

HOMOGENEOUS DIFFERENTIAL EQUATIONS (XII, R. S. AGGARWAL)

EXERCISE 20 (Pg. No.: 961)

In each of the following differential equations show that it is homogenous and solve it

1. $x dy = (x + y) dx$

Sol. $x dy = (x + y) dx$

$$\frac{dy}{dx} = \left(\frac{x+y}{x} \right)$$

$\therefore (x+y)$ and x both are homogeneous function because both function having 1 degree

$$\frac{dy}{dx} = \left(\frac{x+y}{x} \right) \text{ of a homogenous differential equation}$$

$$\frac{dy}{dx} = \left(\frac{x+y}{x} \right) \dots\dots (i)$$

Let $y = vx$

$$\frac{dy}{dx} = \frac{d(vx)}{dx}$$

$$\frac{dy}{dx} = v \frac{dx}{dx} + x \frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \dots\dots (ii)$$

From equation (i) and (ii) $\left(\frac{x+y}{x} \right) = v + x \frac{dv}{dx}$

Putting $y = vx$

$$\frac{x + \sqrt{x}}{x} = v + x \frac{dv}{dx}$$

$$\frac{x(1+v)}{x} = v + x \frac{dv}{dx}$$

$$1 + v = v + x \frac{dv}{dx}$$

$$1 = x \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{1}{x}$$

$$dv = \frac{1}{x} dx$$

Integrating on both side we get $\int dv = \int \frac{1}{x} dx$

$$v = \log|x| + C$$

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Putting $y = vx$

$$v = \frac{y}{x}$$

$$\frac{y}{x} = \log|x| + c$$

$$y = \log|x| + cx$$

2. $(x^2 - y^2)dx + 2xy dy = 0$

Sol. $(x^2 - y^2)dx + 2xy dy = 0$

On separating variables we get $(x^2 - y^2)dx + 2xy dy = 0$

$$2xy dy = -(x^2 - y^2)dx$$

$$\frac{dy}{dx} = \frac{-(x^2 - y^2)}{2xy}$$

$\therefore -(x^2 + y^2)$ and $2xy$ both are homogenous function because both function having 2 degree

$$\frac{dy}{dx} = \frac{-(x^2 - y^2)}{2xy} \text{ of a homogeneous diff equation}$$

$$\frac{dy}{dx} = \frac{-(x^2 - y^2)}{2xy} \dots\dots (i)$$

Let $y = vx \Rightarrow v = \frac{y}{x}$

Diff on the both sides we get $\frac{dy}{dx} = v \frac{dx}{dx} + x \frac{dv}{dx}$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \dots\dots (ii)$$

From equation (i) and (ii)

$$\frac{-(x^2 - y^2)}{2xy} = v + x \frac{dv}{dx}$$

$$\frac{(y^2 - x^2)}{2xy} = v + x \frac{dv}{dx}$$

Putting $y = vx$

$$\frac{v^2x^2 - x^2}{2x(vx)} = v + x \frac{dv}{dx}$$

$$\frac{x^2(v^2 - 1)}{2vx^2} = v + x \frac{dv}{dx}$$

$$\frac{(v^2 - 1)}{2v} = v + x \frac{dv}{dx}$$

$$\frac{(v^2 - 1)}{2v} = v + x \frac{dv}{dx}$$

$$\frac{v^2 - 1}{2v} - v = x \frac{dv}{dx}$$

$$\frac{v^2 - 1 - 2v^2}{2v} = x \frac{dv}{dx}$$

$$\frac{-1 - v^2}{2v} = x \frac{dv}{dx}$$

$$\frac{-(1 + v^2)}{2v} = x \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{-(1 + x)}{2vx}$$

$$\frac{2v}{(1 + v^2)} dv = -\frac{1}{x} dx$$

Integrating on the both side, we get

$$\int \frac{2v}{(1 + v^2)} dv = \int -\frac{1}{x} dx$$

$$2 \int \frac{v}{(1 + v^2)} dv = - \int \frac{1}{x} dx$$

$$\text{Let } (1 + v^2) = t$$

Diff on the both sides wrt v

$$0 + 2v = \frac{dt}{dv}$$

$$2v = \frac{dt}{dv}$$

$$dv = \frac{dt}{2v}$$

$$2 \int \frac{v}{t} \times \frac{dt}{2v} = - \int \frac{1}{x} dx$$

$$\frac{2}{2} \int \frac{1}{t} dt = - \int \frac{1}{x} dx$$

$$\int \frac{1}{t} dt = - \int \frac{1}{x} dx$$

$$\log|t| = -\log|x| + c$$

$$\log|1 + v^2| = -\log|x + c|$$

$$\log|1 + v^2| + \log|x| = c$$

$$\log|(1 + v^2)(x)| = \log c$$

$$(1 + v^2)x = c$$

$$\text{Putting } v = \frac{y}{x} \quad \left(1 + \frac{y^2}{x^2}\right)x = c$$

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$$\left(\frac{x^2 + y^2}{x^2} \right) x = c$$

$$\frac{x^2 + y^2}{x} = c$$

$$x^2 + y^2 = cx$$

3. $x^2 dy + y(x + y) dx = 0$

Sol. $x^2 dy + y(x + y) dx = 0$

On separating variables we get

$$x^2 dy = y(x + y) dx$$

$$\frac{dy}{dx} = \frac{-y(x + y)}{x^2}$$

$$\frac{dy}{dx} = \frac{-(xy + y^2)}{x^2}$$

$\therefore -(xy + y^2)$ and x^2 both are homogeneous function because both function having 2 degree

$$\frac{dy}{dx} = \frac{-(xy + y^2)}{x^2} \text{ of a homogeneous diff equation}$$

$$\frac{dy}{dx} = \frac{-(xy + y^2)}{x^2} \dots\dots (i)$$

Let $y = vx \Rightarrow v = \frac{y}{x}$

Diff on the both sides w.r. to x

$$\frac{dy}{dx} = \frac{v dx}{dx} + x \frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \dots\dots (ii)$$

From equation (i) and (ii)

$$\frac{-(xy + y^2)}{x^2} = v + x \frac{dv}{dx}$$

Putting $y = vx$

$$\frac{-(x(vx) + v^2 x^2)}{x^2} = v + x \frac{dv}{dx}$$

$$\frac{-(vx^2 + v^2 x^2)}{x^2} = v + x \frac{dv}{dx}$$

$$-(v + v^2) = v + x \frac{dv}{dx}$$

$$-v - v^2 - v = x \frac{dv}{dx}$$

$$-(v^2 + 2v) = x \frac{dv}{dx}$$

$$-\frac{(v^2 + 2v)}{x} = \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{-(v^2 + 2v)}{x}$$

$$\frac{dv}{(v^2 + 2v)} = -\frac{1}{x} dx$$

$$\frac{dv}{v(v+2)} = -\frac{1}{x} dx$$

Integrating on the both sides, we get

$$\int \frac{dv}{v(v+2)} = \int -\frac{1}{x} dx$$

$$\int \frac{dv}{v(v+2)} = -\int \frac{1}{x} dx$$

$$\therefore \frac{1}{v(v+2)} = \frac{A}{v} + \frac{B}{v+2} \dots (iii)$$

$$\frac{1}{v(v+2)} = \frac{A(v+2) + Bv}{v(v+2)}$$

$$1 = A(v+2) + Bv$$

Putting $v = -2$

$$1 = A(-2+2) + B(-2)$$

$$1 = 0 + B(-2)$$

$$1 = -2B$$

$$\therefore B = -\frac{1}{2}$$

Putting $v = 0$

$$1 = A(0+2) + B(0)$$

$$1 = 2A + 0$$

$$\therefore A = \frac{1}{2}$$

Putting the value of A and B in equation (iii)

$$\frac{1}{v(v+2)} = \frac{\left(\frac{1}{2}\right)}{v} + \frac{\left(-\frac{1}{2}\right)}{v+2} = \frac{1}{2} \times \frac{1}{v} - \frac{1}{2} \times \frac{1}{v+2}$$

$$\int \frac{dv}{v(v+2)} = -\int \frac{1}{x} dx$$

$$\int \frac{1}{2} \times \frac{1}{v} dv - \int \frac{1}{2} \times \frac{1}{v+2} dv = -\int \frac{1}{x} dx$$

$$\frac{1}{2} \int \frac{1}{v} dv - \frac{1}{2} \int \frac{1}{v+2} dv = -\int \frac{1}{x} dx$$

$$\frac{1}{2} \log|v| - \frac{1}{2} \log|v+2| = -\log|x| + c$$

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$$\frac{1}{2} \log \left| \frac{v}{v+2} \right| = -\log |x| + c$$

