } \tan^2 x = t \Rightarrow 2 \tan x \cdot \sec^2 x \, dx = dt$$

$$\Rightarrow I = \int \frac{dt}{1+t^2} \Rightarrow I = \tan^{-1}(t) + c \therefore I = \tan^{-1}(\tan^2 x) + c$$

36. $\int \frac{(2 \sin 2\phi - \cos \phi)}{(6 - \cos^2 \phi - 4 \sin \phi)} d\phi$

Sol. Let $I = \int \left(\frac{2 \sin 2\phi - \cos \phi}{6 - \cos^2 \phi - 4 \sin \phi} \right) d\phi \Rightarrow I = \int \frac{2 \cdot 2 \sin \phi \cdot \cos \phi - \cos \phi}{6 - (1 - \sin^2 \phi) - 4 \sin \phi} d\phi$

$$\Rightarrow I = \int \frac{(4 \sin \phi - 1) \cos \phi}{6 - 1 + \sin^2 \phi - 4 \sin \phi} d\phi \Rightarrow I = \int \frac{(4 \sin \phi - 1) \cos \phi}{\sin^2 \phi - 4 \sin \phi + 5} d\phi$$

Put $\sin \phi = t \Rightarrow \cos \phi \, d\phi = dt$, we get,

$$I = \int \frac{(4t-1)}{(t^2-4t+5)} dt \text{ Now suppose } 4t-1 = A(2t-4) + B \Rightarrow 4t-1 = 2At-4A+B$$

Equating co-efficient both side, we get, $4 = 2A \therefore A = 2$

And $-1 = -4A + B \Rightarrow B = -1 + 4A = -1 + 4(2) = -1 + 8 = 7$

$$\Rightarrow I = \int \frac{A(2t-4) + B}{(t^2-4t+5)} dt \Rightarrow I = A \int \frac{(2t-4)}{(t^2-4t+5)} dt + B \int \frac{1}{(t^2-4t+5)} dt$$

$$\Rightarrow I = 2I_1 + 7I_2 \quad \dots (1)$$

$$\Rightarrow I_1 = \int \frac{2t-4}{t^2-4t+5} dt$$

Put $t^2 - 4t + 5 = t \Rightarrow (2t - 4)dt = dt$, we get,

$$I_1 = \int \frac{dt}{t} \Rightarrow I_1 = \log|t| + c_1 \Rightarrow I_1 = \log|t^2 - 4t + 5| + c_1$$

$$\text{Now, } I_2 = \int \frac{1}{(t^2 - 4t + 5)} dt \Rightarrow I_2 = \int \frac{1}{(t)^2 - 2 \cdot t \cdot 2 + (2)^2 - (2)^2 + 5} dt$$

$$\Rightarrow I_2 = \int \frac{1}{(t-2)^2 + (1)^2} \Rightarrow I_2 = \tan^{-1}\left(\frac{t-2}{1}\right) + c_2 \Rightarrow I_2 = \tan^{-1}(\sin \phi - 2) + c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$I = 2 \log|t^2 - 4t + 5| + 7 \tan^{-1}(\sin \phi - 2) + c$$

$$\therefore I = 2 \log|\sin^2 \phi - 4 \sin \phi + 5| + 7 \tan^{-1}(\sin \phi - 2) + c$$

37. $\int \frac{dx}{(\sin x - 2 \cos x)(2 \sin x + \cos x)}$

Sol. Let $I = \int \frac{1}{(\sin x - 2 \cos x)(2 \sin x + \cos x)} dx$

$$\Rightarrow I = \int \frac{1}{2 \sin^2 x + \sin x \cos x - 4 \sin x \cos x - 2 \cos^2 x} dx$$

$$\Rightarrow I = \int \frac{1}{2 \sin^2 x - 2 \cos^2 x - 3 \sin x \cos x} dx, \text{ dividing numerator and denominator by } \cos^2 x$$

$$\Rightarrow I = \int \frac{\sec^2 x}{2 \tan^2 x - 2 - 3 \tan x} dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$, we get,

$$I = \int \frac{dt}{2t^2 - 3t - 2} \Rightarrow I = \int \frac{1}{2\left(t^2 - \frac{3}{2}t - 1\right)} dt \Rightarrow I = \frac{1}{2} \int \frac{1}{(t)^2 - 2 \cdot t \cdot \frac{3}{4} + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 - 1} dt$$

$$\Rightarrow I = \frac{1}{2} \int \frac{1}{\left(t - \frac{3}{4}\right)^2 - \frac{25}{16}} dt \Rightarrow I = \frac{1}{2} \int \frac{1}{\left(t - \frac{3}{4}\right)^2 - \left(\frac{5}{4}\right)^2} dt$$

$$\Rightarrow I = \frac{1}{2} \cdot \frac{1}{2 \cdot \left(\frac{5}{4}\right)} \log \left| \frac{t - \frac{3}{4} - \frac{5}{4}}{t - \frac{3}{4} + \frac{5}{4}} \right| + c \Rightarrow I = \frac{1}{5} \log \left| \frac{t-2}{2t+1} \right| + c \therefore I = \frac{1}{5} \log \left| \frac{\tan x - 2}{2 \tan x + 1} \right| + c$$

38. $\int \frac{(1-x^2)}{(1+x^4)} dx$

Sol. Let $I = \int \frac{(1-x^2)}{(1+x^4)} dx \Rightarrow I = \int \frac{\frac{1}{x^2} - 1}{x^2 + \frac{1}{x^2}} dx = \int \frac{\left(1 - \frac{1}{x^2}\right)}{x^2 + \frac{1}{x^2} + 2 - 2} dx$

$$\Rightarrow I = -\int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - (\sqrt{2})^2} dx, \text{ Put } t = x + \frac{1}{x} \Rightarrow dt = \left(1 - \frac{1}{x^2}\right) dx$$

$$I = -\int \frac{dt}{t^2 - (\sqrt{2})^2} - \int \frac{dt}{(\sqrt{2})^2 - t^2} = \frac{1}{2\sqrt{2}} \log \left(\frac{\sqrt{2} + t}{\sqrt{2} - t} \right) + c = \frac{1}{2\sqrt{2}} \log \left(\frac{\sqrt{2} + x + \frac{1}{x}}{\sqrt{2} - x - \frac{1}{x}} \right) + c$$

39. $\int \frac{(x^2 + 1)}{(x^4 + x^2 + 1)} dx$

Sol. Let $I = \int \frac{(x^2 + 1)}{(x^4 + x^2 + 1)} dx \Rightarrow I = \int \frac{x^2 \left(1 + \frac{1}{x^2}\right)}{x^2 \left(x^2 + 1 + \frac{1}{x^2}\right)} dx \Rightarrow I = \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x^2 + \frac{1}{x^2}\right) + 1} dx$

$$\Rightarrow I = \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left[\left(x\right)^2 - 2 \cdot x \cdot \frac{1}{x} + \left(\frac{1}{x}\right)^2 + 2\right] + 1} dx \Rightarrow I = \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + (\sqrt{3})^2} dx$$

Put $\left(x - \frac{1}{x}\right) = t \Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$, we get, $I = \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + (\sqrt{3})^2} dx$

$$\Rightarrow I = \int \frac{dt}{(t)^2 + (\sqrt{3})^2} \Rightarrow I = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) + c$$

$$\Rightarrow I = \frac{1}{\sqrt{3}} \tan^{-1} \left[\frac{\left(x - \frac{1}{x}\right)}{\sqrt{3}} \right] + c \therefore I = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{3}x} \right) + c$$

40. $\int \frac{dx}{\sin^4 x + \cos^4 x}$

Sol. Let $I = \int \frac{1}{\sin^4 x + \cos^4 x} dx$, Dividing numerator and denominator by $\cos^4 x$.

$$\Rightarrow I = \int \frac{\frac{1}{\cos^4 x}}{\frac{\sin^4 x}{\cos^4 x} + \frac{\cos^4 x}{\cos^4 x}} dx \Rightarrow I = \int \frac{\sec^4 x}{\tan^4 x + 1} dx \Rightarrow I = \int \frac{(1 + \tan^2 x) \sec^2 x}{(\tan^4 x + 1)} dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$, we get, $I = \int \frac{t^2 + 1}{t^4 + 1} dt \Rightarrow I = \int \frac{t^2 \left(1 + \frac{1}{t^2}\right)}{t^2 \left(t^2 + \frac{1}{t^2}\right)} dt$

$$\Rightarrow I = \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt \Rightarrow I = \int \frac{1 + \frac{1}{t^2}}{(t)^2 - 2 \cdot t \cdot \frac{1}{t} + \left(\frac{1}{t}\right)^2 + 2} dt \Rightarrow I = \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + (\sqrt{2})^2} dt$$

Put $t - \frac{1}{t} = y \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = dy$, we get,

$$I = \int \frac{dy}{y^2 + (\sqrt{2})^2} \Rightarrow I = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{y}{\sqrt{2}} \right) + c \Rightarrow I = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t - \frac{1}{t}}{\sqrt{2}} \right) + c$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t^2 - 1}{\sqrt{2}t} \right) + c \Rightarrow I = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan^2 x - 1}{\sqrt{2} \tan x} \right) + c$$



EXERCISE 14B (Pg.No.: 732)

1. $\int \frac{1}{\sqrt{16-x^2}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{16-x^2}} dx \Rightarrow I = \int \frac{1}{\sqrt{(4)^2 - (x)^2}} dx \therefore I = \sin^{-1}\left(\frac{x}{4}\right) + c$

2. $\int \frac{1}{\sqrt{1-9x^2}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{1-9x^2}} dx \Rightarrow I = \frac{1}{3} \int \frac{1}{\sqrt{\frac{1}{9} - x^2}} dx \Rightarrow I = \frac{1}{3} \int \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2 - (x)^2}} dx$

$\Rightarrow I = \frac{1}{3} \sin^{-1}\left(\frac{x}{1/3}\right) + c \therefore I = \frac{1}{3} \sin^{-1}(3x) + c$

3. $\int \frac{1}{\sqrt{15-8x^2}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{15-8x^2}} dx \Rightarrow I = \frac{1}{\sqrt{8}} \int \frac{1}{\sqrt{\frac{15}{8} - x^2}} dx \Rightarrow I = \frac{1}{2\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{\sqrt{15}}{2\sqrt{2}}\right)^2 - (x)^2}} dx$

$\Rightarrow I = \frac{1}{2\sqrt{2}} \sin^{-1}\left(\frac{x}{\frac{\sqrt{15}}{2\sqrt{2}}}\right) + c \therefore I = \frac{1}{2\sqrt{2}} \sin^{-1}\left(\frac{2\sqrt{2}}{\sqrt{15}}x\right) + c$

4. $\int \frac{dx}{\sqrt{x^2-4}}$

Sol. Let $I = \int \frac{dx}{\sqrt{x^2-4}} \Rightarrow I = \int \frac{1}{\sqrt{(x)^2 - (2)^2}} dx$

$\therefore \int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left[x + \sqrt{x^2-a^2}\right] + c$

$\Rightarrow I = \log\left|x + \sqrt{x^2 - (2)^2}\right| + c \therefore I = \log\left|x + \sqrt{x^2 - 4}\right| + c$

5. $\int \frac{1}{\sqrt{4x^2-1}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{4x^2-1}} dx \Rightarrow I = \frac{1}{2} \int \frac{1}{\sqrt{(x)^2 - 1/4}} dx \Rightarrow I = \frac{1}{2} \int \frac{1}{\sqrt{(x)^2 - \left(\frac{1}{2}\right)^2}} dx$

$\Rightarrow I = \frac{1}{2} \log\left|x + \sqrt{x^2 - 1/4}\right| + c \therefore I = \frac{1}{2} \log\left|2x + \sqrt{4x^2 - 1}\right| + c$

