

INTEGRATION USING PARTIAL FRACTIONS (XII, R. S. AGGARWAL)

EXERCISE 15A (Pg. no.: 762)

Evaluate

1. $\int \frac{dx}{x(x+2)}$

Sol. Let $I = \int \frac{dx}{x(x+2)}$, Put $\frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$

...(1)

$\Rightarrow A(x+2) + Bx = 1$, Put $x+2=0 \quad \therefore x=-2, B=-\frac{1}{2}$

Put $x=0, A=\frac{1}{2}$, from equation (1), we get, $\frac{1}{x(x+2)} = \frac{1}{2} \cdot \frac{1}{x} - \frac{1}{2} \cdot \frac{1}{x+2}$

$\Rightarrow \int \frac{1}{x(x+2)} dx = \frac{1}{2} \int \frac{1}{x} dx - \frac{1}{2} \int \frac{1}{x+2} dx = \frac{1}{2} \log x - \frac{1}{2} \log(x+2) + c$
 $= \frac{1}{2} [\log x - \log(x+2)] + c = \frac{1}{2} \log \frac{x}{x+2} + c$

2. $\int \frac{2x+1}{(x+2)(x-3)} dx$

Sol. Let $I = \int \frac{2x+1}{(x+2)(x-3)} dx$, Put $\frac{2x+1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$... (1)

$\Rightarrow 2x+1 = A(x-3) + B(x+2)$

Put $x-3=0, x=3; 2 \times 3+1 = A(0) + B(3+2) \Rightarrow B = \frac{7}{5}$

Put $x+2=0 \Rightarrow x=-2; -4+1 = A(-2-3) + B(0) \Rightarrow A = \frac{-3}{-5} = \frac{3}{5}$

Now, From equation (1), we get, $\frac{2x+1}{(x+2)(x-3)} = \frac{3}{5} \cdot \frac{1}{x+2} + \frac{7}{5} \cdot \frac{1}{x-3}$

$\Rightarrow \int \frac{2x+1}{(x+2)(x-3)} dx = \frac{3}{5} \int \frac{1}{x+2} dx + \frac{7}{5} \int \frac{1}{x-3} dx = \frac{3}{5} \log(x+2) + \frac{7}{5} \log(x-3) + c$

3. $\int \frac{x}{(x+2)(3-2x)} dx$

Sol. Let $I = \int \frac{x}{(x+2)(3-2x)} dx$, Put $\frac{x}{(x+2)(3-2x)} = \frac{A}{x+2} + \frac{B}{3-2x}$... (1)

$\Rightarrow A(3-2x) + B(x+2) = x$

Put $3-2x=0 \Rightarrow x=\frac{3}{2}; A(0) + B\left(\frac{3}{2}+2\right) = \frac{3}{2} \Rightarrow B\left(\frac{7}{2}\right) = \frac{3}{2} \therefore B = \frac{3}{7}$

Put $x+2=0 \Rightarrow x=-2; A(7) + B(0) = -2 \Rightarrow A = \frac{-2}{7}$

Now, From equation (1), we get, $\frac{x}{(x+2)(3-2x)} = \frac{-2}{7} \cdot \frac{1}{x+2} + \frac{3}{7} \cdot \frac{1}{3-2x}$

$$\Rightarrow \int \frac{x}{(x+2)(3-2x)} dx = \frac{-2}{7} \int \frac{1}{x+2} dx + \frac{3}{7} \int \frac{1}{3-2x} dx$$

$$= \frac{-2}{7} \log(x+2) + \frac{3}{7} \cdot \frac{1}{-2} \log(3-2x) + c = \frac{-2}{7} \log(x+2) - \frac{3}{14} \log(3-2x) + c$$

4. $\int \frac{dx}{x(x-2)(x-4)}$

Sol. Let $I = \int \frac{dx}{x(x-2)(x-4)}$, Put $\frac{1}{x(x-2)(x-4)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-4}$... (1)

$$\Rightarrow A(x-2)(x-4) + Bx(x-4) + Cx(x-2) = 1$$

Put $x-2=0 \Rightarrow x=2$; $A(0) + B \cdot 2(2-4) + C(0) = 1 \Rightarrow B \cdot 2(-2) = 1 \Rightarrow B = -\frac{1}{4}$

Put $x-4=0 \Rightarrow x=4$; $A(0) + B(0) + C \cdot 4(4-2) = 1 \Rightarrow C \cdot 4(2) = 1, C = \frac{1}{8}$

Put $x=0 \Rightarrow A(0-2)(0-4) + B(0) + C(0) = 1 \Rightarrow A = \frac{1}{8}$

Now, From equation (1), $\frac{1}{x(x-2)(x-4)} = \frac{1}{8} \cdot \frac{1}{x} - \frac{1}{4} \cdot \frac{1}{x-2} + \frac{1}{8} \cdot \frac{1}{x-4}$

$$\Rightarrow \int \frac{1}{x(x-2)(x-4)} dx = \frac{1}{8} \int \frac{1}{x} dx - \frac{1}{4} \int \frac{1}{x-2} dx + \frac{1}{8} \int \frac{1}{x-4} dx = \frac{1}{8} \log x - \frac{1}{4} \log(x-2) + \frac{1}{8} \log(x-4) + c$$

5. $\int \frac{2x-1}{(x-1)(x+2)(x-3)} dx$

Sol. Let $I = \int \frac{2x-1}{(x-1)(x+2)(x-3)} dx$, Put $\frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$... (1)

$$\Rightarrow A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2) = 2x-1$$

Put $x+2=0 \Rightarrow x=-2$; $A(0) + B(-2-1)(-2-3) + C(0) = 2x-1$

$$\Rightarrow B(-3)(-5) = -5, B = \frac{-1}{3}$$

Put $x-3=0 \Rightarrow x=3$; $A(0) + B(0) + C(2)(5) = 5 \Rightarrow C = \frac{1}{2}$

Put $x-1=0 \Rightarrow x=1$; $A(3)(-2) + B(0) + C(0) = 2-1=1 \Rightarrow A = -\frac{1}{6}$

Now, From equation (1), we get, $\frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{-1}{6} \cdot \frac{1}{(x-1)} - \frac{1}{3} \cdot \frac{1}{x+2} + \frac{1}{2} \cdot \frac{1}{x-3}$

$$\Rightarrow \int \frac{2x-1}{(x-1)(x+2)(x-3)} dx = \frac{-1}{6} \int \frac{1}{x-1} dx - \frac{1}{3} \int \frac{1}{x+2} dx + \frac{1}{2} \int \frac{1}{x-3} dx$$

$$= \frac{-1}{6} \log(x-1) - \frac{1}{3} \log(x+2) + \frac{1}{2} \log(x-3) + c$$

6. $\int \frac{2x-3}{(x^2-1)(2x+3)} dx$

Sol. Let $I = \int \frac{2x-3}{(x^2-1)(2x+3)} dx$

Put $\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x-1)(x+1)(2x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(2x+3)} \quad \dots(1)$

$\Rightarrow A(x+1)(2x+3) + B(x-1)(2x+3) + C(x-1)(x+1) = 2x-3$

Put $x+1=0 \Rightarrow x=-1$; $A(0) + B(-1-1)(-2+3) + C(0) = -2-3 \therefore B = \frac{-5}{-2} = \frac{5}{2}$

Put $x-1=0 \Rightarrow x=1$; $A(2)(2+3) + B(0) + C(0) = 2-3 = -1 \therefore A = \frac{-1}{10}$

Put $2x+3=0 \Rightarrow x = -\frac{3}{2}$; $A(0) + B(0) + C\left(-\frac{3}{2}-1\right)\left(-\frac{3}{2}+1\right) = 2\left(-\frac{3}{2}\right) - 3$

$\therefore C\left(-\frac{5}{2}\right)\left(-\frac{1}{2}\right) = -3-3 \Rightarrow C(5) = -24 \Rightarrow C = \frac{-24}{5}$

Now, From equation (1), we get, $\frac{2x-3}{(x^2-1)(2x+3)} = \frac{-1}{10} \cdot \frac{1}{x-1} + \frac{5}{2} \cdot \frac{1}{x+1} - \frac{24}{5} \cdot \frac{1}{2x+3}$

$\Rightarrow \int \frac{2x-3}{(x^2-1)(2x+3)} dx = \frac{-1}{10} \int \frac{1}{x-1} dx + \frac{5}{2} \int \frac{1}{x+1} dx - \frac{24}{5} \int \frac{1}{2x+3} dx$

$= \frac{-1}{10} \log(x-1) + \frac{5}{2} \log(x+1) - \frac{24}{5} \frac{\log(2x+3)}{2} + c = \frac{-1}{10} \log(x-1) + \frac{5}{2} \log(x+1) - \frac{12}{5} \log(2x+3) + c$

7. $\int \frac{2x+5}{x^2-x-2} dx$

Sol. Let $I = \int \frac{2x+5}{x^2-x-2} dx = \int \frac{2x+5}{(x-2)(x+1)} dx$

Put $\frac{2x+5}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)} \Rightarrow A(x+1) + B(x-2) = 2x+5 \quad \dots(1)$

Put $x+1=0 \Rightarrow x=-1$; $A(0) + B(-1-2) = 2(-1) + 5$, $B(-3) = 3 \therefore B = -1$

Put $x-2=0 \Rightarrow x=2$; $A(2+1) + B(0) = 2 \times 2 + 5 = 9$, $A = 3$

Now, From equation (1), we get, $\frac{2x+5}{(x-2)(x+1)} = \frac{3}{x-2} + \frac{-1}{x+1}$

$\Rightarrow \int \frac{2x+5}{(x-2)(x+1)} = \int \frac{3}{x-2} dx + \int \frac{-1}{x+1} dx = 3 \log(x-2) - \log(x+1) + c$

8. $\int \frac{x^2+5x+3}{x^2+3x+2} dx$

Sol. Let $I = \int \frac{x^2+5x+3}{x^2+3x+2} dx = \int \frac{x^2+3x+2+2x+1}{x^2+3x+2} dx \Rightarrow I = \int \frac{x^2+3x+2}{x^2+3x+2} dx + \int \frac{2x+1}{x^2+3x+2} dx$

$\Rightarrow I = \int dx + \int \frac{2x+1}{(x+1)(x+2)} dx \Rightarrow I = x + I_1$, where $I_1 = \int \frac{2x+1}{(x+1)(x+2)} dx$

$$\text{Put } \frac{2x+1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \Rightarrow A(x+2) + B(x+1) = 2x+1$$

$$\text{Put } x+2=0 \Rightarrow x=-2; A(0)+B(-1)=2(-2)+1 \Rightarrow B=3$$

$$\text{Put } x+1=0 \Rightarrow x=-1; A(-1+2)+B(0)=2(-1)+1 \Rightarrow A=-1$$

$$\therefore \frac{2x+1}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{3}{x+2}$$

$$\Rightarrow \int \frac{2x+1}{(x+1)(x+2)} dx = -\int \frac{1}{x+1} dx + 3 \int \frac{1}{x+2} dx = -\log(x+1) + 3\log(x+2) + C$$

$$\text{So, } I = x - \log(x+1) + 3\log(x+2) + C$$

9. $\int \frac{x^2+1}{x^2-1} dx$

Sol. Let $I = \int \frac{x^2+1}{x^2-1} dx \Rightarrow \int \left(1 + \frac{2}{x^2-1}\right) dx \Rightarrow I = \int dx + 2 \int \frac{1}{x^2-1} dx$

$$\Rightarrow I = x + 2 \cdot \frac{1}{2 \cdot 1} \log \left| \frac{x-1}{x+1} \right| + c \quad \therefore I = x + \log \left| \frac{x-1}{x+1} \right| + c$$

10. $\int \frac{x^3}{x^2-4} dx$

Sol. Let $I = \int \frac{x^3}{x^2-4} dx \Rightarrow I = \int \left(x + \frac{4x}{x^2-4}\right) dx$

$$\Rightarrow I = \int x dx + \int \frac{4x}{(x-2)(x+2)} dx = \frac{x^2}{2} + \int \frac{4x}{(x-2)(x+2)} dx \Rightarrow I = x + I_1 \quad \dots(1)$$

$$\therefore I_1 = \int \frac{4x}{x^2-4} dx, \text{ Put } x^2-4=t \Rightarrow 2x dx = dt \Rightarrow I_1 = 2 \int \frac{dt}{t} \Rightarrow I_1 = 2 \log |x^2-4| + c$$

$$\text{Putting the value of } I_1 \text{ in equation (1), } I = \frac{x^2}{2} + 2 \log |x^2-4| + c$$

11. $\int \frac{3+4x-x^2}{(x+2)(x-1)} dx$

Sol. Let $I = \int \frac{3+4x-x^2}{(x+2)(x-1)} dx$

$$= \int \left(-1 + \frac{5x+1}{(x+2)(x-1)}\right) dx = \int -dx + \int \frac{5x+1}{(x+2)(x-1)} dx = -x + I_1 \quad \dots(1)$$

$$\Rightarrow I_1 = \int \frac{5x+1}{(x+2)(x-1)} dx, \text{ Put } \frac{5x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} \quad \dots(2)$$

$$\Rightarrow A(x-1) + B(x+2) = 5x+1$$

$$\text{Put } x-1=0 \Rightarrow x=1; A(0)+B(1+2)=5+1=6 \Rightarrow B=2$$

$$\text{Put } x+2=0 \Rightarrow x=-2; A(-2-1)+B(0)=5 \times (-2)+1 \Rightarrow A(-3)=-9 \Rightarrow A=3$$

$$\text{Now, From equation (2), we get, } \frac{5x+1}{(x+2)(x-1)} = \frac{3}{x+2} + \frac{2}{x-1}$$

$$\Rightarrow I = \int \frac{5x+1}{(x+2)(x-1)} dx = 3 \int \frac{1}{x+2} dx + 2 \int \frac{1}{x-1} dx = 3 \log(x+2) + 2 \log(x-1) + c$$

$\therefore$ From equation (1), we get, $I = -x + 3 \log(x+2) + 2 \log(x-1) + c$

12. $\int \frac{x^3}{(x-1)(x-2)} dx$

Sol. Let, $I = \int \frac{x^3}{(x-1)(x-2)} dx = \int \left\{ (x+3) + \frac{7x-6}{(x-1)(x-2)} \right\} dx$

$$= \frac{x^2}{2} + 3x + \int \frac{7x-6}{(x-1)(x-2)} dx = \frac{x^2}{2} + 3x + I_1 \quad \dots(1), \quad \text{where, } I_1 = \int \frac{7x-6}{(x-1)(x-2)} dx$$

