

DEFINITE INTEGRALS (XII, R. S. AGGARWAL)

EXERCISE 16A (Pg. No.: 799)

Very-Short-Answer Questions

Evaluate :

1. $\int_1^3 x^4 dx$

Sol. Let $I = \int_1^3 x^4 dx \Rightarrow I = \left[\frac{x^5}{5} \right]_1^3 \Rightarrow I = \frac{1}{5} [(3)^5 - (1)^5]$
 $\Rightarrow I = \frac{1}{5} (243 - 1) \Rightarrow I = \frac{1}{5} \times 242 \therefore I = \frac{242}{5}$

2. $\int_1^4 \sqrt{x} dx$

Sol. Let $I = \int_1^4 \sqrt{x} dx \Rightarrow I = \int_1^4 (x)^{1/2} dx \Rightarrow I = \left[\frac{x^{3/2}}{3/2} \right]_1^4 \Rightarrow I = \frac{2}{3} [x^{3/2}]_1^4$
 $\Rightarrow I = \frac{2}{3} [(4)^{3/2} - (1)^{3/2}] \Rightarrow I = \frac{2}{3} [(2^2)^{3/2} - (1^2)^{3/2}] \Rightarrow I = \frac{2}{3} [(2)^3 - 1]$
 $\Rightarrow I = \frac{2}{3} (8 - 1) \Rightarrow I = \frac{2}{3} \times 7 \therefore I = \frac{14}{3}$

3. $\int_1^2 x^{-5} dx$

Sol. Let $I = \int_1^2 x^{-5} dx \Rightarrow I = \left[\frac{x^{-4}}{-4} \right]_1^2 \Rightarrow I = -\frac{1}{4} \left[\frac{1}{(2)^4} - \frac{1}{(1)^4} \right]$
 $\Rightarrow I = -\frac{1}{4} \left[\frac{1}{16} - \frac{1}{1} \right] \Rightarrow I = -\frac{1}{4} \left[\frac{-15}{16} \right] \therefore I = \frac{15}{64}$

4. $\int_0^{16} x^{3/4} dx$

Sol. Let $I = \int_0^{16} x^{3/4} dx \Rightarrow I = \left[\frac{x^{7/4}}{7/4} \right]_0^{16} \Rightarrow I = \frac{4}{7} [x^{7/4}]_0^{16} \Rightarrow I = \frac{4}{7} [(16)^{7/4} - (0)^{7/4}]$
 $\Rightarrow I = \frac{4}{7} [(2^4)^{7/4}] \Rightarrow I = \frac{4}{7} (2)^7 \Rightarrow I = \frac{4}{7} \times 128 \therefore I = \frac{512}{7}$

5. $\int_{-4}^{-1} \frac{dx}{x}$

Sol. Let $I = \int_{-4}^{-1} \frac{dx}{x} \Rightarrow I = [\log |x|]_{-4}^{-1} \Rightarrow I = \log |-1| - \log |-4|$
 $\Rightarrow I = \log |1| - \log |4| \Rightarrow I = 0 - \log 4 \therefore I = -\log 4$

6. $\int_1^4 \frac{1}{\sqrt{x}} dx$

Sol. Let $I = \int_1^4 \frac{1}{\sqrt{x}} dx \Rightarrow I = \int_1^4 x^{-1/2} dx \Rightarrow I = \left[\frac{x^{1/2}}{1/2} \right]_1^4 \Rightarrow I = 2 \left[x^{1/2} \right]_1^4$
 $\Rightarrow I = 2 \left[(4)^{1/2} - (1)^{1/2} \right] \Rightarrow I = 2 \left[(2^2)^{1/2} - (1)^{1/2} \right] \Rightarrow I = 2 \left[2 - 1 \right] \therefore I = 2$

7. $\int_0^1 \frac{1}{\sqrt[3]{x}} dx$

Sol. Let $I = \int_0^1 \frac{1}{\sqrt[3]{x}} dx \Rightarrow I = \int_0^1 x^{-1/3} dx \Rightarrow I = \left[\frac{x^{2/3}}{2/3} \right]_0^1 \Rightarrow I = \frac{3}{2} \left[x^{2/3} \right]_0^1$
 $\Rightarrow I = \frac{3}{2} \left[(1)^{2/3} - (0)^{2/3} \right] \Rightarrow I = \frac{3}{2} \times 1 \therefore I = \frac{3}{2}$

8. $\int_1^8 \frac{1}{x^{2/3}} dx$

Sol. Let $I = \int_1^8 \frac{1}{x^{2/3}} dx \Rightarrow I = \int_1^8 x^{-2/3} dx \Rightarrow I = \left[\frac{x^{1/3}}{1/3} \right]_1^8 \Rightarrow I = 3 \left[x^{1/3} \right]_1^8$
 $\Rightarrow I = 3 \left[(8)^{1/3} - (1)^{1/3} \right] \Rightarrow I = 3 \left[(2^3)^{1/3} - (1)^{1/3} \right] \Rightarrow I = 3(2 - 1) \therefore I = 3$

9. $\int_2^4 3 dx$

Sol. Let $I = \int_2^4 3 dx \Rightarrow I = 3 \int_2^4 dx \Rightarrow I = 3 \left[x \right]_2^4 \Rightarrow I = 3(4 - 2) \Rightarrow I = 3 \times 2 \therefore I = 6$

10. $\int_0^1 \frac{1}{(1+x^2)} dx$

Sol. Let $I = \int_0^1 \frac{1}{(1+x^2)} dx \Rightarrow I = \left[\tan^{-1} x \right]_0^1 \Rightarrow I = \left[\tan^{-1}(1) - \tan^{-1}(0) \right] \therefore I = \frac{\pi}{4}$

11. $\int_0^{\infty} \frac{1}{(1+x^2)} dx$

Sol. Let $I = \int_0^{\infty} \frac{1}{(1+x^2)} dx \Rightarrow I = \left[\tan^{-1} x \right]_0^{\infty} \Rightarrow I = \left[\tan^{-1}(\infty) - \tan^{-1} 0 \right]$
 $\Rightarrow I = \left(\frac{\pi}{2} - 0 \right) \therefore I = \frac{\pi}{2}$

12. $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$

Sol. Let $I = \int_0^1 \frac{1}{\sqrt{1-x^2}} dx \Rightarrow I = \left[\sin^{-1} x \right]_0^1 \Rightarrow I = \left[\sin^{-1}(1) - \sin^{-1}(0) \right]$
 $\Rightarrow I = \left(\frac{\pi}{2} - 0 \right) \therefore I = \frac{\pi}{2}$

13. $\int_0^{\pi/6} \sec^2 x \, dx$

Sol. Let $I = \int_0^{\pi/6} \sec^2 x \, dx \Rightarrow I = [\tan x]_0^{\pi/6} \Rightarrow I = \left[\tan \frac{\pi}{6} - \tan 0 \right]$
 $\Rightarrow I = \left(\frac{1}{\sqrt{3}} - 0 \right) \therefore I = \frac{1}{\sqrt{3}}$

14. $\int_{-\pi/4}^{\pi/4} \operatorname{cosec}^2 x \, dx$

Sol. Let $I = \int_{-\pi/4}^{\pi/4} \operatorname{cosec}^2 x \, dx \Rightarrow I = -[\cot x]_{-\pi/4}^{\pi/4}$
 $\Rightarrow I = -[\cot(\pi/4) - \cot(-\pi/4)] \Rightarrow I = -[1 + 1] \therefore I = -2$

15. $\int_{\pi/4}^{\pi/2} \cot^2 x \, dx$

Sol. Let $I = \int_{\pi/4}^{\pi/2} \cot^2 x \, dx \Rightarrow \int_{\pi/4}^{\pi/2} (\operatorname{cosec}^2 x - 1) \, dx \Rightarrow I = \int_{\pi/4}^{\pi/2} \operatorname{cosec}^2 x \, dx - \int_{\pi/4}^{\pi/2} dx$
 $\Rightarrow I = [-\cot x]_{\pi/4}^{\pi/2} - [x]_{\pi/4}^{\pi/2} \Rightarrow I = -\left[\cot \frac{\pi}{2} - \cot \frac{\pi}{4} \right] - \left[\frac{\pi}{2} - \frac{\pi}{4} \right]$
 $\Rightarrow I = [-0 + 1 - \frac{\pi}{2} + \frac{\pi}{4}] \therefore I = \left(1 - \frac{\pi}{4} \right)$

16. $\int_0^{\pi/4} \tan^2 x \, dx$

Sol. Let $I = \int_0^{\pi/4} \tan^2 x \, dx \Rightarrow I = \int_0^{\pi/4} (\sec^2 x - 1) \, dx \Rightarrow I = \int_0^{\pi/4} \sec^2 x \, dx - \int_0^{\pi/4} dx$
 $\Rightarrow I = [\tan x]_0^{\pi/4} - [x]_0^{\pi/4} \Rightarrow I = \left[\tan \frac{\pi}{4} - \tan 0 \right] - \left[\frac{\pi}{4} - 0 \right]$
 $\Rightarrow I = (1 - 0) - \left(\frac{\pi}{4} \right) \therefore I = \left[1 - \frac{\pi}{4} \right]$

17. $\int_0^{\pi/2} \sin^2 x \, dx$

Sol. Let $I = \int_0^{\pi/2} \sin^2 x \, dx \Rightarrow I = \int_0^{\pi/2} \frac{1 - \cos 2x}{2} \, dx \Rightarrow I = \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2x) \, dx$
 $\Rightarrow I = \frac{1}{2} \left[\int_0^{\pi/2} dx - \int_0^{\pi/2} \cos 2x \, dx \right] \Rightarrow I = \frac{1}{2} \left\{ [x]_0^{\pi/2} - \left[\frac{\sin 2x}{2} \right]_0^{\pi/2} \right\}$
 $\Rightarrow I = \frac{1}{2} \left\{ \left(\frac{\pi}{2} - 0 \right) - \left(\frac{\sin(\pi/2)}{2} - \frac{\sin 2(0)}{2} \right) \right\} \Rightarrow I = \frac{1}{2} \left\{ \frac{\pi}{2} - 0 - 0 \right\} \therefore I = \frac{\pi}{4}$

18. $\int_0^{\pi/4} \cos^2 x \, dx$

Sol. Let $I = \int_0^{\pi/4} \cos^2 x \, dx \Rightarrow I = \int_0^{\pi/4} \frac{1 + \cos 2x}{2} \, dx \Rightarrow I = \frac{1}{2} \left[x + \frac{\sin 2x}{2} \right]_0^{\pi/4}$

$$\Rightarrow I = \frac{1}{2} \left[\left\{ \frac{\pi}{4} + \frac{\sin \left(2 \cdot \frac{\pi}{4} \right)}{2} \right\} - \left(0 + \frac{\sin 2 \cdot 0}{2} \right) \right] \Rightarrow I = \frac{1}{2} \left[\frac{\pi}{4} + \frac{1}{2} \right] \therefore I = \frac{\pi}{8} + \frac{1}{4}$$

19. $\int_0^{\pi/3} \tan x \, dx$

Sol. Let $I = \int_0^{\pi/3} \tan x \, dx \Rightarrow I = \left[-\log |\cos x| \right]_0^{\pi/3} \Rightarrow I = \left[-\log |\cos \pi/3| - (-\log |\cos 0|) \right]$

$$\Rightarrow I = \left(-\log \frac{1}{2} + \log 1 \right) \Rightarrow I = -(\log 1 - \log 2) + \log 1$$

$$\Rightarrow I = -\log 1 + \log 2 + \log 1 \therefore I = \log 2$$

20. $\int_{\pi/6}^{\pi/4} \operatorname{cosec} x \, dx$

Sol. Let $I = \int_{\pi/6}^{\pi/4} \operatorname{cosec} x \, dx \Rightarrow I = \left[\log |\operatorname{cosec} x - \cot x| \right]_{\pi/6}^{\pi/4}$

$$\Rightarrow I = \left[\log \left(\operatorname{cosec} \frac{\pi}{4} - \cot \frac{\pi}{4} \right) - \log \left(\operatorname{cosec} \frac{\pi}{6} - \cot \frac{\pi}{6} \right) \right]$$

$$\Rightarrow I = \log (\sqrt{2} - 1) - \log (2 - \sqrt{3}) \therefore I = \log \left[\frac{\sqrt{2} - 1}{2 - \sqrt{3}} \right]$$

21. $\int_0^{\pi/3} \cos^3 x \, dx$

Sol. Let $I = \int_0^{\pi/3} \cos^3 x \, dx$

We know that $\cos 3x = 4\cos^3 x - 3\cos x \Rightarrow \cos 3x + 3\cos x = 4\cos^3 x \therefore \cos^3 x = \frac{\cos 3x + 3\cos x}{4}$

$$\Rightarrow I = \int_0^{\pi/3} \left(\frac{\cos 3x + 3\cos x}{4} \right) dx \Rightarrow I = \frac{1}{4} \left[\frac{\sin 3x}{3} + 3\sin x \right]_0^{\pi/3}$$

$$\Rightarrow I = \frac{1}{4} \left[\left\{ \frac{\sin \left(3 \cdot \frac{\pi}{3} \right)}{3} + 3\sin \left(\frac{\pi}{3} \right) \right\} - (0) \right] \Rightarrow I = \frac{1}{4} \left[0 + 3 \cdot \frac{\sqrt{3}}{2} \right] \therefore I = \frac{3\sqrt{3}}{8}$$

22. $\int_0^{\pi/2} \sin^3 x \, dx$

Sol. Let $I = \int_0^{\pi/2} \sin^3 x \, dx$

As, we know that $\sin 3x = 3\sin x - 4\sin^3 x \Rightarrow \sin^3 x = \frac{3\sin x - \sin 3x}{4}$ (general trigonometric formula)

$$\begin{aligned}\therefore I &= \int_0^{\pi/2} \sin^3 x \, dx \Rightarrow I = \int_0^{\pi/2} \frac{3 \sin x - \sin 3x}{4} \, dx \Rightarrow I = \frac{1}{4} \int_0^{\pi/2} 3 \sin x - \sin 3x \, dx \\ \Rightarrow I &= \frac{1}{4} \left[-3 \cos x + \frac{\cos 3x}{3} \right]_0^{\pi/2} \Rightarrow I = \frac{1}{4} \left[\left(-3 \cos \frac{\pi}{2} + \cos \frac{3(\pi/2)}{3} \right) - \left(-3 \cos(0) + \frac{\cos(0)}{3} \right) \right] \\ \Rightarrow I &= \frac{1}{4} \left[(0+0) - \left(-3.1 + \frac{1}{3} \right) \right] \Rightarrow I = \frac{1}{4} \left[-\left(\frac{-8}{3} \right) \right] \Rightarrow I = \frac{1}{4} \times \frac{8}{3} \therefore I = \frac{2}{3}\end{aligned}$$

23. $\int_{\pi/4}^{\pi/2} \frac{1-3\cos x}{\sin^2 x} \, dx$

Sol. Let $I = \int_{\pi/4}^{\pi/2} \frac{1-3\cos x}{\sin^2 x} \, dx \Rightarrow \int_{\pi/4}^{\pi/2} \left(\frac{1}{\sin^2 x} - \frac{3\cos x}{\sin^2 x} \right) \, dx \Rightarrow I = \int_{\pi/4}^{\pi/2} (\operatorname{cosec}^2 x - 3 \operatorname{cosec} x \cot x) \, dx$

$$\begin{aligned}\Rightarrow I &= \left[-\cot x - (-3 \operatorname{cosec} x) \right]_{\pi/4}^{\pi/2} \Rightarrow I = \left[-\cot x + 3 \operatorname{cosec} x \right]_{\pi/4}^{\pi/2} \\ \Rightarrow I &= \left[\left(-\cot \frac{\pi}{2} + 3 \operatorname{cosec} \frac{\pi}{2} \right) - \left(-\cot \frac{\pi}{4} + 3 \operatorname{cosec} \frac{\pi}{4} \right) \right] \\ \Rightarrow I &= \left[(0+3.1) - (-1+3\sqrt{2}) \right] \Rightarrow I = 3+1-3\sqrt{2} \therefore I = 4-3\sqrt{2}\end{aligned}$$

24. $\int_0^{\pi/4} \sqrt{1+\cos 2x} \, dx$

Sol. Let $I = \int_0^{\pi/4} \sqrt{1+\cos 2x} \, dx \Rightarrow I = \int_0^{\pi/4} \sqrt{2\cos^2 x} \, dx \Rightarrow I = \sqrt{2} \int_0^{\pi/4} \cos x \, dx$

$$\begin{aligned}\Rightarrow I &= \sqrt{2} [\sin x]_0^{\pi/4} \Rightarrow I = \sqrt{2} \left[\sin \frac{\pi}{4} - \sin 0 \right] \Rightarrow I = \sqrt{2} \left[\frac{1}{\sqrt{2}} - 0 \right] \Rightarrow I = \sqrt{2} \times \frac{1}{\sqrt{2}} \\ \therefore I &= 1\end{aligned}$$

25. $\int_0^{\pi/4} \sqrt{1-\sin 2x} \, dx$

Sol. Let $I = \int_0^{\pi/4} \sqrt{1-\sin 2x} \, dx \Rightarrow I = \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2} \, dx$

$$\begin{aligned}\Rightarrow I &= \int_0^{\pi/4} (\cos x - \sin x) \, dx \Rightarrow [\sin x + \cos x]_0^{\pi/4} \Rightarrow I = \left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - (\sin 0 + \cos 0) \\ \Rightarrow I &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - (0+1) \Rightarrow I = \frac{2}{\sqrt{2}} - 1 \therefore I = \sqrt{2} - 1\end{aligned}$$

26. $\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+\sin x)}$

Sol. Let $I = \int_{-\pi/4}^{\pi/4} \frac{1}{(1+\sin x)} \, dx \Rightarrow \int_{-\pi/4}^{\pi/4} \frac{1}{(1+\sin x)} \times \frac{(1-\sin x)}{(1-\sin x)} \, dx \Rightarrow I = \int_{-\pi/4}^{\pi/4} \frac{(1-\sin x)}{(1-\sin^2 x)} \, dx$

$$\begin{aligned}\Rightarrow I &= \int_{-\pi/4}^{\pi/4} \frac{(1-\sin x)}{(1-\sin^2 x)} \, dx \Rightarrow I = \int_{-\pi/4}^{\pi/4} \frac{1}{(1-\sin^2 x)} - \frac{\sin x}{1-\sin^2 x} \, dx \\ \Rightarrow I &= \int_{-\pi/4}^{\pi/4} \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \, dx \Rightarrow I = \int_{-\pi/4}^{\pi/4} \sec^2 x \, dx - \int_{-\pi/4}^{\pi/4} \tan x \sec x \, dx\end{aligned}$$

$$\Rightarrow I = [\tan x - \sec x]_{-\pi/4}^{\pi/4} \Rightarrow I = \left[\left(\tan \frac{\pi}{4} - \sec \frac{\pi}{4} \right) - \left(\tan \left(-\frac{\pi}{4} \right) - \sec \left(-\frac{\pi}{4} \right) \right) \right]$$

$$\Rightarrow I = (1 - \sqrt{2}) - (-1 - \sqrt{2}) \Rightarrow I = 1 - \sqrt{2} + 1 + \sqrt{2} \quad \therefore I = 2$$

27. $\int_0^{\pi/4} \frac{dx}{1 + \cos 2x}$

Sol. Let $I = \int_0^{\pi/4} \frac{dx}{1 + \cos 2x} \Rightarrow I = \int_0^{\pi/4} \frac{1}{2 \cos^2 x} dx \Rightarrow I = \frac{1}{2} \int_0^{\pi/4} \sec^2 x dx$

$$\Rightarrow I = \frac{1}{2} [\tan x]_0^{\pi/4} \Rightarrow I = \frac{1}{2} \left[\tan \frac{\pi}{4} - \tan 0 \right] \Rightarrow I = \frac{1}{2} [1 - 0] \quad \therefore I = \frac{1}{2}$$

28. $\int_{\pi/4}^{\pi/2} \frac{dx}{1 - \cos x}$

Sol. Let $I = \int_{\pi/4}^{\pi/2} \frac{dx}{1 - \cos x} = \int_{\pi/4}^{\pi/2} \frac{dx}{2 \sin^2 x} = \frac{1}{2} \int_{\pi/4}^{\pi/2} \operatorname{cosec}^2 x dx$

$$= -\frac{1}{2} [\cot x]_{\pi/4}^{\pi/2} = -\frac{1}{2} \left[\cot \frac{\pi}{2} - \cot \frac{\pi}{4} \right] = -\frac{1}{2} [0 - 1] = \frac{1}{2}$$

29. $\int_0^{\pi/4} \sin 2x \sin 3x dx$

Sol. Let $I = \int_0^{\pi/4} \sin 2x \sin 3x dx \Rightarrow I = \frac{1}{2} \int_0^{\pi/4} 2 \sin 2x \sin 3x dx$

$$\Rightarrow I = \frac{1}{2} \int_0^{\pi/4} [\cos(2x - 3x) - \cos(2x + 3x)] dx \Rightarrow I = \frac{1}{2} \int_0^{\pi/4} (\cos(-x) - \cos 5x) dx$$

$$\Rightarrow I = \frac{1}{2} \int_0^{\pi/4} (\cos x - \cos 5x) dx \Rightarrow I = \frac{1}{2} \left[\sin x - \frac{\sin 5x}{5} \right]_0^{\pi/4}$$

$$\Rightarrow I = \frac{1}{2} \left[\sin \left(\frac{\pi}{4} \right) - \frac{\sin 5 \left(\frac{\pi}{4} \right)}{5} \right] - \left[\sin(0) - \frac{\sin 5(0)}{5} \right] \Rightarrow I = \frac{1}{2} \left[\frac{1}{\sqrt{2}} - \frac{1}{5\sqrt{2}} \right]$$

$$\Rightarrow I = \frac{1}{2} \left(\frac{5-1}{5\sqrt{2}} \right) \Rightarrow I = \frac{1}{2} \times \frac{4}{5\sqrt{2}} \quad \therefore I = \frac{2}{5\sqrt{2}}$$

30. $\int_0^{\pi/6} \cos x \cos 2x dx$

Sol. Let $I = \int_0^{\pi/6} \cos x \cos 2x dx \Rightarrow I = \frac{1}{2} \int_0^{\pi/6} 2 \cos x \cos 2x dx$

$$\Rightarrow I = \frac{1}{2} \int_0^{\pi/6} \cos(x + 2x) + \cos(x - 2x) dx \Rightarrow I = \frac{1}{2} \int_0^{\pi/6} \cos 3x + \cos(-x) dx$$

$$\Rightarrow I = \frac{1}{2} \left[\frac{\sin 3x}{3} + \frac{\sin x}{1} \right]_0^{\pi/6} \Rightarrow I = \frac{1}{2} \left[\frac{\sin 3 \left(\frac{\pi}{6} \right)}{3} + \frac{\sin \left(\frac{\pi}{6} \right)}{1} - 0 \right] \Rightarrow I = \frac{1}{2} \left[\frac{\sin \frac{\pi}{2}}{3} + \frac{1}{2} \right]$$

$$\Rightarrow I = \frac{1}{2} \left[\frac{1}{3} + \frac{1}{2} \right] \Rightarrow I = \frac{1}{2} \left[\frac{2+3}{6} \right] \therefore I = \frac{5}{12}$$

31. $\int_0^{\pi} \sin 2x \cos 3x \, dx$

Sol. Let $I = \int_0^{\pi} \sin 2x \cos 3x \, dx \Rightarrow I = \frac{1}{2} \int_0^{\pi} 2 \sin 2x \cos 3x \, dx$

$$\Rightarrow I = \frac{1}{2} \int_0^{\pi} \{ \sin(2x+3x) + \sin(2x-3x) \} \, dx \Rightarrow I = \frac{1}{2} \int_0^{\pi} (\sin 5x - \sin x) \, dx$$

$$\Rightarrow I = \frac{1}{2} \left[\frac{-\cos 5x}{5} + \cos x \right]_0^{\pi} \Rightarrow I = \frac{1}{2} \left[\left\{ \frac{-\cos 5\pi}{5} + \cos \pi \right\} - \left\{ \frac{-\cos 5 \cdot 0}{5} + \cos 0 \right\} \right]$$

$$\Rightarrow I = \frac{1}{2} \left[\left(\frac{1}{5} - 1 \right) - \left(\frac{-1}{5} + 1 \right) \right] \Rightarrow I = \frac{1}{2} \left[\left(\frac{-4}{5} \right) - \left(\frac{4}{5} \right) \right] \Rightarrow I = \frac{1}{2} \left[\frac{-8}{5} \right] \therefore I = \frac{-4}{5}$$

32. $\int_0^{\pi/2} \sqrt{1+\sin x} \, dx$

Sol. Let $I = \int_0^{\pi/2} \sqrt{1+\sin x} \, dx \Rightarrow I = \int_0^{\pi/2} \left(\cos \frac{x}{2} + \sin \frac{x}{2} \right) \, dx \Rightarrow I = \left[\frac{\sin \frac{x}{2}}{\frac{1}{2}} - \frac{\cos \frac{x}{2}}{\frac{1}{2}} \right]_0^{\pi/2}$

$$\Rightarrow I = 2 \left[\sin \frac{x}{2} - \cos \frac{x}{2} \right]_0^{\pi/2} \Rightarrow I = 2 \left[\left\{ \sin \frac{\pi}{4} - \cos \frac{\pi}{4} \right\} - \left\{ \sin 0 - \cos 0 \right\} \right]$$

$$\Rightarrow I = 2 \left[\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) - (0 - 1) \right] \Rightarrow I = 2[0 + 1] \therefore I = 2$$

33. $\int_0^{\pi/2} \sqrt{1+\cos x} \, dx$

Sol. Let $I = \int_0^{\pi/2} \sqrt{1+\cos x} \, dx \Rightarrow I = \int_0^{\pi/2} \sqrt{2 \cos^2 \frac{x}{2}} \, dx \Rightarrow I = \sqrt{2} \int_0^{\pi/2} \cos \frac{x}{2} \, dx$

$$\Rightarrow I = \sqrt{2} \left[\frac{\sin \frac{x}{2}}{\frac{1}{2}} \right]_0^{\pi/2} \Rightarrow I = 2\sqrt{2} \left[\left(\sin \frac{\pi}{4} \right) - (\sin 0) \right] \Rightarrow I = 2\sqrt{2} \left[\frac{1}{\sqrt{2}} - 0 \right] \therefore I = 2$$

34. $\int_0^2 \frac{x^4+1}{x^2+1} \, dx$

Sol. Let $I = \int_0^2 \frac{x^4+1}{x^2+1} \, dx \Rightarrow I = \int_0^2 \left\{ (x^2-1) + \frac{2}{x^2+1} \right\} \, dx \Rightarrow I = \left[\frac{x^3}{3} - x + 2 \tan^{-1}(x) \right]_0^2$

$$\Rightarrow I = \left[\left\{ \frac{(2)^3}{3} - 2 + 2 \tan^{-1}(2) \right\} - \left\{ 0 - 0 + 2 \tan^{-1}(0) \right\} \right]$$

$$\Rightarrow I = \left[\left(\frac{8}{3} - 2 \right) + 2 \tan^{-1}(2) \right] \therefore I = \frac{2}{3} + 2 \tan^{-1}(2)$$

35. $\int_1^2 \frac{dx}{(x+1)(x+2)}$

Sol. Let $I = \int_1^2 \frac{dx}{(x+1)(x+2)} \Rightarrow I = [I_1]_1^2 \dots (1)$

$$I_1 = \int \frac{1}{(x+1)(x+2)} dx \Rightarrow \frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$\Rightarrow \frac{1}{(x+1)(x+2)} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)} \Rightarrow 1 = Ax+2A+Bx+B \Rightarrow 1 = x(A+B)+2A+B$$

Now, Equating co-efficient both side we get

$$A+B=0 \dots (1)$$

$$2A+B=1 \dots (2)$$

By, Solving equation (1) & (2) then we get $A=1, B=-1$

$$I_1 = \int \left(\frac{A}{x+1} + \frac{B}{x+2} \right) dx \Rightarrow I_1 = 1 \cdot \int \frac{1}{x+1} dx - 1 \cdot \int \frac{1}{x+2} dx \Rightarrow I_1 = \log \left| \frac{x+1}{x+2} \right|$$

$$\therefore I = \left[\log \left| \frac{x+1}{x+2} \right| \right]_1^2 \Rightarrow I = \log \left(\frac{3}{4} \right) - \log \left(\frac{2}{3} \right) \Rightarrow I = \log \left(\frac{9}{8} \right) \Rightarrow I = \log 9 - \log 8$$

$$\therefore I = 2 \log 3 - 3 \log 2$$

36. $\int_1^2 \frac{(x+3)}{x(x+2)} dx$

Sol. Let $I = \int_1^2 \frac{(x+3)}{x(x+2)} dx \Rightarrow I = [I_1]_1^2$

$$I_1 = \int \frac{x+3}{x(x+2)} dx \Rightarrow \frac{x+3}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow \frac{x+3}{x(x+2)} = \frac{A(x+2)+Bx}{x(x+2)}$$

$$\Rightarrow x+3 = Ax+2A+Bx \Rightarrow x+3 = x(A+B)+2A$$

By, Equating co-efficient both side we get

$$A+B=1 \dots (1)$$

$$2A=3 \dots (2)$$

Now, solving equation (1) & (2) then we get $A=\frac{3}{2}, B=-\frac{1}{2}$

$$I_1 = \int \left(\frac{A}{x} + \frac{B}{x+2} \right) dx \Rightarrow I_1 = A \int \frac{1}{x} dx + B \int \frac{1}{x+2} dx \Rightarrow I_1 = \frac{3}{2} \log |x| - \frac{1}{2} \log |x+2|$$

$$\therefore I = \left[\frac{3}{2} \log |x| - \frac{1}{2} \log |x+2| \right]_1^2 \Rightarrow I = \left\{ \frac{3}{2} \log 2 - \frac{1}{2} \log 3 \right\} - \left\{ \frac{3}{2} \log 1 - \frac{1}{2} \log 2 \right\}$$

$$\Rightarrow I = \frac{3}{2} \log 2 - \frac{1}{2} \log 3 + \frac{1}{2} \log 2 \therefore I = 2 \log 2 - \frac{1}{2} \log 3$$