6. $\int \frac{1}{\sqrt{9x^2-7}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{9x^2 - 7}} dx \Rightarrow I = \frac{1}{3} \int \frac{1}{\sqrt{x^2 - \frac{7}{9}}} dx \Rightarrow I = \frac{1}{3} \int \frac{1}{\sqrt{(x)^2 - \left(\frac{\sqrt{7}}{3}\right)^2}} dx$

$$\Rightarrow I = \frac{1}{3} \log \left| x + \sqrt{(x)^2 - \left(\frac{\sqrt{7}}{3}\right)^2} \right| + c \Rightarrow I = \frac{1}{3} \log \left| x + \sqrt{x^2 - \frac{7}{9}} \right| + c$$

$$\therefore I = \frac{1}{3} \log \left| 3x + \sqrt{9x^2 - 7} \right| + c$$

7. $\int \frac{1}{\sqrt{x^2 + 9}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{x^2 + 9}} dx \Rightarrow I = \int \frac{1}{\sqrt{(x)^2 + (3)^2}} dx$

$$\Rightarrow I = \log \left| x + \sqrt{x^2 + (3)^2} \right| + c \therefore I = \log \left| x + \sqrt{x^2 + 9} \right| + c$$

8. $\int \frac{1}{\sqrt{1 + 4x^2}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{1 + 4x^2}} dx \Rightarrow I = \frac{1}{2} \int \frac{1}{\sqrt{\frac{1}{4} + x^2}} dx \Rightarrow I = \frac{1}{2} \int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + (x)^2}} dx$

$$\Rightarrow I = \frac{1}{2} \log \left| x + \sqrt{x^2 + \frac{1}{4}} \right| + c \therefore I = \frac{1}{2} \log \left| 2x + \sqrt{4x^2 + 1} \right| + c$$

9. $\int \frac{1}{\sqrt{9 + 4x^2}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{9 + 4x^2}} dx \Rightarrow I = \frac{1}{2} \int \frac{1}{\sqrt{\frac{9}{4} + x^2}} dx \Rightarrow I = \frac{1}{2} \int \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2 + (x)^2}} dx$

$$\Rightarrow I = \frac{1}{2} \log \left| x + \sqrt{x^2 + \frac{9}{4}} \right| + c \therefore I = \frac{1}{2} \log \left| 2x + \sqrt{4x^2 + 9} \right| + c$$

10. $\int \frac{x}{\sqrt{9 - x^4}} dx$

Sol. Let $I = \int \frac{x}{\sqrt{9 - x^4}} dx$

Put $x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{dt}{2}$, we get, $I = \frac{1}{2} \int \frac{dt}{\sqrt{9 - t^2}}$

$$\Rightarrow I = \frac{1}{2} \int \frac{1}{\sqrt{(3)^2 - (t)^2}} dt \Rightarrow I = \frac{1}{2} \sin^{-1} \left(\frac{t}{3} \right) + c \therefore I = \frac{1}{2} \sin^{-1} \left(\frac{x^2}{3} \right) + c$$

11. $\int \frac{3x^2}{\sqrt{9 - 16x^6}} dx$

Sol. Let $I = \int \frac{3x^2}{\sqrt{9-16x^6}} dx \Rightarrow I = \frac{1}{4} \int \frac{3x^2}{\sqrt{\frac{9}{16}-x^6}} dx$

Put $x^3 = t \Rightarrow 3x^2 dx = dt$, we get,

$$I = \frac{1}{4} \int \frac{dt}{\sqrt{(3/4)^2 - (t)^2}} \Rightarrow I = \frac{1}{4} \sin^{-1} \left(\frac{t}{3/4} \right) + c \quad \therefore I = \frac{1}{4} \sin^{-1} \left(\frac{4x^3}{3} \right) + c$$

12. $\int \frac{\sec^2 x}{\sqrt{16 + \tan^2 x}} dx$

Sol. Let $I = \int \frac{\sec^2 x}{\sqrt{16 + \tan^2 x}} dx$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$, we get, $I = \int \frac{1}{\sqrt{(4)^2 + (t)^2}} dt$

$$\Rightarrow I = \log \left| t + \sqrt{t^2 + 4^2} \right| + c \quad \therefore I = \log \left| \tan x + \sqrt{\tan^2 x + 16} \right| + c$$

13. $\int \frac{\sin x}{\sqrt{4 + \cos^2 x}} dx$

Sol. Let $I = \int \frac{\sin x}{\sqrt{4 + \cos^2 x}} dx$, Put $\cos x = t \Rightarrow -\sin x dx = dt \Rightarrow \sin x dx = (-dt)$, we get,

$$I = \int \frac{(-dt)}{\sqrt{4 + t^2}} \Rightarrow I = -\int \frac{1}{\sqrt{(2)^2 + (t)^2}} dt \Rightarrow I = -\log \left| t + \sqrt{(2)^2 + (t)^2} \right| + c$$

$$\therefore I = -\log \left| \cos x + \sqrt{4 + \cos^2 x} \right| + c$$

14. $\int \frac{\cos x}{\sqrt{9 \sin^2 x - 1}} dx$

Sol. Let $I = \int \frac{\cos x}{\sqrt{9 \sin^2 x - 1}} dx$, Put $\sin x = t \Rightarrow \cos x dx = dt$, we get, $I = \int \frac{dt}{\sqrt{9t^2 - 1}}$

$$\Rightarrow I = \frac{1}{3} \int \frac{dt}{\sqrt{(t)^2 - \left(\frac{1}{3}\right)^2}} \Rightarrow I = \frac{1}{3} \log \left| t + \sqrt{(t)^2 - \left(\frac{1}{3}\right)^2} \right| + c$$

$$\Rightarrow I = \frac{1}{3} \log \left| \sin x + \sqrt{\sin^2 x - 1/9} \right| + c \quad \therefore I = \frac{1}{3} \log \left| 3 \sin x + \sqrt{9 \sin^2 x - 1} \right| + c$$

15. $\int \frac{e^x}{\sqrt{4 + e^{2x}}} dx$

Sol. Let $I = \int \frac{e^x}{\sqrt{4 + e^{2x}}} dx \Rightarrow I = \int \frac{e^x}{\sqrt{4 + (e^x)^2}} dx$, Put $e^x = t \Rightarrow e^x dx = dt$, we get,

$$I = \int \frac{dt}{\sqrt{(2)^2 + (t)^2}} \Rightarrow I = \log \left| t + \sqrt{(2)^2 + (t)^2} \right| + c \quad \therefore I = \log \left| e^x + \sqrt{4 + e^{2x}} \right| + c$$

16. $\int \frac{2e^x}{\sqrt{4-e^{2x}}} dx$

Sol. Let $I = \int \frac{2e^x}{\sqrt{4-e^{2x}}} dx \Rightarrow I = \int \frac{2e^x}{\sqrt{4-(e^x)^2}} dx$, Put $e^x = t \Rightarrow e^x dx = dt$, we get,

$$I = 2 \int \frac{dt}{\sqrt{(2)^2 - (t)^2}} \Rightarrow I = 2 \sin^{-1} \left(\frac{t}{2} \right) + c \quad \therefore I = 2 \sin^{-1} \left(\frac{e^x}{2} \right) + c$$

17. $I = \int \frac{1}{\sqrt{1-e^x}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{1-e^x}} dx = \int \frac{1}{\sqrt{e^x(1-e^x)}} dx = \int \frac{dx}{e^{x/2} \sqrt{(e^{-x/2}-1)}} = \int \frac{e^{-x/2}}{\sqrt{(e^{-x/2})^2 - 1}} dx$

Put $e^{-x/2} = t \Rightarrow e^{-x/2} \left(-\frac{1}{2} \right) dx = dt \Rightarrow e^{-x/2} dx = -2dt$, we get,

$$\Rightarrow I = \int \frac{-2dt}{\sqrt{t^2 - 1}} \Rightarrow I = -2 \log |t + \sqrt{t^2 - 1}| + c \quad \therefore I = -2 \log |e^{-x/2} + \sqrt{e^{-x} - 1}| + c$$

18. $\int \sqrt{\frac{a-x}{a+x}} dx$

Sol. Let $I = \int \sqrt{\frac{a-x}{a+x}} dx \Rightarrow I = \int \frac{\sqrt{a-x}}{\sqrt{a+x}} \cdot \frac{\sqrt{a-x}}{\sqrt{a-x}} dx \Rightarrow I = \int \frac{a-x}{\sqrt{a^2-x^2}} dx$

$$\Rightarrow I = a \int \frac{1}{\sqrt{a^2-x^2}} dx - \int \frac{x}{\sqrt{a^2-x^2}} dx \Rightarrow I = a \sin^{-1} \left(\frac{x}{a} \right) - I_1 \quad \dots (1)$$

$$\Rightarrow I_1 = \int \frac{x dx}{\sqrt{a^2-x^2}}, \quad \text{Put } a^2-x^2 = t \Rightarrow -2x dx = dt \Rightarrow x dx = -\frac{dt}{2},$$

We get, $I_1 = -\frac{1}{2} \int \frac{dt}{\sqrt{t}} \Rightarrow I_1 = \frac{-1}{2} \cdot \frac{t^{1/2}}{1/2} + c \quad \therefore I_1 = -\sqrt{a^2-x^2} + c$

Putting the value of I_1 in equation (1), we get, $I = a \sin^{-1} \left(\frac{x}{a} \right) + \sqrt{a^2-x^2} + c$

19. $\int \frac{1}{\sqrt{x^2+6x+5}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{x^2+6x+5}} dx \Rightarrow I = \int \frac{1}{\sqrt{(x)^2 + 2 \cdot x \cdot 3 + (3)^2 - (3)^2 + 5}} dx$

$$\Rightarrow I = \int \frac{1}{\sqrt{(x+3)^2 - (2)^2}} dx \Rightarrow I = \log \left| (x+3) + \sqrt{(x+3)^2 - (2)^2} \right| + c$$

$$\therefore I = \log \left| (x+3) + \sqrt{x^2+6x+5} \right| + c$$

20. $\int \frac{1}{\sqrt{(2-x)^2+1}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{(2-x)^2 + 1}} dx$, Put $2-x=t \Rightarrow -dx=dt \Rightarrow dx=-dt$.