Putting $v = y/x$

$$\frac{1}{2} \log \left[\frac{y/x}{y/x+2} \right] = -\log |x| + \log c$$

$$\log \left[\frac{y/x}{y+2x} \right] = -2 \log |x| + 2 \log c$$

$$\log \left[\frac{y}{y+2x} \right] = -\log |x^2| + \log c^2$$

$$\log \left[\frac{y}{y+2x} \right] + \log |x^2| = \log c^2$$

$$\log \left| \left(\frac{y}{y+2x} \right) x^2 \right| = \log c^2$$

$$\log \left| \left(\frac{vx}{vx+2x} \right) x^2 \right| = \log c^2$$

Putting $v = \frac{y}{x}$

$$\left| \left[\frac{(y/x)x}{(y/x)x+2x} \right] x^2 \right| = \log c^2$$

$$\log \left| \frac{yx^2}{y+2x} \right| = \log c^2$$

$$x^2 y = c^2 (y+2x)$$

4. $(x-y)dy - (x+y)dx = 0$

Sol. $(x-y)dy - (x+y)dx = 0$

On separating variables we get $(x-y)dy = (x+y)dx$

$$\frac{dy}{dx} = \frac{(x+y)}{(x-y)}$$

$\therefore (x+y)$ and $(x-y)$ both are homogenous function because both function having 1 degree

$$\frac{dy}{dx} = \frac{(x+y)}{(x-y)} \text{ of a homogenous differential equation } \frac{dy}{dx} = \frac{(x+y)}{(x-y)} \quad \dots (i)$$

Let $y = vx \Rightarrow v = \frac{y}{x}$

Diff on the both sides w.r. to x

$$\frac{dy}{dx} = v \frac{dx}{dx} + x \frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad \dots (ii)$$

From equation (i) and (ii)

$$\frac{(x+y)}{(x-y)} = v + x \frac{dv}{dx}$$

Putting $y = vx$

$$\frac{(x+vx)}{(x-vx)} = v + x \frac{dv}{dx}$$

$$\frac{x(1+v)}{x(1-v)} = v + x \frac{dv}{dx}$$

$$\frac{(1+v)}{(1-v)} = v + x \frac{dv}{dx}$$

$$\frac{(1+v)}{(1-v)} - v = x \frac{dv}{dx}$$

$$\frac{(1+v) - v(1-v)}{(1-v)} = x \frac{dv}{dx}$$

$$\frac{1+v-v+v^2}{(1-v)} = x \frac{dv}{dx}$$

$$\frac{(1+v^2)}{(1-v)} = x \frac{dv}{dx}$$

$$\frac{(1+v^2)}{(1-v)x} = \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{(1+v^2)}{(1-v)x}$$

$$\frac{(1-v)}{(1+v^2)} dv = \frac{1}{x} dx$$

Integrating on the both side we get

$$\int \frac{(1-v)}{(1+v^2)} dv = \int \frac{1}{x} dx$$

$$\int \frac{1}{1+v^2} dv - \int \frac{v}{1+v^2} dv = \int \frac{1}{x} dx$$

$$\int \frac{1}{1+v^2} dv - \frac{1}{2} \int \frac{2v}{1+v^2} dv = \int \frac{1}{x} dx$$

$$\frac{1}{1} \tan^{-1} v - \frac{1}{2} \int \frac{2v}{1+v^2} dv = \log|x| + c$$

$$\tan^{-1} v - \frac{1}{2} \int \frac{2v}{1+v^2} dv = \log|x| + c$$

Let $1+v^2 = t$

Diff on the both side w.r. to v

$$0 + 2v = \frac{dt}{dv}$$

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$$dv = \frac{dt}{2v}$$

$$\tan^{-1} v - \frac{1}{2} \int \frac{2v}{t} \times \frac{dt}{2v} = \log|x| + c$$

$$\tan^{-1} v - \frac{1}{2} \int \frac{1}{t} dt = \log|x| + c$$

$$\tan^{-1} v - \frac{1}{2} \log t = \log|x| + c$$

$$\tan^{-1} v - \frac{1}{2} \log|1 + v^2| = \log|x| + c$$

Putting $v = \frac{y}{x}$

$$\tan^{-1} \frac{y}{x} - \frac{1}{2} \log\left|1 + \frac{y^2}{x^2}\right| = \log|x| + c$$

$$\tan^{-1} \frac{y}{x} - \frac{1}{2} \log\left|\frac{x^2 + y^2}{x^2}\right| = \log|x| + c$$

$$\tan^{-1} \frac{y}{x} = \frac{1}{2} \log\left|\frac{x^2 + y^2}{x^2}\right| + \log|x| + c$$

$$\tan^{-1} \frac{y}{x} = \log\left|\frac{x^2 + y^2}{x^2}\right|^{1/2} + \log|x| + c$$

$$\tan^{-1} \frac{y}{x} = \log\left|\sqrt{\frac{x^2 + y^2}{x^2}}\right| + \log|x| + c$$

$$\tan^{-1} \frac{y}{x} = \log\left|\frac{\sqrt{x^2 + y^2}}{x} \times x\right| + c$$

$$\tan^{-1} \frac{y}{x} = \log|\sqrt{x^2 + y^2}| + c$$

$$\tan^{-1} \frac{y}{x} = \log|x^2 + y^2|^{1/2} + c$$

$$\tan^{-1} \frac{y}{x} = \frac{1}{2} \log|x^2 + y^2| + c$$

5. $(x + y)dy + (y - 2x)dx = 0$

Sol. $(x + y)dy + (y - 2x)dx = 0$

On separating variables we get $(x + y)dy + (y - 2x)dx = 0$

$$(x + y)dy = -(y - 2x)dx$$

$$\frac{dy}{dx} = \frac{-(y - 2x)}{(x + y)}$$

$\therefore -(y - 2x)$ and $(x + y)$ both are homogeneous function because both function having 1 degree

$$\frac{dy}{dx} = \frac{-(y - 2x)}{(x + y)} \text{ of homogenous differential equation } \frac{dy}{dx} = v \frac{dx}{dx} + x \frac{dv}{dx} \quad (i)$$

Let $y = vx \Rightarrow v = \frac{y}{x}$

Diff on the both side w.r. to x

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad \dots (ii)$$

From equation (i) and (ii)

$$\frac{-(y-2x)}{(x+y)} = v + x \frac{dv}{dx}$$

Putting $y = vx$

$$\frac{-(v-2x)}{(x+vx)} = v + x \frac{dv}{dx}$$

$$\frac{(2x-vx)}{(x+vx)} = v + x \frac{dv}{dx}$$

$$\frac{x(2-v)}{x(1+v)} = v + x \frac{dv}{dx}$$

$$\frac{(2-v)}{(1+v)} = v + x \frac{dv}{dx}$$

$$\frac{(2-v)}{(1+v)} - v = x \frac{dv}{dx}$$

$$\frac{(2-v)-v(1+v)}{(1+v)} = x \frac{dv}{dx}$$

$$\frac{2-v-v-v^2}{1+v} = x \frac{dv}{dx}$$

$$\frac{2-2v-v^2}{1+v} = x \frac{dv}{dx}$$

$$\frac{-(v^2+2v-2)}{(1+v)} = x \frac{dv}{dx}$$

$$\frac{-(v^2+2v-2)}{(1+v)x} = \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{-(v^2+2v-2)}{(v+1)x}$$

$$\frac{(v+1)dv}{v^2+2v-2} = -\frac{1}{x} dx$$

Integrating on the both sides we get

$$\int \frac{(v+1)}{v^2+2v-2} dv = \int -\frac{1}{x} dx$$

$$\frac{1}{2} \int \frac{2(v+1)}{v^2+2v-2} dv = -\int \frac{1}{x} dx$$

$$\frac{1}{2} \int \frac{2v+2}{v^2+2v-2} dv = -\int \frac{1}{x} dx$$

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$$\frac{1}{2} \int \frac{2x+2}{v^2+2v-2} dv = -\log|x| + c$$

$$\text{Let } v^2 + 2v - 2 = t$$

Diff on the both sides w.r. to v

$$2v + 2 - 0 = \frac{dt}{dv}$$

$$(2v + 2) = \frac{dt}{dv}$$

$$dv = \frac{dt}{(2v + 2)}$$

$$\frac{1}{2} \int \frac{(2v + 2)}{t} \times \frac{dt}{(2v + 2)} = -\log|x| + c$$

$$\frac{1}{2} \int \frac{1}{t} dt = -\log|x| + c$$

$$\frac{1}{2} \log|t| + \log|x| = c$$

$$\frac{1}{2} \log|v^2 + 2v - 2| + \log|x| = c$$

$$\text{Putting } v = \frac{y}{x}$$

$$\frac{1}{2} \log \left| \frac{y^2}{x^2} + 2 \frac{y}{x} - 2 \right| + \log|x| = c$$

$$\frac{1}{2} \log \left| \frac{y^2 + 2xy - 2x^2}{x^2} \right| + \log|x| = c$$

$$\log \left| \frac{y^2 + 2xy - 2x^2}{x^2} \right|^{1/2} + \log|x| = c$$

$$\log \left| \sqrt{\frac{y^2 + 2xy - 2x^2}{x^2}} \right| + \log|x| = c$$

$$\log \left| \left(\frac{\sqrt{y^2 + 2xy - 2x^2}}{x} \right) \times x \right| = \log c$$

$$(y^2 + 2xy - 2x^2)^{1/2} = c$$

$$y^2 + 2xy - 2x^2 = c^2$$

$$y^2 + 2xy - 2x^2 = c$$

6. $(x^2 + 3xy + y^2) dx - x^2 dy = 0$

Sol. $(x^2 + 3xy + y^2) dx = x^2 dy = 0$

On separating variables we get $(x^2 + 3xy + y^2) dx = x^2 dy$

$$x^2 dy = (x^2 + 3xy + y^2) dx$$

$$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2}$$

$\therefore (x^2 + 3xy + y^2)$ and x^2 both are homogeneous function because both function having 2 degree

$$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2} \text{ of a homogenous differential equation}$$

$$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2} \dots (i)$$

$$\text{Let } y = vx \Rightarrow v = \frac{y}{x}$$

Diff on the both sides w.r. to x

$$\frac{dy}{dx} = v \frac{dx}{dx} + x \frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \dots (ii)$$