37. $\int_3^4 \frac{dx}{x^2-4}$

Sol. Let $I = \int_3^4 \frac{dx}{x^2-4} \Rightarrow I = \int_3^4 \frac{dx}{(x)^2-(2)^2} \Rightarrow I = \left[\frac{1}{2 \cdot 2} \log \left| \frac{x-2}{x+2} \right| \right]_3^4$

$$\Rightarrow I = \frac{1}{4} \left[\log \left(\frac{1}{3} \right) - \log \left(\frac{1}{5} \right) \right] \Rightarrow I = \frac{1}{4} \left[\log \frac{5}{3} \right] \therefore I = \frac{1}{4} [\log 5 - \log 3]$$

38. $\int_0^4 \frac{dx}{\sqrt{x^2 + 2x + 3}}$

Sol. Let $I = \int_0^4 \frac{dx}{\sqrt{x^2 + 2x + 3}} \Rightarrow I = \int_0^4 \frac{dx}{\sqrt{(x)^2 + 2 \cdot x \cdot 1 + (1)^2 - (1)^2 + 3}}$

$$\Rightarrow I = \int_0^4 \frac{dx}{\sqrt{(x+1)^2 + (\sqrt{2})^2}} \Rightarrow I = \left[\log \left| (x+1) + \sqrt{(x+1)^2 + (\sqrt{2})^2} \right| \right]_0^4$$

$$\Rightarrow I = \left[\log \left| (x+1) + \sqrt{x^2 + 2x + 3} \right| \right]_0^4$$

$$\Rightarrow I = \left[\log \left| (4+1) + \sqrt{(4)^2 + 2 \cdot 4 + 3} \right| - \log \left| (0+1) + \sqrt{0+0+3} \right| \right]$$

$$\Rightarrow I = \log(5 + 3\sqrt{3}) - \log(1 + \sqrt{3}) \therefore I = \log \left(\frac{5 + 3\sqrt{3}}{1 + \sqrt{3}} \right)$$

39. $\int_1^2 \frac{dx}{\sqrt{x^2 + 4x + 3}}$

Sol. Let $I = \int_1^2 \frac{dx}{\sqrt{x^2 + 4x + 3}} \Rightarrow I = \int_1^2 \frac{dx}{\sqrt{(x)^2 + 2 \cdot x \cdot 2 + (2)^2 - (2)^2 + 3}} \Rightarrow I = \int_1^2 \frac{dx}{\sqrt{(x+2)^2 - 1}}$

$$\Rightarrow I = \left[\log \left| (x+2) + \sqrt{(x+2)^2 - 1} \right| \right]_1^2 \Rightarrow I = \left[\log \left| (x+2) + \sqrt{x^2 + 4x + 3} \right| \right]_1^2$$

$$\Rightarrow I = \log(4 + \sqrt{15}) - \log(3 + \sqrt{8})$$

40. $\int_0^1 \frac{dx}{1+x+2x^2}$

Sol. Let $I = \int_0^1 \frac{dx}{1+x+2x^2} \Rightarrow I = \int_0^1 \frac{dx}{2 \left(x^2 + \frac{x}{2} + \frac{1}{2} \right)} \Rightarrow I = \frac{1}{2} \int_0^1 \frac{dx}{x^2 + \frac{x}{2} + \frac{1}{2}}$

$$\Rightarrow I = \frac{1}{2} \int_0^1 \frac{dx}{(x)^2 + 2 \cdot x \cdot \frac{1}{4} + \left(\frac{1}{4} \right)^2 - \left(\frac{1}{4} \right)^2 + \frac{1}{2}}$$

$$\Rightarrow I = \frac{1}{2} \int_0^1 \frac{dx}{\left(x + \frac{1}{4} \right)^2 + \frac{7}{16}}$$

$$\Rightarrow I = \frac{1}{2} \int_0^1 \frac{dx}{\left(x + \frac{1}{4} \right)^2 + \left(\frac{\sqrt{7}}{4} \right)^2} \Rightarrow I = \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{7}}{4}} \left[\tan^{-1} \left(\frac{x + \frac{1}{4}}{\frac{\sqrt{7}}{4}} \right) \right]_0^1$$

$$\Rightarrow I = \frac{2}{\sqrt{7}} \left[\tan^{-1} \left(\frac{4x+1}{\sqrt{7}} \right) \right]_0^1 \therefore I = \frac{2}{\sqrt{7}} \left[\tan^{-1} \left(\frac{5}{\sqrt{7}} \right) - \tan^{-1} \left(\frac{1}{\sqrt{7}} \right) \right]$$

Short-Answer Questions

Evaluate:

41. $\int_0^{\pi/2} (a \cos^2 x + b \sin^2 x) dx$

Sol. Let $I = \int_0^{\pi/2} (a \cos^2 x + b \sin^2 x) dx \Rightarrow I = a \int_0^{\pi/2} \frac{1 + \cos 2x}{2} dx + b \int_0^{\pi/2} \frac{1 - \cos 2x}{2} dx$

$$\Rightarrow I = \frac{a}{2} \left[x + \frac{\sin 2x}{2} \right]_0^{\pi/2} + \frac{b}{2} \left[x - \frac{\sin 2x}{2} \right]_0^{\pi/2}$$

$$\Rightarrow I = \frac{a}{2} \left[\left\{ \frac{\pi}{2} + \frac{\sin 2 \cdot \frac{\pi}{2}}{2} \right\} - 0 \right] + \frac{b}{2} \left[\left\{ \frac{\pi}{2} - \frac{\sin 2 \cdot \frac{\pi}{2}}{2} \right\} - 0 \right]$$

$$\Rightarrow I = \frac{a}{2} \left[\frac{\pi}{2} + 0 \right] + \frac{b}{2} \left[\frac{\pi}{2} - 0 \right] \Rightarrow I = \frac{\pi}{2} \left(\frac{a}{2} + \frac{b}{2} \right) \therefore I = \frac{\pi}{4} (a + b)$$

42. $\int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx$

Sol. Let $I = \int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx \Rightarrow I = \int_{\pi/3}^{\pi/4} (\tan^2 x + 2 \tan x \cot x + \cot^2 x) dx$

$$\Rightarrow I = \int_{\pi/3}^{\pi/4} (\tan^2 x + 2 + \cot^2 x) dx \Rightarrow I = \int_{\pi/3}^{\pi/4} (\sec^2 x - 1 + 2 + \operatorname{cosec}^2 x - 1) dx$$

$$\Rightarrow I = \int_{\pi/3}^{\pi/4} (\sec^2 x + \operatorname{cosec}^2 x) dx \Rightarrow [\tan x - \cot x]_{\pi/3}^{\pi/4}$$

$$\Rightarrow I = \left(\tan \frac{\pi}{4} - \cot \frac{\pi}{4} \right) - \left(\tan \frac{\pi}{3} - \cot \frac{\pi}{3} \right) \Rightarrow I = (1 - 1) - \left(\sqrt{3} - \frac{1}{\sqrt{3}} \right) = \frac{-2}{\sqrt{3}} \therefore I = \frac{-2}{\sqrt{3}}$$

43. $\int_0^{\pi/2} \cos^4 x dx$

Sol. Let $I = \int_0^{\pi/2} \cos^4 x dx \Rightarrow I = \int_0^{\pi/2} (\cos^2 x)^2 dx \Rightarrow I = \int_0^{\pi/2} \left(\frac{1 + \cos 2x}{2} \right)^2 dx$

$$\Rightarrow I = \frac{1}{4} \int_0^{\pi/2} (1 + 2 \cos 2x + \cos^2 2x) dx \Rightarrow I = \frac{1}{4} \int_0^{\pi/2} \left(1 + 2 \cos 2x + \frac{1 + \cos 4x}{2} \right) dx$$

$$\Rightarrow I = \frac{1}{4} \int_0^{\pi/2} \left(\frac{3}{2} + 2 \cos 2x + \frac{\cos 4x}{2} \right) dx \Rightarrow I = \frac{1}{4} \left[\frac{3}{2} x + \frac{2 \sin 2x}{2} + \frac{\sin 4x}{2 \times 4} \right]_0^{\pi/2}$$

$$\Rightarrow I = \frac{1}{4} \left[\frac{3}{2} \cdot \frac{\pi}{2} + \sin \pi + \frac{\sin 2\pi}{8} - 0 \right] = \frac{1}{4} \cdot \frac{3\pi}{4} = \frac{3\pi}{16}$$

44. $\int_0^a \frac{dx}{(ax + a^2 - x^2)}$

Sol. Let $I = \int_0^a \frac{dx}{ax + a^2 - x^2} \Rightarrow I = \int_0^a \frac{dx}{-(x^2 - ax - a^2)}$

$$\Rightarrow I = -\int_0^a \frac{dx}{x^2 - 2 \cdot \frac{a}{2} \cdot x + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 - a^2} \Rightarrow I = -\int_0^a \frac{dx}{\left(x - \frac{a}{2}\right)^2 - \left(\frac{\sqrt{5}a}{2}\right)^2}$$

$$\Rightarrow I = \int_0^a \frac{dx}{\left(\frac{\sqrt{5}a}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2} \Rightarrow I = \frac{1}{2 \cdot \frac{\sqrt{5}}{2} \cdot a} \left[\log \frac{\frac{\sqrt{5}a}{2} + \left(x - \frac{a}{2}\right)}{\frac{\sqrt{5}a}{2} - \left(x - \frac{a}{2}\right)} \right]_0^a$$

$$\Rightarrow I = \frac{1}{\sqrt{5}a} \left[\log \frac{\sqrt{5}a + 2x - a}{\sqrt{5}a - 2x + a} \right]_0^a \quad I = \frac{1}{\sqrt{5}a} \left[\log \frac{\sqrt{5}a + a}{\sqrt{5}a - a} - \log \frac{\sqrt{5}a - a}{\sqrt{5}a + a} \right]$$

$$\Rightarrow I = \frac{1}{\sqrt{5}a} \left[\log \frac{\sqrt{5}+1}{\sqrt{5}-1} - \log \frac{\sqrt{5}-1}{\sqrt{5}+1} \right] \quad I = \frac{1}{\sqrt{5}a} \log \left(\frac{\sqrt{5}+1}{\sqrt{5}-1} \times \frac{\sqrt{5}+1}{\sqrt{5}-1} \right)$$

$$\Rightarrow I = \frac{1}{\sqrt{5}a} \log \left(\frac{\sqrt{5}+1}{\sqrt{5}-1} \right)^2 \Rightarrow \frac{2}{\sqrt{5}a} \log \frac{\sqrt{5}+1}{\sqrt{5}-1} \times \frac{\sqrt{5}+1}{\sqrt{5}+1}$$

$$\Rightarrow I = \frac{2}{\sqrt{5}a} \log \frac{5+1+2\sqrt{5}}{5-1} \Rightarrow I = \frac{2}{\sqrt{5}a} \log \frac{2(3+\sqrt{5})}{4} \Rightarrow I = \frac{2}{\sqrt{5}a} \log \left(\frac{3+\sqrt{5}}{2} \right)$$

45. $\int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}}$

Sol. Let $I = \int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}} \Rightarrow I = \int_{1/4}^{1/2} \frac{dx}{\sqrt{-\left[x^2 - x\right]}} \Rightarrow I = \int_{1/4}^{1/2} \frac{dx}{\sqrt{-\left[\left(x - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right]}}$

$$\Rightarrow I = \int_{1/4}^{1/2} \frac{dx}{\sqrt{-\left[\left(x - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right]}} \Rightarrow I = \int_{1/4}^{1/2} \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}} \Rightarrow I = \left[\sin^{-1} \left(\frac{x - \frac{1}{2}}{\frac{1}{2}} \right) \right]_{1/4}^{1/2}$$

$$\Rightarrow I = \left[\sin^{-1} (2x-1) \right]_{1/4}^{1/2} \Rightarrow I = \left[\sin^{-1} \left(2 \cdot \frac{1}{2} - 1 \right) - \sin^{-1} \left(2 \cdot \frac{1}{4} - 1 \right) \right] \Rightarrow I = \sin^{-1} \left(\frac{-1}{2} \right)$$

$$\Rightarrow I = \sin^{-1} \left(\frac{1}{2} \right) \quad \therefore I = \frac{\pi}{6}$$

46. $\int_0^1 \sqrt{x(1-x)} dx$

Sol. Let $I = \int_0^1 \sqrt{x(1-x)} dx \Rightarrow I = \int_0^1 \sqrt{x-x^2} dx \Rightarrow I = \int_0^1 \sqrt{-\left[x^2 - x\right]} dx$

$$\Rightarrow I = \int_0^1 \sqrt{-\left[\left(x - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right]} dx \Rightarrow I = \int_0^1 \sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} dx$$

$$\begin{aligned}
 \Rightarrow I &= \left[\frac{x - \frac{1}{2}}{2} \sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} + \frac{\left(\frac{1}{2}\right)^2}{2} \sin^{-1} \left(\frac{x - \frac{1}{2}}{\frac{1}{2}} \right) \right]_0^1 \\
 \Rightarrow I &= \left[\frac{2x-1}{4} \sqrt{x-x^2} + \frac{1}{8} \sin^{-1}(2x-1) \right]_0^1 \Rightarrow I = \left[\left\{ \frac{2 \times 1 - 1}{4} \sqrt{1-1} + \frac{1}{8} \sin^{-1}(2 \times 1 - 1) \right\} - \frac{1}{8} \sin^{-1}(-1) \right] \\
 \Rightarrow I &= \frac{1}{8} \sin^{-1}(1) + \frac{1}{8} \sin^{-1}(1) \Rightarrow I = \frac{1}{8} \cdot \frac{\pi}{2} + \frac{1}{8} \cdot \frac{\pi}{2} = \frac{2\pi}{16} \quad \therefore I = \frac{\pi}{8}
 \end{aligned}$$

47. $\int_1^3 \frac{dx}{x^2(x+1)}$

Sol. Let $I = \int_1^3 \frac{dx}{x^2(x+1)} \Rightarrow I = [I_1]_1^3$

$$\therefore I_1 = \int \frac{1}{x^2(x+1)} dx \Rightarrow \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\Rightarrow \frac{1}{x^2(x+1)} = \frac{Ax(x+1) + B(x+1) + C(x^2)}{x^2(x+1)} \Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$$

$$\Rightarrow 1 = x^2(A+C) + (A+B)x + B$$

Now, Equating co-efficient both side we get

$$A+C=0 \quad \dots(1)$$

$$A+B=0 \quad \dots(2)$$

$$B=1 \quad \dots(3)$$

By, Solving equation (1), (2) & (3) we get $A=-1, B=1, C=1$

$$\Rightarrow I_1 = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \right) dx \Rightarrow I_1 = -\int \frac{1}{x} dx + \int \frac{1}{x^2} dx + \int \frac{1}{x+1} dx$$

$$\Rightarrow I_1 = -\log|x| - \frac{1}{x} + \log|x+1| \Rightarrow I_1 = \log\left(\frac{x+1}{x}\right) - \frac{1}{x}$$

$$\therefore I = \left[\log\left(\frac{x+1}{x}\right) - \frac{1}{x} \right]_1^3 \Rightarrow I = \log 4 - \log 3 - \frac{1}{3} - \log 2 + 1 \Rightarrow I = 2\log 2 - \log 3 - \log 2 + \frac{2}{3}$$

$$\therefore I = \log 2 - \log 3 + \frac{2}{3}$$

48. $\int_1^2 \frac{dx}{x(1+2x)^2}$

Sol. Let $I = \int_1^2 \frac{dx}{x(1+2x)^2} \Rightarrow I = [I_1]_1^2$

$$I_1 = \int \frac{1}{x(1+2x)^2} dx \Rightarrow \frac{1}{x(1+2x)^2} = \frac{A}{x} + \frac{B}{1+2x} + \frac{C}{(1+2x)^2}$$

$$\Rightarrow \frac{1}{x(1+2x)^2} = \frac{A(1+2x)^2 + B(1+2x) + Cx}{x(1+2x)^2}$$

$$\Rightarrow 1 = A(1 + 4x + 4x^2) + B(x + 2x^2) + Cx \Rightarrow 1 = A + 4Ax + 4Ax^2 + Bx + 2Bx^2 + Cx$$

$$\Rightarrow 1 = x^2(4A + 2B) + x(4A + B + C) + A$$

Now, Equating co-efficient we get

$$4A + 2B = 0 \Rightarrow 2A + B = 0 \quad \dots(1)$$

$$4A + B + C = 0 \quad \dots(2)$$

$$A = 1 \quad \dots(3)$$

By, Solving equation (1), (2) & (3) then we get $A = 1, B = -2, C = -2$

$$I_1 = \int \left\{ \frac{A}{x} + \frac{B}{1+2x} + \frac{C}{(1+2x)^2} \right\} dx \Rightarrow I_1 = A \int \frac{1}{x} dx + B \int \frac{1}{1+2x} dx + C \int \frac{1}{(1+2x)^2} dx$$

$$\Rightarrow I_1 = \log |x| - 2 \frac{\log(1+2x)}{2} - 2 \left(-\frac{1}{1+2x} \right) \times \frac{1}{2}$$

$$\Rightarrow I_1 = \log |x| - \log(1+2x) + \frac{1}{1+2x} \quad \therefore I = \left[\log |x| - \log(1+2x) + \frac{1}{1+2x} \right]_1^2$$

$$\Rightarrow I = \left(\log 2 - \log(5) + \frac{1}{5} \right) - \left(\log 1 - \log 3 + \frac{1}{3} \right)$$

$$\Rightarrow I = \log 2 - \log 5 + \frac{1}{5} + \log 3 - \frac{1}{3} \quad \therefore I = \log 6 - \log 5 - \frac{2}{15}$$

49. $\int_0^1 x e^x dx$

Sol. Let $I = \int_0^1 x e^x dx \Rightarrow I = [I_1]_0^1$

$$I_1 = \int x e^x dx \Rightarrow I_1 = x \int e^x dx - \int \left[\frac{d(x)}{dx} \int e^x dx \right] dx \Rightarrow I_1 = x e^x - \int 1 \cdot e^x dx$$

$$\Rightarrow I_1 = x e^x - e^x \Rightarrow I_1 = e^x (x - 1)$$

$$I = [e^x (x - 1)]_0^1 \Rightarrow I = [e^1 (1 - 1) - e^0 (0 - 1)] \Rightarrow I = -1(-1) \therefore I = 1$$

50. $\int_0^{\pi/2} x^2 \cos x dx$

Sol. Let $I = \int_0^{\pi/2} x^2 \cos x dx \Rightarrow I = [I_1]_0^{\pi/2}$

$$\therefore I_1 = \int x^2 \cos x dx \Rightarrow I_1 = x^2 \int \cos x dx - \int \left[\frac{d(x^2)}{dx} \int \cos x dx \right] dx$$

$$\Rightarrow I_1 = x^2 (\sin x) - \int 2x \sin x dx \Rightarrow I_1 = x^2 \sin x - 2 \int x \sin x dx \Rightarrow I_1 = x^2 \sin x - 2 \int x \sin x dx$$

$$\Rightarrow I_1 = x^2 \sin x - 2 \left[x \int \sin x dx - \int \left[\frac{d(x)}{dx} \int \sin x dx \right] dx \right]$$

$$\Rightarrow I_1 = x^2 \sin x - 2 \left[-x \cos x + \int \cos x dx \right]$$

$$\Rightarrow I = \left[x^2 \sin x + 2x \cos x - 2 \sin x \right]_0^{\pi/2} \Rightarrow I = \left\{ \left(\frac{\pi}{2} \right)^2 \sin \frac{\pi}{2} + 2 \frac{\pi}{2} \cos \frac{\pi}{2} - 2 \sin \frac{\pi}{2} \right\} - \{0\}$$

$$\therefore I = \frac{\pi^2}{4} - 2$$

51. $\int_0^{\pi/4} x^2 \sin x \, dx$

Sol. Let $I = \int_0^{\pi/4} x^2 \sin x \, dx \Rightarrow I = [I_1]_0^{\pi/4}$

$$\therefore I_1 = \int x^2 \sin x \, dx \Rightarrow I_1 = x^2 \int \sin x \, dx - \int \left\{ \frac{d(x^2)}{dx} \int \sin x \, dx \right\} dx$$

$$\Rightarrow I_1 = -x^2 \cos x + \int 2x \cos x \, dx \Rightarrow I_1 = -x^2 \cos x + 2 \int x \cos x \, dx$$

$$\Rightarrow I_1 = -x^2 \cos x + 2 \left[x \int \cos x \, dx - \int \left\{ \frac{d(x)}{dx} \int \cos x \, dx \right\} dx \right]$$

$$\Rightarrow I_1 = -x^2 \cos x + 2 \left[x \sin x - \int 1 \cdot \sin x \, dx \right] \Rightarrow I_1 = -x^2 \cos x + 2 [x \sin x + \cos x]$$

$$\Rightarrow I_1 = -x^2 \cos x + 2x \sin x + 2 \cos x$$

$$\Rightarrow I = [-x^2 \cos x + 2x \sin x + 2 \cos x]_0^{\pi/4} \Rightarrow I = \left\{ -\left(\frac{\pi}{4} \right)^2 \cos \frac{\pi}{4} + 2 \cdot \frac{\pi}{4} \sin \frac{\pi}{4} + 2 \cos \frac{\pi}{4} \right\} - \{0 + 2 \cos 0\}$$

$$\Rightarrow I = -\frac{\pi^2}{16} \cdot \frac{1}{\sqrt{2}} + \frac{\pi}{2} \cdot \frac{1}{\sqrt{2}} + 2 \cdot \frac{1}{\sqrt{2}} - 2 \therefore I = -\frac{\pi^2}{16\sqrt{2}} + \frac{\pi}{2\sqrt{2}} + \sqrt{2} - 2$$

52. $\int_0^{\pi/2} x^2 \cos 2x \, dx$

Sol. Let $I = \int_0^{\pi/2} x^2 \cos 2x \, dx \Rightarrow I = [I_1]_0^{\pi/2}$

$$I_1 = \int x^2 \cos 2x \, dx \Rightarrow I_1 = x^2 \int \cos 2x \, dx - \int \left\{ \frac{d(x^2)}{dx} \int \cos 2x \, dx \right\} dx$$

$$\Rightarrow I_1 = x^2 \frac{\sin 2x}{2} - \int 2x \cdot \frac{\sin 2x}{2} \, dx \Rightarrow I_1 = \frac{x^2 \sin 2x}{2} - \int x \sin 2x \, dx$$

$$\Rightarrow I_1 = \frac{x^2 \sin 2x}{2} - \left[x \int \sin 2x \, dx - \int \left\{ \frac{d(x)}{dx} \int \sin 2x \, dx \right\} dx \right]$$

$$\Rightarrow I_1 = \frac{x^2 \sin 2x}{2} - \left[-x \frac{\cos 2x}{2} + \int 1 \cdot \frac{\cos 2x}{2} \, dx \right] \Rightarrow I_1 = \frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{1}{2} \cdot \frac{\sin 2x}{2}$$

$$\Rightarrow I_1 = \frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{1}{4} \sin 2x$$

$$\Rightarrow I = \left[\frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{1}{4} \sin 2x \right]_0^{\pi/2}$$

$$\Rightarrow I = \left\{ \frac{\left(\frac{\pi}{2}\right)^2 \sin\left(2 \cdot \frac{\pi}{2}\right)}{2} + \frac{\pi}{2} \frac{\cos\left(2 \cdot \frac{\pi}{2}\right)}{2} - \frac{1}{4} \sin\left(2 \cdot \frac{\pi}{2}\right) \right\} - \{0\} \Rightarrow I = \frac{\pi(-1)}{2} \therefore I = -\frac{\pi}{4}$$

53. $\int_0^{\pi/2} x^3 \sin 3x \, dx$

Sol. Let $I = \int_0^{\pi/2} x^3 \sin 3x \, dx \Rightarrow I = [I_1]_0^{\pi/2}$

$$\begin{aligned} \therefore I_1 &= \int x^3 \sin 3x \, dx \Rightarrow I_1 = x^3 \int \sin 3x \, dx - \int \left[\frac{d(x^3)}{dx} \int \sin 3x \, dx \right] dx \\ \Rightarrow I_1 &= \frac{-x^3 \cos 3x}{3} + \int 3x^2 \cdot \frac{\cos 3x}{3} dx \Rightarrow I_1 = \frac{-x^3 \cos 3x}{3} + \int x^2 \cos 3x \, dx \\ \Rightarrow I_1 &= \frac{-x^3 \cos 3x}{3} + \left[x^2 \int \cos 3x \, dx - \int \left[\frac{d(x^2)}{dx} \int \cos 3x \, dx \right] dx \right] \\ \Rightarrow I_1 &= \frac{-x^3 \cos 3x}{3} + \left[x^2 \frac{\sin 3x}{3} - \int 2x \cdot \frac{\sin 3x}{3} dx \right] \Rightarrow I_1 = \frac{-x^3 \cos 3x}{3} + x^2 \frac{\sin 3x}{3} - \frac{2}{3} \int x \sin 3x \, dx \\ \Rightarrow I_1 &= \frac{-x^3 \cos 3x}{3} + \frac{x^2 \sin 3x}{3} - \frac{2}{3} \left[x \int \sin 3x \, dx - \int \left[\frac{d(x)}{dx} \int \sin 3x \, dx \right] dx \right] \\ \Rightarrow I_1 &= \frac{-x^3 \cos 3x}{3} + \frac{x^2 \sin 3x}{3} - \frac{2}{3} \left[-x \frac{\cos 3x}{3} + \int \frac{\cos 3x}{3} dx \right] \\ \Rightarrow I_1 &= \frac{-x^3 \cos 3x}{3} + \frac{x^2 \sin 3x}{3} + \frac{2x}{9} \cos 3x - \frac{2}{9} \cdot \frac{\sin 3x}{3} \\ \Rightarrow I_1 &= \frac{-x^3 \cos 3x}{3} + \frac{x^2 \sin 3x}{3} + \frac{2x}{9} \cos 3x - \frac{2}{27} \sin 3x \\ \Rightarrow I &= \left[\frac{-x^3 \cos 3x}{3} + \frac{x^2 \sin 3x}{3} + \frac{2x}{9} \cos 3x - \frac{2}{27} \sin 3x \right]_0^{\pi/2} \\ \Rightarrow I &= \left\{ \frac{-\left(\frac{\pi}{2}\right)^3 \cos\left(3 \cdot \frac{\pi}{2}\right)}{3} + \frac{\left(\frac{\pi}{2}\right)^2 \sin\left(3 \cdot \frac{\pi}{2}\right)}{3} + \frac{2}{9} \cdot \frac{\pi}{2} \cos\left(3 \cdot \frac{\pi}{2}\right) - \frac{2}{27} \sin\left(3 \cdot \frac{\pi}{2}\right) \right\} - 0 \\ \Rightarrow I &= \left[\frac{-\pi^2}{12} + \frac{2}{27} \right] \therefore I = \frac{2}{27} - \frac{\pi^2}{12} \end{aligned}$$

54. $\int_0^{\pi/2} x^2 \cos^2 x \, dx$

Sol. Let $I = \int_0^{\pi/2} x^2 \cos^2 x \, dx \Rightarrow I = [I_1]_0^{\pi/2}$

$$I_1 = \int x^2 \cos^2 x \, dx \Rightarrow I_1 = \int x^2 \left(\frac{1 + \cos 2x}{2} \right) dx \Rightarrow I_1 = \frac{1}{2} \left[\int x^2 dx + \int x^2 \cos 2x dx \right]$$

$$\begin{aligned} \Rightarrow I_1 &= \frac{1}{2} \left[\frac{x^3}{3} + \left\{ x^2 \int \cos 2x \, dx - \int \frac{d(x^2)}{dx} \int \cos 2x \, dx \right\} \right] \\ \Rightarrow I_1 &= \frac{x^3}{6} + \frac{1}{2} \left[x^2 \frac{\sin 2x}{2} - \int 2x \frac{\sin 2x}{2} \, dx \right] \Rightarrow I_1 = \frac{x^3}{6} + \frac{1}{2} \left[\frac{x^2 \sin 2x}{2} - \int x \sin 2x \, dx \right] \\ \Rightarrow I_1 &= \frac{x^3}{6} + \frac{x^2 \sin 2x}{4} - \frac{1}{2} \left[x \int \sin 2x \, dx - \int \frac{d(x)}{dx} \int \sin 2x \, dx \right] \\ \Rightarrow I_1 &= \frac{x^3}{6} + \frac{x^2 \sin 2x}{4} - \frac{1}{2} \left[\frac{-x \cos 2x}{2} + \int \frac{\cos 2x}{2} \, dx \right] \\ \Rightarrow I_1 &= \frac{x^3}{6} + \frac{x^2 \sin 2x}{4} + \frac{x}{4} \cos 2x - \frac{1}{4} \frac{\sin 2x}{2} \Rightarrow I_1 = \frac{x^3}{6} + \frac{x^2 \sin 2x}{4} + \frac{x}{4} \cos 2x - \frac{\sin 2x}{8} \\ \Rightarrow I &= \left[\frac{x^3}{6} + \frac{x^2 \sin 2x}{4} + \frac{x}{4} \cos 2x - \frac{\sin 2x}{8} \right]_0^{\pi/2} \\ \Rightarrow I &= \left[\frac{\left(\frac{\pi}{2}\right)^3}{6} + \frac{\left(\frac{\pi}{2}\right)^2 \sin\left(2 \cdot \frac{\pi}{2}\right)}{4} + \frac{\pi}{2} \cdot \frac{1}{4} \cos\left(2 \cdot \frac{\pi}{2}\right) - \frac{\sin\left(2 \cdot \frac{\pi}{2}\right)}{8} \right] - \{0\} \therefore I = \frac{\pi^3}{48} - \frac{\pi}{8} \end{aligned}$$

55. $\int_1^2 \log x \, dx$

Sol. Let $I = \int_1^2 \log x \, dx \Rightarrow I = [I_1]_1^2$

$$\begin{aligned} I_1 &= \int \log x \, dx \Rightarrow I_1 = \log x \int 1 \, dx - \int \left[\frac{d(\log x)}{dx} \int 1 \, dx \right] dx \Rightarrow I_1 = \log x \cdot x - \int \frac{1}{x} \cdot x \, dx \\ \Rightarrow I_1 &= x \log x - \int dx \Rightarrow I_1 = x \log x - x \Rightarrow I = [x \log x - x]_1^2 \\ \Rightarrow I &= (2 \log 2 - 2) - (1 \log 1 - 1) \Rightarrow I = 2 \log 2 - 2 + 1 \therefore I = 2 \log 2 - 1 \end{aligned}$$