Put $\frac{7x-6}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \quad \dots(2)$

$$\Rightarrow A(x-2) + B(x-1) = 7x-6$$

Put $x-2=0$, $x=2$; $A(0) + B(2-1) = 7 \times 2 - 6 \Rightarrow B = 8$

Put $x-1=0 \Rightarrow x=1$; $A(1-2) + B(0) = 7-6=1 \Rightarrow A = -1$

$\therefore$ From equation (2), we get, $\frac{7x-6}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{8}{x-2}$

$$I_1 = \int \frac{7x-6}{(x-1)(x-2)} dx = - \int \frac{1}{x-1} dx + 8 \int \frac{1}{x-2} dx = -\log(x-1) + 8 \log(x-2) + C$$

$\therefore$ From equation (1), we get, $I = \frac{x^2}{2} + 3x - \log(x-1) + 8 \log(x-2) + c$

13. $\int \left(\frac{x^3 - x - 2}{1-x^2} \right) dx$

Sol. Let $I = \int \left(\frac{x^3 - x - 2}{1-x^2} \right) dx \Rightarrow I = \int \left(-x + \frac{-2}{1-x^2} \right) dx \Rightarrow I = \int -x dx + (-2) \int \frac{1}{1-x^2} dx$

$$\Rightarrow I = -\frac{x^2}{2} - 2 \cdot \frac{1}{2 \cdot 1} \log \left| \frac{1+x}{1-x} \right| + c \quad \therefore I = -\frac{x^2}{2} + \log \left| \frac{1-x}{1+x} \right| + c$$

14. $\int \frac{2x+1}{(4-3x-x^2)} dx$

Sol. Let $I = \int \frac{2x+1}{4-3x-x^2} dx \Rightarrow I = \int \frac{2x+1}{(1-x)(4+x)} dx = \int \frac{2x+1}{(1-x)(4+x)} dx$

Put $\frac{2x+1}{(1-x)(4+x)} = \frac{A}{1-x} + \frac{B}{4+x} \quad \dots(1)$

$\therefore A(4+x) + B(1-x) = 2x+1$

Put $1-x=0 \therefore x=1$; $A(5) + B(0) = 3 \therefore A = \frac{3}{5}$

Put $4+x=0 \therefore x=-4$; $A(0) + B(5) = -8+1 = -7 \therefore B = -\frac{7}{5}$

From equation (1), $\frac{2x+1}{(1-x)(4+x)} = \frac{3/5}{1-x} - \frac{7/5}{4+x}$

$$\Rightarrow \int \frac{2x+1}{(1-x)(4+x)} dx = \frac{3}{5} \int \frac{1}{1-x} dx - \frac{7}{5} \int \frac{1}{4+x} dx = \frac{-3}{5} \log(1-x) - \frac{7}{5} \log(4+x) + c$$

$$= -\frac{1}{5} [3 \log(1-x) + 7 \log(4+x)] + c$$

15. $\int \frac{2x}{(x^2+1)(x^2+3)} dx$

Sol. Let $I = \int \frac{2x}{(1+x^2)(3+x^2)} dx$

Put $x^2 = t \Rightarrow 2x dx = dt$

$\therefore I = \int \frac{dt}{(1+t)(3+t)} = \frac{1}{2} \int \left[\frac{1}{1+t} - \frac{1}{3+t} \right] dt$ [Resolving into partial fractions]

$$= \frac{1}{2} [\log|1+t| - \log|3+t|] + c = \frac{1}{2} \log \left| \frac{1+t}{3+t} \right| + c$$

$$= \frac{1}{2} \log \left| \frac{1+x^2}{3+x^2} \right| + c = \frac{1}{2} \log \left(\frac{1+x^2}{3+x^2} \right) + c \quad [\because 1+x^2 > 0, 3+x^2 > 0]$$

16. $\int \frac{\cos x}{(1+\sin x)(2+\sin x)} dx$

Sol. Let $I = \int \frac{\cos x \cdot dx}{(1+\sin x)(2+\sin x)}$

Put $t = \sin x \Rightarrow dt = \cos x dx$; $I = \int \frac{dt}{(1+t)(2+t)}$

Put $\frac{1}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t}$... (1)

$$\Rightarrow \frac{1}{(1+t)(2+t)} = \frac{A(2+t) + B(1+t)}{(1+t)(2+t)} \quad \therefore A(2+t) + B(1+t) = 1$$

Put $t+1=0 \Rightarrow t=-1$; $A(2-1) + B(0) = 1 \Rightarrow A=1$

Put $t+2=0 \Rightarrow t=-2$; $A(0) + B(-2+1) = 1 \Rightarrow B=-1$

$\therefore$ From equation (1), we get, $\frac{1}{(1+t)(2+t)} = \frac{1}{1+t} - \frac{1}{2+t}$

$$\Rightarrow \int \frac{1}{(1+t)(2+t)} dt = \int \frac{1}{1+t} dt - \int \frac{1}{t+2} dt = \log(1+t) - \log(t+2) + c = \log \frac{(1+t)}{t+2} + c$$

So, $I = \int \frac{\cos x \cdot dx}{(1+\sin x)(2+\sin x)} = \log \frac{(1+\sin x)}{2+\sin x} + c$

17. $\int \frac{\sec^2 x}{(2+\tan x)(3x+\tan x)} dx$

Sol. Let $I = \int \frac{\sec^2 x}{(2+\tan x)(3x+\tan x)} dx$

Put $t = \tan x \Rightarrow \frac{dt}{dx} = \sec^2 x$, $dt = \sec^2 x dx$; $I = \int \frac{dt}{(2+t)(3+t)}$

Put $\frac{1}{(2+t)(3+t)} = \frac{A}{2+t} + \frac{B}{3+t} \quad \dots(1)$

$\Rightarrow \frac{1}{(2+t)(3+t)} = \frac{A(3+t) + B(2+t)}{(2+t)(3+t)} \quad \therefore A(3+t) + B(2+t) = 1$

Put $t+2=0 \Rightarrow t=-2$; $A(3-2) + B(0) = 1 \quad \therefore A = 1$

Put $t+3=0 \Rightarrow t=-3$; $A(0) + B(2-3) = 1 \quad \therefore B = -1$

From equation (1), we get, $\frac{1}{(2+t)(3+t)} = \frac{1}{2+t} - \frac{1}{3+t}$

$\Rightarrow \int \frac{1}{(2+t)(3+t)} dt = \int \frac{1}{2+t} dt - \int \frac{1}{3+t} dt = \log(2+t) - \log(3+t) + c = \log\left(\frac{2+t}{3+t}\right) + c$

$\Rightarrow I = \log\left(\frac{2+\tan x}{3+\tan x}\right) + C$

18. $\int \frac{\sin x \cos x}{\cos^2 x - \cos x - 2} dx$

Sol. Let $I = \int \frac{\sin x \cos x}{\cos^2 x - \cos x - 2} dx$

Put $t = \cos x \Rightarrow dt = -\sin x dx$; $I = \int \frac{(-dt)t}{t^2 - t - 2} \Rightarrow I = -\int \frac{t dt}{(t+1)(t-2)}$

Put $\frac{-t}{(t+1)(t-2)} = \frac{A}{t+1} + \frac{B}{t-2} \quad \dots(1)$

$\therefore A(t-2) + B(t+1) = -t$

Put $t-2=0 \Rightarrow t=2$; $A(0) + B(2+1) = -2 \Rightarrow B = \frac{-2}{3}$

Put $t+1=0 \Rightarrow t=-1$; $A(-1-2) + B(0) = 1 \Rightarrow A = \frac{1}{-3} = -\frac{1}{3}$

From equation (1), we get, $\frac{-t}{(t+1)(t-2)} = -\frac{1}{3} \cdot \frac{1}{t+1} - \frac{2}{3} \cdot \frac{1}{t-2}$

$\Rightarrow I = \int \frac{-t}{(t+1)(t-2)} dt = -\frac{1}{3} \int \frac{1}{t+1} dt - \frac{2}{3} \int \frac{1}{t-2} dt$
 $= -\frac{1}{3} \log(t+1) - \frac{2}{3} \log(t-2) + c = -\frac{1}{3} \log(\cos x + 1) - \frac{2}{3} \log(\cos x - 2) + c$

19. $\int \frac{e^x}{(e^{2x} + 5e^x + 6)} dx$

Sol. Let $I = \int \frac{e^x}{e^{2x} + 5e^x + 6} dx$

Put $t = e^x \Rightarrow dt = e^x dx$; $I = \int \frac{dt}{t^2 + 5t + 6} = \int \frac{dt}{(t+2)(t+3)}$

$$\text{Put } \frac{1}{(t+2)(t+3)} = \frac{A}{t+2} + \frac{B}{t+3} \quad \dots(1)$$

$$A(t+3) + B(t+2) = 1$$

$$\text{Put } t+3=0 \Rightarrow t=-3; A(0) + B(-3+2) = 1 \Rightarrow B = -1$$

$$\text{Put } t+2=0 \Rightarrow t=-2; A(-2+3) + B(0) = 1 \Rightarrow A = 1$$

$$\therefore \text{ From equation (1), we get, } \frac{1}{(t+2)(t+3)} = \frac{1}{t+2} - \frac{1}{t+3}$$

$$\Rightarrow \int \frac{dt}{(t+2)(t+3)} = \int \frac{1}{t+2} dt - \int \frac{1}{t+3} dt = \log(t+2) - \log(t+3) + c = \log \frac{t+2}{t+3} + c$$

$$\therefore \int \frac{e^x}{e^{2x} + 5e^x + 6} dx = \log \frac{e^x + 2}{e^x + 3} + c$$

20. $\int \frac{e^x}{e^{3x} - 3e^{2x} - e^x + 3} dx$

Sol. Let $I = \int \frac{e^x}{e^{3x} - 3e^{2x} - e^x + 3} dx$

$$\text{Put } t = e^x, dt = e^x dx; I = \int \frac{dt}{t^3 - 3t^2 - t + 3} \Rightarrow I = \int \frac{dt}{t^2(t-3) - (t-3)} = \int \frac{dt}{(t^2-1)(t-3)}$$

$$\text{Put } \frac{1}{(t-1)(t+1)(t-3)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{C}{t-3} \quad \dots(1)$$

$$\Rightarrow A(t+1)(t-3) + B(t-1)(t-3) + C(t-1)(t+1) = 1$$

$$\text{Put } t+1=0 \Rightarrow t=-1; A(0) + B(-1-1)(-1-3) + C(0) = 1 \Rightarrow B(-2)(-4) = 1 \Rightarrow B = \frac{1}{8}$$

$$\text{Put } t-1=0 \Rightarrow t=1; A(1+1)(1-3) + B(0) + C(0) = 1 \Rightarrow A = -\frac{1}{4}$$

$$\text{Put } t-3=0 \Rightarrow t=3; A(0) + B(0) + C(3-1)(3+1) = 1 \Rightarrow C = \frac{1}{8}$$

$$\therefore \text{ From equation (1), we get, } \frac{1}{(t-1)(t+1)(t-3)} = -\frac{1}{4} \cdot \frac{1}{t-1} + \frac{1}{8} \cdot \frac{1}{t+1} + \frac{1}{8} \cdot \frac{1}{t-3}$$

$$\begin{aligned} \Rightarrow \int \frac{dt}{(t-1)(t+1)(t-3)} &= -\frac{1}{4} \int \frac{1}{t-1} dt + \frac{1}{8} \int \frac{1}{t+1} dt + \frac{1}{8} \int \frac{1}{t-3} dt \\ &= -\frac{1}{4} \log(t-1) + \frac{1}{8} \log(t+1) + \frac{1}{8} \log(t-3) + c \end{aligned}$$

$$\Rightarrow \int \frac{e^x dx}{(e^x-1)(e^x+1)(e^x-3)} = -\frac{1}{4} \log(e^x-1) + \frac{1}{8} \log(e^x+1) + \frac{1}{8} \log(e^x-3) + c$$

21. $\int \frac{2 \log x}{x[2(\log x)^2 - \log x - 3]} dx$

Sol. Let $I = \int \frac{2 \log x}{x[2(\log x)^2 - \log x - 3]} dx$

Put $t = \log x \Rightarrow dt = \frac{1}{x} dx$; $I = \int \frac{2t}{2t^2 - t - 3} dt$

Put $\frac{2t}{2t^2 - t - 3} = \frac{2t}{(2t-3)(t+1)} \Rightarrow \frac{2t}{(2t-3)(t+1)} = \frac{A}{2t-3} + \frac{B}{t+1} \quad \dots(1)$

$\therefore A(t+1) + B(2t-3) = 2t$

Put $2t-3=0 \Rightarrow t = \frac{3}{2}$; $A\left(\frac{3}{2}+1\right) + B(0) = 2 \cdot \frac{3}{2} = 3 \Rightarrow A\left(\frac{5}{2}\right) = 3 \therefore A = \frac{6}{5}$

Put $t+1=0 \Rightarrow t = -1$; $A(0) + B(-2-3) = -2 \therefore B = \frac{-2}{-5} = \frac{2}{5}$

$\therefore$ From equation (1), we get, $\frac{2t}{(2t-3)(t+1)} = \frac{6}{5} \cdot \frac{1}{2t-3} + \frac{2}{5} \cdot \frac{1}{t+1}$

$\Rightarrow \int \frac{2t}{(2t-3)(t+1)} dt = \frac{6}{5} \int \frac{1}{2t-3} dt + \frac{2}{5} \int \frac{1}{t+1} dt = \frac{6}{5} \log \frac{2t-3}{5} + \frac{2}{5} \log(\log x + 1)$

$\Rightarrow \int \frac{2 \log x dx}{x[2(\log x)^2 - \log x - 3]} = \frac{3}{5} \log(2 \log x - 3) + \frac{2}{5} \log(\log x + 1) + c$

22. $\int \frac{\operatorname{cosec}^2 x}{1 - \cot^2 x} dx$

Sol. Let $I = \int \frac{\operatorname{cosec}^2 x}{1 - \cot^2 x} dx$, Put $\cot x = t \Rightarrow -\operatorname{cosec}^2 x = \frac{dt}{dx} \Rightarrow \operatorname{cosec}^2 x dx = -dt$

$I = \int \frac{-dt}{1-t^2} \Rightarrow I = -\int \frac{1}{1-t^2} dt \Rightarrow I = -\frac{1}{2} \log \left| \frac{1+\cot x}{1-\cot x} \right| + c = -\frac{1}{2} \log \left| \frac{1+\cot x}{1-\cot x} \right| + c$

23. $\int \frac{\sec^2 x}{\tan^3 x + 4 \tan x} dx$

Sol. Let $I = \int \frac{\sec^2 x}{\tan^3 x + 4 \tan x} dx$

Put $\tan x = t \Rightarrow dt = \sec^2 x dx$; $I = \int \frac{dt}{t^3 + 4t}$, $I = \int \frac{dt}{t(t^2 + 4)}$