From equation (i) and (ii)

$$\frac{x^2 + 3xy + y^2}{x^2} = v + x \frac{dv}{dx}$$

Putting $y = vx$

$$\frac{x^2 + 3x(vx) + v^2x^2}{x^2} = v + x \frac{dv}{dx}$$

$$\frac{x^2(1 + 3v + v^2)}{x^2} = v + x \frac{dv}{dx}$$

$$(1 + 3v + v^2) = v + x \frac{dv}{dx}$$

$$1 + 3v + v^2 - v = x \frac{dv}{dx}$$

$$(1 + 2v + v^2) = x \frac{dv}{dx}$$

$$(1 + v)^2 = x \frac{dv}{dx}$$

$$x \frac{dv}{dx} = (v + 1)^2$$

$$\frac{dv}{dx} = \frac{(v + 1)^2}{x}$$

$$\frac{dv}{(v + 1)^2} = \frac{1}{x} dx$$

Integrating on the both sides we gets

$$\int \frac{dv}{(v + 1)^2} = \int \frac{1}{x} dx$$

$$\int (v + 1)^{-2} dv = \log|x| + c$$

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$$\frac{(v+1)^{-2+1}}{-2+1} = \log|x| + c$$

$$-(v+1)^{-1} = \log|x| + c$$

$$-\frac{1}{(v+1)} = \log|x| + c$$

Putting $v = \frac{y}{x}$

$$-\frac{1}{\frac{y}{x} + 1} = \log|x| + c$$

$$-\frac{1}{\frac{y+x}{x}} = \log|x| + c$$

$$-\frac{x}{y+x} = \log|x| + c$$

$$-\frac{x}{y+x} - \log|x| = c$$

$$-\left(\log|x| + \frac{x}{y+x}\right) = c$$

$$\log|x| + \frac{x}{y+x} = \frac{c}{-1}$$

$$\log|x| + \frac{x}{y+x} = c$$

7. $2xy dx + (x^2 + 2y^2) dy = 0$

Sol. $2xy dx + (x^2 + 2y^2) dy = 0$

On separating variables we get $(x^2 + 2y^2) dy = -2xy dx$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 + 2y^2}$$

$\therefore -2xy$ and $x^2 + 2y^2$ both are homogeneous function because both function having 2 degree

$$\frac{dy}{dx} = \frac{-2xy}{x^2 + 2y^2} \text{ of a homogeneous differential equation } \frac{dy}{dx} = \frac{-2xy}{x^2 + 2y^2} \dots (i)$$

Let $y = vx \Rightarrow v = \frac{y}{x}$

Diff on the both sides w.r. to x

$$\frac{dy}{dx} = v \frac{dx}{dx} + x \frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \dots (ii)$$

From equation (i) and (ii)

$$\frac{-2xy}{x^2 + 2y^2} = v + x \frac{dv}{dx}$$

Putting $y = vx$

$$\frac{-2x(vx)}{x^2 + 2(vx)^2} = v + x \frac{dv}{dx}$$

$$\frac{-2vx^2}{x^2 + 2v^2x^2} = v + x \frac{dv}{dx}$$

$$\frac{-2vx^2}{x^2(1+2v^2)} = v + x \frac{dv}{dx}$$

$$\frac{-2v}{1+2v^2} = v + x \frac{dv}{dx}$$

$$\frac{-2v}{1+2v^2} - v = x \frac{dv}{dx}$$

$$\frac{-2v - v(1+2v^2)}{1+2v^2} = x \frac{dv}{dx}$$

$$\frac{-2v - v - 2v^3}{1+2v^2} = x \frac{dv}{dx}$$

$$\frac{-3v - 2v^3}{1+2v^2} = x \frac{dv}{dx}$$

$$\frac{-v(2v^2+3)}{1+2v^2} = x \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{-v(2v^2+3)}{x(1+2v^2)}$$

$$\frac{1+2v^2}{v(2v^2+3)} dv = -\frac{1}{x} dx$$

Integrating on the both sides we get

$$\int \frac{1+2v^2}{v(2v^2+3)} dv = \int -\frac{1}{x} dx$$

Left side dividing and multiplying by 3

$$\frac{3}{3} \int \frac{1+2v^2}{v(2v^2+3)} dv = -\int \frac{1}{x} dx \quad \frac{1}{3} \int \left(\frac{3+6v^2}{2v^3+3v} \right) dv = -\int \frac{1}{x} dx$$

$$\text{Let } 2v^3 + 3v = t$$

Diff on the both sides w.r. to v

$$6v^2 + 3 = \frac{dt}{dv}$$

$$dv = \frac{dt}{6v^2 + 3} = \frac{dt}{3+6v^2}$$

$$\frac{1}{3} \int \left(\frac{3+6v^2}{2v^3+3v} \right) dv = -\int \frac{1}{x} dx$$

$$\frac{1}{3} \int \frac{(3+6v^2)}{t} \times \frac{dt}{(3+6v^2)} = -\int \frac{1}{x} dx$$

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$$\frac{1}{3} \int \frac{1}{t} dt = - \int \frac{1}{x} dx$$

$$\frac{1}{3} \log |t| = - \log |x| + c$$

$$\frac{1}{3} \log |2v^3 + 3v| = - \log |x| + c$$

$$\log |2v^3 + 3v| = -3 \log |x| + 3 \log |C|$$

Putting $v = \frac{y}{x}$

$$\log \left| 2 \times \frac{y^3}{x^3} + 3 \times \frac{y}{x} \right| = -3 \log |x| + 3 \log |c|$$

$$\log \left| \frac{2y^3 + 3x^2y}{x^3} \right| = -3 \log |x| + 3 \log |c|$$

$$\log \left| \frac{2y^3 + 3x^2y}{x^3} \right| + 3 \log |x| = 3 \log |c|$$

$$\log \left| \frac{2y^3 + 3x^2y}{x^3} \right| + \log |x|^3 = 3 \log |c|$$

$$\log \left| \left(\frac{2y^3 + 3x^2y}{x^3} \right) \times x^3 \right| = 3 \log |c|$$

$$\log |2y^3 + 3x^2y| = \log |c^3|$$

$$2y^3 + 3x^2y = c \quad [\because c^3 \equiv c \text{ constant}]$$

$$2y^3 + 3x^2y = c$$

$$3x^2y + 2y^3 = c$$

8. $\frac{dy}{dx} + \frac{x-2y}{2x-y} = 0$

Sol. $\frac{dy}{dx} + \frac{x-2y}{2x-y} = 0$

On separating variables we get $\frac{dy}{dx} = - \left(\frac{x-2y}{2x-y} \right)$

$$\frac{dy}{dx} = \frac{2y-x}{2x-y}$$

$\because 2y-x$ and $2x-y$ both are homogenous function because both function having 1 degree

$$\frac{dy}{dx} = \frac{2y-x}{2x-y} \text{ of a homogenous differential equation } \frac{dy}{dx} = \frac{2y-x}{2x-y} \dots\dots (i)$$

Let $y = vx \Rightarrow v = \frac{y}{x}$

Differentiate on the both sides w.r. to x

$$\frac{dy}{dx} = v \frac{dx}{dx} + x \frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad \dots (ii)$$