56. $\int_1^3 \frac{\log x}{(1+x)^2} \, dx$

Sol. Let $I = \int_1^3 \frac{\log x}{(1+x)^2} \, dx \Rightarrow I = [I_1]_1^3$

$$\begin{aligned} \therefore I_1 &= \int \frac{\log x}{(1+x)^2} \, dx \Rightarrow I_1 = \int (\log x) (1+x)^{-2} \, dx \\ \Rightarrow I_1 &= \log x \int (1+x)^{-2} \, dx - \int \left[\frac{d(\log x)}{dx} \int (1+x)^{-2} \, dx \right] dx \\ \Rightarrow I_1 &= \log x \frac{(1+x)^{-1}}{-1} - \int \frac{1}{x} \cdot \frac{(1+x)^{-1}}{-1} \, dx \Rightarrow I_1 = -\frac{1}{(1+x)} \log x + \int \frac{1}{x(1+x)} \, dx \\ \Rightarrow I_1 &= -\frac{1}{(1+x)} \log x + \int \frac{(1+x) - x}{x(1+x)} \, dx \Rightarrow I_1 = -\frac{\log x}{(1+x)} + \int \frac{1}{x} \, dx - \int \frac{1}{(1+x)} \, dx \end{aligned}$$

$$\Rightarrow I_1 = -\frac{\log x}{(1+x)} + \log|x| - \log|x+1| \Rightarrow I_1 = -\frac{\log x}{(1+x)} + \log\left|\frac{x}{x+1}\right|$$

$$I = \left[-\frac{\log x}{(1+x)} + \log\left|\frac{x}{x+1}\right| \right]_1^3 \Rightarrow I = \left[\left\{ -\frac{\log 3}{(1+3)} + \log\left(\frac{3}{3+1}\right) \right\} - \left\{ -\frac{\log 1}{(1+1)} + \log\left(\frac{1}{1+1}\right) \right\} \right]$$

$$\Rightarrow I = \left[-\frac{\log 3}{4} + \log\left(\frac{3}{4}\right) - \log\left(\frac{1}{2}\right) \right] \Rightarrow I = \left[-\frac{\log 3}{4} + \log 3 - \log 4 - \log 1 + \log 2 \right]$$

$$\Rightarrow I = \left[\frac{3}{4} \log 3 - 2 \log 2 + \log 2 \right] \therefore I = \frac{3}{4} \log 3 - \log 2$$

57. $\int_0^{e^2} \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx$

Sol. Let $I = \int_0^{e^2} \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx$, Put $t = \log x \Rightarrow e^t = x \Rightarrow \frac{dt}{dx} = \frac{1}{x} \Rightarrow x dt = dx \Rightarrow e^t dt = dx$

When $x=0$, $t=0$ and when $x=e^2$, $t=2$

$$I = \int_0^2 \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt, \text{ Here } f(t) = \frac{1}{t}, f'(t) = -\frac{1}{t^2} dt$$

$$\Rightarrow I = \int_0^2 \{ f(t) + f'(t) \} e^t dt = [e^t \cdot f(t)]_0^2 = \left[\frac{e^t}{t} \right]_0^2 = \frac{e^2}{2} - \frac{e^0}{0} = \frac{e^2}{2}$$

58. $\int_1^e e^x \left(\frac{1+x \log x}{x} \right) dx$

Sol. Let $I = \int_1^e e^x \left(\frac{1+x \log x}{x} \right) dx \Rightarrow I = [I_1]_1^e$

$$\Rightarrow I_1 = \int e^x \left(\frac{1+x \log x}{x} \right) dx \Rightarrow I_1 = \int e^x \left(\frac{1}{x} + \log x \right) dx$$

Where $f(x) = \log x$, $f'(x) = \frac{1}{x}$; $\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$

$$\therefore I_1 = e^x \cdot \log x \Rightarrow I = [e^x \log x]_1^e \Rightarrow I = (e^e \log e - e^1 \log 1) \therefore I = e^e$$

59. $\int_0^1 \frac{x e^x}{(1+x)^2} dx$

Sol. Let $I = \int_0^1 \frac{x e^x}{(1+x)^2} dx \Rightarrow I = [I_1]_0^1$

$$\Rightarrow I_1 = \int e^x \left\{ \frac{(x+1)-1}{(1+x)^2} \right\} dx \Rightarrow I_1 = \int e^x \left\{ \frac{1}{1+x} - \frac{1}{(1+x)^2} \right\} dx$$

Where $f(x) = \frac{1}{1+x}$, $f'(x) = -\frac{1}{(1+x)^2}$; $\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$

$$I_1 = e^x \left(\frac{1}{1+x} \right) \Rightarrow I = \left[\frac{e^x}{1+x} \right]_0^1 \Rightarrow I = \left(\frac{e^1}{1+1} - \frac{e^0}{1+0} \right) \therefore I = \left(\frac{e}{2} - 1 \right)$$

60. $\int_0^{\pi/2} 2 \tan^3 x \, dx$

Sol. Let $I = \int_0^{\pi/2} 2 \tan^3 x \, dx \Rightarrow I = 2 \int_0^{\pi/2} \tan^2 x \tan x \, dx = 2 \int_0^{\pi/2} (\sec^2 x - 1) \tan x \, dx$

$$\Rightarrow I = 2 \int_0^{\pi/2} (\sec^2 x \tan x - \tan x) \, dx = 2 \int_0^{\pi/2} \sec^2 x \tan x \, dx - 2 \int_0^{\pi/2} \tan x \, dx$$

Put $t = \tan x \Rightarrow dt = \sec^2 x \, dx$

When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = 1; I = 2 \int_0^1 t \, dt - 2 [\log \sec x]_0^{\pi/2}$

$$\Rightarrow I = 2 \left[\frac{t^2}{2} \right]_0^1 - 2 [\log \sec \frac{\pi}{4} - \log \sec 0] \Rightarrow I = 2 \left[\frac{1}{2} - 0 \right] - 2 [\log \sqrt{2} - \log 1]$$

$$\Rightarrow I = 1 - 2 \left[\frac{1}{2} \log 2 - 0 \right] = 1 - \log 2$$

61. $\int_1^2 \frac{5x^2}{x^2 + 4x + 3} \, dx$

Sol. Let $I = \int_1^2 \frac{5x^2}{x^2 + 4x + 3} \, dx \Rightarrow I = [I_1]_1^2$

$$\therefore I_1 = \int \frac{5x^2}{x^2 + 4x + 3} \, dx \Rightarrow I_1 = \int \left(5 + \frac{-20x - 15}{x^2 + 4x + 3} \right) \, dx$$

$$\Rightarrow I_1 = \int 5 \, dx - 5 \int \frac{4x + 3}{(x+3)(x+1)} \, dx \Rightarrow I_1 = 5x - 5 \int \frac{4x + 3}{(x+3)(x+1)} \, dx$$

$$\therefore \frac{4x + 3}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} \Rightarrow \frac{4x + 3}{(x+3)(x+1)} = \frac{A(x+1) + B(x+3)}{(x+3)(x+1)}$$

$$\Rightarrow 4x + 3 = Ax + A + Bx + 3B \Rightarrow 4x + 3 = x(A + B) + A + 3B$$

By, Equating co-efficient both side we get

$$A + B = 4$$

$$\dots (1)$$

$$A + 3B = 3$$

$$\dots (2)$$

Now, Solving equation (1) & (2) then we get, $A = \frac{9}{2}, B = -\frac{1}{2}$

$$I_1 = 5x - 5 \int \left(\frac{A}{x+3} + \frac{B}{x+1} \right) \, dx \Rightarrow I_1 = 5x - 5 \cdot \frac{9}{2} \log |x+3| - 5 \cdot \left(-\frac{1}{2} \right) \log |x+1|$$

$$\Rightarrow I_1 = 5x - \frac{45}{2} \log |x+3| + \frac{5}{2} \log |x+1|$$

$$I = \left[5x - \frac{45}{2} \log |x+3| + \frac{5}{2} \log |x+1| \right]_1^2$$

$$\Rightarrow I = \left[5 \cdot 2 - \frac{45}{2} \log |2+3| + \frac{5}{2} \log (2+1) \right] - \left[5 \cdot 1 - \frac{45}{2} \log (1+3) + \frac{5}{2} \log (1+1) \right]$$

$$\Rightarrow I = \left[10 - \frac{45}{2} \log 5 + \frac{5}{2} \log 3 - 5 + \frac{45}{2} \log 4 - \frac{5}{2} \log 2 \right]$$

$$\Rightarrow I = 5 - \frac{45}{2} \log 5 + \frac{5}{2} \log 3 + 45 \log 2 - \frac{5}{2} \log 2 \quad \therefore I = 5 - \frac{5}{2} \left(9 \log \frac{5}{4} - \log \frac{3}{2} \right)$$

EXERCISE 16B (Pg.No.: 811)

Very-Short-Answer Questions

Evaluate the following integrals:

1. $\int_0^1 \frac{dx}{2x-3}$

Sol. Let $I = \int_0^1 \frac{dx}{2x-3} \Rightarrow I = \left[\frac{\log|2x-3|}{2} \right]_0^1 \Rightarrow I = \log \frac{|2-3|}{2} - \log \frac{|2 \cdot 0 - 3|}{2}$
 $\Rightarrow I = \frac{1}{2}(\log 1 - \log 3) \therefore I = -\frac{1}{2} \log 3$

2. $\int_0^1 \frac{2x}{(1+x^2)} dx$

Sol. Let $I = \int_0^1 \frac{2x}{(1+x^2)} dx$, Put $t = 1+x^2 \Rightarrow dt = 2x dx$

When $x=0$, $t=1$ and when $x=1$, $t=2$

$$I = \int_1^2 \frac{dt}{t} = [\log t]_1^2 = \log 2 - \log 1 = \log 2 - 0 = \log 2$$

3. $\int_1^2 \frac{3x}{(9x^2-1)} dx$

Sol. Let $I = \int_1^2 \frac{3x}{(9x^2-1)} dx$, Put, $9x^2-1=t \Rightarrow 18x dx = dt \Rightarrow 3x dx = \frac{dt}{6}$

$$I_1 = \int \frac{3x}{(9x^2-1)} dx \Rightarrow I_1 = \frac{1}{6} \int \frac{dt}{t} \Rightarrow I_1 = \frac{1}{6} [\log |9x^2-1|]_1^2$$

$$\Rightarrow I_1 = \frac{1}{6} [\log |9(2)^2-1| - \log |9(1)^2-1|] \therefore I_1 = \frac{1}{6} [\log 35 - \log 8]$$

4. $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$

Sol. Let $I = \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$, Put, $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

When $x=0$, $t=0$ and when $x=1$, $t = \frac{\pi}{4}$

$$I = \int_0^{\pi/4} t dt \Rightarrow I = \left[\frac{t^2}{2} \right]_0^{\pi/4} \Rightarrow I = \frac{\left(\frac{\pi}{4}\right)^2}{2} - 0 \therefore I = \frac{\pi^2}{32}$$

5. $\int_0^1 \frac{e^x}{(1+e^{2x})} dx$

Sol. Let $I = \int_0^1 \frac{e^x}{(1+e^{2x})} dx$, Put $t = e^x \Rightarrow dt = e^x dx$

When $x=0$, $t=1$ and when $x=1$, $t=e$

$$\Rightarrow I = \int_1^e \frac{dt}{1+t^2} = [\tan^{-1} t]_1^e = \tan^{-1} e - \tan^{-1} 1 = \tan^{-1} e - \frac{\pi}{4}$$

6. $\int_0^1 \frac{2x}{(1+x^4)} dx$

Sol. Let $I = \int_0^1 \frac{2x}{(1+x^4)} dx$, Put $t = x^2 \Rightarrow dt = 2x dx$

When $x = 0$, $t = 0$ and when $x = 1$, $t = 1$

$$\Rightarrow I = \int_0^1 \frac{dt}{1+t^2} = [\tan^{-1} t]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

7. $\int_0^1 x e^{x^2} dx$

Sol. $\int_0^1 x e^{x^2} dx$, Put $t = x^2 \Rightarrow dx = 2x dx \Rightarrow \frac{dt}{2} = x dx$

When $x = 0$, $t = 0$ and $x = 1$, $t = 1$, $\int_0^1 x e^{x^2} dx = \frac{1}{2} \int_0^1 e^t dt = \frac{1}{2} [e^t]_0^1 = \frac{1}{2} (e - 1)$

8. $\int_1^2 \frac{e^{1/x}}{(x^2)} dx$

Sol. Let $I = \int_1^2 \frac{e^{1/x}}{x^2} dx$, Put $t = \frac{1}{x} \Rightarrow \frac{dt}{dx} = -\frac{1}{x^2} \Rightarrow -dt = \frac{1}{x^2} dx$

When $x = 1$, $t = 1$ and $x = 2$, $t = \frac{1}{2}$

$$\Rightarrow I = \int_1^2 \frac{e^{1/x}}{x^2} dx = \int_1^{1/2} e^t (-dt) = -[e^t]_1^{1/2} = -[e^{1/2} - e] = [e - e^{1/2}] = e - \sqrt{e}$$

9. $\int_0^{\pi/6} \frac{\cos x}{(3+4 \sin x)} dx$

Sol. Let $I = \int_0^{\pi/6} \frac{\cos x}{(3+4 \sin x)} dx$, Put $t = 3+4 \sin x \Rightarrow dt = 4 \cos x dx \Rightarrow \frac{dt}{4} = \cos x dx$

When $x = 0$, $t = 3$ and $x = \frac{\pi}{6}$, $t = 3+4 \cdot \frac{1}{2} = 5$

$$\Rightarrow I = \int_3^5 \frac{dt/4}{t} = \frac{1}{4} [\log t]_3^5 = \frac{1}{4} [\log 5 - \log 3] = \frac{1}{4} \log \frac{5}{3}$$

10. $\int_0^{\pi/2} \frac{\sin x}{(1+\cos^2 x)} dx$

Sol. Let $I = \int_0^{\pi/2} \frac{\sin x}{(1+\cos^2 x)} dx$, Put $t = \cos x \Rightarrow dt = -\sin x dx$

When $x = 0$, $t = 1$ and $x = \frac{\pi}{2}$, $t = 0$

$$\therefore I = \int_1^0 -\frac{dt}{1+t^2} = -\left[\tan^{-1} t\right]_1^0 = -\left[\tan^{-1} 0 - \tan^{-1} 1\right] = -\left[0 - \frac{\pi}{4}\right] = \frac{\pi}{4}$$

11. $\int_0^1 \frac{dx}{(e^x + e^{-x})}$

Sol. Let $I = \int_0^1 \frac{e^x}{e^{2x} + 1} dx$, Put $t = e^x \Rightarrow dt = e^x dx$

When $x = 0$, $t = 1$ and $x = 1$, $t = e$

$$\Rightarrow I = \int_1^e \frac{dt}{1+t^2} = \left[\tan^{-1} t\right]_1^e = \tan^{-1} e - \tan^{-1} 1 = \tan^{-1} e - \frac{\pi}{4}$$

12. $\int_{1/e}^e \frac{dx}{x(\log x)^{1/3}}$

Sol. Let $I = \int_{1/e}^e \frac{dx}{x(\log x)^{1/2}}$, Put $t = \log x \Rightarrow dt = \frac{1}{x} dx$

When $x = \frac{1}{e}$, $t = -1$ and $x = e$, $t = 1$

$$\Rightarrow \int_{-1}^1 \frac{dt}{t^{1/2}} = \int_{-1}^1 t^{-1/2} dt = \left[\frac{t^{1/2}}{1/2}\right]_{-1}^1 = 2\left[t^{1/2}\right]_{-1}^1 = 2\left[(1)^{1/2} - (-1)^{1/2}\right] = 2(1 - 1) = 0$$

Short-Answer Question

Evaluate the following integrals:

13. $\int_0^1 \frac{\sqrt{\tan^{-1} x}}{1+x^2} dx$

Sol. Let $I = \int_0^1 \frac{\sqrt{\tan^{-1} x}}{1+x^2} dx$, Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

When $x = 0$, $t = 0$ and when $x = 1$, $t = \frac{\pi}{4}$

$$\Rightarrow I = \int_0^{\pi/4} \sqrt{t} dt \Rightarrow I = \left[\frac{t^{3/2}}{3/2}\right]_0^{\pi/4} \Rightarrow I = \frac{2}{3} \left[t^{3/2}\right]_0^{\pi/4} \Rightarrow I = \frac{2}{3} \left[\left(\frac{\pi}{4}\right)^{3/2}\right] \therefore I = \frac{1}{12} \pi^{3/2}$$

14. $\int_0^{\pi/2} \frac{\sin x}{\sqrt{1+\cos x}} dx$

Sol. Let $I = \int_0^{\pi/2} \frac{\sin x}{\sqrt{1+\cos x}} dx$, Put $t = 1 + \cos x \Rightarrow dt = -\sin x dx$

When $x = 0$, $t = 2$ and $x = \frac{\pi}{2}$, $t = 1$

$$\Rightarrow I = \int_2^1 \frac{-dt}{\sqrt{t}} = -\int_2^1 t^{-1/2} dt = -\left[2t^{1/2}\right]_2^1 = -2[1 - \sqrt{2}] = 2(\sqrt{2} - 1)$$

15. $\int_0^{\pi/2} \sqrt{\sin x} \cos^5 x dx$

Sol. Let $I = \int_0^{\pi/2} \sqrt{\sin x} \cos^5 x \, dx$

$$\Rightarrow I = \int_0^{\pi/2} \sqrt{\sin x} \cos^4 x \cos x \, dx = \int_0^{\pi/2} \sqrt{\sin x} (1 - \sin^2 x)^2 \cos x \, dx$$

Put $\sin x = t \Rightarrow \cos x \, dx = dt$, When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = 1$

$$\therefore I_1 = \int \sqrt{t} (1 - t^2)^2 dt \Rightarrow I_1 = \int \sqrt{t} (1 - 2t^2 + t^4) dt \Rightarrow \int (\sqrt{t} - 2t^{5/2} + t^{9/2}) dt$$

$$\Rightarrow I_1 = \frac{t^{3/2}}{3/2} - 2 \frac{t^{7/2}}{7/2} + \frac{t^{11/2}}{11/2} \Rightarrow I_1 = \left[\frac{2}{3} t^{3/2} - \frac{4}{7} t^{7/2} + \frac{2}{11} t^{11/2} \right]_0^1$$

$$\Rightarrow I = \frac{2}{3} - \frac{4}{7} + \frac{2}{11} \Rightarrow I = \frac{154 - 132 + 42}{231} \therefore I = \frac{64}{231}$$

16. $\int_0^{\pi/2} \frac{\sin x \cos x}{(1 + \sin^4 x)} dx$

Sol. Let $I = \int_0^{\pi/2} \frac{\sin x \cos x}{(1 + \sin^4 x)} dx \Rightarrow I = [I_1]_0^{\pi/2}$

$$\Rightarrow I_1 = \int \frac{\sin x \cos x}{(1 + \sin^4 x)} dx \Rightarrow I_1 = \int \frac{\sin x \cdot \cos x}{1 + (\sin^2 x)^2} dx$$

Put $\sin^2 x = t \Rightarrow 2 \sin x \cos x \, dx = dt \Rightarrow \sin x \cos x \, dx = \frac{dt}{2}$

When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = 1$

$$\therefore I_1 = \frac{1}{2} \int \frac{dt}{(1 + t^2)} \Rightarrow I_1 = \frac{1}{2} \tan^{-1}(t)$$

$$\Rightarrow I = [I_1]_0^1 \Rightarrow I = \frac{1}{2} [\tan^{-1} 1 - \tan^{-1} 0] = \frac{1}{2} \left[\frac{\pi}{4} - 0 \right] = \frac{\pi}{8}$$

17. $\int_0^a \sqrt{a^2 - x^2} \, dx$

Sol. Let $I = \int_0^a \sqrt{a^2 - x^2} \, dx$

$$\Rightarrow I_1 = \int \sqrt{a^2 - x^2} \, dx \Rightarrow I_1 = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right)$$

$$\Rightarrow I = \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) \right]_0^a \Rightarrow I = \left\{ \frac{a}{2} \sqrt{a^2 - a^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{a}{a} \right) \right\} - \left\{ \frac{a}{2} \sqrt{a^2 - 0} + \frac{a^2}{2} \sin^{-1} (0) \right\}$$

$$\Rightarrow I = \frac{a^2}{2} \sin^{-1}(1) \Rightarrow I = \frac{a^2}{2} \sin^{-1} \left(\sin \frac{\pi}{2} \right) \Rightarrow I = \frac{a^2}{2} \cdot \frac{\pi}{2} \therefore I = \frac{\pi a^2}{4}$$

18. $\int_0^{\sqrt{2}} \sqrt{2 - x^2} \, dx$

Sol. Let $I = \int_0^{\sqrt{2}} \sqrt{2-x^2} dx$

$$\Rightarrow I_1 = \int \sqrt{(\sqrt{2})^2 - (x)^2} dx \Rightarrow I_1 = \frac{x}{2} \sqrt{(\sqrt{2})^2 - (x)^2} + \frac{2}{2} \sin^{-1} \left(\frac{x}{\sqrt{2}} \right)$$

$$\Rightarrow I = [I_1]_0^{\sqrt{2}} \Rightarrow I = \left[\frac{x}{2} \sqrt{2-x^2} + \sin^{-1} \left(\frac{x}{\sqrt{2}} \right) \right]_0^{\sqrt{2}}$$

$$\Rightarrow I = \left[\frac{\sqrt{2}}{2} \sqrt{2-2} + \sin^{-1} \left(\frac{\sqrt{2}}{\sqrt{2}} \right) \right] - \left[\frac{0}{2} \sqrt{2-0} + \sin^{-1}(0) \right] \Rightarrow I = [0 + \sin^{-1}(1)] \therefore I = \frac{\pi}{2}$$

19. $\int_0^a \frac{x^4}{\sqrt{a^2-x^2}} dx$

Sol. Let $I = \int_0^a \frac{x^4}{\sqrt{a^2-x^2}} dx$

Let, $x = a \sin \theta \Rightarrow dx = a \cos \theta d\theta$

$$\Rightarrow I = \int_0^{\pi/2} \frac{(a \sin \theta)^4}{\sqrt{a^2 - a^2 \sin^2 \theta}} \cdot a \cos \theta d\theta$$

When $x=0 \Rightarrow \theta=0 \Rightarrow x=a \therefore \theta = \frac{\pi}{2}$

$$\therefore I = \int_0^{\pi/2} \frac{a^4 \sin^4 \theta}{\sqrt{a^2(1-\sin^2 \theta)}} a \cos \theta d\theta \Rightarrow I = a^4 \int_0^{\pi/2} \sin^4 \theta d\theta \Rightarrow I = a^4 \int \sin^2 \theta \sin^2 \theta d\theta$$

$$\Rightarrow I = a^4 \int \left(\frac{1-\cos 2\theta}{2} \right) \left(\frac{1-\cos 2\theta}{2} \right) d\theta \Rightarrow I = \frac{a^4}{4} \int (1-2\cos 2\theta + \cos^2 2\theta) d\theta$$

$$\Rightarrow I = \frac{a^4}{4} \left[\int d\theta - 2 \int \cos 2\theta d\theta + \int \left(\frac{1+\cos 4\theta}{2} \right) d\theta \right]$$

$$\Rightarrow I = \frac{a^4}{4} \left[\theta - \frac{2 \sin 2\theta}{2} + \frac{1}{2} \left(\theta + \frac{\sin 4\theta}{4} \right) \right]_0^{\pi/2} \Rightarrow I = \frac{a^4}{4} \left[\frac{3\theta}{2} - \sin 2\theta + \frac{\sin 4\theta}{8} \right]_0^{\pi/2}$$

$$\Rightarrow I = \frac{a^4}{4} \left[\left(\frac{3\pi}{4} \right) - \sin^2 \left(\frac{\pi}{2} \right) + \frac{\sin \pi \left(\frac{\pi}{2} \right)}{8} \right] - \left[\frac{3(0)}{2} - \sin(0) + \frac{\sin(4)(0)}{8} \right]$$

$$\Rightarrow I = \frac{a^4}{4} \left[\frac{3\pi}{4} - 0 \right] \therefore I = \frac{3\pi a^4}{16}$$

20. $\int_0^a \frac{x}{\sqrt{a^2+x^2}} dx$

Sol. Let $I = \int_0^a \frac{x}{\sqrt{a^2+x^2}} dx$, Put $a^2+x^2 \Rightarrow t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{dt}{2}$

When $x=0, t=a^2$, & $x=a, t=2a^2$

$$\Rightarrow I = \int_{a^2}^{2a^2} \frac{1}{\sqrt{t}} \cdot \frac{dt}{2} \Rightarrow I = \frac{1}{2} \left[\frac{t^{1/2}}{1/2} \right]_{a^2}^{2a^2} \Rightarrow I = \left[\sqrt{t} \right]_{a^2}^{2a^2} \Rightarrow I = \sqrt{2}a - a \therefore I = a(\sqrt{2} - 1)$$

21. $\int_0^2 x\sqrt{2-x} dx$

Sol. Let $I = \int_0^2 x\sqrt{2-x} dx$, Put $2-x=t \Rightarrow -dx=dt \Rightarrow dx=-dt$

$\therefore x=2-t$, when $x=0, t=2$ & $x=2, t=0$

$$\Rightarrow I = \int_2^0 (2-t)\sqrt{t}(-dt) \Rightarrow I = \int_0^2 (2-t)\sqrt{t} dt \Rightarrow I = 2 \int_0^2 t^{1/2} dt - \int_0^2 t^{3/2} dt$$

$$\Rightarrow I = 2 \left[\frac{t^{3/2}}{3/2} \right]_0^2 - \left[\frac{t^{5/2}}{5/2} \right]_0^2 \Rightarrow I = \frac{4}{3}(2^{3/2}) - \frac{2}{5}(2)^{5/2}$$

$$\Rightarrow I = \frac{4}{3} \left\{ (\sqrt{2})^3 \right\} - \frac{2}{5} \left\{ (\sqrt{2})^5 \right\} \Rightarrow I = \frac{4}{3}(\sqrt{2})^3 - \frac{2}{5}(\sqrt{2})^5$$

$$\Rightarrow I = \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5} = 8\sqrt{2} \left(\frac{1}{3} - \frac{1}{5} \right) = 8\sqrt{2} \left(\frac{5-3}{15} \right) = \frac{8\sqrt{2} \times 2}{15} = \frac{16\sqrt{2}}{15}$$

22. $\int_0^1 \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx$

Sol. Let $I = \int_0^1 \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx \Rightarrow I = [I_1]_0^1$

$$\Rightarrow I_1 = \int \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx,$$

Put $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$. When $x=0, \theta=0$ and when $x=1, \theta = \frac{\pi}{4}$

$\therefore \theta = \tan^{-1} x$

$$\therefore I_1 = \int \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) \sec^2 \theta d\theta \Rightarrow I_1 = \int \sin^{-1} (\sin 2\theta) \sec^2 \theta d\theta$$

$$\Rightarrow I_1 = \int 2\theta \sec^2 \theta d\theta \Rightarrow I_1 = 2 \left[\theta \int \sec^2 \theta d\theta - \int \left[\frac{d\theta}{d\theta} \right] \int \sec^2 \theta d\theta \right] d\theta$$

$$\Rightarrow I_1 = 2 \left[\theta \tan \theta - \int 1 \cdot \tan \theta d\theta \right] \Rightarrow I_1 = 2 \left[\theta \tan \theta + \log |\cos \theta| \right]$$

$$\Rightarrow I_1 = 2 \left[\theta \tan \theta + \log |\cos \theta| \right]_0^{\pi/4} \Rightarrow I_1 = 2 \left[\frac{\pi}{4} \cdot 1 + \log \left(\frac{1}{\sqrt{2}} \right) \right] \Rightarrow I_1 = 2 \left[\frac{\pi}{4} - \frac{1}{2} \log 2 \right] = \frac{\pi}{2} - \log 2$$

23. $\int_0^{\pi/2} \sqrt{1+\cos x} dx$

Sol. Let $I = \int_0^{\pi/2} \sqrt{1+\cos x} dx \Rightarrow I = \int_0^{\pi/2} \sqrt{2 \cos^2 \left(\frac{x}{2} \right)} dx \Rightarrow I = \sqrt{2} \int_0^{\pi/2} \cos \frac{x}{2} dx$

$$\Rightarrow I = \sqrt{2} \cdot \left[\frac{\sin \frac{x}{2}}{\frac{1}{2}} \right]_0^{\pi/2} \Rightarrow I = 2\sqrt{2} \left[\sin \frac{x}{2} \right]_0^{\pi/2} \Rightarrow I = 2\sqrt{2} \left\{ \sin \frac{\pi}{4} - 0 \right\} \Rightarrow I = 2\sqrt{2} \cdot \frac{1}{\sqrt{2}}$$

$$\therefore I = 2$$

24. $\int_0^{\pi/2} \sqrt{1 + \sin x} \, dx$

Sol. Let $I = \int_0^{\pi/2} \sqrt{1 + \sin x} \, dx \Rightarrow I = \int_0^{\pi/2} \sqrt{\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)^2} \, dx \Rightarrow I = \int_0^{\pi/2} \left(\cos \frac{x}{2} + \sin \frac{x}{2}\right) \, dx$

$$\Rightarrow I = \left[\frac{\sin \frac{x}{2}}{\frac{1}{2}} - \frac{\cos \frac{x}{2}}{\frac{1}{2}} \right]_0^{\pi/2} \Rightarrow I = 2 \left[\sin \frac{x}{2} - \cos \frac{x}{2} \right]_0^{\pi/2} \Rightarrow I = 2 \left[\left\{ \sin \frac{\pi}{4} - \cos \frac{\pi}{4} \right\} - \{ \sin 0 - \cos 0 \} \right]$$

$$\therefore I = 2$$

25. $\int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

Sol. Let $I = \int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} \Rightarrow I = [I_1]_0^{\pi/2}$

$$\Rightarrow I_1 = \int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

Dividing numerator and denominator by $\cos^2 x$

$$\therefore I_1 = \int \frac{\frac{1}{\cos^2 x}}{\frac{a^2 \cos^2 x}{\cos^2 x} + \frac{b^2 \sin^2 x}{\cos^2 x}} \, dx \Rightarrow I_1 = \int \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} \, dx$$