Put $\frac{1}{t(t^2 + 4)} = \frac{A}{t} + \frac{Bt+C}{t^2 + 4} \quad \dots(1)$

$\Rightarrow A(t^2 + 4) + (Bt+C)t = 1$. Put $t = 0$, $A(0+4) \times B(0) = 1 \Rightarrow A = \frac{1}{4}$

By, Equating the co-efficient of t^2 and constant here, $A+B=0 \Rightarrow \frac{1}{4} + B = 0 \therefore B = -\frac{1}{4}$, $C = 0$

$\therefore$ From equation (1), we get, $\int \frac{1}{t(t^2 + 4)} dt = \int \left(\frac{1}{4} \cdot \frac{1}{t} + \frac{-\frac{1}{4}t+0}{t^2 + 4} \right) dt = \frac{1}{4} \int \frac{1}{t} dt - \frac{1}{4} \int \frac{t}{t^2 + 4} dt$

$= \frac{1}{4} \log t - \frac{1}{4} \cdot \frac{1}{2} \log(z) + C$, where $Z = t^2 + 4$, $\frac{dZ}{dt} = 2t dt$

$= \frac{1}{4} \log t - \frac{1}{8} \log(t^2 + 4) + c = \frac{1}{4} \log \tan x - \frac{1}{8} \log(\tan^2 x + 4) + c$

24. $\int \frac{\sin 2x}{(1+\sin x)(2+\sin x)} dx$

Sol. Let $I = \int \frac{2 \sin x \cdot \cos x}{(1+\sin x)(2+\sin x)} dx$

Put $t = \sin x \Rightarrow dt = \cos x dx$; $I = \int \frac{2t}{(1+t)(2+t)} dt$

Put $\frac{2t}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t} \quad \dots(1) \quad \therefore A(2+t) + B(1+t) = 2t$

Put $2+t=0 \Rightarrow t=-2$; $A(0) + B(1-2) = -4 \Rightarrow B=4$

Put $1+t=0 \Rightarrow t=-1$; $A(2-1) + B(0) = -2 \Rightarrow A=-2$

$\therefore$ From equation (1), we get, $\frac{2t}{(1+t)(2+t)} = \frac{-2}{1+t} + \frac{4}{2+t}$

$\Rightarrow \int \frac{2t}{(1+t)(2+t)} dt = -2 \int \frac{1}{1+t} dt + 4 \int \frac{1}{2+t} dt = -2 \log(1+t) + 4 \log(2+t) + c$

So, $\int \frac{\sin 2x \cdot dx}{(1+\sin x)(2+\sin x)} = 4 \log(2+\sin x) - 2 \log(1+\sin x) + c$

25. $\int \frac{e^x}{e^x(e^x-1)} dx$

Sol. Let $I = \int \frac{e^x}{e^x(e^x-1)} dx$, Put $t = e^x \Rightarrow dt = e^x dx$; $I = \int \frac{dt}{t(t-1)}$

Put $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1} \quad \dots(1) \quad \therefore A(t-1) + Bt = 1$

Put $t-1=0 \Rightarrow t=1$; $A(0) + B \cdot 1 = 1 \Rightarrow B=1$

Put $t=0$; $A(0-1) + B(0) = 1 \Rightarrow A=-1$

$\therefore$ From equation (1), we get, $\frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$

$\Rightarrow \int \frac{1}{t(t-1)} dt = -\int \frac{1}{t} dt + \int \frac{1}{t-1} dt = -\log t + \log(t-1) + c = \log \frac{t-1}{t} + c$

So, $\int \frac{e^x}{e^x(e^x-1)} dx = \log \left(\frac{e^x-1}{e^x} \right) + c = \log \left(1 - \frac{1}{e^x} \right) + c$

26. $\int \frac{dx}{x(x^4-1)}$

Sol. Let $I = \int \frac{dx}{x(x^4-1)}$, Put $t = x^4 - 1 \Rightarrow 4x^3 dx = dt$

$I = \int \frac{x^3 dx}{x^4(x^4-1)} = \frac{1}{4} \int \frac{dt}{(t+1)t}$

Put $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} \quad \dots(1) \quad \Rightarrow A(t+1) + Bt = 1$

Put $t+1=0 \Rightarrow t=-1$; $A(0)+B(-1)=1 \Rightarrow B=-1$

Put $t=0$; $A(0+1)+0=1 \Rightarrow A=1$

$\therefore$ From equation (1), we get, $\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$

$\Rightarrow \frac{1}{4} \int \frac{1}{t(t+1)} dt = \frac{1}{4} \int \left(\frac{1}{t} - \frac{1}{t+1} \right) dt = \frac{1}{4} [\log t - \log(t+1)] + c$

$\Rightarrow \int \frac{dx}{x(x^4-1)} = \frac{1}{4} [\log(x^4-1) - \log(x^4-1+1)] + c$
 $= \frac{1}{4} [\log(x^4-1) - \log x^4] + c = \frac{1}{4} \log(x^4-1) - \frac{1}{4} 4 \log x + c$

So, $\int \frac{dx}{x(x^4-1)} = \frac{1}{4} \log(x^4-1) - \log x + c$

27. $\int \frac{1-x^2}{x(1-2x)} dx$

Sol. Let $I = \int \frac{-x^2+1}{x(-2x+1)} dx = \int \frac{-x^2+1}{-x(2x-1)} dx = \int \frac{(x^2-1)}{x(2x-1)} dx$
 $= \int \left(\frac{1}{2} + \frac{\frac{1}{2}x-1}{x(2x-1)} \right) dx = \frac{1}{2} \int dx + \frac{1}{2} \int \frac{x}{x(2x-1)} dx - \int \frac{1}{x(2x-1)} dx$

$\Rightarrow I = \frac{1}{2}x + \frac{1}{2} \int \frac{1}{2x-1} dx - I_1 = \frac{1}{2}x + \frac{1}{2} \frac{\log(2x-1)}{2} - I_1 \quad \dots(1)$

$\therefore I_1 = \int \frac{1}{x(2x-1)} dx$, put $\frac{1}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1} \quad \dots(2)$

$\Rightarrow A(2x-1)+Bx=1$, Put $2x-1=0 \Rightarrow x=\frac{1}{2}$, $A(0)+B\left(\frac{1}{2}\right)=1 \Rightarrow B=2$

Put $x=0$, $A(0-1)+B(0)=1 \Rightarrow A=-1$

$\therefore$ From equation (2), we get, $\frac{1}{x(2x-1)} = \frac{-1}{x} + \frac{2}{2x-1}$

$\Rightarrow \int \frac{1}{x(2x-1)} dx = -\int \frac{1}{x} dx + 2 \int \frac{1}{2x-1} dx = -\log x + \frac{2 \log(2x-1)}{2} + c$
 $= \log(2x-1) - \log x + c$

From equation (1),

$I = \frac{1}{2}x + \frac{1}{4} \log(2x-1) - \log(2x-1) + \log x + c = \frac{1}{2}x - \frac{3}{4} \log(1-2x) + \log x + c$

28. $\int \frac{x^2+x+1}{(x+2)(x+1)^2} dx$

Sol. Let $I = \int \frac{x^2+x+1}{(x+2)(x+1)^2} dx$, Put $\frac{x^2+x+1}{(x+2)(x+1)^2} = \frac{A}{(x+2)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} \quad \dots(1)$

$$\Rightarrow A(x+1)^2 + B(x+2)(x+1) + C(x+2) = x^2 + x + 1$$

Put $x+1=0 \Rightarrow x=-1$; $A(0) + B(0) + C(-1+2) = 1-1+1=1 \Rightarrow C=1$

Put $x+2=0 \Rightarrow x=-2$; $A(-2+1)^2 + B(0) + C(0) = 4-2+1=3 \Rightarrow A=3$

Equating the coefficient of x^2 , $A+B=1 \Rightarrow 3+B=1 \Rightarrow B=-2$

$\therefore$ From equation (1), we get, $\frac{x^2+x+1}{(x+2)(x+1)^2} = \frac{3}{(x+2)} - \frac{2}{(x+1)} + \frac{1}{(x+1)^2}$

So, $\int \frac{x^2+x+1}{(x+2)(x+1)^2} dx = \int \frac{3}{(x+2)} dx - 2 \int \frac{1}{x+1} dx + \int \frac{1}{(x+1)^2} dx$
 $= 3 \log(x+2) - 2 \log(x+1) - \frac{1}{x+1} + c$

29. $\int \frac{2x+9}{(x+2)(x-3)^2} dx$

Sol. Let $I = \int \frac{2x+9}{(x+2)(x-3)^2} dx$, Put $\frac{2x+9}{(x+2)(x-3)^2} = \frac{A}{(x+2)} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2}$... (1)

$$\Rightarrow A(x-3)^2 + B(x+2)(x-3) + C(x+2) = 2x+9$$

Put $x-3=0 \Rightarrow x=3$; $A(0) + B(0) + C(3+2) = 6+9=15 \Rightarrow C=3$

Put $x+2=0 \Rightarrow x=-2$; $A(-2-3)^2 + B(0) + C(0) = -4+9=5 \Rightarrow A = \frac{5}{25} = \frac{1}{5}$

By, Equating the coefficient of x^2 , we get, $A+B=0 \Rightarrow \frac{1}{5} + B=0 \Rightarrow B = -\frac{1}{5}$

$\therefore$ From equation (1), we get, $\frac{2x+9}{(x+2)(x-3)^2} = \frac{1}{5} \cdot \frac{1}{x+2} - \frac{1}{5} \cdot \frac{1}{x-3} + \frac{3}{(x-3)^2}$

$$\Rightarrow \int \frac{2x+9}{(x+2)(x-3)^2} dx = \frac{1}{5} \int \frac{1}{x+2} dx - \frac{1}{5} \int \frac{1}{x-3} dx + 3 \int \frac{1}{(x-3)^2} dx$$

 $= \frac{1}{5} \log(x+2) - \frac{1}{5} \log(x-3) - \frac{3}{x-3} + c$

30. $\int \frac{x^2+1}{(x-2)^2(x+3)} dx$

Sol. Let $I = \int \frac{x^2+1}{(x-2)^2(x+3)} dx$, Put $\frac{x^2+1}{(x-2)^2(x+3)} = \frac{A}{x+3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$... (1)

$$\Rightarrow A(x-2)^2 + B(x+3)(x-2) + C(x+3) = x^2+1$$

Put $x-2=0 \Rightarrow x=2$; $A(0) + B(0) + C(5) = 5 \Rightarrow C=1$

Put $x+3=0 \Rightarrow x=-3$; $A(-3-2)^2 + B(0) + C(0) = 9+1=10 \Rightarrow A = \frac{10}{25} = \frac{2}{5} \Rightarrow A = \frac{2}{5}$

By, Equating the coefficient of x^2 , we get, $A+B=1 \Rightarrow \frac{2}{5} + B=1 \Rightarrow B = 1 - \frac{2}{5} = \frac{3}{5}$

$\therefore$ From equation (1), we get, $\frac{x^2+1}{(x-2)^2(x+3)} = \frac{2}{5} \cdot \frac{1}{x+3} + \frac{3}{5} \cdot \frac{1}{x-2} + \frac{1}{(x-2)^2}$

$$\Rightarrow \int \frac{x^2+1}{(x-2)^2(x+3)} dx = \frac{2}{5} \int \frac{1}{x+3} dx + \frac{3}{5} \int \frac{1}{x-2} dx + \int \frac{1}{(x-2)^2} dx$$

$$= \frac{2}{5} \log(x+3) + \frac{3}{5} \log(x-2) - \frac{1}{x-2} + c$$

31. $\int \frac{(x^2+1)}{(x+3)(x-1)^2} dx$

Sol. $\int \frac{x^2+1}{(x-3)(x-1)^2} dx$. Put $\frac{x^2+1}{(x-3)(x-1)^2} = \frac{A}{x-3} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$ (i)

$$\Rightarrow A(x-1)^2 + B(x-3)(x-1) + C(x-3) = x^2 + 1. \text{ Put } x-1=0, x=1$$

$$\Rightarrow A(0) + B(0) + C(1-3) = 1+1 \Rightarrow C = \frac{2}{-2} = -1.$$

Put $x-3=0, x=3. A(3-1)^2 + B(0) + C(0) = 9+1.$

$$A(4) = 10 \therefore A = \frac{10}{4} = \frac{5}{2}. \text{ Equating the coefficient of } x^2$$

$$A+B=1 \Rightarrow \frac{5}{2} + B = 1 \Rightarrow B = 1 - \frac{5}{2} = \frac{-3}{2}.$$

$$\text{From (i)} \int \frac{x^2+1}{(x-3)(x-1)^2} dx = \frac{5}{2} \int \frac{1}{x-3} dx - \frac{3}{2} \int \frac{1}{x-1} dx - \int \frac{1}{(x-1)^2} dx$$

$$= \frac{5}{2} \log|x-3| - \frac{3}{2} \log|x-1| + \frac{1}{x-1} + C$$

32. $\int \frac{x^2+x+1}{(x+2)(x^2+1)} dx$

Sol. $\int \frac{x^2+x+1}{(x+2)(x^2+1)} dx$. Let $I = \int \frac{x^2+x+1}{(x+2)(x^2+1)} dx \Rightarrow \frac{x^2+x+1}{(x+2)(x^2+1)} = \frac{A}{(x+2)} + \frac{Bx+C}{x^2+1}$

$$\Rightarrow x^2+x+1 = (x^2+1)A + (Bx+C)(x+2) \Rightarrow x^2+x+1 = Ax^2 + A + Bx^2 + Cx + 2Bx + 2C$$

$$\Rightarrow x^2+x+1 = (A+B)x^2 + (C+2B)x + (A+2C).$$

$\therefore$ Equating coefficients $A+B=1$ (i) $A+2C=1 \Rightarrow A=1-2C$ (ii)

$$2B+C=1 \Rightarrow 2B=1-C \Rightarrow B = \frac{1-C}{2} \text{(iii). Putting in equation (i)}$$

we have $(1-2C) + \frac{1-C}{2} = 1 \Rightarrow 2-4C+1-C=2 \Rightarrow 3-5C=2 \Rightarrow -5C=-1 \Rightarrow C = \frac{1}{5}$

and $2B=1-\frac{1}{5} = \frac{4}{5} \Rightarrow B = \frac{2}{5}$ and $A=1-2 \times \frac{1}{5} = 1-\frac{2}{5} = \frac{3}{5}.$

$$\therefore I = \int \frac{x^2+x+1}{(x+2)(x^2+1)} dx = \int \frac{A}{(x+2)} dx + \int \frac{Bx+C}{x^2+1} dx = \frac{3}{5} \int \frac{1}{(x+2)} dx + \int \frac{\frac{2}{5}x + \frac{1}{5}}{x^2+1} dx$$

$$\begin{aligned}
 &= \frac{3}{5} \int \frac{1}{(x+2)} + \frac{1}{5} \int \frac{2x+1}{x^2+1} dx = \frac{3}{5} \log(x+2) + \frac{1}{5} I_1 \\
 \therefore I_1 &= \int \frac{4x+1}{x^2+1} = \int \frac{2x}{x^2+1} dx + \int \frac{1}{x^2+1} dx = \log|x^2+1| + \frac{1}{1} \tan^{-1} \frac{x}{1} + C_1 = \log|x^2+1| + \tan^{-1} x + C_1 \\
 \therefore \int \frac{x^2+x+1}{(x+2)(x^2+1)} dx &= \frac{3}{5} \log|x+2| + \frac{1}{5} [\log|x^2+1| + \tan^{-1} x] + C \\
 &= \frac{3}{5} \log|x+2| + \frac{1}{5} \log|x^2+1| + \frac{1}{5} \tan^{-1} x + C
 \end{aligned}$$