From equation (i) and (ii)

$$\frac{2y-x}{2x-y} = v + x \frac{dv}{dx}$$

Putting $y = vx$

$$\frac{2(vx)-x}{2x-vx} = v + x \frac{dv}{dx}$$

$$\frac{x(2v-1)}{x(2-v)} = v + x \frac{dv}{dx}$$

$$\frac{2v-1}{2-v} = v + x \frac{dv}{dx}$$

$$\frac{2v-1}{2-v} - v = x \frac{dv}{dx}$$

$$\frac{2v-1-v(2-v)}{2-v} = x \frac{dv}{dx}$$

$$\frac{2v-1-2v+v^2}{2-v} = x \frac{dv}{dx}$$

$$\frac{(v^2-1)}{(2-v)} = x \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{(v^2-1)}{x(2-v)}$$

$$\frac{(2-v)}{(v^2-1)} dv = \frac{1}{x} dx$$

Integrating on the both sides we get

$$\int \frac{2-v}{v^2-1} dv = \int \frac{1}{x} dx$$

$$\int \frac{2-v}{(v-1)(v+1)} dv = \int \frac{1}{x} dx$$

$$\therefore \frac{2-v}{(v-1)(v+1)} = \frac{A}{(v-1)} + \frac{B}{(v+1)} \quad \dots (iii)$$

$$\Rightarrow \frac{2-v}{(v-1)(v+1)} = \frac{A(v+1) + B(v-1)}{(v-1)(v+1)}$$

$$\Rightarrow 2-v = A(v+1) + B(v-1)$$

Putting $v = 1$

$$\Rightarrow 2-1 = A(1+1) + B(1-1)$$

$$\Rightarrow 1 = 2A + 0 \Rightarrow 1 = 2A$$

$$\therefore A = \frac{1}{2}$$

Putting $v = -1$

HOMOGENEOUS DIFFERENTIAL EQUATIONS (XII, R. S. AGGARWAL)

$$2 - (-1) = A(-1+1) + B(-1-1)$$

$$3 = 0 + B(-2)$$

$$3 = -2B$$

$$\therefore B = -\frac{3}{2}$$

Putting the value of A and B in equation (iii)

$$\frac{2-v}{(v-1)(v+1)} = \frac{\left(\frac{1}{2}\right)}{(v-1)} + \frac{\left(-\frac{3}{2}\right)}{(v+1)} = \frac{1}{2} \times \frac{1}{(v-1)} - \frac{3}{2} \times \frac{1}{(v+1)}$$

$$\int \frac{2-v}{(v-1)(v+1)} dv = \int \frac{1}{x} dx$$

$$\int \frac{1}{2} \times \frac{1}{(v-1)} dv - \int \frac{3}{2} \times \frac{1}{(v+1)} dv = \int \frac{1}{x} dx$$

$$\frac{1}{2} \int \frac{1}{(v-1)} dv - \frac{3}{2} \int \frac{1}{(v+1)} dv = \int \frac{1}{x} dx$$

$$\frac{1}{2} \log|v-1| - \frac{3}{2} \log|v+1| = \log|x| + c$$

Putting $v = \frac{y}{x}$

$$\frac{1}{2} \log \left| \frac{y}{x} - 1 \right| - \frac{3}{2} \log \left| \frac{y}{x} + 1 \right| = \log|x| + c$$

$$\frac{1}{2} \log \left| \frac{y-x}{x} \right| - \frac{3}{2} \log \left| \frac{y+x}{x} \right| = \log|x| + c$$

$$\frac{1}{2} \left[\log \left| \frac{y-x}{x} \right| - 3 \log \left| \frac{y+x}{x} \right| \right] = \log|x| + c$$

$$\log \left| \frac{y-x}{x} \right| - 3 \log \left| \frac{y+x}{x} \right| = 2 \log|x| + c$$

$$\log \left| \frac{y-x}{x} \right| - \log \left| \frac{(y+x)^3}{x^3} \right| = \log|x^2| + c$$

$$\log \left| \frac{y-x}{x} \times \frac{x^3}{(y+x)^3} \right| = \log|x^2| + c$$

$$\log \left| (y-x) \times \frac{x^2}{(y+x)^3} \right| = \log|x^2| + c$$

$$\log \left| (y-x) \times \frac{x^2}{(y+x)^3} \right| = \log|x^2| + c$$

$$\log \left| \frac{(y-x)}{(y+x)^3} \times x^2 \right| - \log|x^2| = c$$

$$\log \left| \frac{(y-x)x^2}{(y+x)^3} \times \frac{1}{x^2} \right| = \log |c|$$

$$\frac{(y-x)}{(y+x)^3} = c$$

$$(y-x) = c(y+x)^3$$

9. $\frac{dy}{dx} + \frac{x^2 - y^2}{3xy} = 0$

Sol. On separating variables we get $\frac{dy}{dx} = -\left(\frac{x^2 - y^2}{3xy}\right)$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{3xy}$$

$\therefore y^2 - x^2$ and $3xy$ both are homogenous functions because both function having 2 degree

$$\frac{dy}{dx} = \frac{y^2 - x^2}{3xy} \quad \dots (i)$$

Let $y = vx \Rightarrow v = \frac{y}{x}$

Diff on the both sides w.r. to x

$$\frac{dy}{dx} = v \frac{dx}{dx} + x \frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad \dots (ii)$$

From equation (i) and (ii)

$$\frac{y^2 - x^2}{3xy} = v + x \frac{dv}{dx}$$

Putting $y = vx$

$$\frac{v^2 x^2 - x^2}{3x(vx)} = v + x \frac{dv}{dx}$$

$$\frac{x^2(v^2 - 1)}{3vx^2} = v + x \frac{dv}{dx}$$

$$\frac{(v^2 - 1)}{3v} = v + x \frac{dv}{dx}$$

$$\frac{v^2 - 1}{3v} - v = x \frac{dv}{dx}$$

$$\frac{(v^2 - 1)}{3v} = v + x \frac{dv}{dx}$$

$$\frac{v^2 - 1}{3v} - v = x \frac{dv}{dx}$$

$$\frac{v^2 - 1 - v(3v)}{3v} = x \frac{dv}{dx}$$

HOMOGENEOUS DIFFERENTIAL EQUATIONS (XII, R. S. AGGARWAL)

$$\frac{v^2 - 1 - 3v^2}{3v} = x \frac{dv}{dx}$$

$$\frac{-1 - 2v^2}{3v} = x \frac{dv}{dx}$$

$$\frac{-(2v^2 + 1)}{3v} = x \frac{dv}{dx}$$

$$\frac{-(2v^2 + 1)}{3v} = x \frac{dv}{dx}$$

$$\frac{-(2v^2 + 1)}{3vx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{-(2v^2 + 1)}{3vx}$$

$$\left(\frac{3v}{2v^2 + 1} \right) dv = -\frac{1}{x} dx$$

Integrating on the both sides we get

$$\int \left(\frac{3v}{2v^2 + 1} \right) dv = \int -\frac{1}{x} dx$$

Dividing and multiplying by 4

$$\frac{3}{4} \int \frac{4v}{2v^2 + 1} dv = - \int \frac{1}{x} dx$$

Let $2v^2 + 1 = t$

Dif on the both sides w.r. to v

$$4v + 0 = \frac{dt}{dv}$$

$$dv = \frac{dt}{4v}$$

$$\frac{3}{4} \int \frac{4v}{t} \times \frac{dt}{4v} = - \int \frac{1}{x} dx$$

$$\frac{3}{4} \int \frac{1}{t} dt = - \int \frac{1}{x} dx$$

$$\frac{3}{4} \log |t| = - \int \frac{1}{x}$$

$$\frac{3}{4} \log |2v^2 + 1| = - \log |x| + c$$

Putting $v = \frac{y}{x}$

$$\frac{3}{4} \log \left| \frac{2y^2}{x^2} + 1 \right| = - \log |x| + c$$

$$\frac{3}{4} \log \left| \frac{2y^2 + x^2}{x^2} \right| = - \log |x| + c$$

$$3 \log \left| \frac{2y^2 + x^2}{x^2} \right| + 4 \log |x| = c$$

$$\log \left| \frac{(2y^2 + x^2)^3}{x^6} \right| + \log |x^4| = c$$

$$\log \left| \frac{(2y^2 + x^2)^3}{x^6} \times x^4 \right| = \log |c|$$

$$\frac{(2y^2 + x^2)^3}{x^2} = c$$

$$(2y^2 + x^2)^3 = cx^2$$

10. $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$

Sol. $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$

On separating variables we get $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$

$\therefore x^2 + y^2$ and $2xy$ both are homogeneous function because both function having 2 degrees

$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ a homogeneous differential equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \quad \dots (i)$$

Let $y = vx \Rightarrow v = \frac{y}{x}$

Diff on the both sides w.r. to x

$$\frac{dy}{dx} = v \frac{dx}{dx} + x \frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad \dots (ii)$$

From equation (i) and (ii)

$$\frac{x^2 + y^2}{2xy} = v + x \frac{dv}{dx}$$

Putting $y = vx$

$$\frac{x^2 + v^2 x^2}{2x(vx)} = v + x \frac{dv}{dx}$$

$$\Rightarrow \frac{x^2(1+v^2)}{2vx^2} = v + x \frac{dv}{dx} \Rightarrow \frac{1+v^2}{2v} = v + x \frac{dv}{dx} \Rightarrow \frac{1+v^2}{2v} - v = x \frac{dv}{dx} \Rightarrow \frac{1+v^2 - v(2v)}{2v} = x \frac{dv}{dx}$$

$$\Rightarrow \frac{1+v^2 - 2v^2}{2v} = x \frac{dv}{dx} \Rightarrow \frac{1-v^2}{2v} = x \frac{dv}{dx} \Rightarrow \frac{(1-v^2)}{2vx} = \frac{dv}{dx} \Rightarrow \frac{dv}{dx} = \frac{(1-v^2)}{2vx} \Rightarrow \frac{2v}{(1-v^2)} dv = \frac{1}{x} dx$$