Put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$, When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = \infty$

$$\therefore I_1 = \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} \Rightarrow I_1 = \frac{1}{b^2} \int_0^{\infty} \frac{1}{\left(\frac{t}{a}\right)^2 + \left(\frac{1}{b}\right)^2} \, dt \Rightarrow I_1 = \frac{1}{ab} \left[\tan^{-1} \left(\frac{b \tan x}{a} \right) \right]_0^{\infty}$$

$$\Rightarrow I = \frac{1}{ab} [\tan^{-1}(\infty) - \tan^{-1}(0)] \Rightarrow I = \frac{1}{ab} \left[\frac{\pi}{2} - 0 \right] \therefore I = \frac{\pi}{2ab}$$

26. $\int_0^{\pi/2} \frac{dx}{1 + \cos^2 x}$

Sol. Let $I = \int_0^{\pi/2} \frac{dx}{1 + \cos^2 x} \Rightarrow I = [I_1]_0^{\pi/2}$

$$\Rightarrow I_1 = \int \frac{1}{1 + \cos^2 x} \, dx$$

By, Dividing numerator and denominator by $\cos^2 x$, $I_1 = \int \frac{\frac{1}{\cos^2 x}}{\frac{1 + \cos^2 x}{\cos^2 x}} \, dx \Rightarrow I_1 = \int \frac{\sec^2 x}{\sec^2 x + 1} \, dx$

$$\Rightarrow I_1 = \int \frac{\sec^2 x}{1 + \tan^2 x + 1} \, dx = \int \frac{\sec^2 x}{2 + \tan^2 x} \, dx$$

Put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$, When $x = 0, t = 0$ and $x = \frac{\pi}{2}, t = \infty$

$$\therefore I_1 = \int_0^{\infty} \frac{dt}{2+t^2} \Rightarrow I_1 = \int_0^{\infty} \frac{dt}{(t)^2 + (\sqrt{2})^2} \Rightarrow I_1 = \frac{1}{\sqrt{2}} \left[\tan^{-1} \left(\frac{t}{\sqrt{2}} \right) \right]_0^{\infty}$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} [\tan^{-1}(\infty) - \tan^{-1}(0)] \Rightarrow I = \frac{1}{\sqrt{2}} \cdot \frac{\pi}{2} \therefore I = \frac{\pi}{2\sqrt{2}}$$

27. $\int_0^{\pi/2} \frac{dx}{4+9\cos^2 x}$

Sol. Let $I = \int_0^{\pi/2} \frac{dx}{4+9\cos^2 x}$

By dividing numerator and denominator by $\cos^2 x$, $I = \int_0^{\pi/2} \frac{\frac{1}{\cos^2 x}}{\frac{4+9\cos^2 x}{\cos^2 x}} dx$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sec^2 x}{4\sec^2 x + 9} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sec^2 x}{4(1+\tan^2 x) + 9} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sec^2 x}{13+4\tan^2 x} dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$, When $x=0$, $t=0$ and when $x=\frac{\pi}{2}$, $t=\infty$

$$\Rightarrow I = \int_0^{\infty} \frac{dt}{13+4t^2} \Rightarrow I = \frac{1}{4} \int_0^{\infty} \frac{1}{\frac{13}{4} + t^2} dt \Rightarrow I = \frac{1}{4} \int_0^{\infty} \frac{1}{\left(\frac{\sqrt{13}}{2}\right)^2 + (t)^2} dt$$

$$\Rightarrow I = \frac{1}{4} \cdot \frac{1}{\frac{\sqrt{13}}{2}} \left[\tan^{-1} \left(\frac{t}{\frac{\sqrt{13}}{2}} \right) \right]_0^{\infty} \Rightarrow I = \frac{1}{2\sqrt{13}} [\tan^{-1}(\infty) - \tan^{-1}(0)]$$

$$\Rightarrow I = \frac{1}{2\sqrt{13}} \cdot \frac{\pi}{2} \therefore I = \frac{\pi}{4\sqrt{13}}$$

28. $\int_0^{\pi/2} \frac{dx}{5+4\sin x}$

Sol. Let $I = \int_0^{\pi/2} \frac{dx}{5+4\sin x} \Rightarrow I = \int_0^{\pi/2} \frac{dx}{5+4\left(\frac{2\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}}\right)}$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\left(1+\tan^2 \frac{x}{2}\right)}{5\left(1+\tan^2 \frac{x}{2}\right)+8\tan \frac{x}{2}} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sec^2 \frac{x}{2}}{5\tan^2 \frac{x}{2}+8\tan \frac{x}{2}+5} dx$$

Put, $\tan \frac{x}{2} = t \Rightarrow \sec^2 \frac{x}{2} \cdot \frac{1}{2} dx = dt \Rightarrow \sec^2 \frac{x}{2} dx = 2dt$

When $x=0$, $t=0$ and when $x=\frac{\pi}{2}$, $t=1$

$$\begin{aligned} \Rightarrow I &= 2 \int_0^{\pi/2} \frac{dt}{5t^2 + 8t + 5} \Rightarrow I = 2 \int_0^{\pi/2} \frac{dt}{5 \left(t^2 + \frac{8}{5}t + 1 \right)} \\ \Rightarrow I &= \frac{2}{5} \int_0^1 \frac{dt}{\left(t + \frac{4}{5} \right)^2 + 2t \cdot \frac{4}{5} + \left(\frac{4}{5} \right)^2 - \left(\frac{4}{5} \right)^2 + 1} \Rightarrow I = \frac{2}{5} \int_0^1 \frac{dt}{\left(t + \frac{4}{5} \right)^2 + \frac{9}{25}} \\ \Rightarrow I &= \frac{2}{5} \int_0^1 \frac{dt}{\left(t + \frac{4}{5} \right)^2 + \left(\frac{3}{5} \right)^2} \Rightarrow I = \frac{2}{5} \cdot \frac{1}{3/5} \left[\tan^{-1} \left(\frac{t + \frac{4}{5}}{\frac{3}{5}} \right) \right]_0^1 \Rightarrow I = \frac{2}{3} \left[\tan^{-1} \left(\frac{5t + 4}{3} \right) \right]_0^1 \\ \Rightarrow I &= \frac{2}{3} \left[\tan^{-1} (3) - \tan^{-1} \left(\frac{4}{3} \right) \right] \Rightarrow I = \frac{2}{3} \tan^{-1} \left(\frac{3 - 4/3}{1 + 3 \times 4/3} \right) \Rightarrow I = \frac{2}{3} \tan^{-1} \left(\frac{9 - 4/3}{1 + 4} \right) \Rightarrow \\ I &= \frac{2}{3} \left[\tan^{-1} \left(\frac{5}{3} \times \frac{1}{5} \right) \right] \therefore I = \frac{2}{3} \tan^{-1} \left(\frac{1}{3} \right) \end{aligned}$$

29. $\int_0^{\pi} \frac{dx}{6 - \cos x}$

Sol. Let $I = \int_0^{\pi} \frac{dx}{6 - \cos x} \Rightarrow I = [I_0]_0^{\pi}$

$$\Rightarrow I_1 = \int \frac{1}{6 - \cos x} dx \Rightarrow I_1 = \int \frac{1}{6 - \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}} dx \Rightarrow I_1 = \int \frac{1 + \tan^2 \frac{x}{2}}{6 \left(1 + \tan^2 \frac{x}{2} \right) - \left(1 - \tan^2 \frac{x}{2} \right)} dx$$

$$\Rightarrow I_1 = \int \frac{\sec^2 \frac{x}{2}}{6 + 6 \tan^2 \frac{x}{2} - 1 + \tan^2 \frac{x}{2}} dx \Rightarrow I_1 = \int \frac{\sec^2 \frac{x}{2}}{7 \tan^2 \frac{x}{2} + 5} dx$$

Put $\tan \frac{x}{2} = t \Rightarrow \sec^2 \frac{x}{2} \cdot \frac{1}{2} dx = dt \Rightarrow \sec^2 \frac{x}{2} dx = 2dt$

When $x = 0, t = 0$ and when $x = \pi, t = \infty$

$$\therefore I_1 = 2 \int \frac{dt}{7t^2 + 5} \Rightarrow I_1 = 2 \int \frac{dt}{7 \left(t^2 + \frac{5}{7} \right)} \Rightarrow I_1 = \frac{2}{7} \int \frac{dt}{\left(t \right)^2 + \left(\frac{\sqrt{5}}{\sqrt{7}} \right)^2}$$

$$\Rightarrow I_1 = \frac{2}{7} \cdot \frac{1}{\sqrt{5}/\sqrt{7}} \tan^{-1} \left(\frac{t}{\sqrt{5}/\sqrt{7}} \right) \Rightarrow I_1 = \frac{2}{\sqrt{35}} \left[\tan^{-1} \left(\frac{\sqrt{7}t}{\sqrt{5}} \right) \right]_0^{\infty}$$

$$\Rightarrow I = \frac{2}{\sqrt{35}} \left[\tan^{-1} (\infty) - \tan^{-1} (0) \right] = \frac{2}{\sqrt{35}} \left[\frac{\pi}{2} - 0 \right] \therefore I = \frac{\pi}{\sqrt{35}}$$

30. $\int_0^{\pi} \frac{dx}{5 + 4 \cos x}$

Sol. Let $I = \int_0^{\pi} \frac{dx}{5+4\cos x} \Rightarrow I = [I_1]_0^{\pi}$

$$\Rightarrow I_1 = \int \frac{dx}{5+4\cos x} \Rightarrow I_1 = \int \frac{dx}{5+4\left(\frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}\right)} \Rightarrow I_1 = \int \frac{1+\tan^2 \frac{x}{2}}{5\left(1+\tan^2 \frac{x}{2}\right)+4\left(1-\tan^2 \frac{x}{2}\right)} dx$$

$$\Rightarrow I_1 = \int \frac{\sec^2 \frac{x}{2}}{5\tan^2 \frac{x}{2}+5+4-4\tan^2 \frac{x}{2}} dx \Rightarrow I_1 = \int \frac{\sec^2 \frac{x}{2}}{\tan^2 \frac{x}{2}+9} dx$$

Put $\tan \frac{x}{2} = t \Rightarrow \sec^2 \frac{x}{2} \cdot \frac{1}{2} dx = dt \Rightarrow \sec^2 \frac{x}{2} dx = 2dt$

When $x=0$, $t=0$ and when $x=\pi$, $t=\infty$

$$\therefore I_1 = 2 \int \frac{dt}{t^2+(3)^2} \Rightarrow I_1 = 2 \cdot \frac{1}{3} \tan^{-1} \left(\frac{t}{3} \right) = \frac{2}{3} \left[\tan^{-1} \frac{t}{3} \right]_0^{\infty}$$

$$\Rightarrow I = \frac{2}{3} \left[\tan^{-1}(\infty) - \tan^{-1}(0) \right] \Rightarrow I = \frac{2}{3} \cdot \frac{\pi}{2} \therefore I = \frac{\pi}{3}$$

31. $\int_0^{\pi/2} \frac{dx}{\cos x + 2 \sin x}$

Sol. Let $I = \int_0^{\pi/2} \frac{dx}{\cos x + 2 \sin x} \Rightarrow I = [I_1]_0^{\pi/2}$

$$\Rightarrow I_1 = \int \frac{dx}{\cos x + 2 \sin x} \Rightarrow I_1 = \int \frac{dx}{\frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}} + 2 \cdot \left(\frac{2 \tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}} \right)} \Rightarrow I_1 = \int \frac{1+\tan^2 \frac{x}{2}}{1-\tan^2 \frac{x}{2} + 4 \tan \frac{x}{2}} dx$$

$$\Rightarrow I_1 = \int \frac{\sec^2 \frac{x}{2}}{1-\tan^2 \frac{x}{2} + 4 \tan \frac{x}{2}} dx$$

Put $\tan \frac{x}{2} = t \Rightarrow \sec^2 \frac{x}{2} \cdot \frac{1}{2} dx = dt \Rightarrow \sec^2 \frac{x}{2} dx = 2dt$

When $x=0$, $t=0$ and when $x=\frac{\pi}{2}$, $t=1$

$$I_1 = \int \frac{2 dt}{-t^2 + 4t + 1} \Rightarrow I_1 = -2 \int \frac{dt}{t^2 - 4t - 1} \Rightarrow I_1 = -2 \int \frac{1}{(t)^2 - 2 \cdot t \cdot 2 + (2)^2 - (2)^2 - 1} dt$$

$$\Rightarrow I_1 = -2 \int \frac{1}{(t-2)^2 - (\sqrt{5})^2} dt \Rightarrow I_1 = 2 \int \frac{1}{(\sqrt{5})^2 - (t-2)^2} dt \Rightarrow I = 2 \cdot \frac{1}{2\sqrt{5}} \log \left| \frac{\sqrt{5}+t-2}{\sqrt{5}-t+2} \right|$$

$$\Rightarrow I = \frac{1}{\sqrt{5}} \log \left(\frac{\sqrt{5}+t-2}{\sqrt{5}-t+2} \right)_0^1 \Rightarrow I = \frac{1}{\sqrt{5}} \left[\log \left| \frac{\sqrt{5}-1}{\sqrt{5}+1} \right| - 0 \right] \Rightarrow I = \frac{1}{\sqrt{5}} \left[\log \left| \frac{(\sqrt{5}-1) \times (\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)} \right| \right]$$

$$\Rightarrow I = \frac{1}{\sqrt{5}} \left[\log \left| \frac{5-2\sqrt{5}+1}{5-1} \right| \right] \Rightarrow I = \frac{1}{\sqrt{5}} \log \left| \frac{6-2\sqrt{5}}{4} \right| \therefore I = \frac{1}{\sqrt{5}} \log \left| \frac{3-\sqrt{5}}{2} \right|$$

32. $\int_0^{\pi} \frac{dx}{3+2\sin x + \cos x}$

Sol. Let $I = \int_0^{\pi} \frac{dx}{3+2\sin x + \cos x} \Rightarrow I = [I_1]_0^{\pi}$

$$\Rightarrow I_1 = \int \frac{dx}{3+2\sin x + \cos x} \Rightarrow I_1 = \int \frac{dx}{3+2 \cdot \frac{2 \tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}} + \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}}$$

$$\Rightarrow I_1 = \int \frac{1+\tan^2 \frac{x}{2}}{3 \left(1+\tan^2 \frac{x}{2} \right) + 4 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx \Rightarrow I_1 = \int \frac{\sec^2 \frac{x}{2}}{3+3 \tan^2 \frac{x}{2} + 4 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx$$

$$\Rightarrow I = \int \frac{\sec^2 \frac{x}{2}}{2 \tan^2 \frac{x}{2} + 4 \tan \frac{x}{2} + 4} dx \Rightarrow I_1 = \frac{1}{2} \int \frac{\sec^2 \frac{x}{2}}{\tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} + 2} dx$$

Put $\tan \frac{x}{2} = t \Rightarrow \sec^2 \frac{x}{2} \cdot \frac{1}{2} dx = dt \Rightarrow \sec^2 \frac{x}{2} dx = 2dt$

When $x=0$, $t=0$ and when $x=\pi$, $t=\infty$

$$\therefore I_1 = \frac{1}{2} \int \frac{2 dt}{t^2 + 2t + 2} \Rightarrow I_1 = \int \frac{dt}{(t)^2 + 2t \cdot 1 + (1)^2 - (1)^2 + 2} \Rightarrow I_1 = \int \frac{dt}{(t+1)^2 + (1)^2} dt$$

$$\Rightarrow I_1 = \tan^{-1}(t+1) \Rightarrow I = [\tan^{-1}(t+1)]_0^{\infty} = [\tan^{-1} \infty - \tan^{-1}(0+1)] = \frac{\pi}{2} - \frac{\pi}{4} \therefore I = \frac{\pi}{4}$$

33. $\int_0^{\pi/4} \frac{\tan^3 x}{1+\cos 2x} dx$

Sol. Let $I = \int_0^{\pi/4} \frac{\tan^3 x}{1+\cos 2x} dx \Rightarrow I = [I_1]_0^{\pi/4}$

$$\therefore I_1 = \int \frac{\tan^3 x}{1+\cos 2x} dx \Rightarrow I_1 = \int \frac{\tan^3 x}{2 \cos^2 x} dx \Rightarrow I_1 = \frac{1}{2} \int \tan^3 x \sec^2 x dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$, when $x=0$, $t=0$ and when $x=\frac{\pi}{4}$, $t=1$

$$\Rightarrow I_1 = \frac{1}{2} \int_0^1 t^3 dt \Rightarrow I_1 = \frac{1}{2} \left[\frac{t^4}{4} \right]_0^1 \therefore I = \frac{1}{8}$$

34. $\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} dx$

Sol. Let $I = \int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} dx \Rightarrow I = [I_1]_0^{\pi/2}; \quad I_1 = \int \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} dx$

Put $\cos x = t \Rightarrow -\sin x dx = dt \Rightarrow \sin x dx = -dt$

When $x = 0, t = 1$ and when $x = \frac{\pi}{2}, t = 0$

$\Rightarrow I_1 = -\int \frac{t}{t^2 + 3t + 2} dt \Rightarrow I_1 = -\int \frac{t}{(t+2)(t+1)} dt$

$\frac{t}{(t+2)(t+1)} = \frac{A}{t+2} + \frac{B}{t+1} \Rightarrow \frac{t}{(t+2)(t+1)} = \frac{A(t+1) + B(t+2)}{(t+2)(t+1)}$

$\Rightarrow t = At + A + Bt + 2B \Rightarrow t = (A+B)t + A + 2B$

By, Equating co-efficient both side we get

$A + B = 1 \quad \dots(1)$

$A + 2B = 0 \quad \dots(2)$

Now, Solving equation (1) & (2) then we get $A = 2, B = -1$

$I_1 = -\left[\int \left(\frac{A}{t+2} + \frac{B}{t+1} \right) dt \right] \Rightarrow I_1 = -\left[A \int \frac{1}{t+2} dt + B \int \frac{1}{t+1} dt \right]$

$\Rightarrow I_1 = -[2 \log(t+2) - \log(t+1)] \Rightarrow I = [\log(t+1) - 2 \log(t+2)]_1^0$

$\Rightarrow I = (\log 1 - 2 \log 2) - (\log 2 - 2 \log 3)$

$\Rightarrow I = -3 \log 2 + 2 \log 3$

$\therefore I = \log 9 - \log 8$

35. $\int_0^{\pi/2} \frac{\sin 2x}{\cos^4 x + \sin^4 x} dx$

Sol. Let $I = \int_0^{\pi/2} \frac{\sin 2x}{\cos^4 x + \sin^4 x} dx \Rightarrow I = [I_1]_0^{\pi/2}$

$I_1 = \int \frac{\sin 2x}{\cos^4 x + \sin^4 x} dx$

By, Dividing numerator and denominator by $\cos^4 x$, $I_1 = \int \frac{\frac{\sin 2x}{\cos^4 x}}{\frac{\cos^4 x}{\cos^4 x} + \frac{\sin^4 x}{\cos^4 x}} dx$

$\Rightarrow I_1 = \int \frac{\sin 2x \cdot \sec^4 x}{1 + \tan^4 x} dx \Rightarrow I_1 = \int \frac{2 \tan x \cdot \sec^2 x \cdot \sec^2 x}{1 + \tan^4 x} dx \Rightarrow I_1 = \int \frac{2 \tan x \sec^2 x}{1 + (\tan^2 x)^2} dx$

Put $\tan^2 x = t \Rightarrow 2 \tan x \sec^2 x dx = dt$. When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = \infty$

$\therefore I_1 = \int \frac{dt}{1+t^2} \Rightarrow I_1 = \tan^{-1}(t) \Rightarrow I = [\tan^{-1} t]_0^{\infty} \Rightarrow I = \tan^{-1}(\infty) - \tan^{-1}(0) \therefore I = \frac{\pi}{2}$

36. $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}} dx$

Sol. Let $I = \int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}} dx \Rightarrow I = [I_1]_{\pi/3}^{\pi/2}; \quad I_1 = \int \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}} dx$

$$\Rightarrow I_1 = \int \frac{\sqrt{2\cos^2 \frac{x}{2}}}{\left(2\sin^2 \frac{x}{2}\right)^{5/2}} dx \Rightarrow I_1 = \int \frac{\sqrt{2}\cos \frac{x}{2}}{2^{5/2}\sin^5 \frac{x}{2}} dx \Rightarrow I_1 = \frac{1}{4} \int \frac{\cos \frac{x}{2}}{\sin^5 \frac{x}{2}} dx$$

Put $\sin \frac{x}{2} = t \Rightarrow \cos \frac{x}{2} \cdot \frac{1}{2} dx = dt \Rightarrow \cos \frac{x}{2} dx = 2dt$

When $x = \frac{\pi}{3}$, $t = \frac{1}{2}$ and when $x = \frac{\pi}{2}$, $t = \frac{1}{\sqrt{2}}$

$$I_1 = \frac{1}{4} \int \frac{2dt}{t^5} \Rightarrow I_1 = \frac{1}{2} \int t^{-5} dt \Rightarrow I_1 = \frac{1}{2} \cdot \frac{t^{-4}}{-4} \Rightarrow I_1 = -\frac{1}{8} \cdot \frac{1}{t^4} \Rightarrow I = -\frac{1}{8} \left[\frac{1}{t^4} \right]_{1/2}^{1/\sqrt{2}}$$

$$\Rightarrow I = -\frac{1}{8} \left[(\sqrt{2})^4 - (2)^4 \right] \Rightarrow I = -\frac{1}{8} [4 - 16] \Rightarrow I = \frac{12}{8} \therefore I = \frac{3}{2}$$

37. $\int_0^1 (\cos^{-1} x)^2 dx$

Sol. Let $I = \int_0^1 (\cos^{-1} x)^2 dx \Rightarrow I = [I_1]_0^1; \quad I_1 = \int (\cos^{-1} x)^2 dx$

Put $\cos^{-1} x = t \therefore x = \cos t$. When $x = 0$, $t = \frac{\pi}{2}$ and when $x = 1$, $t = 0$

$$-\frac{1}{\sqrt{1-x^2}} dx = dt \Rightarrow -dx = \sqrt{1-x^2} dt \Rightarrow dx = -\sqrt{1-\cos^2 t} dt \Rightarrow dx = -\sin t dt$$

$$I_1 = \int t^2 (-\sin t) dt \Rightarrow I_1 = -\int t^2 \sin t dt \Rightarrow I_1 = -\left[t^2 \int \sin t dt - \int \frac{d(t^2)}{dt} \int \sin t dt \right] dt$$

$$\Rightarrow I_1 = -\left[-t^2 \cos t + \int 2t \cos t dt \right] \Rightarrow I_1 = t^2 \cos t - 2 \int t \cos t dt$$

$$\Rightarrow I_1 = t^2 \cos t - 2 \left[t \int \cos t dt - \int \frac{d(t)}{dt} \int \cos t dt \right] dt$$

$$\Rightarrow I_1 = t^2 \cos t - 2 \left[t \sin t - \int 1 \cdot \sin t dt \right] \Rightarrow I_1 = t^2 \cos t - 2 \left[t \sin t + \cos t \right]$$

$$\Rightarrow I_1 = t^2 \cos t - 2t \sin t - 2 \cos t \Rightarrow I = \left[t^2 \cos t - 2t \sin t - 2 \cos t \right]_{\pi/2}^0$$

$$\Rightarrow I = (0 - 2) - (-\pi) \therefore I = \pi - 2$$

38. $\int_0^1 x (\tan^{-1} x)^2 dx$

Sol. Let $I = \int_0^1 x (\tan^{-1} x)^2 dx \Rightarrow I = [I_1]_0^1; \quad I_1 = \int x (\tan^{-1} x)^2 dx$

Put $\tan^{-1} x = t \therefore x = \tan t$. When $x = 0$, $t = 0$ and when $x = 1$, $t = \frac{\pi}{4}$

$$\frac{1}{1+x^2} dx = dt \Rightarrow dx = (1+x^2) dt \Rightarrow dx = (1+\tan^2 t) dt \Rightarrow dx = \sec^2 t dt$$

$$\therefore I_1 = \int \tan t \cdot t^2 \sec^2 t \, dt \Rightarrow I_1 = \int t^2 \tan t \sec^2 t \, dt$$

$$\Rightarrow I_1 = t^2 \int \tan t \sec^2 t \, dt - \int \left[\frac{d(t^2)}{dt} \int \tan t \sec^2 t \, dt \right] dt$$

$$\text{Again put } \tan t = y \Rightarrow \sec^2 t \, dt = dy$$

$$I_1 = t^2 \int y \, dy - \int \left[2t \cdot \int y \, dy \right] dt \Rightarrow I_1 = t^2 \cdot \frac{y^2}{2} - \int 2t \cdot \frac{y^2}{2} \, dt \Rightarrow I_1 = t^2 \cdot \frac{\tan^2 t}{2} - \int t \cdot \tan^2 t \, dt$$

$$\Rightarrow I_1 = \frac{t^2}{2} \tan^2 t - \left[t \int \tan^2 t \, dt - \int \left[\frac{d(t)}{dt} \int \tan^2 t \, dt \right] dt \right]$$

$$\Rightarrow I_1 = \frac{t^2}{2} \tan^2 t - \left[t \int (\sec^2 t - 1) \, dt - \int [1 \cdot \int (\sec^2 t - 1) \, dt] dt \right]$$

$$\Rightarrow I_1 = \frac{t^2}{2} \tan^2 t - \left[t(\tan t - t) - \int (\tan t - t) \, dt \right] \Rightarrow I_1 = \frac{t^2}{2} \tan^2 t - \left[t \tan t - t^2 + \log |\cos t| + \frac{t^2}{2} \right]$$

$$\Rightarrow I_1 = \frac{t^2}{2} \tan^2 t - \left[t \tan t + \log |\cos t| - \frac{t^2}{2} \right] \Rightarrow I = \left[\frac{t^2}{2} \tan^2 t - t \tan t - \log |\cos t| + \frac{t^2}{2} \right]_0^{\frac{\pi}{4}}$$

$$\Rightarrow I = \frac{1}{2} \left(\frac{\pi}{4} \right)^2 - \frac{\pi}{4} + \frac{1}{2} \log 2 + \frac{\pi^2}{32} \therefore I = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2} \log 2$$

39. $\int_0^1 \sin^{-1}(\sqrt{x}) \, dx$

Sol. Let $I = \int_0^1 \sin^{-1}(\sqrt{x}) \, dx$,

$$\text{Put } t = \sin^{-1} \sqrt{x} \Rightarrow \sqrt{x} = \sin t \Rightarrow x = \sin^2 t, \text{ when } x=0, t=0 \text{ and } x=1, t=\frac{\pi}{2}$$

$$\frac{dt}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} \Rightarrow \frac{dt}{dx} = \frac{1}{\sqrt{1-\sin^2 t}} \cdot \frac{1}{2\sin t} \Rightarrow dt = \frac{dx}{2\cos t \sin t} \Rightarrow \sin 2t \, dt = dx$$

$$\therefore I = \int_0^{\pi/2} t \cdot \sin 2t \, dt, \text{ Let } I = [I_1]_0^{\pi/2}$$

$$I_1 = t \int \sin 2t \, dt - \int \left(\frac{dt}{dt} \int \sin 2t \, dt \right) dt = t \cdot \left(\frac{-\cos 2t}{2} \right) + \int 1 \cdot \frac{\cos 2t}{2} \, dt \Rightarrow I_1 = -\frac{t}{2} \cos 2t + \frac{1}{4} \sin 2t$$

$$\therefore I = \left[-\frac{t}{2} \cos 2t + \frac{1}{4} \sin 2t \right]_0^{\pi/2} = \left[-\frac{\pi}{4} \cos \pi + \frac{1}{4} (0) + 0 \right] = -\frac{\pi}{4} (-1) = \frac{\pi}{4}$$

40. $\int_0^a \sin^{-1} \sqrt{\frac{x}{a+x}} \, dx$

Sol. Let $I = \int_0^a \sin^{-1} \sqrt{\frac{x}{a+x}} \, dx \Rightarrow I = [I_1]_0^a; \quad I_1 = \int \sin^{-1} \sqrt{\frac{x}{a+x}} \, dx$

$$\text{Put } x = a \tan^2 \theta \Rightarrow \frac{x}{a} = \tan^2 \theta \Rightarrow \tan \theta = \sqrt{\frac{x}{a}} \Rightarrow \theta = \tan^{-1} \left(\sqrt{\frac{x}{a}} \right)$$

$$\text{When } x=0, \theta=0 \text{ and when } x=a, \theta=\frac{\pi}{4}$$

$$\therefore I_1 = \int \sin^{-1} \sqrt{\frac{a \tan^2 \theta}{a + a \tan^2 \theta}} \cdot 2a \tan \theta \sec^2 \theta d\theta \Rightarrow I_1 = \int \sin^{-1} \sqrt{\frac{\tan^2 \theta}{1 + \tan^2 \theta}} \cdot 2a \tan \theta \sec^2 \theta d\theta$$

$$\Rightarrow I_1 = \int \sin^{-1} \sqrt{\frac{\tan^2 \theta}{\sec^2 \theta}} \cdot 2a \tan \theta \sec^2 \theta d\theta \Rightarrow I_1 = \int \sin^{-1} (\sin \theta) 2a \tan \theta \sec^2 \theta d\theta$$

$$\Rightarrow I_1 = \int \theta 2a \tan \theta \sec^2 \theta d\theta \Rightarrow I_1 = 2a \int \theta \tan \theta \sec^2 \theta d\theta$$

$$\Rightarrow I = 2a \left[\theta \int \tan \theta \sec^2 \theta d\theta - \int \left[\frac{d\theta}{d\theta} \int \tan \theta \sec^2 \theta d\theta \right] d\theta \right]$$