33. $I = \int \frac{2x}{(2x+1)^2} dx$

Sol. Let $I = \int \frac{2x}{(2x+1)^2} dx$, Put $\frac{2x}{(2x+1)^2} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2}$... (1) $\therefore A(2x+1) + B = 2x$

Put $2x+1=0 \Rightarrow x = -\frac{1}{2}$; $A(0) + B = -1 \Rightarrow B = -1$

By, Equating the coefficient of x , $2A = 2 \Rightarrow A = 1$

$\therefore$ From equation (1), we get, $\frac{2x}{(2x+1)^2} = \frac{1}{2x+1} - \frac{1}{(2x+1)^2}$

$$\begin{aligned}
 \Rightarrow \int \frac{2x}{(2x+1)^2} dx &= \int \frac{1}{2x+1} dx - \int \frac{1}{(2x+1)^2} dx \\
 &= \frac{\log(2x+1)}{2} + \frac{1}{2(2x+1)} + c = \frac{1}{2} \left[\log(2x+1) + \frac{1}{2x+1} \right] + c
 \end{aligned}$$

34. $\int \frac{3x+1}{(x+2)(x-2)^2} dx$

Sol. Let $I = \frac{3x+1}{(x+2)(x-2)^2} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

$\therefore A(x-2)^2 + B(x+2)(x-2) + C(x+2) = 3x+1$

Put $x-2=0 \Rightarrow x=2$; $A(0) + B(0) + C(2+1) = 3 \times 2 + 1 \Rightarrow C = \frac{7}{4}$

Put $x+2=0 \Rightarrow x=-2$; $A(-4)^2 + B(0) + C(0) = -6 + 1 = -5 \Rightarrow A = \frac{-5}{16}$

By equation the coefficient of x^2 , we get, $A+B=0 \Rightarrow \frac{-5}{16} + B = 0 \Rightarrow B = \frac{5}{16}$

$\therefore I = -\frac{5}{16} \log|x+2| + \frac{5}{16} \log|x-2| - \frac{7}{4(x-2)} + c$

35. $\int \frac{5x+8}{x^2(3x+8)} dx$

Sol. Let $I = \int \frac{5x+8}{x^2(3x+8)} dx$, Put $\frac{5x+8}{x^2(3x+8)} = \frac{A}{3x+8} + \frac{Bx+C}{x^2}$... (1)

$\therefore Ax^2 + (Bx+C)(3x+8) = 5x+8$

$$\text{Put } 3x+8=0 \Rightarrow x=-\frac{8}{3}; A\left(\frac{64}{9}\right)+B(0)=5\left(-\frac{8}{3}\right)+8=\frac{-40+24}{3}=\frac{-16}{3}$$

$$\Rightarrow A\left(\frac{64}{9}\right)=\frac{-16}{3} \Rightarrow A=\frac{-16 \times 3}{64}=-\frac{3}{4} \Rightarrow A=-\frac{3}{4}$$

By, Equating the coefficient of x^2 and constant term,

$$A+3B=0 \Rightarrow -\frac{3}{4}+3B=0 \Rightarrow 3B=\frac{3}{4} \Rightarrow B=\frac{1}{4}, 8C=8 \Rightarrow C=1$$

$$\therefore \text{ From equation (1), we get, } \frac{5x+8}{x^2(3x+8)}=\frac{-3}{4} \cdot \frac{1}{3x+8}+\frac{1}{4} \cdot \frac{x+1}{x^2}$$

$$\Rightarrow \int \frac{5x+8}{x^2(3x+8)} dx = \frac{-3}{4} \frac{\log(3x+8)}{3} + \frac{1}{4} \int \frac{x}{x^2} dx + \int \frac{1}{x^2} dx = \frac{-1}{4} \log(3x+8) + \frac{1}{4} \log x - \frac{1}{x} + C$$

36. $\int \frac{5x^2-18x+17}{(x-1)^2(2x-3)} dx$

Sol. Let $I = \int \frac{5x^2-18x+17}{(x-1)^2(2x-3)} dx$, Put $\frac{5x^2-18x+17}{(x-1)^2(2x-3)} = \frac{A}{2x-3} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$... (1)

$$\Rightarrow A(x-1)^2 + B(2x-3)(x-1) + C(2x-3) = 5x^2 - 18x + 17$$

$$\text{Put } x-1=0 \Rightarrow x=1; A(0)+B(0)+C(2-3)=5-18+17 \Rightarrow C(-1)=4 \therefore C=-4$$

$$\text{Put } 2x-3=0 \Rightarrow x=\frac{3}{2}; A\left(\frac{3}{2}-1\right)^2+B(0)+C(0)=5\left(\frac{3}{2}\right)^2-18\left(\frac{3}{2}\right)+17$$

$$\Rightarrow A\left(\frac{1}{4}\right)+0=5 \cdot \frac{9}{4}-27+17 \Rightarrow A\left(\frac{1}{4}\right)=\frac{45}{4}-10=\frac{5}{4} \Rightarrow A=5$$

By, Equating the coefficient of x^2 , we get, $A+2B=5 \Rightarrow 5+2B=5 \Rightarrow 2B=0 \Rightarrow B=0$

$$\therefore \text{ From equation (1), we get, } \frac{5x^2-18x+17}{(x-1)^2(2x-3)} = 5 \cdot \frac{1}{2x-3} + 0 - 4 \cdot \frac{1}{(x-1)^2}$$

$$\Rightarrow \int \frac{5x^2-18x+17}{(x-1)^2(2x-3)} = 5 \int \frac{1}{2x-3} dx - 4 \int \frac{1}{(x-1)^2} dx = \frac{5}{2} \log(2x-3) + \frac{4}{x-1} + C$$

37. $\int \frac{8}{(x+2)(x^2+4)} dx$

Sol. Let $I = \int \frac{8}{(x+2)(x^2+4)} dx$, Put $\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$... (1)

$$\Rightarrow A(x^2+4) + (Bx+C)(x+2) = 8$$

$$\text{Put } x+2=0 \Rightarrow x=-2; A(4+4)+0=8 \Rightarrow A=1$$

By equating the coefficient of x^2 and constant term, $A+B=0 \Rightarrow 1+B=0 \Rightarrow B=-1$

$$4A+2C=8 \Rightarrow 4 \cdot 1+2C=8 \Rightarrow 2C=4 \Rightarrow C=2$$

$$\text{From equation (1), we get, } \frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{-x+2}{x^2+4}$$

$$\Rightarrow \int \frac{8}{(x+2)(x^2+4)} dx = \int \frac{1}{x+2} dx - \int \frac{x}{x^2+4} dx + 2 \int \frac{1}{x^2+4} dx$$

$$I = \log(x+2) - \frac{1}{2} \log(x^2+4) + 2 \cdot \frac{1}{2} \tan^{-1} \frac{x}{2}, \text{ where } t = x^2 + 4$$

$$\therefore \int \frac{8}{(x+2)(x^2+4)} dx = \log(x+2) - \frac{1}{2} \log(x^2+4) + \tan^{-1} \frac{x}{2}$$

38. $\int \frac{3x+5}{x^3-x^2+x-1} dx$

Sol. Let $I = \int \frac{3x+5}{x^2(x-1)(x^2+1)} dx = \int \frac{3x+5}{(x-1)(x^2+1)} dx$

Put $\frac{3x+5}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$... (1)

$$\Rightarrow A(x^2+1) + (Bx+C)(x-1) = 3x+5$$

Put $x-1=0 \Rightarrow x=1$; $A(2)+B(0)=3+5=8 \Rightarrow A=4$

By, Equating the coefficient of x^2 and constant term, $A+B=0 \Rightarrow 4+B=0 \Rightarrow B=-4$

$$A-C=5 \Rightarrow 4-C=5 \Rightarrow C=-1$$

$\therefore$ From equation (1), we get, $\frac{3x+5}{(x-1)(x^2+1)} = \frac{4}{x-1} + \frac{-4x-1}{x^2+1}$

$$\Rightarrow \int \frac{3x+5}{(x-1)(x^2+1)} dx = 4 \int \frac{1}{x-1} dx - 4 \int \frac{x}{x^2+1} dx - \int \frac{1}{1+x^2} dx$$

$$= 4 \log(x-1) - \frac{4}{2} \log(x^2+1) - \tan^{-1} x + C = 4 \log(x-1) - 2 \log(x^2+1) - \tan^{-1} x + C$$

39. $\int \frac{2x}{(x^2+1)(x^2+3)} dx$

Sol. Let $I = \int \frac{2x}{(x^2+1)(x^2+3)} dx$, Put $t = x^2 \Rightarrow dt = 2x dx$; $\int \frac{dt}{(t+1)(t+3)}$

$$\frac{1}{(t+1)(t+3)} = \frac{A}{t+1} + \frac{B}{t+3}$$
 ... (1)

$$\Rightarrow A(t+3) + B(t+1) = 1$$

Put $t+3=0 \Rightarrow t=-3$; $A(0)+B(-3+1)=1 \Rightarrow B=-\frac{1}{2}$

Put $t+1=0 \Rightarrow t=-1$; $A(-1+3)+B(0)=1 \Rightarrow A=\frac{1}{2}$

From equation (1), we get, $\frac{1}{(t+1)(t+3)} = \frac{1}{2} \cdot \frac{1}{t+1} - \frac{1}{2} \cdot \frac{1}{t+3}$

$$\Rightarrow \int \frac{1}{(t+1)(t+3)} dt = \frac{1}{2} \int \frac{1}{t+1} dt - \frac{1}{2} \int \frac{1}{t+3} dt = \frac{1}{2} \log(t+1) - \frac{1}{2} \log(t+3) + C$$

$$\Rightarrow \int \frac{2x}{(x^2+1)(x^2+3)} dx = \frac{1}{2} \log(x^2+1) - \frac{1}{2} \log(x^2+3) + C$$

40. $\int \frac{x^2}{(x^4-1)} dx$

Sol. Let $I = \int \frac{x^2}{(x^4-1)} dx$, Put $\frac{x^2}{(x^2-1)(x^2+1)} = \frac{t}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1}$... (1)

$$\Rightarrow A(t+1) + B(t-1) = t$$

Put $t+1=0 \Rightarrow t=-1$; $A(0) + B(-1-1) = -1 \Rightarrow B = \frac{1}{2}$

Put $t-1=0 \Rightarrow t=1$; $A(1+1) + B(0) = 1 \Rightarrow A = \frac{1}{2}$

From equation (1), we get, $\frac{t}{(t-1)(t+1)} = \frac{1}{2} \cdot \frac{1}{t-1} + \frac{1}{2} \cdot \frac{1}{t+1}$

$$\begin{aligned} \Rightarrow \int \frac{x^2}{(x^2-1)(x^2+1)} dx &= \frac{1}{2} \int \frac{1}{x^2-1} dx + \frac{1}{2} \int \frac{1}{x^2+1} dx \\ &= \frac{1}{2} \cdot \frac{1}{2 \cdot 1} \log \frac{x-1}{x+1} + \frac{1}{2} \tan^{-1} x + C = \frac{1}{4} \log \frac{x-1}{x+1} + \frac{1}{2} \tan^{-1} x + C \end{aligned}$$

41. $\int \frac{dx}{x^3-1}$

Sol. Let $I = \int \frac{dx}{x^3-1}$, Put $\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$... (1)

$$\Rightarrow A(x^2+x+1) + (Bx+C)(x-1) = 1$$

Put $x-1=0 \Rightarrow x=1$; $A(1+1+1) + 0 = 1 \therefore A = \frac{1}{3}$

By, Equating the coefficient of x^2 and constant term, $A+B=0 \Rightarrow \frac{1}{3} + B = 0 \therefore B = -\frac{1}{3}$

$$A-C=1 \Rightarrow \frac{1}{3} - C = 1 \Rightarrow C = \frac{1}{3} - 1 = -\frac{2}{3}$$

From equation (1), we get, $\frac{1}{(x-1)(x^2+x+1)} = \frac{1}{3} \cdot \frac{1}{x-1} + \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1}$

$$\Rightarrow I = \int \frac{1}{(x-1)(x^2+x+1)} dx = \frac{1}{3} \int \frac{1}{x-1} dx - \frac{1}{3} \int \frac{x}{x^2+x+1} dx - \frac{2}{3} \int \frac{1}{x^2+x+1} dx$$

$$= \frac{1}{3} \log(x-1) - \frac{1}{6} \int \frac{2x+1-1}{x^2+x+1} dx - \frac{2}{3} \int \frac{1}{x^2+x+1} dx$$

$$= \frac{1}{3} \int (x-1) dx - \frac{1}{6} \int \frac{2x+1}{x^2+x+1} dx + \frac{1}{6} \int \frac{1}{x^2+x+1} dx - \frac{2}{3} \int \frac{1}{x^2+x+1} dx$$

Put $t = x^2 + x + 1 \Rightarrow dt = (2x+1) dx$

$$I = \frac{1}{3} \log(x-1) - \frac{1}{6} \int \frac{dt}{t} + \left(\frac{1}{6} - \frac{2}{3} \right) \int \frac{dx}{x^2+x+1}$$

$$\begin{aligned}
 &= \frac{1}{3} \log(x-1) - \frac{1}{6} \log t + \left(\frac{1-4}{6} \right) \int \frac{dx}{x^2 + 2 \cdot \frac{1}{2} + \left(\frac{1}{2} \right)^2 - \left(\frac{1}{2} \right)^2 + 1} \\
 &= \frac{1}{3} \log(x-1) - \frac{1}{6} \log(x^2 + x + 1) + \frac{1}{2} \int \frac{dx}{(x+1/2)^2 + \left(\frac{\sqrt{3}}{2} \right)^2} \\
 &= \frac{1}{3} \log(x-1) - \frac{1}{6} \log(x^2 + x + 1) - \frac{1}{2} \cdot \frac{1}{\sqrt{3}/2} \tan^{-1} \frac{x+1/2}{\sqrt{3}/2} + c \\
 &= \frac{1}{3} \log(x-1) - \frac{1}{6} \log(x^2 + x + 1) - \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + C
 \end{aligned}$$