We get, $I = \int \frac{-dt}{\sqrt{t^2 + 1}} \Rightarrow I = -\log |t + \sqrt{t^2 + 1}| + c$

$\Rightarrow I = -\log |(2-x) + \sqrt{(2-x)^2 + 1}| + c \quad \therefore I = -\log |(2-x) + \sqrt{x^2 - 4x + 5}| + c$

21. $\int \frac{1}{\sqrt{(x-3)^2 - 1}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{(x-3)^2 - 1}} dx$, Put $x-3=t \Rightarrow dx=dt$

We get, $I = \int \frac{1}{\sqrt{t^2 - 1}} dt \Rightarrow I = \log |t + \sqrt{t^2 - 1}| + c$

$\Rightarrow I = \log |(x-3) + \sqrt{(x-3)^2 - 1}| + c \quad \therefore I = \log |(x-3) + \sqrt{x^2 - 6x + 8}| + c$

22. $\int \frac{1}{\sqrt{x^2 - 6x + 10}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{x^2 - 6x + 10}} dx \Rightarrow I = \int \frac{1}{\sqrt{(x)^2 - 2 \cdot x \cdot 3 + (3)^2 - (3)^2 + 10}} dx$

$\Rightarrow I = \int \frac{1}{\sqrt{(x-3)^2 + (1)^2}} dx \Rightarrow I = \log |(x-3) + \sqrt{(x-3)^2 + (1)^2}| + c$

$\therefore I = \log |(x-3) + \sqrt{x^2 - 6x + 10}| + c$

23. $\int \frac{dx}{\sqrt{2+2x-x^2}}$

Sol. Let $I = \int \frac{dx}{\sqrt{2+2x-x^2}} \Rightarrow I = -\int \frac{1}{\sqrt{-[x^2 - 2x - 2]}} dx$

$\Rightarrow I = \int \frac{1}{\sqrt{-(x)^2 - 2 \cdot x \cdot 1 + (1)^2 - (1)^2 - 2}} dx \Rightarrow I = \int \frac{1}{\sqrt{-(x-1)^2 - (\sqrt{3})^2}} dx$

$\Rightarrow I = \int \frac{1}{\sqrt{(\sqrt{3})^2 - (x-1)^2}} dx \quad \therefore I = \sin^{-1} \left(\frac{x-1}{\sqrt{3}} \right) + c$

24. $\int \frac{1}{\sqrt{8-4x-2x^2}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{8-4x-2x^2}} dx \Rightarrow I = \int \frac{1}{\sqrt{-2[x^2 + 2x - 4]}} dx$

$\Rightarrow I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-(x^2 + 2x - 4)}} dx \Rightarrow I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-(x)^2 + 2 \cdot x \cdot 1 + (1)^2 - (1)^2 - 4}} dx$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-(x+1)^2 - (\sqrt{5})^2}} dx \Rightarrow I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{(\sqrt{5})^2 - (x+1)^2}} dx$$

$$\therefore I = \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{x+1}{\sqrt{5}} \right) + c$$

25. $\int \frac{1}{\sqrt{16-6x-x^2}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{16-6x-x^2}} dx \Rightarrow I = \int \frac{1}{\sqrt{-[x^2+6x-16]}} dx$

$$\Rightarrow I = \int \frac{1}{\sqrt{-(x^2+2 \cdot x \cdot 3 + (3)^2 - (3)^2 - 16)}} dx \Rightarrow I = \int \frac{1}{\sqrt{-(x+3)^2 - (5)^2}} dx$$

$$\Rightarrow I = \int \frac{1}{\sqrt{(5)^2 - (x+3)^2}} dx \quad \therefore I = \sin^{-1} \left(\frac{x+3}{5} \right) + c$$

26. $\int \frac{1}{\sqrt{7-6x-x^2}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{7-6x-x^2}} dx \Rightarrow I = \int \frac{1}{\sqrt{-[x^2+6x-7]}} dx$

$$\Rightarrow I = \int \frac{1}{\sqrt{-(x^2+2 \cdot x \cdot 3 + (3)^2 - (3)^2 - 7)}} dx \Rightarrow I = \int \frac{1}{\sqrt{-(x+3)^2 - (4)^2}} dx$$

$$\Rightarrow I = \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}} dx \quad \therefore I = \sin^{-1} \left(\frac{x+3}{4} \right) + c$$

27. $I = \int \frac{dx}{\sqrt{x-x^2}}$

Sol. Let $I = \int \frac{1}{\sqrt{-[x^2-x]}} dx \Rightarrow I = \int \frac{1}{\sqrt{-(x^2-2 \cdot x \cdot \frac{1}{2} + (\frac{1}{2})^2 - (\frac{1}{2})^2)}} dx$

$$\Rightarrow I = \int \frac{1}{\sqrt{-(x-\frac{1}{2})^2 - (\frac{1}{2})^2}} dx \Rightarrow I = \int \frac{1}{\sqrt{(\frac{1}{2})^2 - (x-\frac{1}{2})^2}} dx$$

$$\Rightarrow I = \sin^{-1} \left(\frac{x-\frac{1}{2}}{\frac{1}{2}} \right) + c \quad \therefore I = \sin^{-1} (2x-1) + c$$

28. $\int \frac{1}{\sqrt{8+2x-x^2}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{8+2x-x^2}} dx \Rightarrow I = \int \frac{1}{\sqrt{-[x^2-2x-8]}} dx$

$$\Rightarrow I = \int \frac{1}{\sqrt{-(x)^2 - 2 \cdot x \cdot 1 + (1)^2 - (1)^2 - 8}} dx \Rightarrow I = \int \frac{1}{\sqrt{-(x-1)^2 - (3)^2}} dx$$

$$\Rightarrow I = \int \frac{1}{\sqrt{(3)^2 - (x-1)^2}} dx \quad \therefore I = \sin^{-1}\left(\frac{x-1}{3}\right) + c$$

29. $\int \frac{1}{\sqrt{x^2 - 3x + 2}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{x^2 - 3x + 2}} dx \Rightarrow I = \int \frac{1}{\sqrt{(x)^2 - 2 \cdot x \cdot \frac{3}{2} + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2}} dx$

$$\Rightarrow I = \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx \Rightarrow I = \log \left| \left(x - \frac{3}{2}\right) + \sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right| + c$$

$$\therefore I = \log \left| \left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2} \right| + c$$

30. $\int \frac{1}{\sqrt{2x^2 + 3x - 2}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{2x^2 + 3x - 2}} dx \Rightarrow I = \int \frac{1}{\sqrt{2\left(x^2 + \frac{3}{2}x - 1\right)}} dx$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{x^2 + \frac{3}{2}x - 1}} dx \Rightarrow I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{(x)^2 + 2 \cdot x \cdot \frac{3}{4} + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 - 1}} dx$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(x + \frac{3}{4}\right)^2 - \left(\frac{5}{4}\right)^2}} dx \Rightarrow I = \frac{1}{\sqrt{2}} \log \left| \left(x + \frac{3}{4}\right) + \sqrt{\left(x + \frac{3}{4}\right)^2 - \left(\frac{5}{4}\right)^2} \right| + c$$

$$\therefore I = \frac{1}{\sqrt{2}} \log \left| \left(x + \frac{3}{4}\right) + \sqrt{x^2 + \frac{3}{2}x - 1} \right| + c$$

31. $\int \frac{1}{\sqrt{2x^2 + 4x + 6}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{2x^2 + 4x + 6}} dx \Rightarrow I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{x^2 + 2x + 3}} dx$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{(x)^2 + 2 \cdot x \cdot 1 + (1)^2 - (1)^2 + 3}} dx \Rightarrow I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{(x+1)^2 + (\sqrt{2})^2}} dx$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \log \left| (x+1) + \sqrt{(x+1)^2 + (\sqrt{2})^2} \right| + c \quad \therefore I = \frac{1}{\sqrt{2}} \log \left| (x+1) + \sqrt{x^2 + 2x + 3} \right| + c$$

32. $\int \frac{1}{\sqrt{1 + 2x - 3x^2}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{1+2x-3x^2}} dx \Rightarrow I = \int \frac{1}{\sqrt{-3\left[x^2 - \frac{2}{3}x - \frac{1}{3}\right]}} dx$

$$\Rightarrow I = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{-\left[(x)^2 - 2x \cdot \frac{1}{3} + \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)\right]}} dx$$

$$\Rightarrow I = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{-\left[\left(x - \frac{1}{3}\right)^2 - \left(\frac{2}{3}\right)^2\right]}} dx \Rightarrow I = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 - \left(x - \frac{1}{3}\right)^2}} dx$$

$$\Rightarrow I = \frac{1}{\sqrt{3}} \sin^{-1} \left[\frac{\left(x - \frac{1}{3}\right)}{\left(\frac{2}{3}\right)} \right] + c \quad \therefore I = \frac{1}{\sqrt{3}} \sin^{-1} \left(\frac{3x-1}{2} \right) + c$$

33. $\int \frac{1}{\sqrt{x}\sqrt{5-x}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{x}\sqrt{5-x}} dx \Rightarrow I = \int \frac{1}{\sqrt{-[x^2-5x]}} dx$

$$\Rightarrow I = \int \frac{1}{\sqrt{-\left[(x)^2 - 2x \cdot \frac{5}{2} + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2\right]}} dx \Rightarrow I = \int \frac{1}{\sqrt{-\left[\left(x - \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2\right]}} dx$$

$$\Rightarrow I = \int \frac{1}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(x - \frac{5}{2}\right)^2}} dx \Rightarrow I = \sin^{-1} \left[\frac{\left(x - \frac{5}{2}\right)}{\left(\frac{5}{2}\right)} \right] + c \quad \therefore I = \sin^{-1} \left(\frac{2x-5}{5} \right) + c$$

34. $\int \frac{1}{\sqrt{3+4x-2x^2}} dx$

Sol. Let $I = \int \frac{1}{\sqrt{3+4x-2x^2}} dx \Rightarrow I = \int \frac{1}{\sqrt{-[2x^2-4x-3]}} dx$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left[x^2 - 2x - \frac{3}{2}\right]}} dx \Rightarrow I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left[(x)^2 - 2x \cdot 1 + (1)^2 - (1)^2 - \frac{3}{2}\right]}} dx$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left[(x-1)^2 - \left(\frac{\sqrt{5}}{\sqrt{2}}\right)^2\right]}} dx \Rightarrow I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{\sqrt{5}}{\sqrt{2}}\right)^2 - (x-1)^2}} dx$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{x-1}{\sqrt{5}/\sqrt{2}} \right) + c \quad \therefore I = \frac{1}{\sqrt{2}} \sin^{-1} \left[\frac{\sqrt{2}(x-1)}{\sqrt{5}} \right] + c$$

35. $\int \frac{x^2}{\sqrt{x^6+2x^3+3}} dx$

Sol. Let $I = \int \frac{x^2}{\sqrt{x^6 + 2x^3 + 3}} dx \Rightarrow I = \int \frac{x^2}{\sqrt{(x^3)^2 + 2x^3 + 3}}$

Put $x^3 = t \Rightarrow 3x^2 dx = dt \Rightarrow x^2 dx = \frac{dt}{3}$

We get, $I = \frac{1}{3} \int \frac{dt}{\sqrt{t^2 + 2t + 3}} \Rightarrow I = \frac{1}{3} \int \frac{1}{\sqrt{(t)^2 + 2 \cdot t \cdot 1 + (1)^2 - (1)^2 + 3}} dt$

$\Rightarrow I = \frac{1}{3} \int \frac{1}{\sqrt{(t+1)^2 + (\sqrt{2})^2}} dt \Rightarrow I = \frac{1}{3} \log \left| (t+1) + \sqrt{(t+1)^2 + (\sqrt{2})^2} \right| + c$

$\therefore I = \frac{1}{3} \log \left| (x^3 + 1) + \sqrt{x^6 + 2x^3 + 3} \right| + c$

36. $\int \frac{(2x+3)}{\sqrt{x^2+x+1}} dx$

Sol. Let $I = \int \frac{(2x+3)}{\sqrt{x^2+x+1}} dx \Rightarrow I = \int \frac{(2x+3)}{\sqrt{x^2+x+1}} dx$

$\Rightarrow 2x+3 = A(2x+1) + B \Rightarrow 2x+3 = 2Ax + A + B$

Equating co-efficient both side, we get, $2 = 2A \therefore A = 1$

And $3 = A + B \Rightarrow B = 3 - A = 3 - 1 = 2$

$\Rightarrow I = \int \frac{A(2x+1) + B}{\sqrt{x^2+x+1}} dx \Rightarrow I = A \int \frac{(2x+1)}{\sqrt{x^2+x+1}} dx + B \int \frac{1}{\sqrt{x^2+x+1}} dx$

$\Rightarrow I = I_1 + 2I_2 \quad \dots (1)$

$\Rightarrow I_1 = \int \frac{2x+1}{\sqrt{x^2+x+1}} dx$, Put $x^2+x+1 = t \Rightarrow (2x+1)dx = dt$