Put $\tan \theta = t \Rightarrow \sec^2 \theta d\theta = dt$

$$\therefore I_1 = 2a \left[\theta \int t dt - \int \left[1 \cdot \int t dt \right] dt \right] \Rightarrow I_1 = 2a \left[\theta \frac{t^2}{2} - \int \frac{t^2}{2} dt \right]$$

$$\Rightarrow I_1 = 2a \left[\theta \frac{\tan^2 \theta}{2} - \frac{1}{2} \int \tan^2 \theta d\theta \right] \Rightarrow I_1 = a\theta \tan^2 \theta - a \int (\sec^2 \theta - 1) d\theta$$

$$\Rightarrow I_1 = a\theta \tan^2 \theta - a(\tan \theta - \theta) \Rightarrow I = \left[a\theta \tan^2 \theta - a(\tan \theta - \theta) \right]_0^{\pi/4} = a \cdot \frac{\pi}{4} \cdot 1 - a \left(1 - \frac{\pi}{4} \right)$$

$$\Rightarrow I = 2a \cdot \frac{\pi}{4} - a \therefore I = a \left(\frac{\pi}{2} - 1 \right)$$

41. $\int_0^9 \frac{dx}{1+\sqrt{x}}$

Sol. Let $I = \int_0^9 \frac{dx}{1+\sqrt{x}} \Rightarrow I = [I_1]_0^9, I_1 = \int \frac{1}{1+\sqrt{x}} dx \Rightarrow I_1 = \int \frac{1}{1+\sqrt{x}} \times \frac{1-\sqrt{x}}{1-\sqrt{x}} dx$

$$\Rightarrow I_1 = \int \frac{1-\sqrt{x}}{1-x} dx \Rightarrow I_1 = \int \frac{1}{1-x} dx - \int \frac{\sqrt{x}}{1-(\sqrt{x})^2} dx$$

Put $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt \Rightarrow dx = 2t dt$

$$\therefore I_1 = -\log |1-x| - \int \frac{t}{1-t^2} \cdot 2t dt \Rightarrow I_1 = -\log |1-t^2| + 2 \int \frac{t^2}{t^2-1} dt$$

$$\Rightarrow I_1 = -\log |1-t^2| + 2 \int \frac{(t^2-1)+1}{t^2-1} dt \Rightarrow I_1 = -\log |1-t^2| + 2 \int \frac{t^2-1}{t^2-1} dt + 2 \int \frac{1}{t^2-1} dt$$

$$\Rightarrow I_1 = -\log |1-t^2| + 2 \int dt + 2 \cdot \frac{1}{2 \cdot 1} \log \left| \frac{t-1}{t+1} \right| \Rightarrow I_1 = -\log |1-t^2| + 2t + \log \left| \frac{t-1}{t+1} \right|$$

$$\Rightarrow I_1 = -\log |1-x| + 2\sqrt{x} + \log \left| \frac{\sqrt{x}-1}{\sqrt{x}+1} \right|$$

$$\therefore I = \left[\log \left| \frac{\sqrt{x}-1}{\sqrt{x}+1} \right| + 2\sqrt{x} - \log |1-x| \right]_0^9$$

$$\Rightarrow I = \left\{ \log \left(\frac{\sqrt{9}-1}{\sqrt{9}+1} \right) + 2\sqrt{9} - \log |1-9| \right\} - \{0\} \Rightarrow I = \log \left(\frac{2}{4} \right) + 2 \cdot 3 - \log 8$$

$$\Rightarrow I = \log \left| \frac{1}{2} \right| + 6 - \log 8 \Rightarrow I = \log 1 - \log 2 + 6 - 3 \log 2 \quad \therefore I = 6 - 4 \log 2$$

42. $\int_0^1 x^3 \sqrt{1+3x^4} dx$

Sol. Let $I = \int_0^1 x^3 \sqrt{1+3x^4} dx \Rightarrow I = [I_1]_0^1$; $I_1 = \int x^3 \sqrt{1+3x^4} dx$

Put $1+3x^4 = t \Rightarrow 12x^3 dx = dt \Rightarrow x^3 dx = \frac{dt}{12}$. When $x=0, t=1$ and $x=1, t=4$

$$\Rightarrow I_1 = \int \sqrt{t} \cdot \frac{dt}{12} \Rightarrow I_1 = \frac{1}{12} \cdot \frac{t^{3/2}}{3/2} \Rightarrow I = \frac{1}{18} [t^{3/2}]_1^4 \Rightarrow I = \frac{1}{18} [(2^2)^{3/2} - 1]$$

$$\Rightarrow I = \frac{1}{18} [8 - 1] \quad \therefore I = \frac{7}{18}$$

43. $\int_0^1 \frac{1-x^2}{(1+x^2)^2} dx$

Sol. Let $I = \int_0^1 \frac{1-x^2}{(1+x^2)^2} dx \Rightarrow I = [I_1]_0^1$; $I_1 = \int \frac{1-x^2}{(1+x^2)^2} dx \Rightarrow I_1 = \int \frac{1-x^2}{(1+x^2)^2} dx$

$$\Rightarrow I_1 = \int \frac{1-x^2}{1+2x^2+x^4} dx \Rightarrow I_1 = \int \frac{x^2 \left(\frac{1}{x^2} - 1 \right)}{x^2 \left(\frac{1}{x^2} + 2 + x^2 \right)} dx \Rightarrow I_1 = \int \frac{\left(\frac{1}{x^2} - 1 \right)}{\left(\frac{1}{x^2} + x^2 + 2 \right)} dx$$

$$\Rightarrow I_1 = - \int \frac{\left(1 - \frac{1}{x^2} \right)}{\left(x^2 + 2 + \frac{1}{x^2} \right)} dx \Rightarrow I_1 = - \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x} \right)^2} dx \Rightarrow I_1 = - \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x} \right)^2} dx$$

Put $x + \frac{1}{x} = t \Rightarrow \left(1 - \frac{1}{x^2} \right) dx = dt$, when $x=0, t=\infty$ and when $x=1, t=2$

$$\Rightarrow I_1 = - \int \frac{dt}{t^2} \Rightarrow I_1 = - \left(-\frac{1}{t} \right) \Rightarrow I_1 = \frac{1}{t} \Rightarrow I = \left[\frac{1}{t} \right]_{\infty}^2 \Rightarrow I = \left(\frac{1}{2} - 0 \right) \quad \therefore I = \frac{1}{2}$$

44. $\int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}}$

Sol. Let $I = \int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}} \Rightarrow I = [I_1]_1^2$; $I_1 = \int \frac{1}{(x+1)\sqrt{x^2-1}} dx$

Put $x+1 = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2} dt$, when $x=0, t=\frac{1}{2}$ and when $x=2, t=\frac{1}{3}$

$$\therefore I_1 = \int \frac{\left(-\frac{1}{t^2} \right) dt}{\frac{1}{t} \sqrt{\frac{1}{t} \left(\frac{1}{t} - 2 \right)}} \quad \therefore x^2 - 1 = (x+1)(x-1) = \frac{1}{t} \left(\frac{1}{t} - 1 - 1 \right) = \frac{1}{t} \left(\frac{1}{t} - 2 \right)$$

$$\Rightarrow I_1 = \int \frac{-dt}{\sqrt{1-2t}} \Rightarrow I_1 = \frac{-(1-2t)^{1/2}}{-\frac{1}{2}} \Rightarrow I_1 = 2(1-2t)^{1/2}$$

$$\Rightarrow I = 2 \left[\sqrt{1-2t} \right]_{1/2}^{1/3} = 2 \left(\sqrt{1-\frac{2}{3}} - \sqrt{1-2 \cdot \frac{1}{2}} \right) = 2\sqrt{\frac{1}{3}} - 0 \quad \therefore I = \frac{2}{\sqrt{3}}$$

45. $\int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x}) dx$

Sol. Let $I = \int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x}) dx \Rightarrow I = \int_0^{\pi/2} \left\{ \frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}} \right\} dx$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx \Rightarrow I = \sqrt{2} \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}} dx$$

Put $(\sin x - \cos x) = t$ and $(\cos x + \sin x) dx = dt$

Squaring both sides, $t^2 = \sin^2 x + \cos^2 x - 2 \sin x \cos x \Rightarrow t^2 = 1 - 2 \sin x \cos x \Rightarrow 2 \sin x \cos x = 1 - t^2$

Also $x=0, t=-1$ and $x=\frac{\pi}{2}, t=1$

$$\therefore I = \sqrt{2} \int_{-1}^1 \frac{dt}{\sqrt{1-t^2}} = \sqrt{2} [\sin^{-1} t]_{-1}^1 \Rightarrow I = \sqrt{2} \{ \sin^{-1}(1) - \sin^{-1}(-1) \}$$

$$\Rightarrow I = \sqrt{2} \{ 2 \sin^{-1}(1) \} \Rightarrow I = \sqrt{2} \times 2 \times \frac{\pi}{2} \therefore I = \sqrt{2} \pi$$

46. $\int_2^3 \frac{2-x}{\sqrt{5x-6-x^2}} dx$

Sol. Let $I = \int_2^3 \frac{2-x}{\sqrt{5x-6-x^2}} dx \Rightarrow I = [I_1]_2^3; I_1 = \int \frac{2-x}{\sqrt{5x-6-x^2}} dx$

$$\Rightarrow 2-x = A \frac{d}{dx} (5x-6-x^2) + B \Rightarrow 2-x = A(5-2x) + B$$

$$\Rightarrow 2-x = 5A - 2Ax + B \Rightarrow 2-x = (5A+B) - 2Ax$$

Equating co-efficient both side we get

$$5A+B=2 \quad \dots(1)$$

$$-2A=-1 \Rightarrow A=\frac{1}{2} \quad \dots(2)$$

Putting the value of A in equation (1) we get, $5A+B=2$

$$\Rightarrow B=2-5A \Rightarrow B=2-5 \cdot \frac{1}{2} \Rightarrow B=2-\frac{5}{2} \therefore B=\frac{-1}{2}$$

$$I_1 = \int \frac{A(5-2x)+B}{\sqrt{5x-6-x^2}} dx$$

$$\Rightarrow I_1 = A \int \frac{5-2x}{\sqrt{5x-6-x^2}} + B \int \frac{1}{\sqrt{5x-6-x^2}} dx \Rightarrow I_1 = \frac{1}{2} I_1 - \frac{1}{2} I_3 \quad \dots(1)$$

$$I_2 = \int \frac{5-2x}{\sqrt{5x-6-x^2}} dx$$

Put $5x - 6 - x^2 = t \Rightarrow (5 - 2x)dx = dt$

$$I_2 = \int \frac{dt}{\sqrt{t}} \Rightarrow I_2 = \frac{t^{1/2}}{1/2} \Rightarrow I_2 = 2\sqrt{5x - 6 - x^2}$$

Again $I_3 = \int \frac{1}{\sqrt{5x - 6 - x^2}} dx \Rightarrow I_3 = \int \frac{1}{\sqrt{-(x^2 - 5x + 6)}} dx$

$$\Rightarrow I_3 = \int \frac{1}{\sqrt{-(x^2 - 2 \cdot x \cdot \frac{5}{2} + (\frac{5}{2})^2 - (\frac{5}{2})^2 + 6)}} dx \Rightarrow I_3 = \int \frac{1}{\sqrt{-(x - \frac{5}{2})^2 - \frac{1}{4}}}$$

$$\Rightarrow I_3 = \int \frac{1}{\sqrt{(\frac{1}{2})^2 - (x - \frac{5}{2})^2}} dx \Rightarrow I_3 = \sin^{-1} \left(\frac{x - \frac{5}{2}}{\frac{1}{2}} \right) \Rightarrow I_3 = \sin^{-1}(2x - 5)$$

Putting the value of I_2 & I_3 in equation (1)

$$I_1 = \frac{1}{2} \cdot 2\sqrt{5x - 6 - x^2} - \frac{1}{2} \sin^{-1}(2x - 5) \Rightarrow I_1 = \sqrt{5x - 6 - x^2} - \frac{1}{2} \sin^{-1}(2x - 5)$$

$$I = \left[\sqrt{5x - 6 - x^2} - \frac{1}{2} \sin^{-1}(2x - 5) \right]_2^3$$

$$\Rightarrow I = \left[\left\{ \sqrt{15 - 6 - 9} - \frac{1}{2} \sin^{-1}(6 - 5) \right\} - \left\{ \sqrt{10 - 6 - 4} - \frac{1}{2} \sin^{-1}(4 - 5) \right\} \right]$$

$$\Rightarrow I = -\frac{1}{2} \sin^{-1}(1) + \frac{1}{2} \sin^{-1}(-1) \Rightarrow I = -\frac{1}{2} \cdot \frac{\pi}{2} + \frac{1}{2} \cdot \frac{\pi}{2} \Rightarrow I = -\frac{\pi}{4} + \frac{\pi}{4} \therefore I = -\frac{\pi}{2}$$

47. $\int_{\pi/4}^{\pi/2} \frac{\cos \theta}{\left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right)^3} d\theta$

Sol. Let $I = \int_{\pi/4}^{\pi/2} \frac{\cos \theta}{\left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right)^3} d\theta$

$$\Rightarrow I = \int_{\pi/4}^{\pi/2} \frac{\cos^2 \left(\frac{\theta}{2} \right) - \sin^2 \left(\frac{\theta}{2} \right)}{\left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right)^3} d\theta = \int_{\pi/4}^{\pi/2} \frac{\left\{ \cos \left(\frac{\theta}{2} \right) - \sin \left(\frac{\theta}{2} \right) \right\} \left\{ \cos \left(\frac{\theta}{2} \right) + \sin \left(\frac{\theta}{2} \right) \right\}}{\left\{ \cos \left(\frac{\theta}{2} \right) + \sin \left(\frac{\theta}{2} \right) \right\}^3} d\theta$$

$$= \int \frac{(\cos \theta / 2 - \sin \theta / 2)}{(\cos \theta / 2 + \sin \theta / 2)} dx$$

Put $t = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \Rightarrow \frac{dt}{d\theta} = -\frac{1}{2} \sin \frac{\theta}{2} + \frac{1}{2} \cos \frac{\theta}{2}$

$$\Rightarrow dt = \frac{1}{2} \left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right) d\theta \Rightarrow 2dt = \left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right) d\theta$$

When $x = \frac{\pi}{4}$, $t = \cos \frac{\pi}{8} + \sin \frac{\pi}{8}$; When $x = \frac{\pi}{2}$, $t = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$$I = \int_{\cos \frac{\pi}{8} + \sin \frac{\pi}{8}}^{\sqrt{2}} \frac{2dt}{t^2} = -2 \left[\frac{1}{t} \right]_{\cos \frac{\pi}{8} + \sin \frac{\pi}{8}}^{\sqrt{2}} = -2 \left[\frac{1}{\sqrt{2}} - \frac{1}{\cos \frac{\pi}{8} + \sin \frac{\pi}{8}} \right]$$

$$\therefore I = \frac{-2}{\sqrt{2}} + \frac{2}{\cos \frac{\pi}{8} + \sin \frac{\pi}{8}} = \frac{2}{\cos \frac{\pi}{8} + \sin \frac{\pi}{8}} - \sqrt{2}$$

48. $\int_0^{(\pi/2)^{1/3}} x^2 \sin x^3 dx$

Sol. Let $I = \int_0^{(\pi/2)^{1/3}} x^2 \sin x^3 dx$, Put $t = x^3 \Rightarrow dt = 3x^2 dx$

When $x = 0, t = 0$; $x = \left(\frac{\pi}{2}\right)^{1/3}, t = \left(\frac{\pi}{2}\right)^{1/3 \times 3} \Rightarrow t = \frac{\pi}{2}$

$\therefore I = \int_0^{\pi/2} \sin t dt / 3 = \frac{1}{3} [-\cos t]_0^{\pi/2} = -\frac{1}{3} \left[\cos \frac{\pi}{2} - \cos 0 \right] = -\frac{1}{3} [0 - 1] = \frac{1}{3}$

49. $\int_1^2 \frac{dx}{x(1+\log x)^2}$

Sol. Let $I = \int_1^2 \frac{dx}{x(1+\log x)^2}$, Put $t = 1 + \log x \Rightarrow dt = \frac{1}{x} dx$

When $x = 1, t = 1$; $x = 2, t = 1 + \log 2$

$\therefore I = \int_1^{1+\log 2} \frac{dt}{t^2} = \left[-\frac{1}{t} \right]_1^{1+\log 2} = -\frac{1}{1+\log 2} + 1 = 1 - \frac{1}{1+\log 2} = \frac{1+\log 2 - 1}{1+\log 2} = \frac{\log 2}{1+\log 2}$

50. $\int_{\pi/6}^{\pi/2} \frac{\operatorname{cosec} x \cot x}{1 + \operatorname{cosec}^2 x} dx$

Sol. Let $I = \int_{\pi/6}^{\pi/2} \frac{\operatorname{cosec} x \cot x}{1 + \operatorname{cosec}^2 x} dx$, Put $t = \operatorname{cosec} x \Rightarrow dt = -\operatorname{cosec} x \cot x dx$

When $x = \frac{\pi}{2}, t = 1$; $x = \frac{\pi}{6}, t = 2$

$\therefore I = \int_2^1 \frac{-dt}{1+t^2} = -\left[\tan^{-1} t \right]_2^1 = -\left[\tan^{-1} 1 - \tan^{-1} 2 \right] = -\left[\frac{\pi}{4} - \tan^{-1} 2 \right] = \tan^{-1} 2 - \frac{\pi}{4}$

EXERCISE 16C (Pg.No.: 834)

Prove that

1. $\int_0^{\pi/2} \frac{\cos x}{(\sin x + \cos x)} dx = \frac{\pi}{4}$

Sol. L.H.S., $I = \int_0^{\pi/2} \frac{\cos x}{(\sin x + \cos x)} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x} dx \quad \dots(2)$$

By, Adding equation (1) and (2), then

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos x}{\sin x + \cos x} + \frac{\sin x}{\cos x + \sin x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos x + \sin x}{\sin x + \cos x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} dx$$

$$\Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \Rightarrow 2I = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4} = \text{R.H.S.}$$

2. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x} + \sqrt{\cos x})} dx = \frac{\pi}{4}$

Sol. L.H.S., $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x} + \sqrt{\cos x})} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \dots(2)$$

By, Adding equation (1) and (2), then

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx \Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \Rightarrow 2I = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4} = \text{R.H.S.}$$

3. (i) $\int_0^{\pi/2} \frac{\sin^3 x}{(\sin^3 x + \cos^3 x)} dx = \frac{\pi}{4}$ (ii) $\int_0^{\pi/2} \frac{\cos^3 x}{(\sin^3 x + \cos^3 x)} dx = \frac{\pi}{4}$

Sol. (i) L.H.S., $I = \int_0^{\pi/2} \frac{\sin^3 x}{(\sin^3 x + \cos^3 x)} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^3 x\left(\frac{\pi}{2} - x\right)}{\sin^3\left(\frac{\pi}{2} - x\right) + \cos^3\left(\frac{\pi}{2} - x\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x} dx \quad \dots(2)$$

By, Adding equation (1) and (2), then, $I_1 = \int_0^{\pi/2} \left(\frac{\sin^3 x}{\sin^3 x + \cos^3 x} + \frac{\cos^3 x}{\cos^3 x + \sin^3 x} \right) dx$

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin^3 x + \cos^3 x}{\sin^3 x + \cos^3 x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} dx \Rightarrow 2I = [x]_0^{\pi/2}$$

$$\Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \Rightarrow 2I = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

(ii) L.H.S, $I = \int_0^{\pi/2} \frac{\cos^3 x}{\sin^3 x + \cos^3 x} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^3 \left(\frac{\pi}{2} - x \right)}{\sin^3 \left(\frac{\pi}{2} - x \right) + \cos^3 \left(\frac{\pi}{2} - x \right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sin^3 x}{\cos^3 x + \sin^3 x} dx \quad \dots(2)$$

By, Adding equation (1) and (2), then

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos^3 x}{\sin^3 x + \cos^3 x} + \frac{\sin^3 x}{\cos^3 x + \sin^3 x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos^3 x + \sin^3 x}{\sin^3 x + \cos^3 x} \right) dx$$

$$\Rightarrow I = \int_0^{\pi/2} dx \Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \Rightarrow 2I = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

4. (i) $\int_0^{\pi/2} \frac{\sin^7 x}{\sin^7 x + \cos^7 x} dx = \frac{\pi}{4}$ (ii) $\int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx = \frac{\pi}{4}$

Sol. (i) L.H.S, $I = \int_0^{\pi/2} \frac{\sin^7 x}{\sin^7 x + \cos^7 x} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^7 \left(\frac{\pi}{2} - x \right)}{\sin^7 \left(\frac{\pi}{2} - x \right) + \cos^7 \left(\frac{\pi}{2} - x \right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\cos^7 x}{\cos^7 x + \sin^7 x} dx \quad \dots(2)$$

By, Adding equation (1) and (2), then

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin^7 x}{\sin^7 x + \cos^7 x} + \frac{\cos^7 x}{\cos^7 x + \sin^7 x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin^7 x + \cos^7 x}{\sin^7 x + \cos^7 x} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx \Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \Rightarrow 2I = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

(ii) L.H.S, $I = \int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^5 \left(\frac{\pi}{2} - x \right)}{\sin^5 \left(\frac{\pi}{2} - x \right) + \cos^5 \left(\frac{\pi}{2} - x \right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\cos^5 x}{\cos^5 x + \sin^5 x} dx \quad \dots(2)$$

By, Adding equation (1) and (2), then

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin^5 x}{\sin^5 x + \cos^5 x} + \frac{\cos^5 x}{\cos^5 x + \sin^5 x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx \Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0\right) \Rightarrow 2I = \frac{\pi}{2} \Rightarrow 2I = \frac{\pi}{2} \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

5. $\int_0^{\pi/2} \frac{\cos^4 x}{(\sin^4 x + \cos^4 x)} dx = \frac{\pi}{4}$

Sol. L.H.S, $I = \int_0^{\pi/2} \frac{\cos^4 x}{(\sin^4 x + \cos^4 x)} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^4 \left(\frac{\pi}{2} - x\right)}{\sin^4 \left(\frac{\pi}{2} - x\right) + \cos^4 \left(\frac{\pi}{2} - x\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sin^4 x}{\cos^4 x + \sin^4 x} dx \quad \dots(2)$$

Adding equation (1) and (2), then

$$2I = \int_0^{\pi/2} \left(\frac{\cos^4 x}{\sin^4 x + \cos^4 x} + \frac{\sin^4 x}{\cos^4 x + \sin^4 x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos^4 x + \sin^4 x}{\sin^4 x + \cos^4 x} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx \Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0\right) \Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4} = \text{R.H.S}$$

6. $\int_0^{\pi/2} \frac{\cos^{1/4} x}{(\sin^{1/4} x + \cos^{1/4} x)} dx = \frac{\pi}{4}$

Sol. L.H.S, $I = \int_0^{\pi/2} \frac{\cos^{1/4} x}{(\sin^{1/4} x + \cos^{1/4} x)} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^{1/4} \left(\frac{\pi}{2} - x\right)}{\sin^{1/4} \left(\frac{\pi}{2} - x\right) + \cos^{1/4} \left(\frac{\pi}{2} - x\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sin^{1/4} x}{(\cos^{1/4} x + \sin^{1/4} x)} dx \quad \dots(2)$$

Adding equation (1) and (2), then

$$2I = \int_0^{\pi/2} \left(\frac{\cos^{1/4} x}{\sin^{1/4} x + \cos^{1/4} x} + \frac{\sin^{1/4} x}{\cos^{1/4} x + \sin^{1/4} x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos^{1/4} x + \sin^{1/4} x}{\sin^{1/4} x + \cos^{1/4} x} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx \Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0\right) \Rightarrow 2I = \frac{\pi}{2} \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

7. $\int_0^{\pi/2} \frac{\sin^{3/2} x}{(\sin^{3/2} x + \cos^{3/2} x)} dx = \frac{\pi}{4}$

Sol. L.H.S, $I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{(\sin^{3/2} x + \cos^{3/2} x)} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^{3/2} \left(\frac{\pi}{2} - x\right)}{\sin^{3/2} \left(\frac{\pi}{2} - x\right) + \cos^{3/2} \left(\frac{\pi}{2} - x\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x} dx \quad \dots(2)$$

By, Adding equation (1) and (2), then

$$2I = \int_0^{\pi/2} \left(\frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} + \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin^{3/2} x + \cos^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx \Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \Rightarrow 2I = \frac{\pi}{2} \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

8. $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \frac{\pi}{4}$

Sol. L.H.S, $I = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^n \left(\frac{\pi}{2} - x \right)}{\sin^n \left(\frac{\pi}{2} - x \right) + \cos^n \left(\frac{\pi}{2} - x \right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x} dx \quad \dots(2)$$

By, Adding equation (1) and (2), then

$$2I = \int_0^{\pi/2} \left(\frac{\sin^n x}{\sin^n x + \cos^n x} + \frac{\cos^n x}{\cos^n x + \sin^n x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx \Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \Rightarrow 2I = \frac{\pi}{2} \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

9. $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx = \frac{\pi}{4}$

Sol. Let $I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\tan \left(\frac{\pi}{2} - x \right)}}{\sqrt{\tan \left(\frac{\pi}{2} - x \right)} + \sqrt{\cot \left(\frac{\pi}{2} - x \right)}} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx \quad \dots(2)$$

By, Adding (1) and (2), we get, $2I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx + \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{\sqrt{\tan x} + \sqrt{\cot x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx \Rightarrow 2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 \Rightarrow I = \frac{\pi}{4}$$

10. $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx = \frac{\pi}{4}$

Sol. Let $I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cot \left(\frac{\pi}{2} - x \right)}}{\sqrt{\tan \left(\frac{\pi}{2} - x \right)} + \sqrt{\cot \left(\frac{\pi}{2} - x \right)}} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx \quad \dots(2)$$

By, Adding (1) and (2), we get,

$$2I = \int_0^{\pi/2} \frac{\sqrt{\tan x} + \sqrt{\cot x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx \Rightarrow 2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

11. $\int_0^{\pi/2} \frac{1}{(1 + \tan x)} dx = \frac{\pi}{4}$

Sol. L.H.S, $I = \int_0^{\pi/2} \frac{1}{(1 + \tan x)} dx \Rightarrow I = \int_0^{\pi/2} \frac{1}{\left(1 + \frac{\sin x}{\cos x}\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx \quad \dots(2)$$

By, Adding equation (1) and (2), then

$$2I = \int_0^{\pi/2} \left(\frac{\cos x}{\cos x + \sin x} + \frac{\sin x}{\sin x + \cos x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos x + \sin x}{\cos x + \sin x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} dx$$

$$\Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \Rightarrow 2I = \frac{\pi}{2} \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

12. $\int_0^{\pi/2} \frac{1}{(1 + \cot x)} dx = \frac{\pi}{4}$

Sol. L.H.S, $I = \int_0^{\pi/2} \frac{1}{(1 + \cot x)} dx \Rightarrow I = \int_0^{\pi/2} \frac{1}{\left(1 + \frac{\cos x}{\sin x}\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \quad \dots(2)$$

By, Adding equation (1) and (2), then

$$2I = \int_0^{\pi/2} \left(\frac{\sin x}{\sin x + \cos x} + \frac{\cos x}{\cos x + \sin x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin x + \cos x}{\sin x + \cos x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} dx$$

$$\Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \Rightarrow 2I = \frac{\pi}{2} \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

13. $\int_0^{\pi/2} \frac{1}{(1 + \tan^3 x)} dx = \frac{\pi}{4}$

Sol. L.H.S, $I = \int_0^{\pi/2} \frac{1}{(1 + \tan^3 x)} dx \Rightarrow I = \int_0^{\pi/2} \frac{1}{\left(1 + \frac{\sin^3 x}{\cos^3 x}\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^3\left(\frac{\pi}{2} - x\right)}{\cos^3\left(\frac{\pi}{2} - x\right) + \sin^3\left(\frac{\pi}{2} - x\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx \quad \dots(2)$$

Adding equation (1) and (2), then

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos^3 x}{\cos^3 x + \sin^3 x} + \frac{\sin^3 x}{\sin^3 x + \cos^3 x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos^3 x + \sin^3 x}{\cos^3 x + \sin^3 x} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx \Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \Rightarrow 2I = \frac{\pi}{2} \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

14. $\int_0^{\pi/2} \frac{1}{(1 + \cot^3 x)} dx = \frac{\pi}{4}$

Sol. L.H.S, $I = \int_0^{\pi/2} \frac{1}{(1 + \cot^3 x)} dx \Rightarrow I = \int_0^{\pi/2} \frac{1}{\left(1 + \frac{\cos^3 x}{\sin^3 x}\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sin^3 x}{(\sin^3 x + \cos^3 x)} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^3 \left(\frac{\pi}{2} - x \right)}{\sin^3 \left(\frac{\pi}{2} - x \right) + \cos^3 \left(\frac{\pi}{2} - x \right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x} dx \quad \dots(2)$$

By, Adding equation (1) and (2), then

$$2I = \int_0^{\pi/2} \left(\frac{\sin^3 x}{\sin^3 x + \cos^3 x} + \frac{\cos^3 x}{\cos^3 x + \sin^3 x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin^3 x + \cos^3 x}{\sin^3 x + \cos^3 x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} dx$$

$$\Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \Rightarrow 2I = \frac{\pi}{2} \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

15. $\int_0^{\pi/2} \frac{dx}{(1 + \sqrt{\tan x})} = \frac{\pi}{4}$

Sol. L.H.S, $I = \int_0^{\pi/2} \frac{dx}{(1 + \sqrt{\tan x})} dx \Rightarrow I = \int_0^{\pi/2} \frac{1}{\left(1 + \frac{\sqrt{\sin x}}{\sqrt{\cos x}}\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos \left(\frac{\pi}{2} - x \right)}}{\sqrt{\cos \left(\frac{\pi}{2} - x \right)} + \sqrt{\sin \left(\frac{\pi}{2} - x \right)}} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \dots(2)$$

By, Adding equation (1) and (2), then

$$2I = \int_0^{\pi/2} \left(\frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} + \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx \Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \Rightarrow 2I = \frac{\pi}{2} \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

16. $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{1 + \sqrt{\cot x}} dx = \frac{\pi}{4}$

Sol. L.H.S, $I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{1 + \sqrt{\cot x}} dx \Rightarrow I = \int_0^{\pi/2} \frac{\left(\frac{\sqrt{\cos x}}{\sqrt{\sin x}} \right)}{\left(1 + \frac{\sqrt{\cos x}}{\sqrt{\sin x}} \right)} dx$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\left(\frac{\sqrt{\cos x}}{\sqrt{\sin x}} \right)}{\left(\frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x}} \right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)} + \sqrt{\cos\left(\frac{\pi}{2}-x\right)}} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \dots(2)$$

By, Adding equation (1) and (2), then

$$2I = \int_0^{\pi/2} \left(\frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} + \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx \Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \Rightarrow 2I = \frac{\pi}{2} \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