Integrating on the both sides we get $\int \frac{2v}{1-v^2} dv = \int \frac{1}{x} dx$

Let $1-v^2 = t$

HOMOGENEOUS DIFFERENTIAL EQUATIONS (XII, R. S. AGGARWAL)

Diff on the both sides w.r. to v

$$0 - 2v = \frac{dt}{dv}$$

$$dv = \frac{dt}{-2v}$$

$$\int \frac{2v}{t} \times \frac{dt}{-2v} = \int \frac{1}{x} dx$$

$$-\log|t| = \log|x| + c$$

$$-\log|1 - v^2| = \log|x| + c$$

$$-\log|1 - v^2| - \log|x| = c$$

$$-\left[\log|1 - v^2| + \log|x|\right] = c$$

$$\log|1 - v^2| + \log|x| = -c$$

$$\log|1 - v^2| + \log|x| = c \quad [\because -c = c \text{ constant}]$$

Putting $v = \frac{y}{x}$

$$\log\left|1 - \frac{y^2}{x^2}\right| + \log|x| = c$$

$$\log\left|\frac{x^2 - y^2}{x^2}\right| + \log|x| = c$$

$$\log\left|\frac{(x^2 - y^2)}{x^2} \times x\right| = \log|c|$$

$$\frac{x^2 - y^2}{x} = c$$

$$x^2 - y^2 = cx$$

11. $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$

Sol. $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$

On separating variables we get $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$

$\because 2xy$ and $(x^2 - y^2)$ both are homogeneous function because both function having 2 degree

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2} \text{ of a homogeneous differential equation } \frac{dy}{dx} = \frac{2xy}{x^2 - y^2} \dots\dots (i)$$

Let $y = vx \Rightarrow v = \frac{y}{x}$

Diff on the both sides w.r. to x

$$\frac{dy}{dx} = v \frac{dv}{dx} + x \frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad \dots\dots (ii)$$

From equation (i) and (ii)

$$\frac{2xy}{x^2 - y^2} = v + x \frac{dv}{dx}$$

Putting $y = vx$

$$\frac{2x(vx)}{x^2 - v^2x^2} = v + x \frac{dv}{dx}$$

$$\frac{2vx^2}{x^2(1 - v^2)} = v + x \frac{dv}{dx}$$

$$\frac{2v}{1 - v^2} = v + x \frac{dv}{dx}$$

$$\frac{2v - v(1 - v^2)}{(1 - v^2)} = x \frac{dv}{dx}$$

$$2v - v + v^3 = x \frac{dv}{dx}$$

$$\frac{(v + v^3)}{(1 - v^2)} = x \frac{dv}{dx}$$

$$\frac{(v + v^3)}{(1 - v^2)x} = \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{(v + v^3)}{x(1 - v^2)}$$

$$\frac{1 - v^2}{v + v^3} dv = \frac{1}{x} dx$$

Integrating on the both sides we get $\int \frac{1 - v^2}{v + v^3} dv = \int \frac{1}{x} dx$

$$\int \frac{(1 - v^2)}{v(1 + v^2)} dv = \int \frac{1}{x} dx$$

$$\therefore \frac{(1 - v^2)}{v(1 + v^2)} = \frac{A}{v} + \frac{(Bv + c)}{1 + v^2} \quad \dots\dots (ii)$$

$$\frac{1 - v^2}{v(1 + v^2)} = \frac{A(1 + v^2) + (Bv + c)v}{v(1 + v^2)}$$

$$1 - v^2 = A + Av^2 + Bv^2 + Cv$$

$$1 + 0v - v^2 = A + Av^2 + Bv^2 + Cv$$

$$1 + 0v - v^2 = v^2(A + B) + Cv + A$$

Comparing on the both sides proper coefficient

$$-v^2 = v^2(A + B) \quad Cv = 0v \quad A = 1$$

$$-1 = A + B \quad c = 0$$

HOMOGENEOUS DIFFERENTIAL EQUATIONS (XII, R. S. AGGARWAL)

$$B = -A - 1$$

Putting the value of $A = 1$ in equation (iv)

$$B = -1 - 1$$

$$B = -2$$

Putting the value of A, B and C in equation (iii)

$$\frac{1-v^2}{v(1+v^2)} = \frac{1}{v} + \frac{(-2v+0)}{1+v^2} = \frac{1}{v} + \frac{(-2v)}{1+v^2}$$

$$\int \frac{1}{v} dv - 2 \int \frac{v}{1+v^2} dv = \int \frac{1}{x} dx$$

$$\int \frac{1}{v} dv - \int \frac{2v}{1+v^2} dv = \int \frac{1}{x} dx$$

$$\text{Let } 1+v^2 = t$$

Diff on the both sides w.r. to v

$$2v = \frac{dt}{dv}$$

$$dv = \frac{dt}{2v}$$

$$\int \frac{1}{v} dv = \int \frac{2v}{t} \times \frac{dt}{2v} = \log|x| + c$$

$$\log|v| - \log|t| = \log|x| + c$$

$$\log|v| - \log|1+v^2| = \log|x| + c$$

$$\log\left|\frac{y}{x}\right| - \log\left|1 + \frac{y^2}{x^2}\right| = \log|x| + c$$

$$\log\left|\frac{y}{x}\right| - \log\left|\frac{x^2+y^2}{x^2}\right| = \log|x| + c$$

$$\log\left|\frac{y}{x}\right| - \log\left|\frac{x^2+y^2}{x^2}\right| = c$$

$$\log\left|\frac{y}{x^2}\right| - \log\left|\frac{x^2+y^2}{x^2}\right| = \log|c|$$

$$\log\left|\frac{\frac{y}{x}}{\frac{x^2+y^2}{x^2}}\right| = \log|c|$$

$$\log\left|\frac{y}{x^2+y^2}\right| = \log|c|$$

$$\frac{y}{x^2+y^2} = c$$

$$y = c(x^2 + y^2), \quad y = c(y^2 + x^2)$$

In each of the following differential equations show that it is homogenous and solve it

12. $\frac{x^2 dy}{dx} = 2xy + y^2$

Sol. $x^2 \frac{dy}{dx} = 2xy + y^2$

On separable variables we get $\frac{dy}{dx} = \frac{2xy + y^2}{x^2}$

$\therefore (2xy + y^2)$ and x^2 both are homogeneous function because both function having 2 degree

$\frac{dy}{dx} = \frac{2xy + y^2}{x^2}$ of a homogenous differential equation $\frac{dy}{dx} = \frac{2xy + y^2}{x^2}$ (i)

Let $y = vx = v = \frac{y}{x}$

Diff on the both sides w.r.to x

$\frac{dy}{dx} = \frac{vdx}{dx} + x \frac{dv}{dx}$

$\frac{dy}{dx} = v + x \frac{dv}{dx}$ (ii)

From equation (i) and (ii)

$\frac{2xy + y^2}{x^2} = v + x \frac{dv}{dx}$

Putting $y = vx$

$\frac{2x(vx) + v^2x^2}{x^2} = v + x \frac{dv}{dx}$

$\frac{2vx^2 + v^2x^2}{x^2} = v + x \frac{dv}{dx}$

$\frac{x^2(2v + v^2)}{x^2} = v + x \frac{dv}{dx}$

$2v + v^2 - v = x \frac{dv}{dx}$

$(v^2 + v) = x \frac{dv}{dx}$

$\frac{dv}{dx} = \frac{v^2 + v}{x}$

$\frac{dv}{dx} = \frac{v(v+1)}{x}$

$\frac{dv}{v(v+1)} = \frac{1}{x} dx$

Integrating on the both sides we get $\int \frac{dv}{v(v+1)} = \int \frac{1}{x} dx$

$\therefore \frac{1}{v(v+1)} = \frac{A}{v} + \frac{B}{v+1}$ (iii)

HOMOGENEOUS DIFFERENTIAL EQUATIONS (XII, R. S. AGGARWAL)

$$\frac{1}{v(v+1)} = \frac{A(v+1) + Bv}{v(v+1)}$$

$$1 = A(v+1) + Bv$$

Putting $1 = -1$

$$1 = A(-1+1) + B(-1)$$

$$1 = 0 + (-B)$$

$$B = -1$$

Putting $v = 0$

$$1 = A(0+1) + B(0)$$

$$A = 1$$

Putting the value of A and B in equation (iii)

$$\frac{1}{v(v+1)} = \frac{1}{v} + \frac{(-1)}{(v+1)}$$

$$\int \frac{1}{v(v+1)} dv = \int \frac{1}{x} dx$$

$$\int \frac{1}{v} dv - \int \frac{1}{(v+1)} dv = \int \frac{1}{x} dx$$

$$\log|v| - \log|v+1| = \log|x| + c$$

Putting $v = \frac{y}{x}$

$$\log\left|\frac{y}{x}\right| - \log\left|\frac{y}{x} + 1\right| = \log|x| + c$$

$$\log\left|\frac{y}{x}\right| - \log\left|\frac{y+x}{x}\right| = \log|x| + c$$

$$\log\left|\frac{\frac{y}{x}}{\frac{y+x}{x}}\right| = \log|x| + c$$

$$\log\left|\frac{y}{y+x}\right| = \log|x| + c$$

$$\log\left|\frac{y}{y+x}\right| - \log|x| = \log|c|$$

$$\log\left|\left(\frac{y}{y+x}\right) \frac{1}{x}\right| = \log|c|$$

$$\log\left|\frac{y}{x(y+x)}\right| = \log|c|$$

$$y = cx(y+x)$$

$$y = cx(x+y)$$

13. $x^2 \frac{dy}{dx} = x^2 + xy + y^2$

Sol. $x^2 \frac{dy}{dx} = x^2 + xy + y^2$

On separating variable we get

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$$

$\therefore (x^2 + xy + y^2)$ and x^2 both are homogenous function because both function having 2 degree