42. $\int \frac{dx}{x^3+1}$

Sol. Let $\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)}$, Put $\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$... (1)

$$\Rightarrow A(x^2-x+1) + (Bx+C)(x+1) = 1$$

Put $x+1=0 \Rightarrow x=-1$; $A(1+1+1) + C(0) = 1 \Rightarrow A = \frac{1}{3}$

By, Equating the coefficient of x^2 and constant term, $A+B=0 \Rightarrow \frac{1}{3} + B = 0 \Rightarrow B = -\frac{1}{3}$,

$$A+C=1 \Rightarrow \frac{1}{3} + C = 1 \Rightarrow C = 1 - \frac{1}{3} = \frac{2}{3}$$

$\therefore$ From equation (1), we get, $\frac{1}{(x+1)(x^2-x+1)} = \frac{1}{3} \cdot \frac{1}{x+1} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1}$

$$\Rightarrow \int \frac{1}{(x+1)(x^2-x+1)} dx = \frac{1}{3} \int \frac{1}{x+1} dx - \frac{1}{3} \int \frac{x}{x^2-x+1} dx + \frac{2}{3} \int \frac{1}{x^2-x+1} dx$$

$$= \frac{1}{3} \log(x+1) - \frac{1}{6} \int \frac{2x-1+1}{x^2-x+1} dx + \frac{2}{3} \int \frac{1}{x^2-x+1} dx$$

$$= \frac{1}{3} \log(x+1) - \frac{1}{6} \int \frac{2x-1}{x^2-x+1} dx - \frac{1}{6} \int \frac{1}{x^2-x+1} dx + \frac{2}{3} \int \frac{dx}{x^2-x+1}$$

$$= \frac{1}{3} \log(x+1) - \frac{1}{6} \log(x^2-x+1) + \left(\frac{2}{3} - \frac{1}{6} \right) \int \frac{dx}{x^2-x+1}$$

$$= \frac{1}{3} \log(x+1) - \frac{1}{6} \log(x^2-x+1) + \frac{1}{2} \int \frac{dx}{x^2 - 2 \cdot \frac{1}{2}x + \left(\frac{1}{2} \right)^2 - \left(\frac{1}{2} \right)^2 + 1}$$

$$= \frac{1}{3} \log(x+1) - \frac{1}{6} \log(x^2-x+1) + \frac{1}{2} \int \frac{dx}{\left(x - \frac{1}{2} \right)^2 + \left(\frac{\sqrt{3}}{2} \right)^2}$$

$$= \frac{1}{3} \log(x+1) - \frac{1}{6} \log(x^2-x+1) + \frac{1}{2} \cdot \frac{1}{\sqrt{3}/2} \tan^{-1} \frac{x-\frac{1}{2}}{\sqrt{3}/2} + c$$

$$= \frac{1}{3} \log(x+1) - \frac{1}{6} \log(x^2 - x + 1) + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}} + c$$

43. $\int \frac{dx}{(x+1)^2(x^2+1)}$

Sol. Let $I = \int \frac{dx}{(x+1)^2(x^2+1)}$, Put $\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$... (1)

$$\Rightarrow A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2 = 1$$

Put $x+1=0 \Rightarrow x=-1$; $A(0)+B(1+1)+0=1 \Rightarrow B=\frac{1}{2}$

By, Equating the coefficient of x^3 , x^2 and constant term.

$$A+C=0 \quad \dots (2)$$

$$A+B+2C=0 \Rightarrow A+2C=-\frac{1}{2} \quad \dots (3)$$

$$A+B+D=1$$

$\therefore$ Solving (2) and (3), we get, $-C=\frac{1}{2} \Rightarrow C=-\frac{1}{2}$, $A-\frac{1}{2}=0 \therefore A=\frac{1}{2}$

$$A+B+D=1 \Rightarrow \frac{1}{2}+\frac{1}{2}+D=1 \Rightarrow 1+D=1 \therefore D=0$$

$\therefore$ From equation (1), we get, $\frac{1}{(x+1)^2(x^2+1)} = \frac{1}{2} \cdot \frac{1}{x+1} + \frac{1}{2} \cdot \frac{1}{(x+1)^2} + \frac{-\frac{1}{2}x+0}{x^2+1}$

$$\begin{aligned} \Rightarrow \int \frac{1}{(x+1)^2(x^2+1)} dx &= \frac{1}{2} \int \frac{1}{x+1} dx + \frac{1}{2} \int \frac{1}{(x+1)^2} dx - \frac{1}{2} \int \frac{x}{x^2+1} dx \\ &= \frac{1}{2} \log(x+1) - \frac{1}{2} \cdot \frac{1}{x+1} - \frac{1}{4} \log(x^2+1) + C \end{aligned}$$

44. $\int \frac{17}{(2x+1)(x^2+4)} dx$

Sol. Let $I = \int \frac{17}{(2x+1)(x^2+4)} dx$, Put $\frac{17}{(2x+1)(x^2+4)} = \frac{A}{2x+1} + \frac{Bx+C}{x^2+4}$... (1)

$$\Rightarrow A(x^2+4) + (Bx+C)(2x+1) = 17$$

Put $2x+1=0 \Rightarrow x=-\frac{1}{2}$; $A(1/4+4)+0=17 \Rightarrow A(17/4)=17 \therefore A=4$

By Equating the coefficient of x^2 and constant term, $A+2B=0 \Rightarrow 4+2B=0 \Rightarrow B=-2$

$$4A+C=17 \Rightarrow 4 \times 4 + C = 17 \therefore C=1$$

$\therefore$ From equation (1), we get, $\frac{17}{(2x+1)(x^2+4)} = \frac{4}{2x+1} + \frac{-2x+1}{x^2+4}$

$$\begin{aligned} \Rightarrow \int \frac{17}{(2x+1)(x^2+4)} dx &= 4 \int \frac{1}{2x+1} dx - 2 \int \frac{2x}{x^2+4} dx + \int \frac{1}{x^2+2^2} dx \\ &= \frac{4 \log(2x+1)}{2} - \log(x^2+4) + \frac{1}{2} \tan^{-1} \frac{x}{2} + C = 2 \log(2x+1) - \log(x^2+4) + \frac{1}{2} \tan^{-1} \frac{x}{2} + C \end{aligned}$$

45. $\int \frac{1}{(x^2+2)(x^2+4)} dx$

Sol. Let $I = \int \frac{1}{(x^2+2)(x^2+4)} dx$, Put $\frac{1}{(x^2+2)(x^2+4)} = \frac{1}{(t+2)(t+4)} = \frac{A}{t+2} + \frac{B}{t+4}$... (1)

$$\Rightarrow A(t+4) + B(t+2) = 1$$

Put $t+4=0 \Rightarrow t=-4$; $A(0) + B(-4+2) = 1 \Rightarrow B = -\frac{1}{2}$

Put $t+2=0 \Rightarrow t=-2$; $A(-2+4) + B(0) = 1 \Rightarrow A = \frac{1}{2}$

From equation (1), we get, $\frac{1}{(t+2)(t+4)} = \frac{1}{2} \cdot \frac{1}{t+2} - \frac{1}{2} \cdot \frac{1}{t+4}$

$$\Rightarrow \int \frac{1}{(x^2+2)(x^2+4)} dx = \frac{1}{2} \int \frac{1}{x^2+2} dx - \frac{1}{2} \int \frac{1}{x^2+4} dx = \frac{1}{2} \int \frac{dx}{x^2+(\sqrt{2})^2} - \frac{1}{2} \int \frac{dx}{x^2+2^2}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{2} \tan^{-1} \frac{x}{2} + C = \frac{1}{2\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} - \frac{1}{4} \tan^{-1} \frac{x}{2} + C$$

46. $\int \frac{x^2}{(x^2+2)(x^2+3)} dx$

Sol. Let $I = \int \frac{x^2}{(x^2+2)(x^2+3)} dx$, Put $\frac{x^2}{(x^2+2)(x^2+3)} = \frac{t}{(t+2)(t+3)}$, where $t = x^2$

$$\Rightarrow \frac{t}{(t+2)(t+3)} = \frac{A}{t+2} + \frac{B}{t+3} \quad \dots (1)$$

$$\Rightarrow A(t+3) + B(t+2) = t$$

Put $t+3=0 \Rightarrow t=-3$; $A(0) + B(-3+2) = -3 \Rightarrow B = 3$

Put $t+2=0 \Rightarrow t=-2$; $A(-2+3) + B(0) = -2 \Rightarrow A = -2$

From equation (1), $\frac{t}{(t+2)(t+3)} = \frac{-2}{t+2} + \frac{3}{t+3}$

$$\Rightarrow \int \frac{x^2}{(x^2+2)(x^2+3)} dx = \int \left(\frac{-2}{x^2+(\sqrt{2})^2} + \frac{3}{x^2+(\sqrt{3})^2} \right) dx$$

$$= -2 \cdot \frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + \frac{3}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + C = -\sqrt{2} \tan^{-1} \frac{x}{\sqrt{2}} + \sqrt{3} \tan^{-1} \frac{x}{\sqrt{3}} + C$$

47. $\int \frac{dx}{(e^x-1)^2}$

Sol. Put $t = e^x - 1 \Rightarrow e^x = t+1 \Rightarrow dt = e^x dx \Rightarrow \frac{dt}{e^x} = dx \Rightarrow \frac{dt}{t+1} = dx \Rightarrow \int \frac{dt}{(t+1)t^2} = dx$

Put $\frac{1}{(t+1)t^2} = \frac{A}{t+1} + \frac{Bt+C}{t^2} \quad \dots (1)$

$$\Rightarrow A(t^2) + (Bt+C)(t+1) = 1, \text{ Put } t+1=0 \Rightarrow t=-1; A=1$$

Equating the coefficient of t^2 and constant term, $A+B=0 \Rightarrow 1+B=0 \therefore B=-1, C=1$

From equation (1), we get, $\frac{1}{(t+1)t^2} dt = \frac{1}{t+1} + \frac{-1t+1}{t^2}$

$$\begin{aligned} \Rightarrow \int \frac{1}{(t+1)t^2} dt &= \int \frac{1}{t+1} dt - \int \frac{t}{t^2} dt + \int \frac{1}{t^2} dt \\ &= \log(t+1) - \int \frac{1}{t} dt + \int \frac{1}{t^2} dt = \log(t+1) - \log t - \frac{1}{t} + C \\ \Rightarrow \int \frac{1}{(e^x-1)^2} dx &= \log e^x - \log(e^x-1) - \frac{1}{e^x-1} + C = x - \log(e^x-1) - \frac{1}{e^x-1} + C \end{aligned}$$

48. $\int \frac{dx}{x(x^5+1)}$

Sol. Let $I = \int \frac{dx}{x(x^5+1)}$, Put $t = x^5 \Rightarrow dt = 5x^4 dx \Rightarrow \frac{dt}{5x^4} = dx$

$$\int \frac{dt}{5x^4} = \frac{1}{5} \int \frac{dt}{x^5(t+1)} = \frac{1}{5} \int \frac{dt}{t(t+1)}$$

Put $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} \dots (1) \therefore A(t+1) + Bt = 1$

Put $t+1=0 \Rightarrow t=-1; A(0) + B(-1) = 1 \Rightarrow B = -1$

Put $t=0; A(0+1) = 1 \Rightarrow A=1,$

$$\begin{aligned} \frac{1}{t(t+1)} &= \frac{1}{t} - \frac{1}{t+1} \Rightarrow \int \frac{1}{t(t+1)} dt = \int \frac{1}{t} dt - \int \frac{1}{t+1} dt = \log t - \log(t+1) + C = \log \frac{t}{t+1} + C \\ \Rightarrow \frac{1}{5} \int \frac{1}{t(t+1)} dt &= \frac{1}{5} \log \frac{t}{t+1} + C \therefore \int \frac{1}{x(x^5+1)} dx = \frac{1}{5} \log \frac{x^5}{x^5+1} + C \end{aligned}$$

$$\log x - \frac{1}{5} \log |x^5+1| + C$$

49. $\int \frac{dx}{x(x^6+1)}$

Sol. Let $I = \int \frac{dx}{x(x^6+1)}$, Put $t = x^6 \Rightarrow \frac{dt}{dx} = 6x^5 \Rightarrow \frac{dt}{6x^5} = dx$

$$I = \int \frac{dt}{6x^5} = \frac{1}{6} \int \frac{dt}{x^6(t+1)} = \frac{1}{6} \int \frac{dt}{t(t+1)}$$

Put $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} \dots (1) \Rightarrow A(t+1) + Bt = 1$

Put $t+1=0 \Rightarrow t=-1; A(0) + B(-1) = 1 \therefore B = -1$

Put $t=0; A(0+1) + 0 = 1 \Rightarrow A=1$

From equation (1), we get, $\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$

$$\Rightarrow \int \frac{1}{t(t+1)} dt = \int \frac{1}{t} dt - \int \frac{1}{t+1} dt = \log t - \log(t+1) + C$$

$$\Rightarrow \int \frac{1}{t(t+1)} dt = \log \frac{t}{t+1} + C \quad \therefore \int \frac{1}{x(x^6+1)} dx = \frac{1}{6} \log \frac{x^6}{x^6+1} + C$$

$$\therefore I = \log|x| - \frac{1}{6} \log|x^6+1| + c$$

50. $\int \frac{dx}{\sin x(3+2\cos x)}$

Sol. Let $I = \int \frac{dx}{\sin x(3+2\cos x)}$, Put $t = \cos x \Rightarrow dt = -\sin x dx \Rightarrow \frac{dt}{-\sin x} = dx$

$$I = \int \frac{dt}{\frac{-\sin x}{\sin x(3+2t)}} = -\int \frac{dt}{\sin^2 x(3+2t)} = -\int \frac{dt}{(1-\cos^2 x)(3+2t)} = -\int \frac{dt}{(1-t^2)(3+2t)}$$

Put $\frac{1}{(1-t^2)(3+2t)} = \frac{1}{(1-t)(1+t)(3+2t)}$

$$\frac{1}{(1-t)(1+t)(3+2t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{3+2t} \quad \dots(1)$$

$$\Rightarrow A(1+t)(3+2t) + B(1-t)(3+2t) + C(1-t^2) = 1$$

Put $1+t=0 \Rightarrow t=-1$; $A(0) + B(2)(3-2) + C(0) = 1$, $B = \frac{1}{2}$

Put $1-t=0 \Rightarrow t=1$; $A(2)(5) + B(0) + C(0) = 1 \Rightarrow A = \frac{1}{10}$

Put $3+2t=0 \Rightarrow t=-\frac{3}{2}$; $A(0) + B(0) + C\left(1-\frac{9}{4}\right) = 1 \Rightarrow C\left(\frac{-5}{4}\right) = 1 \therefore C = -\frac{4}{5}$