17. $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{1 + \sqrt{\tan x}} dx = \frac{\pi}{4}$

Sol. L.H.S, $I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{1 + \sqrt{\tan x}} dx \Rightarrow I = \int_0^{\pi/2} \frac{\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} \right)}{\left(1 + \frac{\sqrt{\sin x}}{\sqrt{\cos x}} \right)} dx$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} \right)}{\left(\frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x}} \right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)} + \sqrt{\sin\left(\frac{\pi}{2}-x\right)}} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \dots(2)$$

By, Adding equation (1) and (2), then

$$\therefore 2I = \int_0^{\pi/2} \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx \Rightarrow 2I = [x]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \Rightarrow 2I = \frac{\pi}{2} \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

18. $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx = 0$

Sol. L.H.S, $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$

... (1)

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1 + \sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx \quad \dots(2)$$

Adding equation (1) and (2), then $2I = \int_0^{\pi/2} \left(\frac{\sin x - \cos x}{1 + \sin x \cos x} + \frac{\cos x - \sin x}{1 + \cos x \sin x} \right) dx$

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin x - \cos x + \cos x - \sin x}{1 + \sin x \cos x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} 0 \Rightarrow 2I = 0 \quad \therefore I = 0 = \text{R.H.S}$$

19. $\int_0^1 x(1-x)^5 dx = \frac{1}{42}$

Sol. Let $I = \int_0^1 x(1-x)^5 dx = \int_0^1 (1-x) \{1-(1-x)\}^5 dx = \int_0^1 (1-x)(x)^5 dx = \int_0^1 (x^5 - x^6) dx$

$$= \left[\frac{x^6}{6} - \frac{x^7}{7} \right]_0^1 = \frac{1}{6} - \frac{1}{7} = 0 = \frac{7-6}{42} = \frac{1}{42}$$

20. $\int_0^2 x\sqrt{2-x} dx = \frac{16\sqrt{2}}{15}$

Sol. Let $I = \int_0^2 x\sqrt{2-x} dx = \int_0^2 (2-x)\sqrt{2-(2-x)} dx = \int_0^2 (2-x)x^{1/2} dx = \int_0^2 (2x^{1/2} - x^{3/2}) dx$

$$= \left[2 \frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2} \right]_0^2 = \left[\frac{4}{3} x^{3/2} - \frac{2}{5} x^{5/2} \right]_0^2 = \frac{4}{3} 2^{3/2} - \frac{2}{5} 2^{5/2} = \frac{4}{3} 2\sqrt{2} - \frac{2}{5} 4\sqrt{2}$$

$$= 8\sqrt{2} \left(\frac{1}{3} - \frac{1}{5} \right) = 8\sqrt{2} \frac{2}{15} = \frac{16\sqrt{2}}{15}$$

21. $\int_0^{\pi} x \cos^2 x dx = \frac{\pi^2}{4}$

Sol. Let $I = \int_0^{\pi} x \cos^2 x dx \quad \dots(1)$

Then, $I = \int_0^{\pi} (\pi-x) \cos^2 (\pi-x) dx$ or $I = \int_0^{\pi} (\pi-x) \cos^2 x dx \quad \dots(2)$

By, Adding (1) and (2) we get, $2I = \int_0^{\pi} \{x \cos^2 x + (\pi-x) \cos^2 x\} dx \Rightarrow 2I = \int_0^{\pi} \pi \cos^2 x dx$

$$\Rightarrow 2I = \pi \int_0^{\pi} \cos^2 x dx \Rightarrow 2I = \pi \int_0^{\pi} \left(\frac{1 + \cos 2x}{2} \right) dx \Rightarrow 2I = \frac{\pi}{2} \left[x + \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$\Rightarrow 2I = \frac{\pi}{2} \left[\left\{ \pi + \frac{\sin 2\pi}{2} \right\} - (0) \right] \Rightarrow 2I = \frac{\pi}{2} [\pi + 0] \Rightarrow 2I = \frac{\pi^2}{2} \quad \therefore I = \frac{\pi^2}{4}$$

22. $\int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x} dx = \frac{\pi^2}{4}$

Sol. Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x} dx \quad \dots(1)$

$$\text{Then, } I = \int_0^{\pi} \frac{(\pi-x) \tan(\pi-x)}{\sec(\pi-x) \operatorname{cosec}(\pi-x)} dx \text{ or } I = \int_0^{\pi} \frac{(\pi-x) \tan x}{\sec x \operatorname{cosec} x} dx \quad \dots (2)$$

Adding (1) and (2) we get

$$2I = \int_0^{\pi} \left[\frac{x \tan x}{\sec x \operatorname{cosec} x} + \frac{(\pi-x) \tan x}{\sec x \operatorname{cosec} x} \right] dx \Rightarrow 2I = \pi \int_0^{\pi} \sin^2 x \, dx \Rightarrow 2I = \pi \int_0^{\pi} \frac{1 - \cos 2x}{2} dx$$

$$\Rightarrow 2I = \frac{\pi}{2} \left[x - \frac{\sin 2x}{2} \right]_0^{\pi} \Rightarrow 2I = \frac{\pi}{2} \left[\left\{ \pi - \frac{\sin 2\pi}{2} \right\} - \{0\} \right] \Rightarrow 2I = \frac{\pi}{2} [\pi - 0] \therefore I = \frac{\pi^2}{4}$$

$$23. \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$$

$$\text{Sol. Let } I = \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x} dx \quad 1)$$

$$\text{Then, } I = \int_0^{\pi/2} \frac{\cos^2 \left(\frac{\pi}{2} - x \right)}{\sin \left(\frac{\pi}{2} - x \right) + \cos \left(\frac{\pi}{2} - x \right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx \quad \dots (2)$$

$$\text{By, Adding (1) \& (2) we get, } 2I = \int_0^{\pi/2} \left(\frac{\cos^2 x}{\sin x + \cos x} + \frac{\sin^2 x}{\cos x + \sin x} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos^2 x + \sin^2 x}{\sin x + \cos x} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx \Rightarrow 2I = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{1}{\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x} dx$$

$$\Rightarrow 2I = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\cos \frac{\pi}{4} \sin x + \sin \frac{\pi}{4} \cos x} = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\sin \left(x + \frac{\pi}{4} \right)}$$

$$\Rightarrow 2I = \frac{1}{\sqrt{2}} \left[\log \left| \operatorname{cosec} \left(\frac{\pi}{2} + \frac{\pi}{4} \right) - \cot \left(\frac{\pi}{2} + \frac{\pi}{4} \right) \right| - \log \left| \operatorname{cosec} \left(0 + \frac{\pi}{4} \right) - \cot \left(0 + \frac{\pi}{4} \right) \right| \right]$$

$$\Rightarrow 2I = \frac{1}{\sqrt{2}} \left[\log \left| \operatorname{cosec} \frac{3\pi}{4} - \cot \frac{3\pi}{4} \right| - \log \left| \operatorname{cosec} \frac{\pi}{4} - \cot \frac{\pi}{4} \right| \right]$$

$$\Rightarrow 2I = \frac{1}{\sqrt{2}} \left[\log(\sqrt{2} + 1) - \log(\sqrt{2} - 1) \right] \Rightarrow 2I = \frac{1}{\sqrt{2}} \log \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)$$

$$\Rightarrow 2I = \frac{1}{\sqrt{2}} \log \left[\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1} \right] \Rightarrow 2I = \frac{1}{\sqrt{2}} \log \left[\frac{(\sqrt{2} + 1)^2}{2 - 1} \right]$$

$$\Rightarrow 2I = \frac{2}{\sqrt{2}} \log(\sqrt{2} + 1) \therefore I = \frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$$

$$24. \int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx = \frac{\pi^2}{4}$$

Sol. Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx \quad \dots(1)$

Then, $I = \int_0^{\pi} \frac{(\pi-x) \tan(\pi-x)}{\sec(\pi-x) + \cos(\pi-x)} dx$ or $I = \int_0^{\pi} \frac{(\pi-x) \tan x}{\sec x + \cos x} dx \quad \dots(2)$

Adding (1) and (2) we get, $2I = \int_0^{\pi} \left[\frac{x \tan x}{\sec x + \cos x} + \frac{(\pi-x) \tan x}{\sec x + \cos x} \right] dx$

$\Rightarrow 2I = \int_0^{\pi} \frac{x \tan x + \pi \tan x - x \tan x}{\sec x + \cos x} dx \Rightarrow 2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \cos x} dx \Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$

Put $\cos x = t \Rightarrow -\sin x dx = dt \Rightarrow \sin x dx = -dt$

Clearly $x=0, t=1$ and $x=\pi, t=-1$

$2I = \pi \int_1^{-1} \frac{-dt}{1+t^2} \Rightarrow 2I = \pi \int_{-1}^1 \frac{1}{1+t^2} dt \Rightarrow 2I = \pi [\tan^{-1}(t)]_{-1}^1 \Rightarrow 2I = \pi [\tan^{-1}(1) - \tan^{-1}(-1)]$

$\Rightarrow 2I = \pi [\tan^{-1}(1) + \tan^{-1}(1)] \Rightarrow 2I = \pi [2 \tan^{-1}(1)] \therefore I = \frac{\pi^2}{4}$

25. $\int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx = \pi \left(\frac{\pi}{2} - 1 \right)$

Sol. Let $I = \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx \quad \dots(1)$

Then $I = \int_0^{\pi} \frac{(\pi-x) \sin(\pi-x)}{1 + \sin(\pi-x)} dx$ or $I = \int_0^{\pi} \frac{(\pi-x) \sin x}{1 + \sin x} dx \quad \dots(2)$

Adding (1) and (2) we get

$2I = \int_0^{\pi} \left[\frac{x \sin x}{1 + \sin x} + \frac{(\pi-x) \sin x}{1 + \sin x} \right] dx \Rightarrow 2I = \int_0^{\pi} \frac{x \sin x + \pi \sin x - x \sin x}{1 + \sin x} dx$

$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx \Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x} \times \frac{1 - \sin x}{1 - \sin x} dx \Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x (1 - \sin x)}{1 - \sin^2 x} dx$

$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x (1 - \sin x)}{\cos^2 x} dx \Rightarrow 2I = \pi \left[\int_0^{\pi} \tan x \sec x dx - \int_0^{\pi} \tan^2 x dx \right]$

$\Rightarrow 2I = \pi \left[[\sec x]_0^{\pi} - \int_0^{\pi} (\sec^2 x - 1) dx \right] \Rightarrow 2I = \pi [\sec \pi - \tan \pi + \pi] - [\sec 0 - \tan 0 + 0]$

$\Rightarrow 2I = \pi [-1 + \pi - 1] \Rightarrow 2I = \pi (\pi - 2) \Rightarrow I = \frac{\pi}{2} (\pi - 2) \therefore I = \pi \left(\frac{\pi}{2} - 1 \right)$

26. $\int_0^{\pi} \frac{x}{1 + \sin^2 x} dx = \frac{\pi^2}{2\sqrt{2}}$

Sol. Let $I = \int_0^{\pi} \frac{x}{1 + \sin^2 x} dx \quad \dots(1)$

Then $I = \int_0^{\pi} \frac{\pi-x}{1 + \sin^2(\pi-x)} dx$ or $I = \int_0^{\pi} \frac{\pi-x}{1 + \sin^2 x} dx \quad \dots(2)$

Adding (1) and (2) we get,

$$2I = \int_0^{\pi} \left\{ \frac{x}{1+\sin^2 x} + \frac{\pi-x}{1+\sin^2 x} \right\} dx \Rightarrow 2I = \int_0^{\pi} \left(\frac{x+\pi-x}{1+\sin^2 x} \right) dx \Rightarrow 2I = \pi \int_0^{\pi} \frac{1}{1+\sin^2 x} dx$$

$$\Rightarrow 2I = 2\pi \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin^2 x} \cdot dx \Rightarrow I = \pi \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin^2 x} \cdot dx = \pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\sec^2 x + \tan^2 x} dx$$

$$\Rightarrow I = \pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x dx}{1+2\tan^2 x} \text{ put } \tan x = t \Rightarrow \sec^2 x \cdot dx = dt$$

$$\Rightarrow I = \pi \int_0^{\infty} \frac{dt}{1+\sqrt{2}t^2} = \pi \left[\frac{\tan^{-1}(\sqrt{2}t)}{\sqrt{2}} \right]_0^{\infty} = \frac{\pi}{\sqrt{2}} \left[\tan^{-1} \infty - \tan^{-1} 0 \right] = \frac{\pi}{\sqrt{2}} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi^2}{2\sqrt{2}}$$

$$27. \int_0^{\pi/2} (2 \log \cos x - \log \sin 2x) dx = -\frac{\pi}{2} (\log 2)$$

$$\text{Sol. Let } I = \int_0^{\pi/2} \{2 \log(\cos x) - \log(\sin 2x)\} dx \Rightarrow I = \int_0^{\pi/2} \{2 \log \cos x - \log(2 \sin x \cos x)\} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \{2 \log \cos x - \log 2 - \log \sin x - \log \cos x\} dx$$

$$\Rightarrow I = \int_0^{\pi/2} (\log \cos x - \log \sin x - \log 2) dx \quad \dots (1)$$

$$\Rightarrow I = \int_0^{\pi/2} (\log \sin x - \log \cos x - \log 2) dx \quad \dots (2)$$

$$\text{Adding (1) and (2), we get, } 2I = \int_0^{\pi/2} -2 \log 2 dx = -2 \log 2 [x]_0^{\pi/2} = -2 \cdot \frac{\pi}{2} \log 2$$

$$\Rightarrow 2I = -\pi \log 2 \Rightarrow I = -\frac{\pi}{2} \log 2$$

$$28. \int_0^{\infty} \frac{x}{(1+x)(1+x^2)} dx = \frac{\pi}{4}$$

$$\text{Sol. Let } I = \int_0^{\infty} \frac{x}{(1+x)(1+x^2)} dx$$

$$\text{Put } x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta. \text{ Clearly } x=0, \theta=0 \text{ and } x=\infty, \theta=\frac{\pi}{2}.$$

$$I = \int_0^{\pi/2} \frac{\tan \theta \sec^2 \theta d\theta}{(1+\tan \theta)(1+\tan^2 \theta)} \Rightarrow I = \int_0^{\pi/2} \frac{\tan \theta}{1+\tan \theta} d\theta \Rightarrow I = \int_0^{\pi/2} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta \quad \dots (1)$$

$$\text{Then, } I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right) + \cos\left(\frac{\pi}{2}-\theta\right)} d\theta \text{ or } I = \int_0^{\pi/2} \frac{\cos \theta}{\cos \theta + \sin \theta} d\theta \quad \dots (2)$$

By, Adding (1) and (2) we get, $2I = \int_0^{\pi/2} \left(\frac{\sin \theta}{\sin \theta + \cos \theta} + \frac{\cos \theta}{\sin \theta + \cos \theta} \right) d\theta$

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin \theta + \cos \theta}{\sin \theta + \cos \theta} \right) d\theta \Rightarrow 2I = \int_0^{\pi/2} d\theta \Rightarrow 2I = [\theta]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \therefore I = \frac{\pi}{4}$$

29. $\int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}} = \frac{\pi}{4}$

Sol. Let $I = \int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}}$

Let $x = a \sin \theta$, $dx = a \cos \theta d\theta$. Clearly $x = 0$, $\theta = 0$, and $x = a$, $\theta = \frac{\pi}{2}$.

$$I = \int_0^{\pi/2} \frac{a \cos \theta d\theta}{a \sin \theta + \sqrt{a^2 - a^2 \sin^2 \theta}} \Rightarrow I = \int_0^{\pi/2} \frac{a \cos \theta}{a \sin \theta + \sqrt{a^2 (1 - \sin^2 \theta)}} d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{a \cos \theta}{a \sin \theta + a \cos \theta} d\theta \Rightarrow I = \int_0^{\pi/2} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta \quad \dots(1)$$

$$\text{Then } I = \int_0^{\pi/2} \frac{\cos \left(\frac{\pi}{2} - \theta \right)}{\sin \left(\frac{\pi}{2} - \theta \right) + \cos \left(\frac{\pi}{2} - \theta \right)} d\theta \text{ or } I = \int_0^{\pi/2} \frac{\sin \theta}{\cos \theta + \sin \theta} d\theta \quad \dots(2)$$

By, Adding (1) and (2) we get, $2I = \int_0^{\pi/2} \left(\frac{\cos \theta}{\sin \theta + \cos \theta} + \frac{\sin \theta}{\cos \theta + \sin \theta} \right) d\theta$

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos \theta + \sin \theta}{\sin \theta + \cos \theta} \right) d\theta \Rightarrow 2I = \int_0^{\pi/2} d\theta \Rightarrow 2I = [\theta]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \therefore I = \frac{\pi}{4}$$

30. $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx = \frac{a}{2}$

Sol. Let $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx \quad \dots(1)$

$$\Rightarrow I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{a-(a-x)}} dx \Rightarrow I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx \quad \dots(2)$$

By, Adding (1) and (2) we get, $2I = \int_0^a \left(\frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} + \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} \right) dx$

$$\Rightarrow 2I = \int_0^a \left[\frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} \right] dx \Rightarrow 2I = \int_0^a dx \Rightarrow 2I = [x]_0^a \Rightarrow 2I = a \therefore I = \frac{a}{2}$$

31. $\int_0^{\pi} \sin^2 x \cos^3 x dx = 0$

Sol. Let $I = \int_0^{\pi} \sin^2 x \cos^3 x dx \quad \dots(1)$

$$\text{Then } I = \int_0^{\pi} \sin^2(\pi - x) \cos^3(\pi - x) dx \Rightarrow I = - \int_0^{\pi} \sin^2 x \cos^3 x dx \quad \dots(2)$$

$$\text{By, Adding (1) and (2) we get, } 2I = \int_0^{\pi} (\sin^2 x \cos^3 x - \sin^2 x \cos^3 x) dx \Rightarrow 2I = 0 \therefore I = 0$$

$$32. \int_0^{\pi} \sin^{2m} x \cos^{2m+1} x dx = 0, \text{ where } m \text{ is a positive integer}$$

$$\text{Sol. Let } I = \int_0^{\pi} \sin^{2m} x \cos^{2m+1} x dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^{\pi} \sin^{2m} x (\pi - x) \cos^{2m+1}(\pi - x) dx$$

$$\text{When } m \text{ is positive integer then, } I = - \int_0^{\pi} \sin^{2m} x \cos^{2m+1} x dx \quad \dots(2)$$

$$\text{By, Adding (1) and (2) we get, } 2I = \int_0^{\pi} (\sin^{2m} x \cos^{2m+1} x - \sin^{2m} x \cos^{2m+1} x) dx \Rightarrow 2I = 0$$

$$\therefore I = 0$$

$$33. \int (\sin x - \cos x) \log(\sin x + \cos x) dx = 0$$

$$\text{Sol. Let } I = \int_0^{\pi/2} (\sin x - \cos x) \log(\sin x + \cos x) dx \quad \dots(1)$$

$$\text{Then } I = \int \left\{ \sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right) \right\} \log \left\{ \sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right) \right\}$$

$$\text{or } I = \int_0^{\pi/2} (\cos x - \sin x) \log(\cos x + \sin x) dx \quad \dots(2)$$

By, Adding (1) and (2) we get

$$2I = \int_0^{\pi/2} \{ (\sin x - \cos x) \log(\sin x + \cos x) + (\cos x - \sin x) \log(\cos x + \sin x) \} dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log(\sin x + \cos x) \{ \sin x - \cos x + \cos x - \sin x \} dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log(\sin x + \cos x) \cdot 0 dx \Rightarrow 2I = 0 \therefore I = 0$$

$$34. \int_0^{\pi/2} \log(\sin 2x) dx$$

$$\text{Sol. Let } I = \int_0^{\pi/2} \log(\sin 2x) dx$$

$$\text{Put } 2x = t \Rightarrow 2 dx = dt \Rightarrow dx = \frac{dt}{2}. \text{ Clearly } x = 0, t = 0 \text{ and } x = \frac{\pi}{2}, t = \pi.$$

$$I = \int_0^{\pi} \log(\sin t) \cdot \frac{dt}{2} \Rightarrow I = \frac{1}{2} \int_0^{\pi} \log(\sin t) dt$$

$$\Rightarrow I = \frac{1}{2} \times 2 \cdot \int_0^{\pi/2} \log(\sin t) dt \Rightarrow I = \int_0^{\pi/2} \log(\sin t) dt \quad \dots(1)$$

$$\text{Then } I = \int_0^{\pi/2} \log\left\{\sin\left(\frac{\pi}{2}-t\right)\right\} dt \text{ or } I = \int_0^{\pi/2} \log(\cos t) dt \quad \dots(2)$$

Adding (1) and (2) we get,

$$2I = \int_0^{\pi/2} (\log \sin t + \log \cos t) dt = \int_0^{\pi/2} \log(\sin t \cos t) dt = \int_0^{\pi/2} \log\left(\frac{\sin 2t}{2}\right) dt$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log(\sin 2t) dt - \int_0^{\pi/2} (\log 2) dt$$

$$\Rightarrow 2I = \frac{1}{2} \int_0^{\pi} \log \sin t dt - \log 2 \int_0^{\pi/2} dt \quad [\text{Putting } 2x = t \text{ in the 1st integral}]$$

$$\Rightarrow 2I = \frac{1}{2} \int_0^{\pi} \log \sin t dt - \log 2 [t]_0^{\pi/2} \Rightarrow 2I = \int_0^{\pi/2} \log(\sin t) dt - \frac{\pi}{2} \log 2$$

$$\Rightarrow 2I = I - \frac{\pi}{2} \log 2 \quad \therefore I = -\frac{\pi}{2} \log 2$$

$$35. \int_0^{\pi} x \log(\sin x) dx = -\frac{\pi^2}{2} \log 2$$

$$\text{Sol. Let } I = \int_0^{\pi} x \log(\sin x) dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^{\pi} (\pi - x) \log \sin(\pi - x) dx \Rightarrow I = \int_0^{\pi} (\pi - x) \log \sin x dx \quad \dots(2)$$

Adding (1) and (2), we get,

$$2I = \int_0^{\pi} x \log \sin x dx + \int_0^{\pi} (\pi - x) \log \sin x dx = \int_0^{\pi} (x + \pi - x) \log \sin x dx = \int_0^{\pi} \pi \log \sin x dx$$

$$\Rightarrow 2I = -2\pi \int_0^{\pi/2} \log \sin x dx \Rightarrow I = \pi \int_0^{\pi/2} \log \sin x dx \quad \dots(3)$$

$$\Rightarrow I = \pi \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx \Rightarrow I = \pi \int_0^{\pi/2} \log \cos x dx \quad \dots(4)$$

By, Adding equation (3) and (4), we have

$$2I = \pi \int_0^{\pi/2} (\log \sin x + \log \cos x) dx = \pi \int_0^{\pi/2} \log \sin x \cos x dx = \pi \int_0^{\pi/2} \log \frac{2 \sin x \cos x}{2} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi/2} \log \frac{\sin 2x}{2} dx = \pi \int_0^{\pi/2} (\log \sin 2x - \log 2) dx = \pi \int_0^{\pi/2} \log \sin 2x dx - \pi \int_0^{\pi/2} \log 2 dx$$

$$\text{Put } t = 2x \Rightarrow \frac{dt}{2} = dx. \text{ When } x = 0, t = 0 \text{ and when } x = \frac{\pi}{2}, t = \pi$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \log \sin t dt / 2 - \pi \log 2 [x]_0^{\pi/2}$$

$$\Rightarrow 2I = \frac{\pi}{2} \times 2 \int_0^{\pi/2} \log \sin t \, dt - \pi \log 2 \left[\frac{\pi}{2} - 0 \right] = \pi \int_0^{\pi/2} \log \sin x \, dx - \frac{\pi^2}{2} \log 2 \left[\because \int_a^b f(x) \, dx = \int_a^b f(t) \, dt \right]$$

$$\Rightarrow 2I = I - \frac{\pi^2}{2} \log 2 \quad [\text{From (3)}] \Rightarrow I = -\frac{\pi^2}{2} \log 2$$

36. $\int_0^{\pi} \log(1 + \cos x) \, dx = -\pi(\log 2)$

Sol. Let $I = \int_0^{\pi} \log(1 + \cos x) \, dx \quad \dots(1)$

$$\Rightarrow I = \int_0^{\pi} \log\{1 + \cos(\pi - x)\} \, dx \Rightarrow I = \int_0^{\pi} \log(1 - \cos x) \, dx \quad \dots(2)$$

By, Adding (1) and (2) we get, $2I = \int_0^{\pi} \{\log(1 + \cos x) + \log(1 - \cos x)\} \, dx$

$$\Rightarrow 2I = \int_0^{\pi} \log\{(1 + \cos x)(1 - \cos x)\} \, dx \Rightarrow 2I = \int_0^{\pi} \log(1 - \cos^2 x) \, dx$$

$$\Rightarrow 2I = \int_0^{\pi} \log(\sin^2 x) \, dx \Rightarrow 2I = 2 \int_0^{\pi} \log(\sin x) \, dx \Rightarrow I = \int_0^{\pi} \log(\sin x) \, dx$$

$$\Rightarrow I = 2 \int_0^{\pi/2} \log(\sin x) \, dx \Rightarrow I = 2 \int_0^{\pi/2} \log(\sin x) \, dx \Rightarrow I = 2I_1 \quad \dots(3)$$

$$\Rightarrow I_1 = \int_0^{\pi/2} \log(\sin x) \, dx \quad \dots(4)$$

Then $I_1 = \int_0^{\pi/2} \log\left\{\sin\left(\frac{\pi}{2} - x\right)\right\} \, dx$ or $I_1 = \int_0^{\pi/2} \log(\cos x) \, dx \quad \dots(5)$

By, Adding (4) and (5) we get, $2I_1 = \int_0^{\pi/2} \{\log(\sin x) + \log(\cos x)\} \, dx$

$$\Rightarrow 2I_1 = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) \, dx \Rightarrow 2I_1 = \int_0^{\pi/2} \{\log(\sin 2x) - \log 2\} \, dx$$

$$\Rightarrow 2I_1 = \frac{1}{2} \int_0^{\pi} \log \sin t \, dt - (\log 2) \int_0^{\pi/2} dx \quad [\text{Putting } 2x = t \text{ in the 1st integral}]$$

$$\Rightarrow 2I_1 = \frac{1}{2} \int_0^{\pi} \log(\sin t) \, dt - \log 2 [x]_0^{\pi/2} \Rightarrow 2I_1 = \left(\frac{1}{2} \times 2\right) \int_0^{\pi/2} \log(\sin t) \, dt - \log 2 \left(\frac{\pi}{2} - 0\right)$$

$$\Rightarrow 2I_1 = \int_0^{\pi/2} \log(\sin t) \, dt - \frac{\pi}{2} \log 2 \Rightarrow 2I_1 = \int_0^{\pi/2} \log(\sin x) \, dx - \frac{\pi}{2} \log 2 \left[\because \int_a^b f(x) \, dx = \int_a^b f(t) \, dt \right]$$

$$\Rightarrow 2I_1 = I_1 - \frac{\pi}{2} \log 2 \quad \therefore I_1 = -\frac{\pi}{2} \log 2$$

Putting the value of I_1 in equation (3), $I = 2\left(-\frac{\pi}{2} \log 2\right) \therefore I = -\pi \log 2$

37. $\int \log(\tan x + \cot x) \, dx = \pi(\log 2)$

Sol. Let $I = \int_0^{\pi/2} \log(\tan x + \cot x) dx \Rightarrow I = \int_0^{\pi/2} \log\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right) dx \Rightarrow I = \int_0^{\pi/2} \log\left(\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right) dx$

$$\Rightarrow I = \int_0^{\pi/2} \log\left(\frac{1}{\sin x \cos x}\right) dx \Rightarrow I = \int_0^{\pi/2} \log\left(\frac{1}{\sin x \cos x}\right) dx \Rightarrow I = \int_0^{\pi/2} \log\left(\frac{2}{2 \sin x \cos x}\right) dx$$

$$\Rightarrow I = \int_0^{\pi/2} \log\left(\frac{2}{\sin 2x}\right) dx \Rightarrow I = \int_0^{\pi/2} \log 2 dx - \int_0^{\pi/2} \log(\sin 2x) dx \Rightarrow I = \log 2 \int_0^{\pi/2} dx - I_1$$

$$\Rightarrow I = \log 2 [x]_0^{\pi/2} - I_1 \Rightarrow I = \log 2 \left(\frac{\pi}{2} - 0\right) - I_1 \Rightarrow I = \log 2 \left(\frac{\pi}{2} - 0\right) - I_1$$

$$\Rightarrow I = \frac{\pi}{2} \log 2 - I_1 \quad \dots(1)$$

Now, $I_1 = \int_0^{\pi/2} \log(\sin 2x) dx$

Put $2x = t \Rightarrow 2dx = dt \Rightarrow dx = \frac{dt}{2}$. Clearly $x = 0, t = 0$ and $x = \frac{\pi}{2}, t = \pi$

$$I_1 = \int_0^{\pi} \log(\sin t) \cdot \frac{dt}{2} \Rightarrow I_1 = \frac{1}{2} \times 2 \int_0^{\pi} \log(\sin t) dt \Rightarrow I = \int_0^{\pi} \log(\sin t) dt \quad \dots(2)$$

Then $I_1 = \int_0^{\pi/2} \log\left\{\sin\left(\frac{\pi}{2} - t\right)\right\} dt$ or $I_1 = \int_0^{\pi/2} \log(\cos t) dt \quad \dots(3)$

By, Adding (2) and (3) we get, $2I_1 = \int_0^{\pi/2} (\log \sin t + \log \cos t) dt \Rightarrow 2I_1 = \int_0^{\pi/2} \log(\sin t \cdot \cos t) dt$

$$\Rightarrow 2I_1 = \int_0^{\pi/2} \log\left(\frac{\sin 2t}{2}\right) dt \Rightarrow 2I_1 = \int_0^{\pi/2} \log(\sin 2t) dt - \int_0^{\pi/2} (\log 2) dt$$

$$\Rightarrow 2I_1 = \frac{1}{2} \int_0^{\pi} \log(\sin t) dt - (\log 2) \int_0^{\pi/2} dt \quad [\text{Putting } 2x = t \text{ in the 1st integral}]$$

$$\Rightarrow 2I_1 = \frac{1}{2} \int_0^{\pi} \log \sin t dt - \log 2 [t]_0^{\pi/2} \Rightarrow 2I_1 = \frac{1}{2} \times 2 \int_0^{\pi/2} \log(\sin t) dt - \log 2 \left(\frac{\pi}{2} - 0\right)$$

$$\Rightarrow 2I_1 = \int_0^{\pi/2} \log(\sin t) dt - \frac{\pi}{2} \log 2 \Rightarrow 2I_1 = I_1 - \frac{\pi}{2} \log 2 \Rightarrow I_1 = -\frac{\pi}{2} \log 2$$