$\Rightarrow I_1 = \int \frac{1}{\sqrt{t}} dt \Rightarrow I_1 = \frac{t^{1/2}}{1/2} + c_1 \Rightarrow I_1 = 2\sqrt{x^2+x+1} + c_1$

Now, $I_2 = \int \frac{1}{\sqrt{x^2+x+1}} dx \Rightarrow I_2 = \int \frac{1}{\sqrt{(x)^2 + 2 \cdot x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1}} dx$

$\Rightarrow I_2 = \int \frac{1}{\sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}} dx \Rightarrow I_2 = \log \left| \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right| + c$

$\Rightarrow I_2 = \log \left| \left(x + \frac{1}{2}\right) + \sqrt{x^2+x+1} \right| + c$

Putting the value of I_1 & I_2 in equation (1), we get, $I = 2\sqrt{x^2+x+1} + 2\log \left| \left(x + \frac{1}{2}\right) + \sqrt{x^2+x+1} \right| + c$

37. $\int \frac{(2x+3)}{\sqrt{x^2+4x+3}} dx$

Sol. Let $I = \int \frac{(2x+3)}{\sqrt{x^2+4x+3}} dx \Rightarrow 2x+3 = A(2x+4) + B \Rightarrow 2x+3 = 2Ax + 4A + B$

Equating co-efficient both side, we get, $2 = 2A \therefore A = 1$

And $3 = 4A + B \Rightarrow B = 3 - 4A = 3 - 4(1) = -1$

$$\Rightarrow I = \int \frac{A(2x+4)+B}{\sqrt{x^2+4x+3}} dx \Rightarrow I = A \int \frac{(2x+4)}{\sqrt{x^2+4x+3}} dx + B \int \frac{1}{\sqrt{x^2+4x+3}} dx$$

$$\Rightarrow I = I_1 - I_2 \quad \dots(1)$$

$$\Rightarrow I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+3}} dx$$

Put $x^2+4x+3=t \Rightarrow (2x+4)dx=dt$, we get,

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} \Rightarrow I_1 = 2\sqrt{t} + c_1 \Rightarrow I_1 = 2\sqrt{x^2+4x+3} + c_1$$

$$\text{Now, } I_2 = \int \frac{1}{\sqrt{x^2+4x+3}} dx \Rightarrow I_2 = \int \frac{1}{\sqrt{(x)^2+2 \cdot x \cdot 2+(2)^2-(2)^2+3}} dx$$

$$\Rightarrow I_2 = \int \frac{1}{\sqrt{(x+2)^2-(1)^2}} dx \Rightarrow I_2 = \log \left| x+2+\sqrt{x^2+4x+3} \right| + c$$

Putting the values of I_1 & I_2 in equation (1), we get,

$$\Rightarrow I = 2\sqrt{x^2+4x+3} + (-1) \log \left| (x+2)+\sqrt{x^2+4x+3} \right| + c$$

$$\therefore I = 2\sqrt{x^2+4x+3} - \log \left| (x+2)+\sqrt{x^2+4x+3} \right| + c$$

38. $\int \frac{(4x+3)}{\sqrt{2x^2+2x-3}} dx$

Sol. Let $I = \int \frac{(4x+3)}{\sqrt{2x^2+2x-3}} dx \Rightarrow 4x+3 = A(4x+2)+B \Rightarrow 4x+3 = 4Ax+2A+B$

Equating co-efficient both side, we get, $4 = 4A \therefore A = 1$ and $3 = 2A + B$

$$\Rightarrow B = 3 - 2A = 3 - 2(1) = 1 \Rightarrow I = \int \frac{A(4x+2)+B}{\sqrt{2x^2+2x-3}} dx$$

$$\Rightarrow I = A \int \frac{(4x+2)}{\sqrt{2x^2+2x-3}} dx + B \int \frac{1}{\sqrt{2x^2+2x-3}} dx$$

$$\Rightarrow I = I_1 + I_2 \quad \dots(1)$$

$$\Rightarrow I_1 = \int \frac{4x+2}{\sqrt{2x^2+2x-3}} dx, \text{ Put } 2x^2+2x-3=t \Rightarrow (4x+2)dx=dt$$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} \Rightarrow I_1 = 2\sqrt{t} + c_1 \Rightarrow I_1 = 2\sqrt{2x^2+2x-3} + c_1$$

$$\text{Now, } I_2 = \int \frac{1}{\sqrt{2x^2+2x-3}} dx \Rightarrow I_2 = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{x^2+x-3/2}} dx$$

$$\Rightarrow I_2 = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{(x)^2+2 \cdot x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)}} dx$$

$$\Rightarrow I_2 = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{\sqrt{7}}{4}\right)^2}} dx \Rightarrow I_2 = \frac{1}{\sqrt{2}} \log \left| \left(x+\frac{1}{2}\right) + \sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{\sqrt{7}}{4}\right)^2} \right| + c$$

$$\Rightarrow I_2 = \frac{1}{\sqrt{2}} \log \left| \left(x+\frac{1}{2}\right) + \sqrt{x^2 + x - \frac{3}{2}} \right| + c$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore I = 2\sqrt{2x^2 + 2x - 3} + \frac{1}{\sqrt{2}} \log \left| \left(x+\frac{1}{2}\right) + \sqrt{x^2 + x - \frac{3}{2}} \right| + c$$

39. $\int \frac{(3-2x)}{\sqrt{2+x-x^2}} dx$

Sol. Let $I = \int \frac{(3-2x)}{\sqrt{2+x-x^2}} dx \Rightarrow 3-2x = A(1-2x) + B \Rightarrow 3-2x = A-2Ax+B$

Equating co-efficient both side, we get, $-2 = -2A \therefore A=1$

And $3 = A+B \Rightarrow B = 3-A = 3-1 = 2$

$$\Rightarrow I = \int \frac{A(1-2x)+B}{\sqrt{2+x-x^2}} dx \Rightarrow I = A \int \frac{(1-2x)}{\sqrt{2+x-x^2}} dx + B \int \frac{1}{\sqrt{2+x-x^2}} dx$$

$$\Rightarrow I = AI_1 + BI_2 \quad \dots(1)$$

$$\Rightarrow I_1 = \int \frac{1-2x}{\sqrt{2+x-x^2}} dx \quad \text{Put } 2+x-x^2 = t \Rightarrow (1-2x) dx = dt, \text{ we get,}$$

$$I_1 = \int \frac{dt}{\sqrt{t}} \Rightarrow I_1 = 2\sqrt{t} + c_1 \Rightarrow I_1 = 2\sqrt{2+x-x^2} + c_1$$

Now, $I_2 = \int \frac{1}{\sqrt{2+x-x^2}} dx \Rightarrow I_2 = \int \frac{1}{\sqrt{-(x^2-x-2)}} dx$

$$\Rightarrow I_2 = \int \frac{1}{\sqrt{-(x^2-2x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 2}} dx \Rightarrow I_2 = \int \frac{1}{\sqrt{-(x-\frac{1}{2})^2 - \left(\frac{3}{2}\right)^2}} dx$$

$$\Rightarrow I_2 = \int \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}} dx \Rightarrow I_2 = \sin^{-1} \left[\frac{\left(x-\frac{1}{2}\right)}{\left(\frac{3}{2}\right)} \right] + c \Rightarrow I_2 = \sin^{-1} \left(\frac{2x-1}{3} \right) + c$$

Putting the value of I_1 & I_2 in equation (1), we get, $I = 2\sqrt{2+x-x^2} + 2 \sin^{-1} \left(\frac{2x-1}{3} \right) + c$

40. $\int \frac{(x+2)}{\sqrt{x^2+2x-1}} dx$

Sol. $I = \int \frac{(x+2)}{\sqrt{x^2+2x-1}} dx \Rightarrow x+2 = A(2x+2) + B \Rightarrow x+2 = 2Ax+2A+B$

Equating co-efficient both side, we get, $1 = 2A \therefore A = \frac{1}{2}$

$$\text{And } 2 = 2A + B \Rightarrow B = 2 - 2A = 2 - 2\left(\frac{1}{2}\right) = 2 - 1 = 1$$

$$\Rightarrow I = \int \frac{A(2x+2)+B}{\sqrt{x^2+2x-1}} dx \Rightarrow I = A \int \frac{(2x+2)}{\sqrt{x^2+2x-1}} dx + B \int \frac{1}{\sqrt{x^2+2x-1}} dx$$

$$\Rightarrow I = \frac{1}{2} I_1 + I_2 \quad \dots (1)$$

$$\Rightarrow I_1 = \int \frac{2x+2}{\sqrt{x^2+2x-1}} dx, \text{ Put } x^2+2x-1=t \Rightarrow (2x+2)dx=dt, \text{ we get,}$$

$$\therefore I_1 = \int \frac{dt}{\sqrt{t}} \Rightarrow I_1 = 2\sqrt{t} + c_1 \Rightarrow I_1 = 2\sqrt{x^2+2x-1} + c_1$$

$$\text{Now, } I_2 = \int \frac{1}{\sqrt{x^2+2x-1}} dx \Rightarrow I_2 = \int \frac{1}{\sqrt{(x)^2+2 \cdot x \cdot 1+(1)^2-(1)^2-1}} dx$$

$$\Rightarrow I_2 = \int \frac{1}{\sqrt{(x+1)^2-(\sqrt{2})^2}} dx \Rightarrow I_2 = \log \left| (x+1) + \sqrt{x^2+2x-1} \right| + c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$I = 2\left(\frac{1}{2}\right)\sqrt{x^2+2x-1} + \log \left| (x+1) + \sqrt{x^2+2x-1} \right| + c$$

$$\therefore I = \sqrt{x^2+2x-1} + \log \left| (x+1) + \sqrt{x^2+2x-1} \right| + c$$

41. $\int \frac{(3x+1)}{\sqrt{5-2x-x^2}} dx$

Sol. Let $I = \int \frac{(3x+1)}{\sqrt{5-2x-x^2}} dx \Rightarrow 3x+1 = A(-2-2x) + B \Rightarrow 3x+1 = -2A-2Ax+B$

Equating co-efficient both side, we get, $3 = -2A \therefore A = -\frac{3}{2}$ and $1 = -2A+B$

$$\Rightarrow B = 1 + 2A = 1 + 2\left(-\frac{3}{2}\right) = 1 - 3 = -2 \Rightarrow I = \int \frac{A(-2-2x)+B}{\sqrt{5-2x-x^2}} dx$$

$$\Rightarrow I = A \int \frac{(-2-2x)}{\sqrt{5-2x-x^2}} dx + B \int \frac{1}{\sqrt{5-2x-x^2}} dx \Rightarrow I = -\frac{3}{2} I_1 - 2I_2 \quad \dots (1)$$

$$\therefore I_1 = \int \frac{(-2-2x)}{\sqrt{5-2x-x^2}} dx, \text{ Put } 5-2x-x^2=t \Rightarrow (-2-2x)dx=dt$$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} \Rightarrow I_1 = 2\sqrt{t} + c_1 \Rightarrow I_1 = 2\sqrt{5-2x-x^2} + c_1$$

$$\text{Now, } I_2 = \int \frac{1}{\sqrt{5-2x-x^2}} dx \Rightarrow I_2 = \int \frac{1}{\sqrt{-[x^2+2x-5]}} dx$$

$$\Rightarrow I_2 = \int \frac{1}{\sqrt{-(x)^2+2 \cdot x \cdot 1+(1)^2-(1)^2-5}} dx \Rightarrow I_2 = \int \frac{1}{\sqrt{-(x+1)^2-(\sqrt{6})^2}} dx$$

$$\Rightarrow I_2 = \int \frac{1}{\sqrt{(\sqrt{6})^2 - (x+1)^2}} dx \Rightarrow I_2 = \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + c$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore I = 2\left(-\frac{3}{2}\right)\sqrt{5-2x-x^2} + (-2)\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + c \therefore I = -3\sqrt{5-2x-x^2} - 2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + c$$