$\therefore$ From equation (1), we get $\frac{1}{(1-t)(1+t)(3+2t)} = \frac{1}{10} \cdot \frac{1}{1-t} + \frac{1}{2} \cdot \frac{1}{1+t} - \frac{4}{5} \cdot \frac{1}{3+2t}$

$$\begin{aligned} \Rightarrow \int \frac{1}{(1-t)(1+t)(3+2t)} dt &= \frac{1}{10} \int \frac{1}{1-t} dt + \frac{1}{2} \int \frac{1}{1+t} dt - \frac{4}{5} \int \frac{1}{3+2t} dt \\ &= \frac{-1}{10} \log(1-t) + \frac{1}{2} \log(1+t) - \frac{4}{5} \frac{\log(3+2t)}{2} + C = \int \frac{dx}{\sin x(3+2\cos x)} \\ &= \frac{-1}{10} (1-\cos x) + \frac{1}{2} \log(1+\cos x) - \frac{2}{5} \log(3+2\cos x) + C \end{aligned}$$

51. $\int \frac{dx}{\cos x(5-4\sin x)}$

Sol. Let $I = \int \frac{dx}{\cos x(5-4\sin x)}$, Put $t = \sin x$, $dt = \cos x dx$; $I = \int \frac{dt}{\cos^2 x(5-4t)}$

$$\Rightarrow I = \int \frac{dt}{(1-\sin^2 x)(5-4t)} = \int \frac{dt}{(1-t^2)(5-4t)}$$

Put $\frac{1}{(1-t)(1+t)(5-4t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{5-4t} \quad \dots(1)$

$$\Rightarrow A(1+t)(5-4t) + B(1-t)(5-4t) + C(1-t^2) = 1$$

$$\text{Put } 1+t=0 \Rightarrow t=-1; A(0)+B(2)(9)=1 \Rightarrow B=\frac{1}{18}$$

$$\text{Put } 1-t=0 \Rightarrow t=1; A(2)+B(0)+C(0)=1 \Rightarrow A=\frac{1}{2}$$

$$\text{Put } 5-4t=0 \Rightarrow t=\frac{5}{4}; A(0)+B(0)+C\left(1-\frac{25}{16}\right)=1 \Rightarrow C\left(\frac{-9}{16}\right)=1 \therefore C=\frac{-16}{9}$$

$$\text{From equation (1), we get, } \frac{1}{(1-t^2)(5-4t)} = \frac{1}{2} \cdot \frac{1}{1-t} + \frac{1}{18} \cdot \frac{1}{1+t} + \frac{-16}{9} \cdot \frac{1}{5-4t}$$

$$\begin{aligned} \Rightarrow \int \frac{1}{(1-t^2)(5-4t)} dt &= \frac{1}{2} \int \frac{1}{1-t} dt + \frac{1}{18} \int \frac{1}{1+t} dt - \frac{16}{9} \int \frac{1}{5-4t} dt \\ &= \frac{-1}{2} \log(1-t) + \frac{1}{18} \log(1+t) - \frac{16 \log(5-4t)}{9 \cdot -4} + C \\ &= \frac{-1}{2} \log(1-t) + \frac{1}{18} \log(1+t) + \frac{4}{9} \log(5-4t) + C \\ \Rightarrow \int \frac{1}{\cos x(5-4 \sin x)} dx &= \frac{-1}{2} \log(1-\sin x) + \frac{1}{18} \log(1+\sin x) + \frac{4}{9} \log(5-4 \sin x) + C \end{aligned}$$

52. $\int \frac{dx}{\sin x \cos^2 x}$

Sol. Let $I = \int \frac{1}{\sin x \cos^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^2 x} dx = \int \frac{\sin^2 x}{\sin x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin x \cos^2 x} dx$

$$= \int \frac{\sin x}{\cos x \cos x} dx + \int \frac{1}{\sin x} dx = \int (\tan x \sec x + \operatorname{cosec} x) dx$$

$$= \sec x - \frac{1}{2} \log \cot^2 \frac{x}{2} = \sec x - \frac{1}{2} \log \left(\frac{1+\cos x}{1-\cos x} \right) + C$$

53. $\int \frac{\tan x}{(1-\sin x)} dx$

Sol. Let $I = \int \frac{\tan x}{1-\sin x} dx \Rightarrow I = \int \frac{\sin x}{\cos x(1-\sin x)} dx$

$$\text{Put } t = \sin x \Rightarrow dt = \cos x dx; I = \int \frac{\sin x \cdot \cos x dx}{\cos^2 x(1-\sin x)} = \int \frac{t dt}{(1-\sin^2 x)(1-t)} = \int \frac{t dt}{(1-t^2)(1-t)}$$

$$\text{Put } \frac{t}{(1-t)(1+t)(1-t)} = \frac{t}{(1-t)^2(1+t)} = \frac{A}{1+t} + \frac{B}{1-t} + \frac{C}{(1-t)^2} \quad \dots(1)$$

$$\Rightarrow A(1-t)^2 + B(1-t)(1+t) + C(1+t) = t$$

$$\text{Put } 1-t=0 \Rightarrow t=1; A(0)+B(0)+C(1+1)=1 \Rightarrow C=\frac{1}{2}$$

$$\text{Put } 1+t=0 \Rightarrow t=-1; A(2)^2+B(0)+C(0)=-1 \Rightarrow A=-\frac{1}{4}$$

$$\text{By, Equating the coefficient of } t^2, \text{ we get, } A-B=0 \Rightarrow \frac{-1}{4}-B=0 \Rightarrow B=-\frac{1}{4}$$

∴ From equation (1), we get, $\frac{t}{(1-t)^2(1+t)} = \frac{-1}{4} \cdot \frac{1}{1+t} - \frac{1}{4} \cdot \frac{1}{1-t} + \frac{1}{2} \cdot \frac{1}{(1-t)^2}$

$$\Rightarrow \int \frac{t}{(1-t)^2(1+t)} dt = -\frac{1}{4} \int \frac{1}{1+t} dt - \frac{1}{4} \int \frac{1}{1-t} dt + \frac{1}{2} \int \frac{1}{(1-t)^2} dt$$

$$= -\frac{1}{4} \log(1+t) + \frac{1}{4} \log(1-t) - \frac{1}{2} \cdot \frac{1}{1-t} + C$$

$$\Rightarrow \int \frac{\tan x}{1-\sin x} dx = -\frac{1}{4} \log(1+\sin x) + \frac{1}{4} \log(1-\sin x) - \frac{1}{2} \cdot \frac{1}{1-\sin x} + C$$

54. $\int \frac{dx}{\sin x + \sin 2x}$

Sol. Let $I = \int \frac{dx}{\sin x + 2 \sin x \cdot \cos x} = \int \frac{dx}{\sin x(1+2 \cos x)}$

Now, Put $t = \cos x \Rightarrow dt = -\sin x dx \Rightarrow \frac{-dt}{\sin x} = dx$

$$I = \int \frac{-dt}{\sin^2 x(1+2t)} = -\int \frac{dt}{(1-\cos^2 t)(1+2t)} \Rightarrow I = -\int \frac{dt}{(1-t^2)(1+2t)}$$

Put $\frac{1}{(1-t)(1+t)(1+2t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{1+2t} \dots (1)$

$$\Rightarrow A(1+t)(1+2t) + B(1-t)(1+2t) + C(1-t^2) = 1$$

Put $1+t=0 \Rightarrow t=-1; A(0)+B(2)(1-2)+C(0)=1 \Rightarrow B = \frac{-1}{2}$

Put $1-t=0 \Rightarrow t=1; A(2)(3)+B(0)+C(0)=1, \Rightarrow A = \frac{1}{6}$

Put $1+2t=0 \Rightarrow t=-\frac{1}{2}, A(0)+B(0)+C\left(1-\frac{1}{4}\right)=1 \Rightarrow C\left(\frac{3}{4}\right)=1 \Rightarrow C = \frac{4}{3}$

From equation (1), we get, $\frac{1}{(1-t^2)(1+2t)} = \frac{1}{6} \cdot \frac{1}{1-t} - \frac{1}{2} \cdot \frac{1}{1+t} + \frac{4}{3} \cdot \frac{1}{1+2t}$

$$\Rightarrow \int \frac{1}{(1-t)^2(1+2t)} dt = \frac{1}{6} \int \frac{1}{1-t} dt - \frac{1}{2} \int \frac{1}{1+t} dt + \frac{4}{3} \int \frac{1}{1+2t} dt$$

$$= \frac{-1}{6} \log(1-t) - \frac{1}{2} \log(1+t) + \frac{4}{3} \frac{\log(1+2t)}{2} + C$$

$$= \frac{-1}{6} \log(1-t) - \frac{1}{2} \log(1+t) + \frac{2}{3} \log(1+2t) + C$$

$$\Rightarrow \int \frac{1}{\sin x + \sin 2x} dx = \frac{-1}{6} \log(1-\cos x) - \frac{1}{2} \log(1+\cos x) + \frac{2}{3} \log(1+\cos x) + C$$

55. $\int \frac{x^2}{x^4 - x^2 - 12} dx$

Sol. Let $I = \int \frac{x^2}{x^4 - x^2 - 12} dx$, Put $\frac{x^2}{x^4 - x^2 - 12} = \frac{t}{t^2 - t - 12}$, where $t = x^2$

$$\frac{t}{t^2 - t - 12} = \frac{t}{(t-4)(t+3)} = \frac{A}{t-4} + \frac{B}{t+3} \dots (1)$$

$$\therefore A(t+3) + B(t-4) = t$$

$$\text{Put } t+3=0 \Rightarrow t=-3; A(0)+B(-7)=-3 \Rightarrow B=\frac{3}{7}$$

$$\text{Put } t-4=0 \Rightarrow t=4; A(4+3)+B(0)=4 \Rightarrow A=\frac{4}{7}$$

$$\text{From (1), } \frac{t}{(t-4)(t+3)} = \frac{4}{7} \cdot \frac{1}{t-4} + \frac{3}{7} \cdot \frac{1}{t+3}$$

$$\begin{aligned} \Rightarrow \int \frac{x^2}{(x^2-4)(x^2+3)} dx &= \frac{4}{7} \int \frac{1}{x^2-2^2} dx + \frac{3}{7} \int \frac{1}{x^2+(\sqrt{3})^2} \\ &= \frac{4}{7} \cdot \frac{1}{2 \cdot 2} \log \frac{x-2}{x+2} + \frac{3}{7} \cdot \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + C = \frac{1}{7} \log \left(\frac{x-2}{x+2} \right) + \frac{\sqrt{3}}{7} \tan^{-1} \frac{x}{\sqrt{3}} + C \end{aligned}$$

$$56. \int \frac{x^4}{(x^2+1)(x^2+9)(x^2+16)} dx$$

$$\text{Sol. Let } I = \int \frac{x^4}{(x^2+1)(x^2+9)(x^2+16)} dx$$

$$I = \int \frac{(x^2)^2}{(x^2+1)(x^2+9)(x^2+16)} dx$$

$$\text{Again Let } x^2 = t$$

$$I = \int \frac{t^2}{(t+1)(t+9)(t+16)} dt$$

$$\frac{t^2}{(t+1)(t+9)(t+16)} = \frac{A}{(t+1)} + \frac{B}{(t+9)} + \frac{C}{(t+16)} \quad \dots (i)$$

$$\frac{t^2}{(t+1)(t+9)(t+16)} = \frac{A(t+9)(t+16) + B(t+1)(t+16) + C(t+1)(t+9)}{(t+1)(t+9)(t+16)}$$

$$t^2 = A(t+9)(t+16) + B(t+1)(t+16) + C(t+1)(t+9)$$

$$\text{Putting } t = -1$$

$$(-1)^2 = A(-1+9)(-1+16) + B(-1+1)(-1+16) + C(-1+1)(-1+9)$$

$$1 = A(8)(15) + 0 + 0$$

$$I = 120A$$

$$\therefore A = \frac{1}{120}$$

$$\text{Putting } t = -9$$

$$(-9)^2 = A(-9+9)(-9+16) + B(-9+1)(-9+16) + C(-9+1)(-9+9)$$

$$81 = 0 + B(-8)(7) + 0$$

$$81 = -56B$$

$$\therefore B = \frac{-81}{56}$$

$$\text{Putting } t = -16$$

$$(-16)^2 = A(-16+9)(-16+16) + B(-8)B(-16+1)(-16+16) + C(-16+1)(-16+9)$$

$$256 = 0 + 0 + C(-15)(-7)$$

$$256 = C(105)$$

$$C = \frac{256}{105}$$

Putting the value of A, B and C in equation (i) we get

$$\frac{t^2}{(t+1)(t+9)(t+16)} = \frac{A}{(t+1)} + \frac{B}{(t+9)} + \frac{C}{(t+16)}$$

$$\int \frac{t^2}{(t+1)(t+9)(t+16)} dt = \int \left[\frac{A}{(t+1)} + \frac{B}{(t+9)} + \frac{C}{(t+16)} \right] dt$$

$$I = \int \left[\frac{1}{120} \frac{1}{(t+1)} + \left(\frac{-81}{56} \right) \frac{1}{(t+9)} + \left(\frac{256}{105} \right) \frac{1}{(t+16)} \right] dt$$

$$I = \frac{1}{120} \int \frac{1}{(t+1)} dt - \frac{81}{56} \int \frac{1}{(t+9)} dt + \frac{256}{105} \int \frac{1}{(t+16)} dt$$

Putting $t = x^2$

$$I = \frac{1}{120} \int \frac{1}{x^2+1} dx - \frac{81}{56} \int \frac{1}{x^2+9} dx + \frac{256}{105} \int \frac{1}{x^2+16} dx$$

$$I = \frac{1}{120} \int \frac{1}{x^2+1} dx - \frac{81}{56} \int \frac{1}{x^2+(3)^2} dx + \frac{256}{105} \int \frac{1}{x^2+(4)^2} dx$$

$$I = \frac{1}{120} \tan^{-1}(x) = \frac{81}{56} \times \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + \frac{256}{105} \times \frac{1}{4} \tan^{-1}\left(\frac{x}{4}\right) + c$$

$$I = \frac{1}{120} \tan^{-1} x - \frac{27}{56} \tan^{-1}\left(\frac{x}{3}\right) + \frac{64}{105} \tan^{-1}\left(\frac{x}{4}\right) + c$$

57. $\int \frac{\sin 2x}{(1-\cos 2x)(2-\cos 2x)} dx$

Sol. Let $I = \int \frac{\sin 2x}{(1-\cos 2x)(2-\cos 2x)} dx$

Put $t = \cos 2x \Rightarrow dt = -\sin 2x dx$; $\int \frac{-dt/2}{(1-t)(2-t)} = \frac{1}{2} \int \frac{dt}{(t-2)(1-t)}$

Put $\frac{1}{(t-2)(1-t)} = \frac{A}{t-2} + \frac{B}{1-t} \quad \dots (1) \quad \therefore A(1-t) + B(t-2) = 1$