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} \text{ of a homogenous differential equation } \frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} \dots (i)$$

Let $y = vx \Rightarrow v = \frac{y}{x}$

Diff on the both sides w.r to x

$$\frac{dy}{dx} = v \frac{dx}{dx} + x \frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \dots (ii)$$

From equation (i) and (ii)

$$\frac{x^2 + xy + y^2}{x^2} = v + x \frac{dv}{dx}$$

Putting $y = vx$

$$\frac{x^2 + x(vx) + v^2x^2}{x^2} = v + x \frac{dv}{dx}$$

$$\frac{x^2(1 + v + v^2)}{x^2} = v + x \frac{dv}{dx}$$

$$v^2 + v + 1 = v + x \frac{dv}{dx}$$

$$v^2 + v - v + 1 = x \frac{dv}{dx}$$

$$v^2 + 1 = x \frac{dv}{dx}$$

$$\frac{v^2 + 1}{x} = \frac{dv}{dx}$$

$$\frac{dv}{v^2 + 1} = \frac{1}{x} dx$$

Integrating on the both sides we get

$$\int \frac{dv}{v^2 + 1} = \int \frac{1}{x} dx$$

$$\frac{1}{1} \tan^{-1} \frac{v}{1} = \log|x| + c$$

$$\tan^{-1} v = \log|x| + c$$

Putting $v = \frac{y}{x}$

HOMOGENEOUS DIFFERENTIAL EQUATIONS (XII, R. S. AGGARWAL)

$$\tan^{-1}\left(\frac{y}{x}\right) = \log|x| + c$$

$$\tan^{-1} \frac{y}{x} = \log|x| + c$$

14. $y^2 + (x^2 - xy) \frac{dy}{dx} = 0$

Sol. $y^2 + (x^2 - xy) \frac{dy}{dx} = 0$

On separating variables we get $(x^2 - xy) \frac{dy}{dx} = -y^2$

$$\frac{dy}{dx} = \frac{-y^2}{x^2 - xy}$$

$\therefore -y^2$ and $(x^2 - xy)$ both are homogeneous function because both function having 2 degree

$$\frac{dy}{dx} = \frac{-y^2}{x^2 - xy} \text{ of a homogeneous differential equation } \frac{dy}{dx} = \frac{-y^2}{x^2 - xy} \dots (i)$$

Let $y = vx \Rightarrow v = \frac{y}{x}$

Diff on the both sides w.r. to x

$$\frac{dy}{dx} = v \frac{dx}{dx} + x \frac{dv}{dx}$$

$$\frac{dxy}{dx} = v + x \frac{dv}{dx} \dots (ii)$$

From equation (i) and (ii)

$$\frac{-y^2}{x^2 - xy} = v + x \frac{dv}{dx}$$

Putting $y = vx$

$$\frac{-v^2 x^2}{x^2 - x(vx)} = v + x \frac{dv}{dx}$$

$$\frac{-v^2 x^2}{x^2(1-v)} = v + x \frac{dv}{dx}$$

$$\frac{-v^2}{(1-v)} = v + x \frac{dv}{dx}$$

$$\frac{-v^2 - v}{(1-v)} = x \frac{dv}{dx}$$

$$\frac{-v^2 - v(1-v)}{(1-v)} = x \frac{dv}{dx}$$

$$\frac{-v - v(1-v)}{(1-v)} = x \frac{dv}{dx}$$

$$\frac{-v^2 - v + v^2}{(1-v)} = x \frac{dv}{dx}$$

$$\frac{-v}{-(v-1)} = x \frac{dv}{dx}$$

$$\frac{v}{(v-1)} = x \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{v}{x(v-1)}$$

$$\frac{(v-1)}{v} dv = \frac{1}{x} dx$$

$$\left(\frac{v}{v} - \frac{1}{v}\right) dv = \frac{1}{x} dx$$

$$\left(1 - \frac{1}{v}\right) dv = \frac{1}{x} dx$$

Integrating on the both sides we get

$$\int \left(1 - \frac{1}{v}\right) dv = \int \frac{1}{x} dx$$

$$\int dv - \int \frac{1}{v} dv = \int \frac{1}{x} dx$$

$$v - \log|v| = \log|x| + c$$

Putting $v = \frac{y}{x}$

$$\frac{y}{x} - \log\left|\frac{y}{x}\right| = \log|x| + c$$

$$\frac{y}{x} - \log\left|\frac{y}{x}\right| - \log|x| = c$$

$$\frac{y}{x} - \log\left|\frac{y}{x} \times x\right| = c$$

$$\frac{y}{x} - \log|y| = c$$

$$\frac{y}{x} = c + \log|y|$$

$$y = x \{c + \log|y|\}$$

5. $x \frac{dy}{dx} - y = 2\sqrt{y^2 - x^2}$

Sol. $x \frac{dy}{dx} - y = 2\sqrt{y^2 - x^2}$

On separating variables we get $x \frac{dy}{dx} = 2\sqrt{y^2 - x^2} + y$

$$\frac{dy}{dx} = \frac{2\sqrt{y^2 - x^2} + y}{x}$$

$\therefore (2\sqrt{y^2 - x^2} + y)$ and x both are homogeneous function because

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$$\frac{dy}{dx} = \frac{2\sqrt{y^2 - x^2} + y}{x} \text{ of a homogeneous differential equation}$$

$$\frac{dy}{dx} = \frac{2\sqrt{y^2 - x^2} + y}{x} \dots\dots (i)$$

$$\text{Let } y = vx \Rightarrow v = \frac{y}{x}$$

Differentiation on the both sides w.r. to x

$$\frac{dy}{dx} = v \frac{dx}{dx} + x \frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \dots\dots (ii)$$

From equation (i) and (ii)

$$\frac{2\sqrt{y^2 - x^2} + y}{x} = v + x \frac{dv}{dx}$$

Putting $y = vx$

$$\frac{2\sqrt{v^2 x^2 - x^2} + vx}{x} = v + x \frac{dv}{dx}$$

$$\frac{2x\sqrt{v^2 - 1} + vx}{x} = v + x \frac{dv}{dx}$$

$$\frac{x(v\sqrt{v^2 - 1} + v)}{x} = v + x \frac{dv}{dx}$$

$$2\sqrt{v^2 - 1} + v - v = x \frac{dv}{dx}$$

$$2\sqrt{v^2 - 1} = x \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{2\sqrt{v^2 - 1}}{x}$$

$$\frac{dv}{2\sqrt{v^2 - 1}} = \frac{1}{x} dx$$

$$\text{Integrating on the both sides we get } \int \frac{dv}{2\sqrt{v^2 - 1}} = \int \frac{1}{x} dx$$

$$\frac{1}{2} \log |v + \sqrt{v^2 - 1}| = \log |x| + c$$

$$\text{Putting } v = \frac{y}{x}$$

$$\frac{1}{2} \log \left| \frac{y}{x} + \sqrt{\frac{y^2}{x^2} - 1} \right| = \log |x| + c$$

$$\frac{1}{2} \log \left| \frac{y}{x} + \sqrt{\frac{y^2 - x^2}{x^2}} \right| = \log |x| + c$$

$$\frac{1}{2} \log \left| \frac{y}{x} + \frac{\sqrt{y^2 - x^2}}{x} \right| = \log |x| + c$$

$$\frac{1}{2} \log \left| \frac{y + \sqrt{y^2 - x^2}}{x} \right| = \log |x| + c$$

$$\log \left| \frac{y + \sqrt{y^2 - x^2}}{x} \right| = 2 \log |x| + c$$

$$\log \left| \frac{y + \sqrt{y^2 - x^2}}{x} \right| = \log |x^2| + c$$

$$\log |y + \sqrt{y^2 - x^2}| - \log |x| = \log |x^2| + c$$

$$\log |y + \sqrt{y^2 - x^2}| = \log |x^2 + x| + \log |c|$$

$$\log |y + \sqrt{y^2 - x^2}| = \log |x^3| + \log |c|$$

$$\log |y + \sqrt{y^2 - x^2}| = \log |x^3 + c|$$

$$y + \sqrt{y^2 - x^2} = cx^3$$

16. $y^2 dx + (x^2 + xy + y^2) dy = 0$

Sol. The given equation may be written as $\frac{dy}{dx} = -\frac{y^2}{x^2 + xy + y^2}$

For checking homogeneous putting tx for x and ty for y .