42. $\int \frac{(6x+5)}{\sqrt{6+x-2x^2}} dx$

Sol. Let $I = \int \frac{(6x+5)}{\sqrt{6+x-2x^2}} dx \Rightarrow 6x+5 = A(1-4x) + B \Rightarrow 6x+5 = A-4Ax+B$

Equating co-efficient both side, we get, $6 = -4A \therefore A = -\frac{3}{2}$

And $5 = A+B \Rightarrow B = 5-A = 5+\frac{3}{2} \therefore B = \frac{13}{2}$

$$\Rightarrow I = \int \frac{A(1-4x)+B}{\sqrt{6+x-2x^2}} dx \Rightarrow I = A \int \frac{(1-4x)}{\sqrt{6+x-2x^2}} dx + B \int \frac{1}{\sqrt{6+x-2x^2}} dx$$

$$\Rightarrow I = -\frac{3}{2}I_1 + \frac{13}{2}I_2 \quad \dots(1)$$

$$\Rightarrow I_1 = \int \frac{1-4x}{\sqrt{6+x-2x^2}} dx, \text{ Put } 6+x-2x^2 = t \Rightarrow (1-4x)dx = dt, \text{ we get,}$$

$$\therefore I_1 = \int \frac{dt}{\sqrt{t}} \Rightarrow I_1 = 2\sqrt{t} + c_1 \Rightarrow I_1 = 2\sqrt{6+x-2x^2} + c_1$$

$$\text{Now, } I_2 = \int \frac{1}{\sqrt{6+x-2x^2}} dx \Rightarrow I_2 = \int \frac{1}{\sqrt{-(2x^2-x-6)}} dx$$

$$\Rightarrow I_2 = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left[x^2 - \frac{x}{2} - 3\right]}} dx \Rightarrow I_2 = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left[\left(x\right)^2 - 2 \cdot x \cdot \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - 3\right]}} dx$$

$$\Rightarrow I_2 = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left[\left(x-\frac{1}{4}\right)^2 - \left(\frac{7}{4}\right)^2\right]}} dx \Rightarrow I_2 = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{7}{4}\right)^2 - \left(x-\frac{1}{4}\right)^2}} dx$$

$$\Rightarrow I_2 = \frac{1}{\sqrt{2}} \sin^{-1}\left[\frac{\left(x-\frac{1}{4}\right)}{\left(\frac{7}{4}\right)}\right] + c \Rightarrow I_2 = \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{4x-1}{7}\right) + c$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$I = 2\left(-\frac{3}{2}\right)\sqrt{6+x-2x^2} + \frac{13}{2}\left(\frac{1}{\sqrt{2}}\right)\sin^{-1}\left(\frac{4x-1}{7}\right) + c$$

$$\therefore I = -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right) + c$$

43. $I = \int \sqrt{\frac{1+x}{x}} dx$

Sol. Let $I = \int \sqrt{\frac{1+x}{x}} dx \Rightarrow I = \int \frac{\sqrt{1+x}}{\sqrt{x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}} dx \Rightarrow I = \int \frac{1+x}{\sqrt{x^2+x}} dx$

$(1+x) = A(2x+1) + B \Rightarrow 1+x = 2Ax + A + B$

Equating co-efficient both side, we get, $A+B=1 \dots (1) \quad 2A=1 \quad \therefore A=\frac{1}{2}$

Putting the value of A in equation (1), we get, $B=\frac{1}{2}$

$\Rightarrow I = \int \frac{A(2x+1)+B}{\sqrt{x^2+x}} dx \Rightarrow I = A \int \frac{2x+1}{\sqrt{x^2+x}} dx + B \int \frac{1}{\sqrt{x^2+x}} dx$

$\Rightarrow I = \frac{1}{2} I_1 + \frac{1}{2} I_2 \dots (2)$

$\Rightarrow I_1 = \int \frac{2x+1}{\sqrt{x^2+x}} dx$, Put $x^2+x=t \Rightarrow (2x+1)dx = dt$

We get, $I_1 = \int \frac{1}{\sqrt{t}} dt \Rightarrow I_1 = 2\sqrt{t} + c_1 \Rightarrow I_1 = 2\sqrt{x^2+x} + c_1$

$\Rightarrow I_2 = \int \frac{1}{\sqrt{x^2+x}} dx \Rightarrow I_2 = \int \frac{1}{\sqrt{(x)^2 + 2 \cdot x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$

$\Rightarrow I_2 = \int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx \Rightarrow I_2 = \log \left| x + \frac{1}{2} + \sqrt{x^2+x} \right| + c_2$

Putting the value of I_1 & I_2 in equation (2), we get, $I = \sqrt{x^2+x} + \frac{1}{2} \log \left| x + \frac{1}{2} + \sqrt{x^2+x} \right| + c$

44. $\int \frac{(x+2)}{\sqrt{x^2+5x+6}} dx$

Sol. Let $I = \int \frac{(x+2)}{\sqrt{x^2+5x+6}} dx$

$x+2 = A \frac{d}{dx}(x^2+5x+6) + B$

$x+2 = A(2x+5) + B \dots (i)$

$x+2 = 2Ax + (5A+B)$

Comparing on the side with proper co-efficient we get $x = 2Ax$

$1 = 2A$

$\therefore A = \frac{1}{2}$

$2 = (5A+B)$

$$2 = 5 \times \frac{1}{2} + B$$

$$2 = \frac{5}{2} + B$$

$$2 - \frac{5}{2} = B$$

$$\frac{4-5}{2} = B$$

$$\therefore B = \frac{1}{2}$$

Putting the value of A and B in equation (i) we get

$$x+2 = \frac{1}{2}(2x+5) + \left(-\frac{1}{2}\right)$$

$$x+2 = \frac{1}{2}(2x+5) - \frac{1}{2}$$

$$I = \int \frac{\frac{1}{2}(2x+5) - \frac{1}{2}}{\sqrt{x^2+5x+6}} dx$$

$$I = \int \frac{\frac{1}{2}(2x+5)}{\sqrt{x^2+5x+6}} dx - \int \frac{\frac{1}{2}}{\sqrt{x^2+5x+6}} dx$$

$$I = \frac{1}{2} \int \frac{2x+5}{\sqrt{x^2+5x+6}} dx - \frac{1}{2} \int \frac{1}{\sqrt{x^2+5x+6}} dx$$

$$I = I_1 - I_2$$

$$I_1 = \frac{1}{2} \int \frac{2x+5}{\sqrt{x^2+5x+6}} dx$$

$$\text{Let } x^2+5x+6 = t$$

$$\text{Diff on the both side w.r. to } x, \text{ we get } \frac{d}{dx}(x^2+5x+6) = \frac{d}{dx}(t)$$

$$2x+5 = \frac{dt}{dx}$$

$$dx = \frac{dt}{2x+5}$$

$$I_1 = \frac{1}{2} \int \frac{2x+5}{\sqrt{t}} \times \frac{dt}{2x+5}$$

$$I_1 = \frac{1}{2} \int t^{-1/2} dt$$

$$I_1 = \frac{1}{2} \times \frac{t^{1/2}}{1/2} + c$$

$$I_2 = \frac{2}{2} \times t^{1/2} + c$$

$$I_1 = \sqrt{t} + c$$

$$I_1 = \sqrt{x^2+5x+6} + c$$

$$I_2 = \frac{1}{2} \int \frac{1}{\sqrt{x^2 + 5x + 6}} dx$$

$$I_2 = \frac{1}{2} \int \frac{1}{\sqrt{x^2 + 2x \cdot \frac{x}{5} + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 6}} dx$$

$$I_2 = \frac{1}{2} \int \frac{1}{\sqrt{\left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + 6}} dx$$

$$I_2 = \frac{1}{2} \int \frac{1}{\sqrt{\left(x + \frac{5}{2}\right)^2 - \frac{25 + 24}{4}}} dx$$

$$I_2 = \frac{1}{2} \int \frac{1}{\sqrt{\left(x + \frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

$$I_2 = \frac{1}{2} \log \left| \left(x + \frac{5}{2}\right) + \sqrt{\left(x + \frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right| + c$$

$$I_2 = \frac{1}{2} \log \left| \left(x + \frac{5}{2}\right) + \sqrt{x^2 + 5x + 6} \right| + c$$

From equation (i)

$$I = I_1 - I_2$$

$$I = \sqrt{x^2 + 5x + 6} - \frac{1}{2} \log \left| \left(x + \frac{5}{2}\right) + \sqrt{x^2 + 5x + 6} \right| + c$$

EXERCISE 14C (Pg.No.: 744)

Evaluate the following integrals :

1. $\int \sqrt{4-x^2} dx$

Sol. Let $I = \int \sqrt{4-x^2} dx \Rightarrow I = \int \sqrt{(2)^2 - (x)^2} dx$

$$\Rightarrow I = \frac{x}{2} \sqrt{4-x^2} + \frac{(2)^2}{2} \sin^{-1} \left(\frac{x}{2} \right) + c \quad \therefore I = \frac{x}{2} \sqrt{4-x^2} + 2 \sin^{-1} \left(\frac{x}{2} \right) + c$$

2. $\int \sqrt{4-9x^2} dx$

Sol. Let $I = \int \sqrt{4-9x^2} dx \Rightarrow I = \int \sqrt{9 \left(\frac{4}{9} - x^2 \right)} dx \Rightarrow I = 3 \int \sqrt{\left(\frac{2}{3} \right)^2 - (x)^2} dx$

$$\Rightarrow I = 3 \left[\frac{x}{2} \sqrt{\left(\frac{2}{3} \right)^2 - x^2} + \frac{\left(\frac{2}{3} \right)^2}{2} \sin^{-1} \left(\frac{x}{2/3} \right) \right] + c \Rightarrow I = 3 \left[\frac{x}{2} \sqrt{\frac{4}{9} - x^2} + \frac{4}{9 \cdot 2} \sin^{-1} \left(\frac{3x}{2} \right) \right] + c$$

$$\therefore I = \frac{x}{2} \sqrt{4-9x^2} + \frac{2}{3} \sin^{-1} \left(\frac{3x}{2} \right) + c$$

3. $\int \sqrt{x^2-2} dx$

Sol. Let $I = \int \sqrt{x^2-2} dx \Rightarrow I = \int \sqrt{(x)^2 - (\sqrt{2})^2} dx$

$$\Rightarrow I = \frac{x}{2} \sqrt{(x)^2 - (\sqrt{2})^2} - \frac{(\sqrt{2})^2}{2} \log \left| x + \sqrt{(x)^2 - (\sqrt{2})^2} \right| + c \quad \therefore I = \frac{x}{2} \sqrt{x^2-2} - \log \left| x + \sqrt{x^2-2} \right| + c$$

4. $\int \sqrt{2x^2-3} dx$

Sol. Let $I = \int \sqrt{2 \left(x^2 - \frac{3}{2} \right)} dx \Rightarrow I = \int \sqrt{2 \left(x^2 - \frac{3}{2} \right)} dx \Rightarrow I = \sqrt{2} \int \sqrt{\left(x^2 - \frac{3}{2} \right)} dx$

$$\Rightarrow I = \sqrt{2} \int \sqrt{(x)^2 - \left(\frac{\sqrt{3}}{\sqrt{2}} \right)^2} dx \Rightarrow I = \sqrt{2} \left[\frac{x}{2} \sqrt{(x)^2 - \left(\frac{\sqrt{3}}{\sqrt{2}} \right)^2} - \frac{\left(\frac{\sqrt{3}}{\sqrt{2}} \right)^2}{2} \log \left| x + \sqrt{(x)^2 - \left(\frac{\sqrt{3}}{\sqrt{2}} \right)^2} \right| \right] + c$$

$$\therefore I = \frac{x}{2} \sqrt{2x^2-3} - \frac{3}{2\sqrt{2}} \log \left| \sqrt{2}x + \sqrt{2x^2-3} \right| + c$$