Put $1-t=0 \Rightarrow t=1$; $A(0) + B(1-2) = 1 \Rightarrow B = -1$

Put $t-2=0 \Rightarrow t=2$; $A(1-2) + B(0) = 1 \Rightarrow A = -1$

From (1), $\frac{1}{(t-2)(1-t)} = \frac{-1}{t-2} + \frac{-1}{1-t}$

$$\Rightarrow \int \frac{1}{(t-2)(1-t)} dt = \int \frac{1}{2-t} dt + \int \frac{1}{t-1} dt$$

$$= -\log(2-t) + \log(t-1) + C = \log(t-1) - \log(2-t) + C$$

$$\Rightarrow \int \frac{\sin 2x}{(1-\cos 2x)(2-\cos 2x)} dx = \log(\cos 2x - 1) - \log(2 - \cos 2x) + C$$

58. $\int \frac{2}{(1-x)(1+x^2)} dx$

Sol. Let $\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx}{1+x^2} + \frac{C}{1+x^2}$

$$\Rightarrow 2 = A(1+x^2) + Bx(1-x) + C(1-x)$$

Putting $x=1$, we have $2 = 2A + 0 + 0$

$$\Rightarrow A = 1$$

Putting $x=0$ we have $2 = A + C$

$$\Rightarrow C = 2 - A \Rightarrow C = 2 - 1 = 1$$

Putting $x=2$ we have $2 = 5A - 2B - C$

$$\Rightarrow 2 = 5 \times 1 - 2B - 1 \Rightarrow 2B = 2 \Rightarrow B = 1$$

$$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x}{1+x^2} + \frac{1}{1+x^2}$$

Integrating both sides we have

$$\int \frac{2 dx}{(1-x)(1+x^2)} = \int \frac{dx}{1-x} + \int \frac{x}{1+x^2} + \int \frac{dx}{1+x^2}$$

$$= -\log|1-x| + \frac{1}{2} \log(1+x^2) + \tan^{-1} x + C$$

59. $\int \frac{2x^2+1}{x^2(x^2+4)} dx$

Sol. Let $I = \int \frac{2x^2+1}{x^2(x^2+4)} dx$

Again Let $x^2 = t$

$$\frac{2t+1}{t(t+4)} = \frac{A}{t} + \frac{B}{(t+4)} \quad \dots (i)$$

$$\frac{2t+1}{t(t+4)} = \frac{A(t+4) + B(t)}{t(t+4)}$$

$$2t+1 = A(t+4) + B(t)$$

Putting $t = -4$

$$2(-4)+1 = A(-4+4) + B(-4)$$

$$-8+1 = 0 - 4B$$

$$-7 = -4B$$

$$\therefore B = \frac{7}{4}$$

Putting $t = 0$

$$2(0)+1=A(0+4)+B(0)$$

$$1=4A$$

$$\therefore A=\frac{1}{4}$$

$$\text{Now } \frac{2t+1}{t(t+4)}=\frac{A}{t}+\frac{B}{(t+4)}$$

Putting the value of A and B in equation (i) we get $\frac{2t+1}{t(t+4)}=\frac{1}{4t}+\frac{7}{4(t+4)}$

$$\int \frac{2t+1}{t(t+4)} dt = \int \left(\frac{1}{4t} + \frac{7}{4(t+4)} \right) dt$$

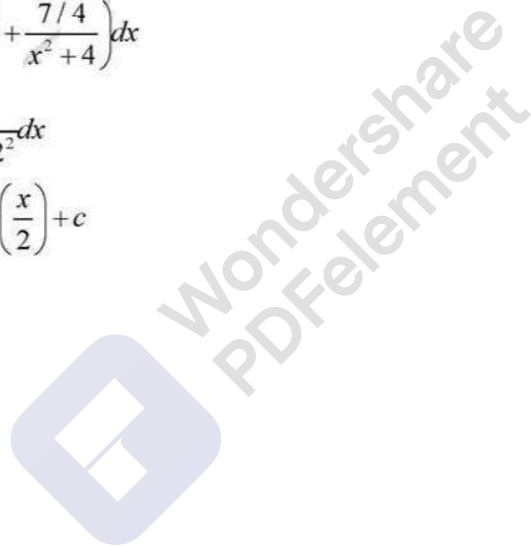
Putting $t = x^2$

$$\int \frac{2x^2+1}{x^2(x^2+4)} dx = \int \left(\frac{1/4}{x^2} + \frac{7/4}{x^2+4} \right) dx$$

$$I = \frac{1}{4} \int x^{-2} dx + \frac{7}{4} \int \frac{1}{x^2+2^2} dx$$

$$I = \frac{1}{4} \times \frac{x^{-1}}{-1} + \frac{7}{4} \times \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + c$$

$$I = \frac{-1}{4x} + \frac{7}{8} \tan^{-1} \frac{x}{2} + c$$



EXERCISE 4B (Pg. No.: 770)

Very-short Answer question

Evaluate:

1. $\int x^{-6} dx$

Sol. Let $I = \int x^{-6} dx \Rightarrow I = \frac{x^{-6+1}}{-6+1} + c \therefore I = \frac{x^{-5}}{-5} + c = -\frac{1}{5x^5} + c$

2. $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx$

Sol. Let $I = \int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx \Rightarrow I = \int x^{1/2} dx + \int x^{-1/2} dx \Rightarrow I = \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + c$
 $\therefore I = \frac{2}{3} x^{3/2} + 2x^{1/2} + c = \frac{2}{3} x^{3/2} + 2\sqrt{x} + c$

3. $\int \sin 3x dx$

Sol. Let $I = \int \sin 3x dx \therefore I = -\frac{\cos 3x}{3} + c$

4. $\int \frac{x^2}{1+x^3} dx$

Sol. Let $I = \int \frac{x^2}{1+x^3} dx$, Put $1+x^3 = t \Rightarrow 3x^2 dx = dt \Rightarrow x^2 dx = \frac{dt}{3}$
 $\Rightarrow I = \int \frac{1}{3} \cdot \frac{dt}{t} \Rightarrow I = \frac{1}{3} \log |t| + c \therefore I = \frac{1}{3} \log |1+x^3| + c$

5. $\int \frac{2 \cos x}{3 \sin^2 x} dx$

Sol. Let $I = \int \frac{2 \cos x}{3 \sin^2 x} dx \Rightarrow I = \frac{2}{3} \int \cot x \operatorname{cosec} x dx \Rightarrow I = \frac{2}{3} (-\operatorname{cosec} x) + c$
 $\therefore I = -\frac{2}{3} \operatorname{cosec} x + c$

6. $\int \frac{(3 \sin \phi - 2) \cos \phi}{(5 - \cos^2 \phi - 4 \sin \phi)} d\phi$

Sol. Let $I = \int \frac{(3 \sin \phi - 2) \cos \phi}{(5 - \cos^2 \phi - 4 \sin \phi)} d\phi = \int \frac{(3 \sin \phi - 2) \cos \phi}{\{5 - (1 - \sin^2 \phi) - 4 \sin \phi\}} d\phi$
 $\Rightarrow I = \int \frac{(3 \sin \phi - 2) \cos \phi}{(5 - 1 + \sin^2 \phi - 4 \sin \phi)} d\phi \Rightarrow I = \int \frac{(3 \sin \phi - 2) \cos \phi}{\sin^2 \phi - 4 \sin \phi + 4} d\phi$

Put $\sin \phi = t \Rightarrow \cos \phi d\phi = dt$

$\Rightarrow I = \int \frac{(3t - 2)}{t^2 - 4t + 4} dt$

Let $3t - 2 = A \frac{d}{dt}(t^2 - 4t + 4) + B \Rightarrow 3t - 2 = A(2t - 4) + B \Rightarrow 3t - 2 = 2At - 4A + B$

Equating co-efficient we get, $2A = 3 \therefore A = \frac{3}{2}$ & $-4A + B = -2 \Rightarrow B = -2 + 4A \therefore B = 4$

$$\Rightarrow I = \int \frac{A(2t-4)+B}{t^2-4t+4} dt \Rightarrow I = A \int \frac{2t-4}{t^2-4t+4} dt + B \int \frac{1}{t^2-4t+4} dt$$

$$\Rightarrow I = \frac{3}{2} I_1 + 4 I_2 \quad \dots (1)$$

$$I_1 = \int \frac{2t-4}{t^2-4t+4} dt, \text{ Put } t^2-4t+4 = y \Rightarrow (2t-4) dt = dy$$

$$\Rightarrow I_1 = \int \frac{dy}{y} \Rightarrow I_1 = \log |y| + c_1 \Rightarrow I_1 = \log |t^2-4t+4| + c_1$$

$$\Rightarrow I_1 = \log |\sin^2 \phi - 4 \sin \phi + 4| + c_1$$

$$I_2 = \int \frac{1}{t^2-4t+4} dt \Rightarrow I_2 = \int \frac{1}{(t-2)^2} dt \Rightarrow I_2 = -\frac{1}{t-2} + c_2$$

$$\therefore I_2 = -\frac{1}{\sin \phi - 2} + c_2$$

Putting the value of I_1 & I_2 in equation (1)

$$\Rightarrow I = \frac{3}{2} \log |\sin^2 \phi - 4 \sin \phi + 4| - \frac{4}{\sin \phi - 2} + c$$

$$I = \frac{3}{2} \log |\sin^2 \phi - 2 \cdot 2 \sin \phi + 2^2| - \frac{4}{\sin \phi - 2} + c$$

$$= \frac{3}{2} \log |\sin \phi - 2|^2 - \frac{4}{\sin \phi - 2} + c$$

$$= 3 \log |\sin \phi - 2| - \frac{4}{\sin \phi - 2} + c$$

$$= 3 \log |\sin \phi - 2| - \frac{4}{\sin \phi - 2} + c$$

$$I = \frac{3}{2} \log |\sin^2 \phi - 2 \sin \phi + 2^2| - \frac{4}{\sin \phi - 2} + c$$

$$\therefore I = 3 \log |\sin \phi - 2| - \frac{4}{\sin \phi - 2} + c$$

7. $\int \sin^2 x \, dx$

Sol. Let $I = \int \sin^2 x \, dx \Rightarrow I = \int \frac{1 - \cos 2x}{2} \, dx$

$$\Rightarrow I = \frac{1}{2} \int (1 - \cos 2x) \, dx \Rightarrow I = \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right] + c \quad \therefore I = \frac{x}{2} - \frac{\sin 2x}{4} + c$$

8. $\int \frac{(\log x)^2}{x} \, dx$

Sol. Let $I = \int \frac{(\log x)^2}{x} \, dx$, Put $\log x = t \Rightarrow \frac{1}{x} \, dx = dt$

$$\Rightarrow I = \int t^2 dt \Rightarrow I = \frac{t^3}{3} + c \quad \therefore I = \frac{(\log x)^3}{3} + c$$

9. $\int \frac{(x+1)(x+\log x)^2}{x} dx$

Sol. Let $I = \int \frac{(x+1)(x+\log x)^2}{x} dx$, Put $x + \log x = t \Rightarrow \left(1 + \frac{1}{x}\right) dx = dt \Rightarrow \frac{x+1}{x} dx = dt$

$$\Rightarrow I = \int t^2 dt \Rightarrow I = \frac{t^3}{3} + c \quad \therefore I = \frac{(x+\log x)^3}{3} + c$$

10. $\int \frac{\sin x}{1+\cos x} dx$

Sol. Let $I = \int \frac{\sin x}{1+\cos x} dx$, Put $1+\cos x = t \Rightarrow -\sin x dx = dt \Rightarrow \sin x dx = -dt$

$$\Rightarrow I = -\int \frac{dt}{t} \Rightarrow I = -\log |t| + c \quad \therefore I = -\log |1+\cos x| + c$$

11. $\int \frac{1+\tan x}{1-\tan x} dx$

Sol. Let $I = \int \frac{1+\tan x}{1-\tan x} dx \Rightarrow I = \int \frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}} dx \Rightarrow I = \int \frac{\cos x + \sin x}{\cos x - \sin x} dx$

Put $\cos x - \sin x = t \Rightarrow (-\sin x - \cos x) dx = dt \Rightarrow (\sin x + \cos x) dx = -dt$

$$\Rightarrow I = -\int \frac{dt}{t} \Rightarrow I = -\log |t| + c \quad \therefore I = -\log |\cos x - \sin x| + c$$

12. $\int \frac{1-\cot x}{1+\cot x} dx$

Sol. Let $I = \int \frac{1-\cot x}{1+\cot x} dx \Rightarrow I = \int \frac{1-\frac{\cos x}{\sin x}}{1+\frac{\cos x}{\sin x}} dx \Rightarrow I = \int \frac{\sin x - \cos x}{\sin x + \cos x} dx$

Put $\sin x + \cos x = t \Rightarrow (\cos x - \sin x) dx = dt \Rightarrow -(\sin x - \cos x) dx = dt \Rightarrow (\sin x - \cos x) dx = -dt$

$$\Rightarrow I = -\int \frac{dt}{t} \Rightarrow I = -\log |t| + c \quad \therefore I = -\log |\sin x + \cos x| + c$$

13. $\int \frac{1+\cot x}{x+\log(\sin x)} dx$

Sol. Let $I = \int \frac{1+\cot x}{x+\log(\sin x)} dx$

Put $x + \log(\sin x) = t \Rightarrow \left(1 + \frac{1}{\sin x} \cdot \cos x\right) dx = dt \Rightarrow (1 + \cot x) dx = dt$

$$\Rightarrow I = \int \frac{dt}{t} \Rightarrow I = \log |t| + c \quad \therefore I = \log |x + \log(\sin x)| + c$$

14. $\int \frac{1 - \sin 2x}{x + \cos^2 x} dx$

Sol. Let $I = \int \frac{1 - \sin 2x}{x + \cos^2 x} dx$, Put $x + \cos^2 x = t \Rightarrow 1 + 2 \cos x (-\sin x) = \frac{dt}{dx} \Rightarrow (1 - \sin 2x) dx = dt$
 $\Rightarrow I = \int \frac{dt}{t} \Rightarrow I = \log |t| + c \therefore I = \log |x + \cos^2 x| + c$

15. $\int \frac{\sec^2(\log x)}{x} dx$

Sol. Let $I = \int \frac{\sec^2(\log x)}{x} dx$, Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$
 $\Rightarrow I = \int \sec^2(t) dt \Rightarrow I = \tan(t) + c \therefore I = \tan(\log x) + c$

16. $\int \frac{\sin(2 \tan^{-1} x)}{1 + x^2} dx$

Sol. Let $I = \int \frac{\sin(2 \tan^{-1} x)}{1 + x^2} dx$, Put $2 \tan^{-1} x = t \Rightarrow 2 \cdot \frac{1}{1 + x^2} dx = dt \Rightarrow \frac{1}{1 + x^2} dx = \frac{dt}{2}$
 $\Rightarrow I = \int \sin(t) \frac{dt}{2} \Rightarrow I = -\frac{1}{2} \cos(2 \tan^{-1} x) + c \therefore I = -\frac{1}{2} \cos(2 \tan^{-1} x) + c$