Putting the value of I_1 in equation (1), $I = \frac{\pi}{2} \log 2 + \frac{\pi}{2} \log 2 \therefore I = \pi \log 2$

38. $\int_{\pi/8}^{3\pi/8} \frac{\cos x}{\cos x + \sin x} dx = \frac{\pi}{4}$

Sol. Let $I = \int_{\pi/8}^{3\pi/8} \frac{\cos x}{\cos x + \sin x} dx \quad \dots(1)$

$$\Rightarrow I = \int_{\pi/8}^{3\pi/8} \frac{\cos\left\{\left(\frac{\pi}{8} + \frac{3\pi}{8}\right) - x\right\}}{\cos\left\{\left(\frac{\pi}{8} + \frac{3\pi}{8}\right) - x\right\} + \sin\left\{\left(\frac{\pi}{8} + \frac{3\pi}{8}\right) - x\right\}} dx$$

$$\Rightarrow I = \int_{\pi/8}^{3\pi/8} \frac{\cos\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right) + \sin\left(\frac{\pi}{2}-x\right)} dx \Rightarrow I = \int_{\pi/8}^{3\pi/8} \frac{\sin x}{\sin x + \cos x} dx \quad \dots(2)$$

$$\text{By, Adding (1) and (2) we get, } 2I = \int_{\pi/8}^{3\pi/8} \left(\frac{\cos x}{\cos x + \sin x} + \frac{\sin x}{\sin x + \cos x} \right) dx \Rightarrow 2I = \int_{\pi/8}^{3\pi/8} \frac{\cos x + \sin x}{\cos x + \sin x} dx$$

$$\Rightarrow 2I = \int_{\pi/8}^{3\pi/8} dx \Rightarrow 2I = [x]_{\pi/8}^{3\pi/8} \Rightarrow 2I = \left(\frac{3\pi}{8} - \frac{\pi}{8} \right) \Rightarrow 2I = \frac{2\pi}{8} \therefore I = \frac{\pi}{8}$$

39. $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan x}} dx = \frac{\pi}{12}$

Sol. Let $I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan x}} dx \Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}} dx \Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \dots(1)$

$$\text{Then, } I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left\{\left(\frac{\pi}{6}+\frac{\pi}{3}\right)-x\right\}}}{\sqrt{\cos\left\{\left(\frac{\pi}{6}+\frac{\pi}{3}\right)-x\right\}} + \sqrt{\sin\left\{\left(\frac{\pi}{6}+\frac{\pi}{3}\right)-x\right\}}} dx$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)} + \sqrt{\sin\left(\frac{\pi}{2}-x\right)}} dx \Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \dots(2)$$

$$\text{By, Adding (1) and (2) we get, } 2I = \int_{\pi/6}^{\pi/3} \left(\frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} + \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \right) dx$$

$$\Rightarrow 2I = \int_{\pi/6}^{\pi/3} \left(\frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} \right) dx \Rightarrow 2I = \int_{\pi/6}^{\pi/3} dx \Rightarrow 2I = [x]_{\pi/6}^{\pi/3}$$

$$\Rightarrow 2I = \left(\frac{\pi}{3} - \frac{\pi}{6} \right) \Rightarrow 2I = \frac{2\pi - \pi}{6} \Rightarrow 2I = \frac{\pi}{6} \therefore I = \frac{\pi}{12}$$

40. $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x} = 2$

Sol. Let $I = \int_{\pi/4}^{3\pi/4} \frac{1}{1+\cos x} dx \Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{1}{1+\cos x} \times \frac{1-\cos x}{1-\cos x} dx \Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{1-\cos x}{1-\cos^2 x} dx$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{1-\cos x}{\sin^2 x} dx \Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{1-\cos x}{\sin^2 x} dx \Rightarrow I = \int_{\pi/4}^{3\pi/4} (\operatorname{cosec}^2 x - \cot x \operatorname{cosec} x) dx$$

$$\Rightarrow I = [-\cot x + \operatorname{cosec} x]_{\pi/4}^{3\pi/4} \Rightarrow I = [\operatorname{cosec} x - \cot x]_{\pi/4}^{3\pi/4}$$

$$\Rightarrow I = \left[\left\{ \operatorname{cosec}\left(\frac{3\pi}{4}\right) - \cot\left(\frac{3\pi}{4}\right) \right\} - \left\{ \operatorname{cosec}\left(\frac{\pi}{4}\right) - \cot\left(\frac{\pi}{4}\right) \right\} \right]$$

$$\Rightarrow I = \{\sqrt{2} - (-1)\} - \{\sqrt{2} - 1\} \Rightarrow I = \sqrt{2} + 1 - \sqrt{2} + 1 \therefore I = 2$$

41. $\int_{\pi/4}^{3\pi/4} \frac{x}{1+\sin x} dx = \pi(\sqrt{2}-1)$

Sol. Let $I = \int_{\pi/4}^{3\pi/4} \frac{x}{1+\sin x} dx \quad \dots(1)$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{\left(\frac{\pi}{4} + \frac{3\pi}{4}\right) - x}{1 + \sin\left\{\left(\frac{\pi}{4} + \frac{3\pi}{4}\right) - x\right\}} dx \Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{\pi - x}{1 + \sin(\pi - x)} dx$$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{\pi - x}{1 + \sin x} dx \quad \dots(2)$$

Adding (1) and (2) we get, $2I = \int_{\pi/4}^{3\pi/4} \left(\frac{x}{1+\sin x} + \frac{\pi-x}{1+\sin x} \right) dx \Rightarrow 2I = \int_{\pi/4}^{3\pi/4} \left(\frac{x+\pi-x}{1+\sin x} \right) dx$

$$\Rightarrow 2I = \pi \int_{\pi/4}^{3\pi/4} \frac{1}{1+\sin x} dx \Rightarrow 2I = \pi \int_{\pi/4}^{3\pi/4} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx \Rightarrow 2I = \pi \int_{\pi/4}^{3\pi/4} \frac{1-\sin x}{1-\sin^2 x} dx$$

$$\Rightarrow 2I = \pi \int_{\pi/4}^{3\pi/4} \frac{1-\sin x}{\cos^2 x} dx \Rightarrow 2I = \pi \int_{\pi/4}^{3\pi/4} (\sec^2 x - \tan x \sec x) dx \Rightarrow 2I = \pi [\tan x - \sec x]_{\pi/4}^{3\pi/4}$$

$$\Rightarrow 2I = \pi \left[\tan\left(\frac{3\pi}{4}\right) - \sec\left(\frac{3\pi}{4}\right) \right] - \left[\tan\frac{\pi}{4} - \sec\frac{\pi}{4} \right]$$

$$\Rightarrow 2I = \pi [-1 + \sqrt{2} - 1 + \sqrt{2}] \Rightarrow 2I = \pi [2\sqrt{2} - 2] \therefore I = \pi(\sqrt{2}-1)$$

42. $\int_{a/4}^{3a/4} \frac{\sqrt{x}}{\sqrt{a-x}+\sqrt{x}} dx = \frac{a}{4}$

Sol. Let $I = \int_{a/4}^{3a/4} \frac{\sqrt{x}}{\sqrt{a-x}+\sqrt{x}} dx \quad \dots(1)$

$$\text{Then, } I = \int_{a/4}^{3a/4} \frac{\sqrt{\left(\frac{a+3a}{4}\right) - x}}{\sqrt{a - \left\{\left(\frac{a+3a}{4}\right) - x\right\}} + \sqrt{\left\{\left(\frac{a+3a}{4}\right) - x\right\}}} dx$$

$$\Rightarrow I = \int_{a/4}^{3a/4} \frac{\sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} dx \quad \dots(2)$$

Adding (1) and (2) we get, $2I = \int_{a/4}^{3a/4} \left(\frac{\sqrt{x}}{\sqrt{a-x}+\sqrt{x}} + \frac{\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}} \right) dx$

$$\Rightarrow 2I = \int_{a/4}^{3a/4} \left(\frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} \right) dx \Rightarrow 2I = \int_{a/4}^{3a/4} dx \Rightarrow 2I = [x]_{a/4}^{3a/4}$$

$$\Rightarrow 2I = \left(\frac{3a}{4} - \frac{a}{4} \right) \Rightarrow 2I = \frac{2a}{4} \therefore I = \frac{a}{4}$$

43. $\int_1^4 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}} dx = \frac{3}{2}$

Sol. Let $I = \int_1^4 \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} dx$... (1)

Then $I = \int_1^4 \frac{\sqrt{(1+4)-x}}{\sqrt{5-\{(1+4)-x\}} + \sqrt{(1+4)-x}} dx$ or $I = \int_1^4 \frac{\sqrt{5-x}}{\sqrt{5-(5-x)} + \sqrt{5-x}} dx$

$\Rightarrow I = \int_1^4 \frac{\sqrt{5-x}}{\sqrt{x} + \sqrt{5-x}} dx$... (2)

By, Adding (1) and (2) we get, $2I = \int_1^4 \left(\frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} + \frac{\sqrt{5-x}}{\sqrt{x} + \sqrt{5-x}} \right) dx$

$\Rightarrow 2I = \int_1^4 \left(\frac{\sqrt{x} + \sqrt{5-x}}{\sqrt{5-x} + \sqrt{x}} \right) dx \Rightarrow 2I = \int_1^4 1 dx \Rightarrow 2I = [x]_1^4 \Rightarrow 2I = (4-1) \therefore I = \frac{3}{2}$

44. $\int_0^{\pi/2} x \cot x dx = \frac{\pi}{2} \log 2$

Sol. Let $I = \int_0^{\pi/2} x \cot x dx \Rightarrow I = \int_0^{\pi/2} x \cot x dx \Rightarrow I = x \int_0^{\pi/2} \cot x dx - \int_0^{\pi/2} \left[\frac{d(x)}{dx} \int \cot x dx \right] dx$

$\Rightarrow I = x [\log(\sin x)]_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot \log(\sin x) dx$

$\Rightarrow I = \left\{ \frac{\pi}{2} \log \left(\sin \frac{\pi}{2} \right) - 0 \right\} - \int_0^{\pi/2} \log(\sin x) dx \Rightarrow I = - \int_0^{\pi/2} \log(\sin x) dx$

$\Rightarrow I = -I_1$... (1)

$I_1 = \int_0^{\pi/2} \log(\sin x) dx$... (2)

Then $I_1 = \int_0^{\pi/2} \log \left[\sin \left(\frac{\pi}{2} - x \right) \right] dx$ or $I_1 = \int_0^{\pi/2} \log(\cos x) dx$... (3)

By, Adding (2) and (3) we get, $2I_1 = \int_0^{\pi/2} [\log(\sin x) + \log(\cos x)] dx$

$\Rightarrow 2I_1 = \int_0^{\pi/2} \log(\sin x \cdot \cos x) dx \Rightarrow 2I_1 = \int_0^{\pi/2} \log \left(\frac{\sin 2x}{2} \right) dx$

$\Rightarrow 2I_1 = \int_0^{\pi/2} \log(\sin 2x) dx - \int_0^{\pi/2} \log 2 dx \Rightarrow 2I_1 = \frac{1}{2} \int_0^{\pi} \log(\sin t) dt - (\log 2) \int_0^{\pi/2} dx$

[Putting $2x = t$ in the 1st integral]

$2I_1 = \frac{1}{2} \int_0^{\pi} \log(\sin t) dt - (\log 2) [x]_0^{\pi/2} \Rightarrow 2I_1 = \left(\frac{1}{2} \times 2 \right) \int_0^{\pi/2} \log \sin t dt - (\log 2) \left(\frac{\pi}{2} - 0 \right)$

$\Rightarrow 2I_1 = \int_0^{\pi/2} \log(\sin x) dx - \frac{\pi}{2} \log 2 \Rightarrow 2I_1 = I_1 - \frac{\pi}{2} \log 2 \Rightarrow I_1 = -\frac{\pi}{2} \log 2$

Putting the value of I_1 in equation (1), $I = \frac{\pi}{2} \log 2$

45. $\int_0^1 \left(\frac{\sin^{-1} x}{x} \right) dx = \frac{\pi}{2} (\log 2)$

Sol. Let $I = \int_0^1 \left(\frac{\sin^{-1}(x)}{x} \right) dx$

Put $x = \sin \theta \Rightarrow dx = \cos \theta d\theta$. Clearly $x = 0, \theta = 0$, and $x = 1, \theta = \frac{\pi}{2}$.

Now $I = \int_0^{\pi/2} \frac{\sin^{-1}(\sin \theta)}{\sin \theta} \cdot \cos \theta d\theta \Rightarrow I = \int_0^{\pi/2} \theta \cdot \cot \theta d\theta$

$\Rightarrow I = \left[\theta \log(\sin \theta) \right]_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot \log(\sin \theta) d\theta$

$\Rightarrow I = \theta \left\{ \frac{\pi}{2} \log \left(\sin \frac{\pi}{2} \right) - 0^* \right\} - \int_0^{\pi/2} \log(\sin \theta) d\theta$

$\Rightarrow I = 0 - \int_0^{\pi/2} \log(\sin \theta) d\theta$

$\Rightarrow I = - \int_0^{\pi/2} \log(\sin \theta) d\theta \Rightarrow I = -I_1 \quad \dots (1)$

$I_1 = \int_0^{\pi/2} \log(\sin \theta) d\theta \quad \dots (2)$

Then $I_1 = \int_0^{\pi/2} \log \left[\sin \left(\frac{\pi}{2} - \theta \right) \right] d\theta$ or $I_1 = \int_0^{\pi/2} \log(\cos \theta) d\theta$

By, Adding (2) and (3) we get, $2I_1 = \int_0^{\pi/2} [\log(\sin \theta) + \log(\cos \theta)] d\theta$

$\Rightarrow 2I_1 = \int_0^{\pi/2} \log(\sin \theta \cdot \cos \theta) d\theta \Rightarrow 2I_1 = \int_0^{\pi/2} \log \left(\frac{\sin 2\theta}{2} \right) d\theta$

$\Rightarrow 2I_1 = \int_0^{\pi/2} \log(\sin 2\theta) d\theta - \int_0^{\pi/2} \log 2 d\theta \Rightarrow 2I_1 = \frac{1}{2} \int_0^{\pi} \log(\sin t) dt - (\log 2) \int_0^{\pi/2} d\theta$

[Putting $2\theta = t$ in the 1st integral]

$2I_1 = \left(\frac{1}{2} \times 2 \right) \cdot \int_0^{\pi/2} \log(\sin t) dt - \log 2 \left(\frac{\pi}{2} - 0 \right) \quad \left[\because \int_a^b f(x) dx = \int_a^b f(t) dt \right]$

$\Rightarrow 2I_1 = \int_0^{\pi/2} \log \sin \theta d\theta - \frac{\pi}{2} \log 2 \Rightarrow 2I_1 = I_1 - \frac{\pi}{2} \log 2 \Rightarrow I_1 = -\frac{\pi}{2} \log 2$

Putting value of I_1 in equation (1), $I = \frac{\pi}{2} \log 2$

46. $\int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx = -\frac{\pi}{2} \log 2$

Sol. Let $I = \int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx$

Put $x = \sin \theta \Rightarrow dx = \cos \theta d\theta$. Clearly $x = 0, \theta = 0$, and $x = 1, \theta = \frac{\pi}{2}$.

$$I = \int_0^{\pi/2} \frac{\log(\sin \theta)}{\sqrt{1 - \sin^2 \theta}} \cos \theta d\theta \Rightarrow I = \int_0^{\pi/2} \log(\sin \theta) d\theta$$

$$I_1 = \int_0^{\pi/2} \log(\sin \theta) d\theta \quad \dots(2)$$

$$\text{Then } I_1 = \int_0^{\pi/2} \log \left[\sin \left(\frac{\pi}{2} - \theta \right) \right] d\theta \text{ or } I_1 = \int_0^{\pi/2} \log(\cos \theta) d\theta$$

$$\text{By, Adding (2) and (3) we get, } 2I_1 = \int_0^{\pi/2} [\log(\sin \theta) + \log(\cos \theta)] d\theta$$

$$\Rightarrow 2I_1 = \int_0^{\pi/2} \log(\sin \theta \cdot \cos \theta) d\theta \Rightarrow 2I_1 = \int_0^{\pi/2} \log \left(\frac{\sin 2\theta}{2} \right) d\theta$$

$$\Rightarrow 2I_1 = \int_0^{\pi/2} \log(\sin 2\theta) d\theta - \int_0^{\pi/2} \log 2 d\theta \Rightarrow 2I_1 = \frac{1}{2} \int_0^{\pi} \log(\sin t) dt - (\log 2) \int_0^{\pi/2} d\theta$$

[Putting $2\theta = t$ in the 1st integral]

$$2I_1 = \left(\frac{1}{2} \times 2 \right) \int_0^{\pi/2} \log(\sin t) dt - \log 2 \left(\frac{\pi}{2} - 0 \right) \Rightarrow 2I_1 = \int_0^{\pi/2} \log \sin \theta d\theta - \frac{\pi}{2} \log 2$$

$$\Rightarrow 2I_1 = I_1 - \frac{\pi}{2} \log 2 \Rightarrow I_1 = -\frac{\pi}{2} \log 2$$

Now, Putting value of I_1 in equation (1), $I = \frac{\pi}{2} \log 2$

47. $\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8} \log 2$

Sol. Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$

Put $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$. Clearly $x = 0, \theta = 0$ and $x = 1, \theta = \frac{\pi}{4}$.

$$\therefore I = \int_0^{\pi/4} \frac{\log(1+\tan \theta)}{1+\tan^2 \theta} \cdot \sec^2 \theta d\theta \Rightarrow I = \int_0^{\pi/4} \log(1+\tan \theta) d\theta \quad \dots(1)$$

$$\text{Then } I = \int_0^{\pi/4} \log \left[1 + \tan \left(\frac{\pi}{4} - \theta \right) \right] d\theta \text{ or } I = \int_0^{\pi/4} \log \left[1 + \frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \tan \theta} \right] d\theta$$

$$\Rightarrow I = \int_0^{\pi/4} \log \left[1 + \frac{1 - \tan \theta}{1 + \tan \theta} \right] d\theta \Rightarrow I = \int_0^{\pi/4} \log \left(\frac{2}{1 + \tan \theta} \right) d\theta$$

$$\Rightarrow I = \int_0^{\pi/4} \log \left(\frac{2}{1 + \tan \theta} \right) d\theta \Rightarrow I = \int_0^{\pi/4} [\log 2 - \log(1 + \tan \theta)] d\theta \quad \dots(2)$$

By, Adding (1) and (2) we get

$$2I = \int_0^{\pi/4} [\log(1 + \tan \theta) + \log 2 - \log(1 + \tan \theta)] d\theta \Rightarrow 2I = \int_0^{\pi/4} \log 2 d\theta$$

$$\Rightarrow 2I = \log 2 \int_0^{\pi/4} d\theta \Rightarrow 2I = \log 2 [\theta]_0^{\pi/4} \Rightarrow 2I = \log 2 \left(\frac{\pi}{4} - 0 \right) \therefore I = \frac{\pi}{8} \log 2$$

48. $\int_{-a}^a x^3 \sqrt{a^2 - x^2} dx = 0$

Sol. Let $f(x) = x^3 \sqrt{a^2 - x^2}$

Then, $f(-x) = (-x)^3 \sqrt{a^2 - (-x)^2} = -x^3 \sqrt{a^2 - x^2} = -f(x)$

$\therefore f(x)$ is an odd function of x .

But $\int_{-a}^a f(x) dx = 0$, when $f(x)$ is odd. $\therefore \int_{-a}^a x^3 \sqrt{a^2 - x^2} dx = 0$

49. $\int_{-\pi}^{\pi} (\sin^{75} x + x^{125}) dx = 0$

Sol. Let $f(x) = \sin^{75} x + x^{125}$

Then, $f(-x) = \sin^{75}(-x) + (-x)^{125} = -\sin^{75} x - x^{125} = -(\sin^{75} x + x^{125}) = -f(x)$

$\therefore f(x)$ is an odd function of x .

But $\int_{-a}^a f(x) dx = 0$, where $f(x)$ is odd. $\therefore \int_{-\pi}^{\pi} (\sin^{75} x + x^{125}) dx = 0$

50. $\int_{-\pi}^{\pi} x^{12} \sin^9 x dx = 0$

Sol. Let $f(x) = x^{12} \sin^9 x$

Then, $f(-x) = (-x)^{12} \sin^9(-x) = -x^{12} \sin^9 x = -f(x)$.

$\therefore f(x)$ is an odd function of x .

But $\int_{-a}^a f(x) dx = 0$, when $f(x)$ is odd. $\therefore \int_{-\pi}^{\pi} x^{12} \sin^9 x dx = 0$

51. $\int_{-1}^1 e^{|x|} dx = 2(e - 1)$

Sol. Clearly, $|x| = \begin{cases} -x & \text{when } -1 \leq x \leq 0 \\ x & \text{when } 0 \leq x \leq 1 \end{cases}$

$$\begin{aligned} \therefore \int_{-1}^1 e^{|x|} dx &= \int_{-1}^0 e^{|x|} dx + \int_0^1 e^{|x|} dx = \int_{-1}^0 e^{-x} dx + \int_0^1 e^x dx = -[e^{-x}]_{-1}^0 + [e^x]_0^1 \\ &= -[e^0 - e^1] + e^1 - e^0 = -[1 - e] + e^1 = -1 + e + e - 1 = 2e - 2 = 2(e - 1) \end{aligned}$$

52. $\int_{-2}^2 |x+1| dx = 5$

Sol. Clearly $|x+1| = \begin{cases} -(x+1) & \text{when } -2 \leq x \leq -1 \\ x+1 & \text{when } -1 \leq x \leq 2 \end{cases}$

$$\begin{aligned} \therefore \int_{-2}^2 |x+1| dx &= \int_{-2}^{-1} |x+1| dx + \int_{-1}^2 |x+1| dx = -\int_{-2}^{-1} (x+1) dx + \int_{-1}^2 (x+1) dx \\ &= -\left[\frac{x^2}{2} + x\right]_{-2}^{-1} + \left[\frac{x^2}{2} + x\right]_{-1}^2 = -\left[\left(\frac{1}{2} - 1\right) - \left(\frac{4}{2} - 2\right)\right] + \left[\frac{4}{2} + 2 - \left(\frac{1}{2} - 1\right)\right] \\ &= -\left[-\frac{1}{2} - 0\right] + \left[4 + \frac{1}{2}\right] = \frac{1}{2} + \frac{9}{2} = 5 \end{aligned}$$

53. $\int_0^8 |x-5| dx = 17$

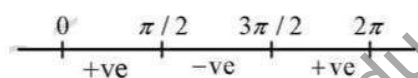
Sol. Clearly $|x-5| = \begin{cases} -(x-5) & \text{when } 0 \leq x \leq 5 \\ (x-5) & \text{when } 5 \leq x \leq 8 \end{cases}$

$$\begin{aligned} \therefore \int_0^8 (x-5) dx &= \int_0^5 |x-5| dx + \int_5^8 |x-5| dx = \int_0^5 -(x-5) dx + \int_5^8 (x-5) dx = -\left[\frac{x^2}{2} - 5x\right]_0^5 + \left[\frac{x^2}{2} - 5x\right]_5^8 \\ &= -\left[\left\{\frac{(5)^2}{2} - 5 \times 5\right\} - (0)\right] + \left[\left\{\frac{(8)^2}{2} - 5 \times 8\right\} - \left\{\frac{(5)^2}{2} - (5 \times 5)\right\}\right] \\ &= -\left[\left(\frac{25}{2} - 25\right)\right] + \left[(32 - 40) - \left(\frac{25}{2} - 25\right)\right] \\ &= -\left(-\frac{25}{2}\right) + \left[-8 - \left(-\frac{25}{2}\right)\right] = \frac{25}{2} - 8 + \frac{25}{2} = \frac{25}{2} - \frac{16}{2} + \frac{25}{2} = \frac{34}{2} = 17 \end{aligned}$$

54. $\int_0^{2\pi} (\cos x) dx = 4$

Sol. Given $\int_0^{2\pi} (\cos x) dx = 4$; $|\cos x| = \begin{cases} \cos x, & \text{when } 0 \leq x \leq \frac{\pi}{2} \\ -\cos x, & \text{when } \frac{\pi}{2} \leq x \leq \frac{3\pi}{2} \\ \cos x, & \text{when } \frac{3\pi}{2} \leq x \leq 2\pi \end{cases}$

$$\begin{aligned} \int_0^{2\pi} \cos x dx &= \int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{3\pi/2} |\cos x| dx + \int_{3\pi/2}^{2\pi} \cos x dx \\ &= \int_0^{\pi/2} \cos x dx - \int_{\pi/2}^{3\pi/2} \cos x dx + \int_{3\pi/2}^{2\pi} \cos x dx \\ &= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{3\pi/2} + [\sin x]_{3\pi/2}^{2\pi} \\ &= \left[\sin \frac{\pi}{2} - \sin 0\right] - \left[\sin \frac{3\pi}{2} - \sin \frac{\pi}{2}\right] + \left[\sin 2\pi - \sin \frac{3\pi}{2}\right] \\ &= [1 - 0] - [-1 - 1] + [0 + 1] = 1 + 2 + 1 = 4 \end{aligned}$$



55. $\int_{-\pi/4}^{\pi/4} |\sin x| dx = 2 - 2\sqrt{2}$

Sol. Clearly $|\sin x| = \begin{cases} -\sin x & \text{when } -\frac{\pi}{4} \leq x \leq 0 \\ \sin x & \text{when } 0 \leq x \leq \frac{\pi}{4} \end{cases}$

$$\begin{aligned} \therefore \int_{-\pi/4}^{\pi/4} |\sin x| dx &= \int_{-\pi/4}^0 |\sin x| dx + \int_0^{\pi/4} (\sin x) dx = \int_{-\pi/4}^0 -\sin x dx + \int_0^{\pi/4} \sin x dx \\ &= \int_0^{\pi/4} \sin x dx + \int_0^{\pi/4} \sin x dx = [-\cos x]_0^{\pi/4} + [-\cos x]_0^{\pi/4} \\ &= \left[-\cos\left(-\frac{\pi}{4}\right) - (-\cos 0) \right] + \left[-\cos \frac{\pi}{4} - (-\cos 0) \right] \\ &= \left(-\frac{1}{\sqrt{2}} + 1 \right) + \left(-\frac{1}{\sqrt{2}} + 1 \right) = 2 - \frac{2}{\sqrt{2}} = 2 - \sqrt{2} \end{aligned}$$

56. $f(x) = \begin{cases} 2x+1 & \text{when } 1 \leq x \leq 2 \\ x^2+1 & \text{when } 2 \leq x \leq 3 \end{cases}$, show that $\int_1^3 f(x) dx = \frac{34}{3}$.

Sol. $\int_1^3 f(x) dx = \int_1^2 f(x) dx + \int_2^3 f(x) dx = \int_1^2 (2x+1) dx + \int_2^3 (x^2+1) dx = \left[2 \frac{x^2}{2} + x \right]_1^2 + \left[\frac{x^3}{3} + x \right]_2^3$

$$\begin{aligned} &= \left[x^2 + x \right]_1^2 + \left[\frac{x^3}{3} + x \right]_2^3 = \left[\{(2)^2 + 2\} - \{(1)^2 + 1\} \right] + \left[\left\{ \frac{(3)^3}{3} + 3 \right\} - \left\{ \frac{(2)^3}{3} + 2 \right\} \right] \\ &= [(4+2) - (2)] + \left[(9+3) - \left(\frac{8}{3} + 2 \right) \right] = 6 - 2 + 12 - \frac{14}{3} = 16 - \frac{14}{3} = \frac{34}{3} \end{aligned}$$

57. Let $f(x) = \begin{cases} 3x^2+4 & \text{when } 0 \leq x \leq 2 \\ 9x-2 & \text{when } 2 \leq x \leq 4 \end{cases}$ show that $\int_0^4 f(x) dx = 66$

Sol. $\int_0^4 f(x) dx = \int_0^2 (3x^2+4) dx + \int_2^4 (9x-2) dx = \left[3 \frac{x^3}{3} + 4x \right]_0^2 + \left[9 \frac{x^2}{2} - 2x \right]_2^4 = [x^3 + 4x]_0^2 + \left[\frac{9x^2}{2} - 2x \right]_2^4$

$$\begin{aligned} &= \left[\{(2)^3 + 4(2)\} - \{(0)^3 + 4(0)\} \right] + \left[\left\{ \frac{9(4)^2}{2} - 2(4) \right\} - \left\{ \frac{9(2)^2}{2} - 2(2) \right\} \right] \\ &= [(8+8) - 0] + [(72-8) - (18-4)] = 16 + [64-14] = 16 + 50 = 66 \end{aligned}$$

58. Prove that $\int_0^4 \{|x| + |x-2| + |x-4|\} dx = 20$

Sol. L.H.S = $\int_0^4 |x| dx + \int_0^4 |x-2| dx + \int_0^4 |x-4| dx = \int_0^4 x dx + \int_0^2 (2-x) dx + \int_2^4 (x-2) dx + \int_4^4 (4-x) dx$

$$\begin{aligned} &= \left[\frac{x^2}{2} \right]_0^4 + \left[2x - \frac{x^2}{2} \right]_0^2 + \left[\frac{x^2}{2} - 2x \right]_2^4 + \left[4x - \frac{x^2}{2} \right]_4^4 \\ &= 8 + (4-2) + (8-8) - (2-4) + (16-8) = 20 \end{aligned}$$

EXERCISE 16D (Pg. No.: 843)

Evaluate each of the following integrals as the limits of sums.