5. $\int \sqrt{x^2+5} dx$

Sol. Let $I = \int \sqrt{x^2+5} dx \Rightarrow I = \int \sqrt{(x)^2 + (\sqrt{5})^2} dx$

$$\Rightarrow I = \frac{x}{2} \sqrt{(x)^2 + (\sqrt{5})^2} + \frac{(\sqrt{5})^2}{2} \log \left| x + \sqrt{(x)^2 + (\sqrt{5})^2} \right| + c$$

$$\therefore I = \frac{x}{2} \sqrt{x^2+5} + \frac{5}{2} \log \left| x + \sqrt{x^2+5} \right| + c$$

6. $\int \sqrt{4x^2 + 9} \, dx$

Sol. Let $I = \int \sqrt{4x^2 + 9} \, dx \Rightarrow I = \int \sqrt{4\left(x^2 + \frac{9}{4}\right)} \, dx \Rightarrow I = 2 \int \sqrt{\left(x\right)^2 + \left(\frac{3}{2}\right)^2} \, dx$

$$\Rightarrow I = 2 \left[\frac{x}{2} \sqrt{\left(x\right)^2 + \left(\frac{3}{2}\right)^2} + \frac{\left(\frac{3}{2}\right)^2}{2} \log \left| x + \sqrt{\left(x\right)^2 + \left(\frac{3}{2}\right)^2} \right| \right] + c$$

$$\therefore I = \frac{x}{2} \sqrt{4x^2 + 9} + \frac{9}{4} \log \left| 2x + \sqrt{4x^2 + 9} \right| + c$$

7. $\int \sqrt{3x^2 + 4} \, dx$

Sol. Let $I = \int \sqrt{3x^2 + 4} \, dx \Rightarrow I = \int \sqrt{3\left(x^2 + \frac{4}{3}\right)} \, dx \Rightarrow I = \sqrt{3} \int \sqrt{\left(x\right)^2 + \left(\frac{2}{\sqrt{3}}\right)^2} \, dx$

$$\Rightarrow I = \sqrt{3} \left[\frac{x}{2} \sqrt{\left(x\right)^2 + \left(\frac{2}{\sqrt{3}}\right)^2} + \frac{\left(\frac{2}{\sqrt{3}}\right)^2}{2} \log \left| x + \sqrt{\left(x\right)^2 + \left(\frac{2}{\sqrt{3}}\right)^2} \right| \right] + c$$

$$\therefore I = \frac{x}{2} \sqrt{3x^2 + 4} + \frac{2}{\sqrt{3}} \log \left| \sqrt{3}x + \sqrt{3x^2 + 4} \right| + c$$

8. $\int \cos x \sqrt{9 - \sin^2 x} \, dx$

Sol. Let $I = \int \cos x \sqrt{9 - \sin^2 x} \, dx$

Now, Put $\sin x = t \Rightarrow \cos x \, dx = dt$, we get, $I = \int \sqrt{9 - t^2} \, dt \Rightarrow I = \int \sqrt{(3)^2 - (t)^2} \, dt$

$$\Rightarrow I = \frac{t}{2} \sqrt{(3)^2 - (t)^2} + \frac{(3)^2}{2} \sin^{-1} \left(\frac{t}{3} \right) + c \quad \therefore I = \frac{\sin x}{2} \sqrt{9 - \sin^2 x} + \frac{9}{2} \sin^{-1} \left(\frac{\sin x}{3} \right) + c$$

9. $\int \sqrt{x^2 - 4x + 2} \, dx$

Sol. Let $I = \int \sqrt{x^2 - 4x + 2} \, dx \Rightarrow I = \int \sqrt{(x)^2 - 2 \cdot x \cdot 2 + (2)^2 - (2)^2 + 2} \, dx$

$$\Rightarrow I = \int \sqrt{(x-2)^2 - 2} \, dx \Rightarrow I = \int \sqrt{(x-2)^2 - (\sqrt{2})^2} \, dx$$

$$\Rightarrow I = \frac{x-2}{2} \sqrt{(x-2)^2 - (\sqrt{2})^2} - \frac{(\sqrt{2})^2}{2} \log \left| x-2 + \sqrt{(x-2)^2 - (\sqrt{2})^2} \right| + c$$

$$\therefore I = \frac{(x-2)}{2} \sqrt{x^2 - 4x + 2} - \log \left| (x-2) + \sqrt{x^2 - 4x + 2} \right| + c$$

10. $\int \sqrt{x^2 + 6x - 4} \, dx$

Sol. Let $I = \int \sqrt{x^2 + 6x - 4} \, dx \Rightarrow I = \int \sqrt{(x)^2 + 2 \cdot x \cdot 3 + (3)^2 - (3)^2 - 4} \, dx$

$$\Rightarrow I = \frac{x+3}{2} \sqrt{(x+3)^2 - (\sqrt{13})^2} - \frac{(\sqrt{13})^2}{2} \log \left| (x+3) + \sqrt{(x+3)^2 - (\sqrt{13})^2} \right| + c$$

$$\therefore I = \frac{(x+3)}{2} \sqrt{x^2+6x-4} - \frac{13}{2} \log \left| (x+3) + \sqrt{x^2+6x-4} \right| + c$$

11. $\int \sqrt{2x-x^2} dx$

Sol. Let $I = \int \sqrt{2x-x^2} dx \Rightarrow I = \int \sqrt{-[x^2-2x]} dx$

$$\Rightarrow I = \int \sqrt{-(x^2-2x+1)-(1)^2} dx \Rightarrow I = \int \sqrt{-(x-1)^2-(1)^2} dx$$

$$\Rightarrow I = \int \sqrt{(1)^2-(x-1)^2} dx \Rightarrow I = \frac{x-1}{2} \sqrt{(1)^2-(x-1)^2} + \frac{(1)^2}{2} \sin^{-1} \left(\frac{x-1}{1} \right) + c$$

$$\therefore I = \frac{(x-1)}{2} \sqrt{2x-x^2} + \frac{1}{2} \sin^{-1}(x-1) + c$$

12. $\int \sqrt{1-4x-x^2} dx$

Sol. Let $I = \int \sqrt{1-4x-x^2} dx \Rightarrow I = \int \sqrt{-[x^2+4x-1]} dx$

$$\Rightarrow I = \int \sqrt{-(x^2+2x+2)-(2)^2-1} dx \Rightarrow I = \int \sqrt{-(x+2)^2-5} dx$$

$$\Rightarrow I = \int \sqrt{(\sqrt{5})^2-(x+2)^2} dx \Rightarrow I = \frac{x+2}{2} \sqrt{(\sqrt{5})^2-(x+2)^2} + \frac{(\sqrt{5})^2}{2} \sin^{-1} \left(\frac{x+2}{\sqrt{5}} \right) + c$$

$$\therefore I = \frac{(x+2)}{2} \sqrt{1-4x-x^2} + \frac{5}{2} \sin^{-1} \left(\frac{x+2}{\sqrt{5}} \right) + c$$

13. $\int \sqrt{2ax-x^2} dx$

Sol. Let $I = \int \sqrt{2ax-x^2} dx \Rightarrow I = \int \sqrt{-[x^2-2ax]} dx$

$$\Rightarrow I = \int \sqrt{-(x^2-2xa+(a)^2)-(a)^2} dx \Rightarrow I = \int \sqrt{-(x-a)^2-a^2} dx$$

$$\Rightarrow I = \int \sqrt{(a)^2-(x-a)^2} dx \Rightarrow I = \frac{x-a}{2} \sqrt{(a)^2-(x-a)^2} + \frac{(a)^2}{2} \sin^{-1} \left(\frac{x-a}{a} \right) + c$$

$$\therefore I = \frac{(x-a)}{2} \sqrt{2ax-x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x-a}{a} \right) + c$$

14. $\int \sqrt{2x^2+3x+4} dx$

Sol. Let $I = \int \sqrt{2x^2+3x+4} dx \Rightarrow I = \int \sqrt{2 \left(x^2 + \frac{3}{2}x + 2 \right)} dx$

$$\Rightarrow I = \sqrt{2} \int \sqrt{x^2 + \frac{3}{2}x + 2} dx \Rightarrow I = \sqrt{2} \int \sqrt{(x)^2 + 2x \cdot \frac{3}{4} + \left(\frac{3}{4} \right)^2 - \left(\frac{3}{4} \right)^2 + 2} dx$$

$$\Rightarrow I = \sqrt{2} \int \sqrt{\left(x + \frac{3}{4} \right)^2 - \left(\frac{\sqrt{23}}{4} \right)^2} dx$$

$$\Rightarrow I = \sqrt{2} \left[\frac{x + \frac{3}{4}}{2} \sqrt{\left(x + \frac{3}{4}\right)^2 - \left(\frac{\sqrt{23}}{4}\right)^2} - \frac{\left(\frac{\sqrt{23}}{4}\right)^2}{2} \cdot \log \left| \left(x + \frac{3}{4}\right) + \sqrt{\left(x + \frac{3}{4}\right)^2 - \left(\frac{\sqrt{23}}{4}\right)^2} \right| \right] + c$$

$$\therefore I = \frac{(4x+3)}{8} \sqrt{2x^2+3x+4} + \frac{23\sqrt{2}}{32} \log \left| \left(x + \frac{3}{4}\right) + \frac{1}{\sqrt{2}} \sqrt{2x^2+3x+4} \right| + c$$

15. $\int \sqrt{x^2+x} \, dx$

Sol. Let $I = \int \sqrt{x^2+x} \, dx \Rightarrow I = \int \sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \, dx \Rightarrow I = \int \sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \, dx$

$$\Rightarrow I = \frac{x + \frac{1}{2}}{2} \sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} - \frac{\left(\frac{1}{2}\right)^2}{2} \log \left| \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right| + c$$

$$\therefore I = \frac{(2x+1)}{4} \sqrt{x^2+x} - \frac{1}{8} \log \left| \left(x + \frac{1}{2}\right) + \sqrt{x^2+x} \right| + c$$

16. $\int \sqrt{x^2+x+1} \, dx$

Sol. Let $I = \int \sqrt{x^2+x+1} \, dx \Rightarrow I = \int \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \, dx \Rightarrow I = \int \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \, dx$

$$\Rightarrow I = \frac{x + \frac{1}{2}}{2} \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2} \log \left| \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right| + c$$

$$\therefore I = \frac{(2x+1)}{4} \sqrt{x^2+x+1} + \frac{3}{8} \log \left| \left(x + \frac{1}{2}\right) + \sqrt{x^2+x+1} \right| + c$$

17. $\int (2x-5) \sqrt{x^2-4x+3} \, dx$

Sol. Let $I = \int (2x-5) \sqrt{x^2-4x+3} \, dx$

Using, $(2x-5) = A \frac{d}{dx} (x^2-4x+3) + B$

$$\Rightarrow 2x-5 = A(2x-4) + B \Rightarrow 2x-5 = 2Ax-4A+B$$

Now, Equating co-efficient both side we get, $2A=2 \Rightarrow A=1$ and $-4A+B=-5$

$$\Rightarrow B=-5+4A \Rightarrow B=-5+4(1) \therefore B=-1$$

$$\Rightarrow I = \int \{A(2x-4) + B\} \sqrt{x^2-4x+3} \, dx$$

$$\Rightarrow I = A \int (2x-4) \sqrt{x^2-4x+3} \, dx + B \int \sqrt{x^2-4x+3} \, dx$$

$$\Rightarrow I = I_1 - I_2 \quad \dots (1), \quad \text{where } I_1 = \int (2x-4) \sqrt{x^2-4x+3} \, dx$$