17. $\int \frac{\tan x \sec^2 x}{(1 - \tan^2 x)} dx$

Sol. Let $I = \int \frac{\tan x \sec^2 x}{1 - \tan^2 x} dx$, Put $1 - \tan^2 x = t \Rightarrow -2 \tan x \sec^2 x = \frac{dt}{dx} \Rightarrow \tan x \sec^2 x dx = -\frac{dt}{2}$
 $\Rightarrow I = \int \frac{1}{t} \left(\frac{dt}{-2} \right) \Rightarrow I = -\frac{1}{2} \log |t| + c \therefore I = -\frac{1}{2} \log |1 - \tan^2 x| + c$

18. $\int \frac{x^4 + 1}{x^2 + 1} dx$

Sol. Let $I = \int \frac{x^4 + 1}{x^2 + 1} dx \Rightarrow I = \int \frac{x^4 - 1 + 2}{x^2 + 1} dx \Rightarrow I = \int \frac{x^4 - 1}{x^2 + 1} dx + 2 \int \frac{1}{x^2 + 1} dx$
 $\Rightarrow I = \int \frac{(x^2 - 1)(x^2 + 1)}{(x^2 + 1)} dx + 2 \tan^{-1} x \Rightarrow I = \int (x^2 - 1) dx + 2 \tan^{-1} x \therefore I = \frac{x^3}{3} - x + 2 \tan^{-1} x + c$

19. $\int \tan^{-1} \sqrt{\frac{1 - \sin x}{1 + \sin x}} dx$

Sol. Let $I = \int \tan^{-1} \sqrt{\frac{1 - \sin x}{1 + \sin x}} dx$
 $\Rightarrow I = \int \tan^{-1} \sqrt{\frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2}}} dx \Rightarrow I = \int \tan^{-1} \left(\frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2}} \right) dx$

Dividing numerator and denominator by $\cos \frac{x}{2}$ we get,

$$I = \int \tan^{-1} \left(\frac{1 - \tan \frac{x}{2}}{1 + \tan \frac{x}{2}} \right) dx \Rightarrow I = \int \tan^{-1} \left\{ \tan \left(\frac{\pi}{4} - \frac{x}{2} \right) \right\} dx$$

$$\Rightarrow I = \int \left(\frac{\pi}{4} - \frac{x}{2} \right) dx \Rightarrow I = \frac{\pi}{4} \cdot x - \frac{1}{2} \cdot \frac{x^2}{2} + c \quad \therefore I = \frac{\pi x}{4} - \frac{x^2}{4} + c$$

20. $\int \log(1+x^2) dx$

Sol. Let $I = \int \log(1+x^2) dx$

$$\Rightarrow I = \log(1+x^2) \int dx - \int \left[\frac{d\{\log(1+x^2)\}}{d(1+x^2)} \times \frac{d(1+x^2)}{dx} \int dx \right] dx$$

$$\Rightarrow I = \log(1+x^2) \cdot x - \int \frac{1}{1+x^2} \cdot 2x \cdot x dx \Rightarrow I = x \log(1+x^2) - 2 \int \frac{x^2}{x^2+1} dx$$

$$I = x \log(1+x^2) - 2 \int \frac{(x^2+1)-1}{x^2+1} dx \Rightarrow I = x \log(1+x^2) - 2 \int \frac{x^2+1}{x^2+1} dx + 2 \int \frac{1}{x^2+1} dx$$

$$\Rightarrow x \log(1+x^2) - 2 \int dx + 2 \tan^{-1} x + c \quad \therefore I = x \log(1+x^2) - 2x + 2 \tan^{-1} x + c$$

21. $\int \cos x \cos 3x dx$

Sol. Let $I = \int \cos x \cos 3x dx \Rightarrow I = \frac{1}{2} \int 2 \cos x \cos 3x dx$

$$\Rightarrow I = \frac{1}{2} \int \{\cos(x+3x) + \cos(x-3x)\} dx \Rightarrow I = \frac{1}{2} \int (\cos 4x + \cos 2x) dx$$

$$\Rightarrow I = \frac{1}{2} \left[\frac{\sin 4x}{4} + \frac{\sin 2x}{2} \right] + c \quad \therefore I = \frac{\sin 4x}{8} + \frac{\sin 2x}{4} + c$$

22. $\int \sin 3x \sin x dx$

Sol. Let $I = \int \sin 3x \sin x dx \Rightarrow I = \frac{1}{2} \int 2 \sin 3x \sin x dx$

$$\Rightarrow I = \frac{1}{2} \int \{\cos(3x-x) - \cos(3x+x)\} dx \Rightarrow I = \frac{1}{2} \int (\cos 2x - \cos 4x) dx$$

$$\Rightarrow I = \frac{1}{2} \left[\frac{\sin 2x}{2} - \frac{\sin 4x}{4} \right] + c \quad \therefore I = \frac{\sin 2x}{4} - \frac{\sin 4x}{8} + c$$

23. $\int \frac{x e^x}{(x+1)^2} dx$

Sol. Let $I = \int \frac{x e^x}{(x+1)^2} dx \Rightarrow I = \int e^x \left\{ \frac{(x+1)-1}{(x+1)^2} \right\} dx \Rightarrow I = \int e^x \left\{ \frac{1}{x+1} - \frac{1}{(x+1)^2} \right\} dx$

where $f(x) = \frac{1}{x+1}$, $f'(x) = -\frac{1}{(x+1)^2}$

$$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow I = e^x f(x) + c \quad \therefore I = e^x \cdot \frac{1}{x+1} + c$$

24. $\int e^x (\tan x - \log \cos x) dx$

Sol. Let $I = \int e^x (\tan x - \log \cos x) dx$

where $f(x) = -\log \cos x$, $f'(x) = \tan x$

$\Rightarrow I = \int e^x [f(x) + f'(x)] dx \Rightarrow e^x f(x) + c \quad \therefore I = -e^x \log(\cos x) + c$

25. $\int \frac{1}{1 - \sin x} dx$

Sol. Let $I = \int \frac{1}{1 - \sin x} dx \Rightarrow I = \int \frac{1}{1 - \sin x} \times \frac{1 + \sin x}{1 + \sin x} dx \Rightarrow I = \int \frac{1 + \sin x}{1 - \sin^2 x} dx$

$\Rightarrow I = \int \frac{1 + \sin x}{\cos^2 x} dx \Rightarrow I = \int \frac{1}{\cos^2 x} dx + \int \frac{\sin x}{\cos^2 x} dx$

$\Rightarrow I = \int \sec^2 x dx + \int \tan x \sec x dx \quad \therefore I = \tan x + \sec x + c$

26. $\int x \cos(x^2) dx$

Sol. Let $I = \int x \cos(x^2) dx$, Put $x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{dt}{2}$

$\Rightarrow I = \int \cos(t) \cdot \frac{dt}{2} \Rightarrow I = \frac{1}{2} \int \cos(t) dt \Rightarrow I = \frac{1}{2} \sin(t) + c \quad \therefore I = \frac{1}{2} \sin(x^2) + c$

27. $\int \frac{\cot x}{\sqrt{\sin x}} dx$

Sol. Let $I = \int \frac{\cot x}{\sqrt{\sin x}} dx \Rightarrow I = \int \frac{\cos x}{\sin x \sqrt{\sin x}} dx \Rightarrow I = \int \frac{\cos x}{(\sin x)^{3/2}} dx$

Put $\sin x = t \Rightarrow \cos x dx = dt$

$\Rightarrow I = \int \frac{dt}{t^{3/2}} \Rightarrow I = \int t^{-3/2} dt \Rightarrow I = \frac{t^{-1/2}}{-1/2} + c \Rightarrow I = \frac{-2}{\sqrt{t}} + c \quad \therefore I = \frac{-2}{\sqrt{\sin x}} + c$

28. $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$

Sol. Let $I = \int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx \Rightarrow I = \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} dx \Rightarrow I = \int \frac{\sin^2 x}{\cos^2 x} dx \Rightarrow I = \int \tan^2 x dx$

$\Rightarrow I = \int (\sec^2 x - 1) dx \quad \therefore I = \tan x - x + c$

29. $\int \sin^{-1}(\cos x) dx$

Sol. Let $I = \int \sin^{-1}(\cos x) dx \Rightarrow I = \int \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - x\right)\right\} dx$

$\Rightarrow I = \int \left(\frac{\pi}{2} - x\right) dx \Rightarrow I = \frac{\pi}{2}x - \frac{x^2}{2} + c \quad \therefore I = \frac{\pi}{2}x - \frac{x^2}{2} + c$

30. $\int \frac{dx}{\sqrt{x+2} + \sqrt{x+1}}$

Sol. Let $I = \int \frac{1}{\sqrt{x+2} + \sqrt{x+1}} dx \Rightarrow I = \int \frac{1}{\sqrt{x+2} + \sqrt{x+1}} \times \frac{\sqrt{x+2} - \sqrt{x+1}}{\sqrt{x+2} - \sqrt{x+1}} dx$

$$\Rightarrow I = \int \frac{\sqrt{x+2} - \sqrt{x+1}}{(\sqrt{x+2})^2 - (\sqrt{x+1})^2} dx \Rightarrow I = \int \frac{\sqrt{x+2} - \sqrt{x+1}}{(x+2) - (x+1)} dx$$

$$\Rightarrow I = \int \sqrt{x+2} dx - \int \sqrt{x+1} dx \Rightarrow I = \frac{(x+2)^{3/2}}{3/2} - \frac{(x+1)^{3/2}}{3/2} + c$$

$$\therefore I = \frac{2}{3} \left[(x+2)^{3/2} - (x+1)^{3/2} \right] + c$$

31. $\int 2^x dx$

Sol. Let $I = \int 2^x dx \therefore I = \frac{2^x}{\log 2} + c$

32. $\int \frac{1 + \tan x}{(x + \log \sec x)} dx$

Sol. Let $I = \int \frac{1 + \tan x}{x + \log(\sec x)} dx$

Put $x + \log(\sec x) = t \Rightarrow 1 + \frac{1}{\sec x} \cdot \sec x \tan x = \frac{dt}{dx} \Rightarrow (1 + \tan x) dx = dt$

$\Rightarrow I = \int \frac{dt}{t} \Rightarrow I = \log |t| + c \therefore I = \log |x + \log(\sec x)| + c$

33. $\int \frac{\sec^2(\log x)}{x} dx$

Sol. Let $I = \int \frac{\sec^2(\log x)}{x} dx$, Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$\Rightarrow I = \int \sec^2(t) dt \Rightarrow I = \tan(t) + c \therefore I = \tan(\log x) + c$

34. $\int (2x+1)\sqrt{x^2+x+1} dx$

Sol. Let $I = \int (2x+1)\sqrt{x^2+x+1} dx$, Put $x^2+x+1 = t \Rightarrow (2x+1) dx = dt$

$\Rightarrow I = \int \sqrt{t} dt \Rightarrow I = \frac{t^{3/2}}{3/2} + c \therefore I = \frac{2}{3} (x^2+x+1)^{3/2} + c$

35. $\int \frac{dx}{\sqrt{9x^2+16}}$

Sol. Let $I = \int \frac{dx}{\sqrt{9x^2+16}} \Rightarrow I = \int \frac{dx}{\sqrt{9\left(x^2 + \frac{16}{9}\right)}} \Rightarrow I = \frac{1}{3} \int \frac{1}{\sqrt{\left(x\right)^2 + \left(\frac{4}{3}\right)^2}} dx$

$\Rightarrow I = \frac{1}{3} \log \left| x + \sqrt{\left(x\right)^2 + \left(\frac{4}{3}\right)^2} \right| + c \therefore I = \frac{1}{3} \log \left| 3x + \sqrt{9x^2+16} \right| + c$

36. $\int \frac{dx}{\sqrt{4-9x^2}}$

Sol. Let $I = \int \frac{dx}{\sqrt{4-9x^2}} \Rightarrow I = \int \frac{1}{\sqrt{9\left(\frac{4}{9}-x^2\right)}} dx$

$\Rightarrow I = \frac{1}{3} \int \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 - (x)^2}} dx \Rightarrow I = \frac{1}{3} \sin^{-1}\left(\frac{x}{2/3}\right) + c \therefore I = \frac{1}{3} \sin^{-1}\left(\frac{3x}{2}\right) + c$

37. $\int \frac{dx}{\sqrt{4x^2-25}}$

Sol. Let $I = \int \frac{1}{\sqrt{4x^2-25}} dx \Rightarrow I = \int \frac{1}{\sqrt{4\left(x^2-\frac{25}{4}\right)}} dx \Rightarrow I = \frac{1}{2} \int \frac{1}{\sqrt{(x)^2 - \left(\frac{5}{2}\right)^2}} dx$

$\Rightarrow I = \frac{1}{2} \log \left| x + \sqrt{(x)^2 - \left(\frac{5}{2}\right)^2} \right| + c \therefore I = \frac{1}{2} \log \left| 2x + \sqrt{4x^2-25} \right| + c$

38. $\int \sqrt{4-x^2} dx$

Sol. Let $I = \int \sqrt{4-x^2} dx \Rightarrow I = \int \sqrt{(2)^2 - (x)^2} dx$

$\Rightarrow I = \frac{x}{2} \sqrt{(2)^2 - (x)^2} + \frac{(2)^2}{2} \sin^{-1}\left(\frac{x}{2}\right) + c \therefore I = \frac{x}{2} \sqrt{4-x^2} + 2 \sin^{-1}\left(\frac{x}{2}\right) + c$

39. $\int \sqrt{9+x^2} dx$

Sol. Let $I = \int \sqrt{9+x^2} dx \Rightarrow I = \int \sqrt{(3)^2 + (x)^2} dx$

$\Rightarrow I = \frac{x}{2} \sqrt{(3)^2 + (x)^2} + \frac{(3)^2}{2} \log \left| x + \sqrt{(3)^2 + x^2} \right| + c \therefore I = \frac{x}{2} \sqrt{9+x^2} + \frac{9}{2} \log \left| x + \sqrt{9+x^2} \right| + c$

40. $\int \sqrt{x^2-16} dx$

Sol. Let $I = \int \sqrt{x^2-16} dx \Rightarrow I = \int \sqrt{(x)^2 - (4)^2} dx$

$\Rightarrow I = \frac{x}{2} \sqrt{x^2-16} - \frac{(4)^2}{2} \log \left| x + \sqrt{x^2-4^2} \right| + c = \frac{x}{2} \sqrt{x^2-16} - 8 \log \left| x + \sqrt{x^2-16} \right| + c$