The given equation remains unchanged so it is a homogeneous equation.

Put $y = vx$, we get, $\frac{dy}{dx} = v + x \frac{dv}{dx} \Rightarrow v + x \frac{dv}{dx} = -\frac{v^2 x^2}{x^2 + x \cdot vx + v^2 x^2} \Rightarrow v + x \frac{dv}{dx} = -\frac{v^2}{1 + v + v^2}$

$$\Rightarrow x \frac{dv}{dx} = -\frac{v^2}{1 + v + v^2} - \frac{v}{1} \Rightarrow x \frac{dv}{dx} = -\left[\frac{v^2 + v + v^2 + v^3}{1 + v + v^2} \right] \Rightarrow \int \frac{1 + v + v^2}{v^3 + 2v^2 + v} dv = -\int \frac{dx}{x}$$

$$\Rightarrow \int \frac{v^2 + v + 1}{v(v^2 + 2v + 1)} dv = -\log |x| + \log c \Rightarrow \int \frac{v^2 + v + 1}{v(v+1)^2} dv = -\log |x| + \log c$$

$$\Rightarrow I_1 = -\log |x| + \log c \quad \dots (1)$$

$$I_1 = \int \frac{v^2 + v + 1}{v(v+1)^2} dv \Rightarrow \frac{v^2 + v + 1}{v(v+1)^2} = \frac{A}{v} + \frac{B}{v+1} + \frac{C}{(v+1)^2}$$

$$\Rightarrow \frac{v^2 + v + 1}{v(v+1)^2} = \frac{A(v+1)^2 + Bv(v+1) + Cv}{v(v+1)^2} \Rightarrow v^2 + v + 1 = A(v^2 + 2v + 1) + B(v^2 + v) + Cv$$

$$\Rightarrow v^2 + v + 1 = v^2(A+B) + v(2A+B+C) + A$$

Equating co-efficient both side we get,

$$A + B = 1 \quad \dots (2)$$

$$2A + B + C = 1 \quad \dots (3)$$

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$$A=1 \quad \dots(4)$$

From (3), $A=1 \Rightarrow A+B=1 \therefore B=0$

Putting the value of A, B in equation (3), $2.1+0+C=1 \Rightarrow C=-1 \therefore A=1, B=0, C=-1$

$$\Rightarrow I_1 = \int \left(\frac{A}{v} + \frac{B}{v+1} + \frac{C}{(v+1)^2} \right) dv \Rightarrow I = \log v$$

$$\Rightarrow I_1 = \log \left(\frac{y}{x} \right) + \frac{1}{\frac{y}{x}+1} + c \quad \therefore I_1 = \log \left(\frac{y}{x} \right) + \frac{x}{x+y} + c$$

Putting the value of I_1 in equation (1), $\log \left(\frac{y}{x} \right) + \frac{x}{x+y} - \log |x| + \log c \therefore \log |y| + \frac{x}{x+y} = c$

17. $(x-y) \frac{dy}{dx} = x+3y$

Sol. The given equation may be written as $\frac{dy}{dx} = \frac{x+3y}{x-y}$

For checking homogeneous putting tx for x and ty for y .

The given equation remains unchanged so it is a homogeneous equation.

Put $y = vx$, we get, $\frac{dy}{dx} = v + x \frac{dv}{dx} \Rightarrow v + x \frac{dv}{dx} = \frac{x+3.vx}{x-vx} \Rightarrow v + x \frac{dv}{dx} = \frac{x+3.vx}{x-vx}$

$$\Rightarrow x \frac{dv}{dx} = \frac{1+3v}{1-v} - v \Rightarrow x \frac{dv}{dx} = \frac{1+3v-v+v^2}{1-v} \Rightarrow x \frac{dv}{dx} = \frac{1+2v+v^2}{1-v}$$

$$\Rightarrow \int \frac{1-v}{v^2+2v+1} dv = \int \frac{dx}{x} \Rightarrow I_1 = \log |x| + \log c \quad \dots(1)$$

$$I_1 = \int \frac{1-v}{v^2+2v+1} dv$$

Putting $1-v = A \frac{d}{dv}(v^2+2v+1) + B$, we get,

$$1-v = A(2v+2) + B \Rightarrow 1-v = 2Av + 2A + B \Rightarrow 2A = -1 \text{ \& } 2A+B=1$$

$$A = -\frac{1}{2} \text{ \& } B = 1-2A \Rightarrow B = 1+1 \therefore B = 2$$

$$\Rightarrow I_1 = \int \frac{A(2v+2) + B}{v^2+2v+1} dv \Rightarrow I_1 = A \int \frac{2v+2}{v^2+2v+1} dv + B \int \frac{1}{v^2+2v+1} dv$$

Put $v^2+2v+1=t \Rightarrow (2v+2) = \frac{dt}{dv} \Rightarrow (2v+2)dv = dt$

$$\Rightarrow I_1 = -\frac{1}{2} \int \frac{1}{t} dt + 2 \int \frac{1}{(v)^2+2.v.1+(1)^2-1} dv$$

$$\Rightarrow I_1 = -\frac{1}{2} \log |v^2+2v+1| + 2 \int \frac{1}{(v+1)^2} dv \Rightarrow I_1 = -\frac{1}{2} \log |v^2+2v+1| - 2 \cdot \frac{1}{v+1}$$

$$\Rightarrow I_1 = -\frac{1}{2} \log |(v+1)^2| - 2 \cdot \frac{1}{v+1} + c \Rightarrow I_1 = -\log |v+1| - \frac{2}{v+1} + c$$

$$\Rightarrow I_1 = -\log \left| \frac{y}{x} + 1 \right| - \frac{2}{\frac{y}{x} + 1} + c \quad \Rightarrow I_1 = -\log \left| \frac{x+y}{x} \right| - \frac{2x}{x+y} + c$$

Putting the value of I_1 in equation (1),

$$-\log \left(\frac{x+y}{x} \right) - \frac{2x}{x+y} = \log x + \log c \quad \therefore \log |c(x+y)| + \frac{2x}{x+y} = 0$$

18. $(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy = 0$

Sol. The given equation may be written as $\frac{dy}{dx} = \frac{-(x^3 + 3xy^2)}{y^3 + 3x^2y}$

For checking homogeneous putting tx for x and ty for y .

The given equation remains unchanged so it is a homogeneous equation.

Put $y = vx$, we get, $\frac{dy}{dx} = v + x \frac{dv}{dx} \Rightarrow v + x \frac{dv}{dx} = \frac{-(x^3 + 3x \cdot v^2 x^2)}{v^3 x^3 + 3x^2 \cdot vx}$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{-(1 + 3v^2)}{v^3 + 3v} \Rightarrow x \frac{dv}{dx} = \frac{-1 - 3v^2}{v^3 + 3v} - v \Rightarrow x \frac{dv}{dx} = \frac{-1 - 3v^2 - v^4 - 3v^2}{v^3 + 3v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{-1 - 6v^2 - v^4}{v^3 + 3v} \Rightarrow \int \frac{v^3 + 3v}{v^4 + 6v^2 + 1} dv = -\int \frac{dx}{x}$$

Put $v^4 + 6v^2 + 1 = t \Rightarrow 4v^3 + 12v = \frac{dt}{dv} \Rightarrow (v^3 + 3v)dv = \frac{dt}{4}$

$$\Rightarrow \frac{1}{4} \int \frac{1}{t} dt = -\log |x| + \log(c) \Rightarrow \frac{1}{4} \log |t| = 4 \log xc - \log |x| + \log c$$

$$\Rightarrow \frac{1}{4} \log |v^4 + 6v^2 + 1| = -\log |x| + \log c \Rightarrow \log \left(\frac{y^4}{x^4} + \frac{6y^2}{x^2} + 1 \right) = \log x^{-1} \cdot c$$

$$\Rightarrow \log \frac{(y^4 + 6x^2 y^2 + x^4)}{x^4} = 4 \log x^{-1} c = \log x^{-4} c \Rightarrow \frac{y^4 + 6x^2 y^2 + x^4}{x^4} = x^{-4} c$$

$$\Rightarrow y^4 + 6x^2 y^2 + x^4 = c$$

19. $(x - \sqrt{xy})dy = ydx$

Sol. The given equation may be written as $(x - \sqrt{xy})dx = ydx \Rightarrow \frac{dy}{dx} = \frac{y}{x - \sqrt{xy}}$

For checking homogeneous putting tx for x and ty for y .

The given equation remains unchanged so it is a homogeneous equation.