Put $x^2-4x+3=t \Rightarrow (2x-4) \, dx = dt$, we get,

$$\begin{aligned}\therefore I_1 &= \int \sqrt{t} \, dt \Rightarrow I_1 = \frac{t^{3/2}}{3/2} + c_1 \Rightarrow I_1 = \frac{2}{3}(x^2 - 4x + 3)^{3/2} + c_1 \Rightarrow I_2 = \int \sqrt{x^2 - 4x + 3} \, dx \\ \Rightarrow I_2 &= \int \sqrt{(x)^2 - 2 \cdot x \cdot 2 + (2)^2 - (2)^2 + 3} \, dx \Rightarrow I_2 = \int \sqrt{(x-2)^2 - (1)^2} \, dx \\ \Rightarrow I_2 &= \frac{x-2}{2} \sqrt{(x-2)^2 - (1)^2} - \frac{(1)^2}{2} \log \left| (x-2) + \sqrt{(x-2)^2 - (1)^2} \right| + c \\ \Rightarrow I_2 &= \frac{x-2}{2} \sqrt{x^2 - 4x + 3} - \frac{1}{2} \log \left| (x-2) + \sqrt{x^2 - 4x + 3} \right| + c\end{aligned}$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore I = \frac{2}{3}(x^2 - 4x + 3)^{3/2} - \frac{(x-2)}{2} \sqrt{x^2 - 4x + 3} + \frac{1}{2} \log \left| (x-2) + \sqrt{x^2 - 4x + 3} \right| + c$$

18. $\int (x+2)\sqrt{x^2+x+1} \, dx$

Sol. Let $I = \int (x+2)\sqrt{x^2+x+1} \, dx$

By Using, $x+2 = A \cdot \frac{d}{dx}(x^2+x+1) + B$

$$\Rightarrow x+2 = A(2x+1) + B \Rightarrow x+2 = 2Ax + A + B$$

Now, Equating co-efficient both side, we get, $2A = 1 \therefore A = \frac{1}{2}$

And $A+B=2 \Rightarrow B=2-A \Rightarrow B=2-\frac{1}{2} \therefore B=\frac{3}{2}$

$$\Rightarrow I = \int \left\{ A(2x+1) + B \right\} \sqrt{x^2+x+1} \, dx$$

$$\Rightarrow I = A \int (2x+1)\sqrt{x^2+x+1} \, dx + B \int \sqrt{x^2+x+1} \, dx \Rightarrow I = \frac{1}{2} I_1 + \frac{3}{2} I_2 \quad \dots(1)$$

$$\Rightarrow I_1 = \int (2x+1)\sqrt{x^2+x+1} \, dx$$

Put $x^2+x+1=t \Rightarrow (2x+1) \, dx = dt$, we get,

$$\therefore I_1 = \int \sqrt{t} \, dt \Rightarrow I_1 = \frac{t^{3/2}}{3/2} + c_1 \Rightarrow I_1 = \frac{2}{3}(x^2+x+1)^{3/2} + c_1$$

$$\Rightarrow I_2 = \int \sqrt{x^2+x+1} \, dx \Rightarrow I_2 = \int \sqrt{(x)^2 + 2 \cdot x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1} \, dx$$

$$\Rightarrow I_2 = \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \, dx \Rightarrow I_2 = \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \, dx$$

$$\Rightarrow I_2 = \frac{x+\frac{1}{2}}{2} \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2} \log \left| \left(x+\frac{1}{2}\right) + \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right| + c$$

$$\therefore I_2 = \frac{2x+1}{4} \sqrt{x^2+x+1} + \frac{3}{8} \log \left| \left(x+\frac{1}{2}\right) + \sqrt{x^2+x+1} \right| + c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore I = \frac{1}{3}(x^2+x+1)^{3/2} + \frac{3}{8}(2x+1)\sqrt{x^2+x+1} + \frac{9}{16} \log \left| \left(x + \frac{1}{2}\right) + \sqrt{x^2+x+1} \right| + c$$

19. $\int (x-5)\sqrt{x^2+x} \, dx$

Sol. Let $I = \int (x-5)\sqrt{x^2+x} \, dx$

Using, $(x-5) = A \cdot \frac{d}{dx}(x^2+x) + B$

$$\Rightarrow x-5 = A(2x+1) + B \Rightarrow x-5 = 2Ax + A + B$$

Equating co-efficient we get, $2A=1 \quad \therefore A = \frac{1}{2}$

And $A+B=-5 \Rightarrow B=-5-A \Rightarrow B=-5-\frac{1}{2} \quad \therefore B = -\frac{11}{2}$

$$\Rightarrow I = \int \{A(2x+1) + B\} \sqrt{x^2+x} \, dx \Rightarrow I = A \int (2x+1) \sqrt{x^2+x} \, dx + B \int \sqrt{x^2+x} \, dx$$

$$\Rightarrow I = \frac{1}{2} I_1 - \frac{11}{2} I_2 \quad \dots (1)$$

$$I_1 = \int (2x+1) \sqrt{x^2+x} \, dx$$

Now, Put $x^2+x=t \Rightarrow (2x+1) \, dx = dt$, we get,

$$\therefore I_1 = \int \sqrt{t} \, dt \Rightarrow I_1 = \frac{t^{3/2}}{3/2} + c_1 \Rightarrow I_1 = \frac{2}{3} (x^2+x)^{3/2} + c_1$$

$$\Rightarrow I_2 = \int \sqrt{x^2+x} \, dx \Rightarrow I_2 = \int \sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \, dx \Rightarrow I_2 = \int \sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \, dx$$

$$\Rightarrow I_2 = \frac{x + \frac{1}{2}}{2} \sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} - \frac{\left(\frac{1}{2}\right)^2}{2} \log \left| x + \frac{1}{2} + \sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right| + c$$

$$\Rightarrow I_2 = \frac{2x+1}{4} \sqrt{x^2+x} - \frac{1}{8} \log \left| \left(x + \frac{1}{2}\right) + \sqrt{x^2+x} \right| + c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore I = \frac{1}{3} (x^2+x)^{3/2} - \frac{11}{8} (2x+1) \sqrt{x^2+x} + \frac{11}{16} \log \left| \left(x + \frac{1}{2}\right) + \sqrt{x^2+x} \right| + c$$

20. $\int (4x+1)\sqrt{x^2-x-2} \, dx$

Sol. Let $I = \int (4x+1)\sqrt{x^2-x-2} \, dx$

By Using, $4x+1 = A \cdot \frac{d}{dx}(x^2-x-2) + B$

$$\Rightarrow 4x+1 = A(2x-1) + B \Rightarrow 4x+1 = 2Ax - A + B$$

Now Equating co-efficient both side we get, $2A=4 \quad \therefore A=2$

And $-A+B=1 \Rightarrow B=1+A \Rightarrow B=1+2 \quad \therefore B=3$

$$\Rightarrow I = \int \{A(2x-1) + B\} \sqrt{x^2-x-2} \, dx$$

$$\Rightarrow I = A \int (2x-1) \sqrt{x^2-x-2} dx + B \int \sqrt{x^2-x-2} dx \Rightarrow I = 2I_1 + 3I_2 \quad \dots(1)$$

$$\therefore I_1 = \int (2x-1) \sqrt{x^2-x-2} dx$$

Put $x^2-x-2=t \Rightarrow (2x-1) dx = dt$, we get,

$$\therefore I_1 = \int \sqrt{t} dt \Rightarrow I_1 = \frac{t^{3/2}}{3/2} + c_1 \Rightarrow I_1 = \frac{2}{3} (x^2-x-2)^{3/2} + c_1$$

$$\Rightarrow I_2 = \int \sqrt{x^2-x-2} dx$$

$$\Rightarrow I_2 = \int \sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} dx \Rightarrow I_2 = \int \sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} dx$$

$$\Rightarrow I_2 = \frac{x - \frac{1}{2}}{2} \sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} - \frac{\left(\frac{3}{2}\right)^2}{2} \log \left| \left(x - \frac{1}{2}\right) + \sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} \right| + c_2$$

$$\Rightarrow I_2 = \frac{2x-1}{4} \sqrt{x^2-x-2} - \frac{9}{8} \log \left| \left(x - \frac{1}{2}\right) + \sqrt{x^2-x-2} \right| + c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore I = \frac{4}{3} (x^2-x-2)^{3/2} + \frac{3}{4} (2x-1) \sqrt{x^2-x-2} - \frac{27}{8} \log \left| \left(x - \frac{1}{2}\right) + \sqrt{x^2-x-2} \right| + c$$

21. $\int (2x+1) \sqrt{2x^2+3} dx$

Sol. Let $I = \int (2x+1) \sqrt{2x^2+3} dx$

Put $2x+1 = A \frac{d}{dx} (2x^2+3) + B \Rightarrow 2x+1 = A(4x) + B$

Now Equating co-efficient both side we get, $4A=2 \therefore A = \frac{1}{2}$ & $B=1$

$$I = \int \{A(4x) + B\} \sqrt{2x^2+3} dx \Rightarrow I = A \int 4x \sqrt{2x^2+3} dx + B \int \sqrt{2x^2+3} dx$$

$$\Rightarrow I = \frac{1}{2} I_1 + I_2 \quad \dots(1)$$

$$I_1 = \int 4x \sqrt{2x^2+3} dx, \text{ Put } 2x^2+3=t \Rightarrow 4x dx = dt, \text{ we get,}$$

$$I_1 = \int \sqrt{t} dt \Rightarrow I_1 = \frac{t^{3/2}}{3/2} + c_1 \Rightarrow I_1 = \frac{2}{3} (2x^2+3)^{3/2} + c_1$$

$$I_2 = \int \sqrt{2x^2+3} dx \Rightarrow I_2 = \int \sqrt{2 \left(x^2 + \frac{3}{2}\right)} dx \Rightarrow I_2 = \sqrt{2} \int \sqrt{\left(x\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2} dx$$

$$\Rightarrow I_2 = \sqrt{2} \left[\frac{x}{2} \sqrt{\left(x\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2} \log \left| x + \sqrt{\left(x\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2} \right| \right] + c$$

$$\Rightarrow I_2 = \frac{x}{2} \sqrt{2x^2+3} + \frac{3}{2\sqrt{2}} \log \left| \sqrt{2}x + \sqrt{2x^2+3} \right| + c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore I = \frac{1}{3}(2x^2 + 3)^{3/2} + \frac{x}{2}\sqrt{2x^2 + 3} + \frac{3}{2\sqrt{2}} \log \left| \sqrt{2}x + \sqrt{2x^2 + 3} \right| + c$$

22. $\int x\sqrt{1+x-x^2} dx$

Sol. Let $I = \int x\sqrt{1+x-x^2} dx$

Using $x = A \frac{d}{dx}(1+x-x^2) + B$

$$\Rightarrow x = A(1-2x) + B \Rightarrow x = A - 2Ax + B$$

By Equating co-efficient both side, we get, $-2A = 1 \therefore A = -\frac{1}{2}$

And $A+B=0 \Rightarrow B=-A \therefore B=\frac{1}{2}$

$$\Rightarrow I = \int \{A(1-2x) + B\} \sqrt{1+x-x^2} dx$$

$$\Rightarrow I = A \int (1-2x) \sqrt{1+x-x^2} dx + B \int \sqrt{1+x-x^2} dx \Rightarrow I = -\frac{1}{2} I_1 + \frac{1}{2} I_2 \quad \dots (1)$$

$$\Rightarrow I_1 = \int (1-2x) \sqrt{1+x-x^2} dx, \text{ Put } 1+x-x^2 = t \Rightarrow (1-2x)dx = dt, \text{ we get.}$$

$$\therefore I_1 = \int \sqrt{t} dt \Rightarrow I_1 = \frac{t^{3/2}}{3/2} + c_1 \Rightarrow I_1 = \frac{2}{3} (1+x-x^2)^{3/2} + c_1$$