1. $\int_0^2 (x+4) dx$

Sol. Let $f(x) = x+4$ i.e., $a=0$, $b=2$, $nh=2-0=2$

$$\begin{aligned} \therefore \int_0^2 (x+4) dx &= \lim_{h \rightarrow 0} h [f(0) + f(0+h) + \dots + f(0+(n-1)h)] \\ &= \lim_{h \rightarrow 0} h [4 + (h+4) + (2h+4) + \dots + ((n-1)h+4)] \\ &= \lim_{h \rightarrow 0} h [4n + h(1+2+3+\dots+(n-1))] = \lim_{h \rightarrow 0} h \left[4n + h \cdot \frac{n(n-1)}{2} \right] \\ &= \lim_{h \rightarrow 0} \left[4nh + \frac{h^2(n^2-n)}{2} \right] = \lim_{h \rightarrow 0} \left[4nh + \frac{n^2 h^2}{2} - \frac{nh^2}{2} \right] = 4(2) + \frac{(2)^2}{2} - \frac{2 \cdot 0}{2} = 8+2=10 \end{aligned}$$

2. $\int_1^2 (3x-2) dx$

Sol. Let $f(x) = 3x-2$, $a=1$, $b=2$, $nh=2-1=1$

$$\begin{aligned} \therefore \int_1^2 (3x-2) dx &= \lim_{h \rightarrow 0} h [f(1) + f(1+h) + \dots + f(1+(n-1)h)] \\ &= \lim_{h \rightarrow 0} h [\{3(1)-2\} + \{3(1+h)-2\} + \{3(1+2h)-2\} + \dots + \{3(1+(n-1)h)-2\}] \\ &= \lim_{h \rightarrow 0} h [1 + (1+3h) + (1+6h) + \dots + \{1+3(n-1)h\}] \\ &= \lim_{h \rightarrow 0} h [n + 3h(1+2+3+\dots+(n-1))] = \lim_{h \rightarrow 0} h \left[n + 3h \cdot \frac{n(n-1)}{2} \right] \\ &= \lim_{h \rightarrow 0} \left[nh + \frac{3h^2(n^2-n)}{2} \right] = \lim_{h \rightarrow 0} \left[nh + \frac{3 \cdot h^2 n^2}{2} - \frac{3 \cdot h^2 n}{2} \right] = \lim_{h \rightarrow 0} \left[nh + \frac{3(nh)^2}{2} - \frac{3(nh) \cdot h}{2} \right] \\ &= 1 + \frac{3(1)^2}{2} - \frac{3 \cdot 1 \cdot 0}{2} = 1 + \frac{3}{2} = \frac{5}{2} \end{aligned}$$

3. $\int_1^3 x^2 dx$

Sol. Let $f(x) = x^2$ i.e. $a=1$, $b=3$, $nh=3-1=2$

$$\begin{aligned} \therefore \int_1^3 x^2 dx &= \lim_{h \rightarrow 0} h [f(1) + f(1+h) + \dots + f(1+(n-1)h)] \\ &= \lim_{h \rightarrow 0} h [(1)^2 + (1+h)^2 + (1+2h)^2 + \dots + (1+(n-1)h)^2] \\ &= \lim_{h \rightarrow 0} h [1 + (1+2h+h^2) + (1+4h+4h^2) + \dots + \{1+2(n-1)h+(n-1)^2 h^2\}] \\ &= \lim_{h \rightarrow 0} h [n + 2h(1+2+3+\dots+(n-1)) + h^2(1^2+2^2+3^2+\dots+(n-1)^2)] \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} h \left[n + 2h \cdot \frac{n(n-1)}{2} + h^2 \cdot \frac{n(n-1)(2n-1)}{6} \right] = \lim_{h \rightarrow 0} \left[nh + h^2(n^2 - n) + h^3 \frac{(2n^3 - 3n^2 + n)}{6} \right] \\
 &= \lim_{h \rightarrow 0} \left[nh + (nh)^2 - (nh) \cdot h + \frac{2(nh)^3}{6} - \frac{3(nh)^2 \cdot h}{6} + \frac{(nh) \cdot h^2}{6} \right] \\
 &= 2 + (2)^2 - 2 \cdot 0 + \frac{(2)^3}{3} - \frac{3(2)^2 \cdot 0}{6} + \frac{2 \cdot 0}{6} = 6 + \frac{8}{3} = \frac{26}{3}
 \end{aligned}$$

4. $\int_0^3 (x^2 + 1) dx$

Sol. Let $f(x) = x^2 + 1$ i.e. $a = 0$, $b = 3$, $nh = 3 - 0 = 3$

$$\begin{aligned}
 \therefore \int_0^3 (x^2 + 1) dx &= \lim_{h \rightarrow 0} h \left[f(0) + f(0+h) + f(0+2h) + \dots + f(0+(n-1)h) \right] \\
 &= \lim_{h \rightarrow 0} h \left[1 + (h^2 + 1) + (4h^2 + 1) + (9h^2 + 1) + \dots + \{(n-1)^2 h^2 + 1\} \right] \\
 &= \lim_{h \rightarrow 0} h \left[n + h^2 (1 + 4 + 9 + \dots + (n-1)^2) \right] = \lim_{h \rightarrow 0} \left[h + h^2 \cdot \frac{n(n-1)(2n-1)}{6} \right] \\
 &= \lim_{h \rightarrow 0} \left[nh + h^3 \cdot \frac{(2n^3 - 3n^2 + n)}{6} \right] = \lim_{h \rightarrow 0} \left[nh + 2 \cdot \frac{(nh)^3}{6} - 3 \cdot \frac{(nh)^2 \cdot h}{6} + \frac{(nh) \cdot h^2}{6} \right] \\
 &= 3 + \frac{(3)^3}{3} - \frac{3 \cdot (3)^2 \cdot 0}{6} + \frac{3 \cdot 0}{6} = 3 + 9 = 12
 \end{aligned}$$

5. $\int_2^5 (3x^2 - 5) dx$

Sol. Let $f(x) = 3x^2 - 5$ i.e. $a = 2$, $b = 5$, $nh = 5 - 2 = 3$

$$\begin{aligned}
 \therefore \int_2^5 (3x^2 - 5) dx &= \lim_{h \rightarrow 0} h \left[f(2) + f(2+h) + f(2+2h) + \dots + f(2+(n-1)h) \right] \\
 &= \lim_{h \rightarrow 0} h \left[7 + (7 + 12h + 3h^2) + (7 + 24h + 12h^2) + \dots + \{7 + 12(n-1)h + 3(n-1)^2 h^2\} \right] \\
 &= \lim_{h \rightarrow 0} h \left[7n + 12h(1 + 2 + 3 + \dots + (n-1)) + 3h^2(1^2 + 2^2 + \dots + (n-1)^2) \right] \\
 &= \lim_{h \rightarrow 0} h \left[7n + 12h \cdot \frac{n(n-1)}{2} + 3h^2 \cdot \frac{n(n-1)(2n-1)}{6} \right] \\
 &= \lim_{h \rightarrow 0} \left[7nh + 6h^2(n^2 - n) + \frac{h^3(2n^3 - 3n^2 + n)}{2} \right] \\
 &= \lim_{h \rightarrow 0} \left[7nh + 6(nh)^2 - 6(nh) \cdot h + (nh)^3 - \frac{3(nh)^2 \cdot h}{2} + \frac{(nh) \cdot h^2}{2} \right] \\
 &= \left[7 \cdot 3 + 6(3)^2 - 6 \cdot 3 \cdot 0 + (3)^3 - \frac{3(3)^2 \cdot 0}{2} + \frac{3 \cdot 0}{2} \right] = 21 + 54 + 27 = 102
 \end{aligned}$$

6. $\int_0^3 (x^2 + 2x) dx$

Sol. Let $f(x) = x^2 + 2x$ i.e. $a = 0$, $b = 3$, $nh = 3 - 0 = 3$

$$\begin{aligned} \therefore \int_0^3 (x^2 + 2x) dx &= \lim_{h \rightarrow 0} h [f(0) + f(0+h) + f(0+2h) + \dots + f(0+(n-1)h)] \\ &= \lim_{h \rightarrow 0} h [0 + (h^2 + 2h) + (4h^2 + 4h) + (9h^2 + 6h) + \dots + \{(n-1)^2 h^2 + 2(n-1)h\}] \\ &= \lim_{h \rightarrow 0} h [2h(1+2+3+\dots+(n-1)) + h^2(1^2 + 2^2 + 3^2 + \dots + (n-1)^2)] \\ &= \lim_{h \rightarrow 0} h \left[2h \cdot \frac{n(n-1)}{2} + h^2 \cdot \frac{n(n-1)(2n-1)}{6} \right] = \lim_{h \rightarrow 0} \left[h^2(n^2 - n) + h^3 \cdot \frac{(2n^3 - 3n^2 + n)}{6} \right] \\ &= \lim_{h \rightarrow 0} \left[(nh)^2 - (nh)h + \frac{2(nh)^3}{6} - \frac{3(nh)^2 \cdot h}{6} + \frac{(nh)h^2}{6} \right] \\ &= \left[(3)^2 - 3 \cdot 0 + \frac{(3)^3}{3} - \frac{3 \cdot (3)^2 \cdot 0}{6} + \frac{3 \cdot 0}{6} \right] = 9 + \frac{27}{3} = 9 + 9 = 18 \end{aligned}$$

7. $\int_1^4 (3x^2 + 2x) dx$

Sol. Let $f(x) = 3x^2 + 2x$ i.e. $a = 1$, $b = 4$, $nh = 4 - 1 = 3$

$$\begin{aligned} \therefore \int_1^4 (3x^2 + 2x) dx &= \lim_{h \rightarrow 0} h [f(1) + f(1+h) + f(1+2h) + \dots + f(1+(n-1)h)] \\ &= \lim_{h \rightarrow 0} h [5 + (5 + 8h + 3h^2) + (5 + 16h + 12h^2) + \dots + \{5 + 8(n-1)h + 3(n-1)^2 h^2\}] \\ &= \lim_{h \rightarrow 0} h [5n + 8h(1+2+3+\dots+(n-1)) + 3h^2(1^2 + 2^2 + 3^2 + \dots + (n-1)^2)] \\ &= \lim_{h \rightarrow 0} h \left[5n + 8h \cdot \frac{n(n-1)}{2} + 3h^2 \cdot \frac{n(n-1)(2n-1)}{6} \right] \\ &= \lim_{h \rightarrow 0} \left[5nh + 4h^2(n^2 - n) + \frac{h^3(2n^3 - 3n^2 + n)}{2} \right] \\ &= \lim_{h \rightarrow 0} \left[5(nh) + 4(nh)^2 - 4(nh)h + (nh)^3 - \frac{3(nh)^2 \cdot h}{2} + \frac{(nh)h^2}{2} \right] \\ &= 5(3) + 4(3)^2 - 4 \cdot 3 \cdot 0 + (3)^3 - \frac{3(3)^2 \cdot 0}{2} + \frac{3 \cdot 0}{2} = 15 + 36 + 27 = 78 \end{aligned}$$

8. $\int_1^3 (x^2 + 5x) dx$

Sol. Let $f(x) = x^2 + 5x$ i.e. $a = 1$, $b = 3$, $nh = 3 - 1 = 2$

$$\therefore \int_1^3 (x^2 + 5x) dx = \lim_{h \rightarrow 0} h [f(1) + f(1+h) + f(1+2h) + \dots + f(1+(n-1)h)]$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} h \left[6 + (6 + 7h + h^2) + (6 + 14h + 4h^2) + \dots + \{6 + 7(n-1)h + (n-1)^2 h^2\} \right] \\
 &= \lim_{h \rightarrow 0} h \left[6n + 7h(1 + 2 + 3 + \dots + (n-1)) + h^2(1^2 + 2^2 + 3^2 + \dots + (n-1)^2) \right] \\
 &= \lim_{h \rightarrow 0} h \left[6n + 7h \cdot \frac{n(n-1)}{2} + h^2 \cdot \frac{n(n-1)(2n-1)}{6} \right] \\
 &= \lim_{h \rightarrow 0} \left[6nh + 7h^2 \cdot \frac{(n^2 - n)}{2} + h^3 \cdot \frac{(2n^3 - 3n^2 + n)}{6} \right] \\
 &= \lim_{h \rightarrow 0} \left[6nh + \frac{7(nh)^2}{2} - \frac{7(nh) \cdot h}{2} + \frac{(nh)^3}{3} - \frac{(nh)^2 \cdot h}{2} + \frac{(nh)h^2}{6} \right] \\
 &= 6(2) + \frac{7(2)^2}{2} - \frac{7 \cdot 2 \cdot 0}{2} + \frac{(2)^3}{3} - \frac{(2)^2 \cdot 0}{2} + \frac{2 \cdot 0}{6} = 12 + 14 + \frac{8}{3} = 26 + \frac{8}{3} = \frac{86}{3}
 \end{aligned}$$

9. $\int_1^3 (2x^2 + 3x) dx$

Sol. Let $I = \int_1^3 (2x^2 + 3x) dx$, i.e., $a = 1$, $b = 3$, $nh = b - a \Rightarrow nh = 3 - 1 = 2$

$$f(x) = 2x^2 + 3x$$

$$f(1) = 2 + 3 = 5$$

$$f(1+h) = 2(1+h)^2 + 3(1+h)$$

$$f(1+2h) = 2(1+2h)^2 + 3(1+2h)$$

$$f(1+3h) = 2(1+3h)^2 + 3(1+3h)$$

$$\dots\dots\dots$$

$$f\{1+(n-1)h\} = 2\{1+(n-1)h\}^2 + 3\{1+(n-1)h\}$$

Now, $\int_1^3 (2x^2 + 3x) dx = \lim_{h \rightarrow 0} h [f(1) + f(1+h) + f(1+2h) + \dots + f\{1+(n-1)h\}]$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} h \left[5 + \{2(1+h)^2 + 3(1+h)\} + \{2(1+2h)^2 + 3(1+2h)\} \right. \\
 &\quad \left. + \{2(1+3h)^2 + 3(1+3h)\} + \dots + 2\{1+(n-1)h\}^2 \right] \\
 &= \lim_{h \rightarrow 0} \left[5n + 2\{h^2 + (2h)^2 + (3h)^2 + \dots + (n-1)^2 h^2\} + 2\{2 \cdot 1 \cdot h + 2 \cdot 1 \cdot 2h + 2 \cdot 1 \cdot 3h + \dots + 2 \cdot 1 \cdot (n-1)h\} \right. \\
 &\quad \left. + \{3h + 3 \cdot 2h + 3 \cdot 3h + 3 \cdot 4h + \dots + 3 \cdot (n-1)h\} \right] \\
 &= \lim_{h \rightarrow 0} h \left[5n + 2h^2 \{1^2 + 2^2 + 3^2 + \dots + (n-1)^2\} + 2 \cdot 2h \{1 + 2 + 3 + \dots + (n-1)\} \right. \\
 &\quad \left. + 3h \{1 + 2 + 3 + \dots + (n-1)\} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} h \left[5n + 2h^2 \cdot \frac{n(n-1)(2n-1)}{6} + 7h \cdot \frac{n(n-1)}{2} \right] \\
 &= \lim_{h \rightarrow 0} \left[5nh + \frac{nh(nh-h)(2nh-h)}{3} + \frac{7}{2}nh(nh-h) \right] \\
 &= \left[5 \cdot 2 + \frac{2(2-0)(2 \cdot 2 - 0)}{3} + \frac{7}{2} \cdot 2(2-0) \right] = 10 + \frac{16}{3} + 14 = 24 + \frac{16}{3} = \frac{72+16}{3} = \frac{88}{3}
 \end{aligned}$$

10. $\int_0^2 x^3 dx$

Sol. Let $f(x) = x^3$ i.e. $a=0$, $b=2$, $nh=2-0=2$

$$\begin{aligned}
 \therefore \int_0^2 x^3 dx &= \lim_{h \rightarrow 0} h \left[f(0) + f(0+h) + f(0+2h) + \dots + f(0+(n-1)h) \right] \\
 &= \lim_{h \rightarrow 0} h \left[0 + h^3 + 8h^3 + 27h^3 + \dots + (n-1)^3 h^3 \right] \\
 &= \lim_{h \rightarrow 0} \left[h \cdot h^3 (1^3 + 2^3 + 3^3 + \dots + (n-1)^3) \right] = \lim_{h \rightarrow 0} h^4 \left(\frac{n(n-1)}{2} \right)^2 = \lim_{h \rightarrow 0} h^4 \cdot \frac{n^2(n-1)^2}{4} \\
 &= \lim_{h \rightarrow 0} \left[h^4 \cdot \frac{n^2(n^2-2n+1)}{4} \right] = \lim_{h \rightarrow 0} \left[\frac{(nh)^4}{4} - 2 \cdot \frac{(nh)^3 \cdot h}{4} + \frac{(nh)^2 \cdot h^2}{4} \right] \\
 &= \frac{(2)^4}{4} - \frac{2 \cdot (2)^3 \cdot 0}{4} + \frac{(2)^2 \cdot 0}{4} = \frac{16}{4} - 0 + 0 = 4
 \end{aligned}$$

11. $\int_2^4 (x^2 - 3x + 2) dx$

Sol. Let $f(x) = x^2 - 3x + 2$ i.e. $a=2$, $b=4$, $nh=4-2=2$

$$\begin{aligned}
 \therefore \int_2^4 (x^2 - 3x + 2) dx &= \lim_{h \rightarrow 0} h \left[f(2) + f(2+h) + f(2+2h) + \dots + f(2+(n-1)h) \right] \\
 &= \lim_{h \rightarrow 0} h \left[0 + (h+h^2) + (2h+4h^2) + (2h+9h^2) + \dots + \{(n-1)h + (n-1)^2 h^2\} \right] \\
 &= \lim_{h \rightarrow 0} h \left[h(1+2+3+\dots+(n-1)) + h^2(1^2+2^2+3^2+\dots+(n-1)^2) \right] \\
 &= \lim_{h \rightarrow 0} h \left[h \cdot \frac{n(n-1)}{2} + h^2 \cdot \frac{n(n-1)(2n-1)}{6} \right] = \lim_{h \rightarrow 0} \left[\frac{h^2(n^2-n)}{2} + \frac{h^3(2n^3-3n^2+n)}{6} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{(nh)^2}{2} - \frac{(nh) \cdot h}{2} + \frac{(nh)^3}{3} - \frac{(nh)^2 \cdot h}{2} + \frac{(nh) \cdot h^2}{6} \right] \\
 &= \frac{(2)^2}{2} - \frac{2 \cdot 0}{2} + \frac{(2)^3}{3} - \frac{(2)^2 \cdot 0}{2} + \frac{2 \cdot 0}{6} = 2 + \frac{8}{3} = \frac{14}{3}
 \end{aligned}$$

12. $\int_0^2 (x^2 + x) dx$

Sol. Let $f(x) = x^2 + x$ i.e. $a=0$, $b=2$, $nh=2-0=2$

$$\begin{aligned}
 \therefore \int_0^2 (x^2 + x) dx &= \lim_{h \rightarrow 0} h [f(0) + f(0+h) + f(0+2h) + \dots + f(0+(n-1)h)] \\
 &= \lim_{h \rightarrow 0} h [0 + (h^2 + h) + (4h^2 + h) + (9h^2 + 3h) + \dots + \{(n-1)^2 h^2 + (n-1)h\}] \\
 &= \lim_{h \rightarrow 0} h [h^2 (1^2 + 2^2 + 3^2 + \dots + (n-1)^2) + h(1 + 2 + 3 + \dots + (n-1))] \\
 &= \lim_{h \rightarrow 0} h \left[h^2 \cdot \frac{n(n-1)(2n-1)}{6} + h \cdot \frac{n(n-1)}{2} \right] = \lim_{h \rightarrow 0} \left[h^3 \frac{(2n^3 - 3n^2 + n)}{6} + \frac{h^2 (n^2 - n)}{2} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{(nh)^3}{3} - \frac{3(nh)^2 \cdot h}{6} + \frac{(nh) \cdot h^2}{6} + \frac{(nh)^2}{2} - \frac{(nh) \cdot h}{2} \right] \\
 &= \frac{(2)^3}{3} - \frac{3(2)^2 \cdot 0}{6} + \frac{2 \cdot 0}{6} + \frac{(2)^2}{2} - \frac{2 \cdot 0}{2} = \frac{8}{3} + 2 = \frac{14}{3}
 \end{aligned}$$

13. $\int_0^3 (2x^2 + 3x + 5) dx$

Sol. Let $f(x) = 2x^2 + 3x + 5$ i.e. $a = 0$, $b = 3$, $nh = 3 - 0 = 3$

$$\begin{aligned}
 \therefore \int_0^3 (2x^2 + 3x + 5) dx &= \lim_{h \rightarrow 0} h [f(0) + f(0+h) + f(0+2h) + \dots + f(0+(n-1)h)] \\
 &= \lim_{h \rightarrow 0} h [5 + (2h^2 + 3h + 5) + (8h^2 + 6h + 5) + \dots + \{2(n-1)^2 h^2 + 3(n-1)h + 5\}] \\
 &= \lim_{h \rightarrow 0} h [5n + \{3h(1 + 2 + 3 + \dots + (n-1))\} + 2h^2 (1^2 + 2^2 + 3^2 + \dots + (n-1)^2)] \\
 &= \lim_{h \rightarrow 0} h \left[5n + 3h \cdot \frac{n(n-1)}{2} + 2h^2 \cdot \frac{n(n-1)(2n-1)}{6} \right] \\
 &= \lim_{h \rightarrow 0} \left[5nh + \frac{3h^2 (n^2 - n)}{2} + \frac{h^3 (2n^3 - 3n^2 + n)}{3} \right] \\
 &= \lim_{h \rightarrow 0} \left[5nh + \frac{3(nh)^2}{2} - \frac{3(nh) \cdot h}{2} + \frac{2(nh)^3}{3} - \frac{3(nh)^2 \cdot h}{3} + \frac{(nh) \cdot h^2}{3} \right] \\
 &= 5(3) + \frac{3(3)^2}{2} - \frac{3 \cdot 3 \cdot 0}{2} + \frac{2(3)^3}{3} - \frac{3(3)^2 \cdot 0}{3} + \frac{3 \cdot 0}{3} = 15 + \frac{27}{2} + 18 = 33 + \frac{27}{2} = \frac{93}{2}
 \end{aligned}$$

14. $\int_0^1 |3x-1| dx$

Sol. Let $I = \int_0^1 |3x-1| dx$

$$\begin{array}{c}
 \frac{1}{3} \\
 -\infty \quad -ve \quad +ve \quad +\infty
 \end{array}$$

$$3x-1=0 \Rightarrow x = \frac{1}{3}$$

$$I = \int_0^1 |3x-1| dx \Rightarrow I = \int_0^{1/3} |3x-1| dx + \int_{1/3}^1 |3x-1| dx \Rightarrow I = \int_0^{1/3} -(3x-1) dx + \int_{1/3}^1 (3x-1) dx$$

$$\Rightarrow I = \int_0^{1/3} (1-3x) dx + \int_{1/3}^1 (3x-1) dx$$

$$\Rightarrow I = I_1 + I_2, \text{ where } I_1 = \int_0^{1/3} (1-3x) dx, \quad I_2 = \int_{1/3}^1 (3x-1) dx$$

$$I_1 = \int_0^{1/3} (1-3x) dx, \quad nh = \frac{1}{3} - 0 = \frac{1}{3}$$

$$f(x) = 1 - 3x$$

$$f(0) = 1$$

$$f(0+h) = 1 - 3(0+h)$$

$$f(0+2h) = 1 - 3(0+2h)$$

$$f(0+3h) = 1 - 3(0+3h)$$

$$\dots\dots\dots$$

$$f\{1+(n-1)h\} = 1 - 3(n-1)h$$

$$\int_0^{1/3} (1-3x) dx = \lim_{h \rightarrow 0} h \left[f(0) + f(0+h) + f(0+2h) + \dots + f(0+(n-1)h) \right]$$

$$= \lim_{h \rightarrow 0} h \left[1 + (1-3h) + 1-3.2h + (1-3.3h) + \dots + 1-3(n-1)h \right]$$

$$= \lim_{h \rightarrow 0} h \left[(1+1+1+\dots) - 3h\{1+2+3+\dots+(n-1)\} \right]$$

$$= \lim_{h \rightarrow 0} h \left[n - 3h \cdot \frac{n(n+1)}{2} \right] = \lim_{h \rightarrow 0} \left[nh - 3nh \frac{(nh+h)}{2} \right] = \frac{1}{3} - \frac{3 \cdot \frac{1}{3} \left(\frac{1}{3} + 0 \right)}{2} = \frac{1}{3} - \frac{1}{6} = \frac{2-1}{6} = \frac{1}{6}$$

$$I_2 = \int_{1/3}^1 (3x-1) dx, \quad nh = 1 - \frac{1}{3} = \frac{2}{3}$$

$$f(x) = 3x - 1$$

$$f\left(\frac{1}{3}\right) = 3 \cdot \frac{1}{3} - 1 = 0$$

$$f\left(\frac{1}{3}+h\right) = 3\left(\frac{1}{3}+h\right) - 1 = 3h$$

$$f\left(\frac{1}{3}+2h\right) = 3\left(\frac{1}{3}+2h\right) - 1 = 3.2h$$

$$\dots\dots\dots$$

$$f\left\{\frac{1}{3}+(n-1)h\right\} = 3\left\{\frac{1}{3}+(n-1)h\right\} - 1 = 3(n-1)h$$

$$\int_{1/3}^1 (3x-1) dx = \lim_{h \rightarrow 0} h \left[f\left(\frac{1}{3}\right) + f\left(\frac{1}{3}+h\right) + f\left(\frac{1}{3}+2h\right) + \dots + f\left\{\frac{1}{3}+(n-1)h\right\} \right]$$

$$= \lim_{h \rightarrow 0} h \left[0 + 3h + 3.2h + 3.3h + \dots + 3(n-1)h \right] = \lim_{h \rightarrow 0} h.3h \left[1 + 2 + 3 + \dots + (n-1) \right]$$

$$= \lim_{h \rightarrow 0} 3h.h \cdot \frac{n(n+1)}{2} = \lim_{h \rightarrow 0} \frac{3nh(nh+h)}{2} = \frac{3 \cdot \frac{2}{3} \left(\frac{2}{3} - 0 \right)}{2} = \frac{2}{3}$$

$$\text{So, } I = \frac{1}{6} + \frac{2}{3} = \frac{1+4}{6} = \frac{5}{6}$$

15. $\int_0^2 e^x dx$

Sol. Let $f(x) = e^x$ i.e. $a = 0$, $b = 2$, $nh = 2 - 0 = 2$

$$\begin{aligned} \therefore \int_0^2 e^x dx &= \lim_{h \rightarrow 0} h [f(0) + f(0+h) + f(0+2h) + \dots + f(0+(n-1)h)] \\ &= \lim_{h \rightarrow 0} h [e^0 + e^{0+h} + e^{0+2h} + \dots + e^{0+(n-1)h}] = \lim_{h \rightarrow 0} h [e^0 + e^0 \cdot e^h + e^0 \cdot e^{2h} + \dots + e^0 \cdot e^{(n-1)h}] \\ &= \lim_{h \rightarrow 0} h [1 + e^h + (e^h)^2 + \dots + (e^h)^{n-1}] = \lim_{h \rightarrow 0} h \left[1 \cdot \frac{(e^h)^n - 1}{e^h - 1} \right] = \lim_{h \rightarrow 0} h \left(\frac{e^{nh} - 1}{e^h - 1} \right) = \lim_{h \rightarrow 0} \frac{e^2 - 1}{\frac{e^h - 1}{h}} = e^2 - 1 \end{aligned}$$

16. $\int_1^3 e^{-x} dx$

Sol. Let $f(x) = e^{-x}$ i.e. $a = 1$, $b = 3$, $nh = 3 - 1 = 2$

$$\begin{aligned} \therefore \int_1^3 e^{-x} dx &= \lim_{h \rightarrow 0} h [f(1) + f(1+h) + \dots + f(1+(n-1)h)] \\ &= \lim_{h \rightarrow 0} h [e^{-1} + e^{-(1+h)} + e^{-(1+2h)} + \dots + e^{-\{1+(n-1)h\}}] = \lim_{h \rightarrow 0} h [e^{-1} + e^{-1} \cdot e^{-h} + e^{-1} \cdot e^{-2h} + \dots + e^{-1} \cdot e^{-(n-1)h}] \\ &= \lim_{h \rightarrow 0} e^{-1} h [1 + (e^{-h}) + (e^{-h})^2 + \dots + (e^{-h})^{n-1}] = \lim_{h \rightarrow 0} h e^{-1} \left\{ \frac{(e^{-h})^n - 1}{e^{-h} - 1} \right\} \\ &= e^{-1} \left(\frac{e^{-nh} - 1}{-1} \right) = - \left(\frac{e^{-2} - 1}{e^{-1} - 1} \right) = - \left(\frac{e^2 - 1}{e - 1} \right) = \frac{e^2 - 1}{e^3} \end{aligned}$$

17. $\int_a^b \cos x dx$

Sol. Let $f(x) = \cos x$ i.e. $a = a$, $b = b$, $nh = b - a$

$$\begin{aligned} \therefore \int_a^b \cos x dx &= \lim_{h \rightarrow 0} h [f(a) + f(a+h) + \dots + f(a+(n-1)h)] \\ &= \lim_{h \rightarrow 0} h [\cos a + \cos(a+h) + \cos(a+2h) + \dots + \cos(a+(n-1)h)] \\ &= \lim_{h \rightarrow 0} h \frac{\sin \frac{nh}{2} \cos \left\{ a + \left(\frac{n-1}{2} \right) h \right\}}{\sin \left(\frac{h}{2} \right)} = \lim_{h \rightarrow 0} \frac{h \sin \frac{nh}{2} \cos \left(\frac{2a + nh - h}{2} \right)}{\sin(h/2) \cdot \frac{h}{2}} \\ &= 2 \sin \left(\frac{b-a}{2} \right) \cos \left(\frac{2a + b - a - 0}{2} \right) = 2 \sin \left(\frac{b-a}{2} \right) \cos \left(\frac{a+b}{2} \right) \\ &= \sin \left(\frac{b-a}{2} + \frac{a+b}{2} \right) + \sin \left(\frac{b-a}{2} - \frac{a+b}{2} \right) = \sin \left(\frac{b-a+a+b}{2} \right) + \sin \left(\frac{b-a-a-b}{2} \right) \\ &= \sin(b) + \sin(-a) = \sin b - \sin a \end{aligned}$$