Now, $I_2 = \int \sqrt{1+x-x^2} dx$

$$\Rightarrow I_2 = \int \sqrt{-[x^2 - x - 1]} dx \Rightarrow I_2 = \int \sqrt{-\left[(x)^2 - 2x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 1\right]} dx$$

$$\Rightarrow I_2 = \int \sqrt{-\left[\left(x - \frac{1}{2}\right)^2 - \frac{5}{4}\right]} dx \Rightarrow I_2 = \int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} dx$$

$$\Rightarrow I_2 = \frac{x - \frac{1}{2}}{2} \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} + \frac{\left(\frac{\sqrt{5}}{2}\right)^2}{2} \sin^{-1} \frac{\left(x - \frac{1}{2}\right)}{\frac{\sqrt{5}}{2}} + c_2$$

$$\Rightarrow I_2 = \frac{2x-1}{4} \sqrt{1+x-x^2} + \frac{5}{8} \sin^{-1} \left(\frac{2x-1}{\sqrt{5}} \right) + c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore I = -\frac{1}{3} (1+x-x^2)^{3/2} + \frac{1}{8} (2x-1) \sqrt{1+x-x^2} + \frac{5}{16} \sin^{-1} \left(\frac{2x-1}{\sqrt{5}} \right) + c$$

23. $\int (2x-5) \sqrt{2+3x-x^2} dx$

Sol. Let $I = \int (2x-5) \sqrt{2+3x-x^2} dx$

Using $2x-5 = A \frac{d}{dx}(2+3x-x^2) + B$

$$\Rightarrow 2x-5 = A(3-2x) + B \Rightarrow 2x-5 = 3A-2Ax+B$$

By Equating co-efficient both side, we get, $-2A = 2 \therefore A = -1$

$$\text{And } 3A+B = -5 \Rightarrow B = -5-3A \therefore B = -2$$

$$\Rightarrow I = \int \{A(3-2x) + B\} \sqrt{2+3x-x^2} dx$$

$$\Rightarrow I = A \int (3-2x) \sqrt{2+3x-x^2} dx + B \int \sqrt{2+3x-x^2} dx \Rightarrow I = -I_1 - 2I_2 \quad \dots(1)$$

$$\Rightarrow I_1 = \int (3-2x) \sqrt{2+3x-x^2} dx, \text{ Put } 2+3x-x^2 = t \Rightarrow (3-2x) dx = dt, \text{ we get,}$$

$$\therefore I_1 = \int \sqrt{t} dt \Rightarrow I_1 = \frac{t^{3/2}}{3/2} + c_1 \Rightarrow I_1 = \frac{2}{3} (2+3x-x^2)^{3/2} + c_1$$

$$\text{Now, } I_2 = \int \sqrt{2+3x-x^2} dx$$

$$\Rightarrow I_2 = \int \sqrt{-(x^2-3x+2)} dx \Rightarrow I_2 = \int \sqrt{-\left[(x)^2 - 2 \cdot x \cdot \frac{3}{2} + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2\right]} dx$$

$$\Rightarrow I_2 = \int \sqrt{-\left[\left(x-\frac{3}{2}\right)^2 - \frac{17}{4}\right]} dx \Rightarrow I_2 = \int \sqrt{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2} dx$$

$$\Rightarrow I_2 = \frac{x-\frac{3}{2}}{2} \sqrt{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2} + \frac{\left(\frac{\sqrt{17}}{2}\right)^2}{2} \sin^{-1} \left(\frac{x-\frac{3}{2}}{\frac{\sqrt{17}}{2}} \right) + c_2$$

$$\Rightarrow I_2 = \frac{2x-3}{4} \sqrt{2+3x-x^2} + \frac{17}{8} \sin^{-1} \left(\frac{2x-3}{\sqrt{17}} \right) + c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore I = -\frac{2}{3} (2+3x-x^2)^{3/2} - \frac{1}{2} (2x-3) \sqrt{2+3x-x^2} - \frac{17}{4} \sin^{-1} \left(\frac{2x-3}{\sqrt{17}} \right) + c$$

24. $\int (6x+5) \sqrt{6+x-2x^2} dx$

Sol. Let $I = \int (6x+5) \sqrt{6+x-2x^2} dx$

$$\text{Using, } 6x+5 = A \frac{d}{dx} (6+x-2x^2) + 5$$

$$\Rightarrow 6x+5 = A(1-4x) + B \Rightarrow 6x+5 = A-4Ax+B$$

By Equating co-efficient both side, we get, $-4A = 6 \therefore A = -\frac{3}{2}$

$$\text{And } A+B = 5 \Rightarrow B = 5-A \therefore B = \frac{13}{2}$$

$$\Rightarrow I = \int \{A(1-4x) + B\} \sqrt{6+x-2x^2} dx$$

$$\Rightarrow I = A \int (1-4x) \sqrt{6+x-2x^2} dx + B \int \sqrt{6+x-2x^2} dx \Rightarrow I = -\frac{3}{2} I_1 + \frac{13}{2} I_2 \quad \dots(1)$$

$$\Rightarrow I_1 = \int (1-4x) \sqrt{6+x-2x^2} dx, \text{ Put } 6+x-2x^2 = t \Rightarrow (1-4x) dx = dt, \text{ we get,}$$

$$\therefore I_1 = \int \sqrt{t} \, dt \Rightarrow I_1 = \frac{t^{3/2}}{3/2} + c_1 \Rightarrow I_1 = \frac{2}{3}(6+x-2x^2)^{3/2} + c_1$$

$$\text{Now, } I_2 = \int \sqrt{6+x-2x^2} \, dx$$

$$\Rightarrow I_2 = \int \sqrt{-2\left[x^2 - \frac{x}{2} - 3\right]} \, dx \Rightarrow I_2 = \sqrt{2} \int \sqrt{-\left[\left(x\right)^2 - 2x \cdot \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - 3\right]} \, dx$$

$$\Rightarrow I_2 = \sqrt{2} \int \sqrt{-\left[\left(x - \frac{1}{4}\right)^2 - \frac{49}{16}\right]} \, dx \Rightarrow I_2 = \sqrt{2} \int \sqrt{\left(\frac{7}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2} \, dx$$

$$\Rightarrow I_2 = \sqrt{2} \left[\frac{x - \frac{1}{4}}{2} \sqrt{\left(\frac{7}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2} + \frac{\left(\frac{7}{4}\right)^2}{2} \sin^{-1} \left(\frac{x - \frac{1}{4}}{\frac{7}{4}} \right) \right] + c_2$$

$$\Rightarrow I_2 = \frac{4x-1}{8} \sqrt{6+x-2x^2} + \frac{49}{16\sqrt{2}} \sin^{-1} \left(\frac{4x-1}{7} \right) + c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore I = -(6+x-2x^2)^{3/2} + \frac{13(4x-1)}{16} \sqrt{6+x-2x^2} + \frac{637}{32\sqrt{2}} \sin^{-1} \left(\frac{4x-1}{7} \right) + c$$

25. $\int (x+1)\sqrt{1-x-x^2} \, dx$

Sol. Let $I = \int (x+1)\sqrt{1-x-x^2} \, dx$

$$\text{Using, } x+1 = A \frac{d}{dx}(1-x-x^2) + B$$

$$\Rightarrow x+1 = A(-1-2x) + B \Rightarrow x+1 = -A-2Ax+B \Rightarrow x+1 = (-A+B) - 2Ax$$

$$\text{By Equating co-efficient both side, we get, } -2A=1 \therefore A = -\frac{1}{2}$$

$$\text{And } -A+B=1 \Rightarrow B=1+A \Rightarrow B=1-\frac{1}{2} \therefore B=\frac{1}{2}$$

$$\Rightarrow I = \int \{A(-1-2x) + B\} \sqrt{1-x-x^2} \, dx$$

$$\Rightarrow I = A \int (-1-2x) \sqrt{1-x-x^2} \, dx + B \int \sqrt{1-x-x^2} \, dx \Rightarrow I = \frac{-1}{2} I_1 + \frac{1}{2} I_2 \quad \dots(1)$$

$$\Rightarrow I_1 = \int (-1-2x) \sqrt{1-x-x^2} \, dx \quad \text{Put } 1-x-x^2 = t \Rightarrow (-1-2x) \, dx = dt, \text{ we get,}$$

$$\therefore I_1 = \int \sqrt{t} \, dt \Rightarrow I_1 = \frac{t^{3/2}}{3/2} + c_1 \Rightarrow I_1 = \frac{2}{3}(1-x-x^2)^{3/2} + c_1$$

$$\text{Now, } I_2 = \int \sqrt{1-x-x^2} \, dx$$

$$\Rightarrow I_2 = \int \sqrt{-\left[x^2 + x - 1\right]} \, dx \Rightarrow I_2 = \int \sqrt{-\left[\left(x\right)^2 + 2x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 1\right]} \, dx$$

$$\Rightarrow I_2 = \int \sqrt{-\left[\left(x + \frac{1}{2}\right)^2 - \frac{5}{4}\right]} \, dx \Rightarrow I_2 = \int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2} \, dx$$

$$\Rightarrow I_2 = \frac{x+\frac{1}{2}}{2} \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x+\frac{1}{2}\right)^2} + \frac{\left(\frac{\sqrt{5}}{2}\right)^2}{2} \sin^{-1} \frac{\left(x+\frac{1}{2}\right)}{\sqrt{5}/2} + c_2$$

$$\Rightarrow I_2 = \frac{2x+1}{4} \sqrt{1-x-x^2} + \frac{5}{8} \sin^{-1} \left(\frac{2x+1}{\sqrt{5}} \right) + c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore I = \frac{-1}{3} (1-x-x^2)^{3/2} + \frac{2x+1}{8} (\sqrt{1-x-x^2}) + \frac{5}{16} \sin^{-1} \left(\frac{2x+1}{\sqrt{5}} \right) + c$$

26. $\int (x-3)\sqrt{x^2+3x-18} dx$

Sol. Let $x-3 = \lambda \frac{d}{dx} (x^2+3x-18)$

$$\Rightarrow x-3 = \lambda (2x+3) + \mu$$

Comparing coefficient of x and constant term we get

$$1 = 2\lambda, -3 = 3\lambda + \mu \Rightarrow \lambda = \frac{1}{2}, \mu = -3 - \frac{3}{2} = -\frac{9}{2}$$

$$\therefore \text{Given integral} = \int \left[\frac{1}{2} (2x+3) + \left(-\frac{9}{2} \right) \right] dx$$

$$= \frac{1}{2} \int (2x+3) \sqrt{x^2+3x-18} dx + \left(-\frac{9}{2} \right) \int \sqrt{x^2+3x-18} dx$$

$$= \frac{1}{2} \left[\frac{(x^2+3x-18)^{3/2}}{\frac{3}{2}} \right] - \frac{9}{2} \int \sqrt{\left(x+\frac{3}{2}\right)^2 - \frac{9}{4} - 18} dx$$

$$= \frac{1}{3} (x^2+3x-18)^{3/2} - \frac{9}{2} \int \sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2} dx$$

$$= \frac{1}{3} (x^2+3x-18)^{3/2} - \frac{9}{8} \left[(2x+3) \sqrt{x^2+3x-18} - \frac{81}{2} \log \left[x + \frac{3}{2} + \sqrt{x^2+3x-18} \right] \right]$$