Put $y = vx$, we get, $\frac{dy}{dx} = v + x \frac{dv}{dx} \Rightarrow v + x \frac{dv}{dx} = \frac{vx}{x - \sqrt{x \cdot vx}} \Rightarrow v + x \frac{dv}{dx} = \frac{v}{1 - \sqrt{v}}$

$$\Rightarrow x \frac{dv}{dx} = \frac{v}{1 - \sqrt{v}} - v \Rightarrow x \frac{dv}{dx} = \frac{v - v + v\sqrt{v}}{1 - \sqrt{v}} \Rightarrow \int \frac{1 - \sqrt{v}}{v\sqrt{v}} dv = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{1}{v^{3/2}} dv - \int \frac{1}{v} dv = \log |x| \Rightarrow \frac{v^{-1/2}}{-1/2} - \log |v| = \log |x| + c \Rightarrow -\frac{2}{\sqrt{v}} - \log |v| = \log |x| + c$$

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$$\Rightarrow -\frac{2\sqrt{x}}{\sqrt{y}} - \log \left| \frac{y}{x} \right| = \log |x| + c \Rightarrow -2\sqrt{\frac{x}{y}} - \log y = c$$

$$\therefore 2\sqrt{\frac{x}{y}} + \log |y| = c x^2 \quad [\text{Answer given in the book is wrong.}]$$

3. $x^2 \frac{dy}{dx} + y^2 = xy$

sol. The given equation may be written as $x \frac{dy}{dx} = y - \frac{y^2}{x} \Rightarrow \frac{dy}{dx} = \frac{xy - y^2}{x^2}$

For checking homogeneous putting tx for x and ty for y .

The given equation remains unchanged so it is a homogeneous equation.

Put $y = vx$, we get, $\frac{dy}{dx} = v + x \frac{dv}{dx} \Rightarrow v + x \frac{dv}{dx} = \frac{x \cdot vx - v^2 x^2}{x^2} \Rightarrow v + x \frac{dv}{dx} = v - v^2$

$$\Rightarrow x \frac{dv}{dx} = -v^2 \Rightarrow \int \frac{dv}{v^2} = -\int \frac{dx}{x} \Rightarrow -\frac{1}{v} = -\log |x| + c \Rightarrow \frac{1}{v} = \log |x| - c$$

$$\Rightarrow \frac{x}{y} = \log |x| - c \quad \therefore c = \log |x| - \frac{x}{y}$$

1. $x \frac{dy}{dx} = y(\log y - \log x + 1)$

sol. The given equation may be written as $\frac{dy}{dx} = \frac{y(\log y - \log x + 1)}{x}$

For checking homogeneous putting tx for x and ty for y .

The given equation remains unchanged so it is a homogeneous equation.

Put $y = vx$, we get $\frac{dy}{dx} = v + x \frac{dv}{dx} \Rightarrow v + x \frac{dv}{dx} = \frac{vx(\log(vx) - \log x + 1)}{x}$

$$\Rightarrow v + x \frac{dv}{dx} = v(\log v + \log x - \log x + 1) \Rightarrow v + x \frac{dv}{dx} = v(\log v + 1)$$

$$\Rightarrow x \frac{dv}{dx} = v(\log v + 1) \Rightarrow x \frac{dv}{dx} = v \log v \Rightarrow \int \frac{1}{v \log v} dv = \int \frac{dx}{x}$$

$$\Rightarrow \log(\log v) = \log(x) + \log c \Rightarrow \log v = xc \Rightarrow \log \left(\frac{y}{x} \right) = xc \Rightarrow \frac{y}{x} = e^{xc} \quad \therefore y = x e^{xc}$$

2. $x \frac{dy}{dx} - y + x \sin \frac{y}{x} = 0$

sol. Given differential equation is $x \cdot \frac{dy}{dx} - y + x \cdot \sin \frac{y}{x} = 0$

$$\Rightarrow x \cdot \frac{dy}{dx} = y - x \cdot \sin \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \sin \frac{y}{x} \quad \dots (i)$$

This is of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

This the given differential equation is homogeneous differential equation

$$\text{Let } \frac{y}{x} = v \Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

From (i)

$$v + x \frac{dv}{dx} = v - \sin v$$

$$\Rightarrow x \cdot \frac{dv}{dx} = -\sin v \Rightarrow \operatorname{cosec} v dv = -\frac{dx}{x}$$

$$\text{Integrating both sides we get } \int \operatorname{cosec} v dv = -\int \frac{dx}{x}$$

$$\Rightarrow \log \left| \tan \frac{v}{2} \right| = -\log |x| + \log C_1$$

$$\Rightarrow \log |x| \log \left| \tan \frac{v}{2} \right| = \log C_1$$

$$\Rightarrow \log \left| x \cdot \tan \frac{v}{2} \right| = \log C_1$$

$$\Rightarrow \left| x \cdot \tan \frac{v}{2} \right| = C_1$$

$$\Rightarrow x \cdot \tan \left(\frac{y}{2x} \right) = \pm C_1$$

$$\Rightarrow x \cdot \tan \left(\frac{y}{2x} \right) = C \quad \{\text{Let } \pm C_1 = C\}$$

This is the required solution of given differential equation

23. $x \frac{dy}{dx} = y - x \cos^2 \left(\frac{y}{x} \right)$

Sol. Given differential equation is

$$x \cdot \frac{dy}{dx} = y - x \cdot \cos^2 \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \cos^2 \frac{y}{x} \quad \dots (i)$$

$$\text{This is of the form } \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Thus the given differential equation is a homogenous differential equation

$$\text{Let } \frac{y}{x} = v$$

$$\Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

$$\text{Now, from (i) we have } v + x \cdot \frac{dv}{dx} = v - \cos^2 v$$

$$\Rightarrow x \cdot \frac{dv}{dx} = -\cos^2 v$$

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$$\Rightarrow \sec^2 v \, dv = -\frac{dx}{x}$$

Integrating both sides we have $\int \sec^2 v \, dv = -\int \frac{dx}{x}$

$$\Rightarrow \tan v = -\log|x| + C$$

$$\Rightarrow \tan\left|\frac{y}{x}\right| + \log|x| = C$$

This is the required solution of given differential equation

24. $\left(x \cos \frac{y}{x}\right) \frac{dy}{dx} = \left(y \cos \frac{y}{x}\right) + x$

Sol. Find $\frac{dy}{dx}$

$$x \cos\left(\frac{y}{x}\right) \frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x$$

$$\frac{dy}{dx} = \frac{y \cos\left(\frac{y}{x}\right) + x}{x \cos\left(\frac{y}{x}\right)} = \frac{y}{x} + \sec\left(\frac{y}{x}\right) = F\left(\frac{y}{x}\right)$$

Thus the given differential equation is homogeneous differential equation

Let $\frac{y}{x} = v$

$$\Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Now the differential equation is,

$$v + x \frac{dv}{dx} = v + \sec v$$

$$\cos v \, dv = \frac{dx}{x}$$

Integrating both sides

$$\int \cos v \, dv = \int \frac{dx}{x}$$

$$\sin v = \log|x| + c_1$$

Putting $v = \frac{y}{x}$ & $c_1 = \log c$

$$\sin \frac{y}{x} = \log|x| + \log|c|$$

$$\sin \frac{y}{x} = \log|cx|$$

25. Find the particular solution of the differential equation $2xy + y^2 - 2x^2 \frac{dy}{dx} = 0$, it being given that

$y = 2$ when $x = 1$

Sol. Given differential equation is

$$2xy + y^2 - 2x^2 \frac{dy}{dx} = 0$$

$$\Rightarrow 2x^2 \frac{dy}{dx} = 2xy + y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{1}{2} \left(\frac{y}{x} \right)^2 \quad \dots (i)$$

This is of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

This the given differential is homogeneous differential equation

Let $\frac{y}{x} = v$

$$\Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Now, form (i) we have $v + x \frac{dv}{dx} = v + \frac{v^2}{2}$

$$\Rightarrow x \frac{dv}{dx} = \frac{v^2}{2}$$

$$\Rightarrow \frac{2}{v^2} dv = \frac{dx}{x}$$

Integrating both sides we have $2 \int \frac{dv}{v^2} = \int \frac{dx}{x}$

$$\Rightarrow \frac{-2}{v} = \log|x| + C$$

$$\Rightarrow -\frac{2x}{y} = \log|x| + C \quad \dots (ii)$$

Putting $x = 1$ and $y = 2$ we have $-\frac{2 \times 1}{2} = \log|1| + C$

$$\Rightarrow C = -1$$

Putting $C = -1$ in equation (i) we have

$$-\frac{2x}{y} = \log|x| - 1$$

$$\Rightarrow -2x = y \log|x| - y$$

$$\Rightarrow -2x = y - y \log|x|$$

$$\Rightarrow y = \frac{2x}{1 - \log|x|}$$

This is the required solution of given differential equation

26. Find the particular solution of the differential equation $\left\{ x \sin^2 \frac{y}{x} - y \right\} dx + x dy = 0$, it being given that

$$y = \frac{\pi}{4} \text{ when } x = 1$$

Sol. Given differential equation is $\left\{ x \cdot \sin^2 \frac{y}{x} - y \right\} dx + x dy = 0$

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$$\Rightarrow x dy = \left(y - x \sin^2 \frac{y}{x} \right) dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \sin^2 \frac{y}{x} \quad \dots (i)$$

This is of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

This the given differential equation is homogeneous differential equation

Let $\frac{y}{x} = v$

$$\Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

Now, from (ii)

$$v + x \cdot \frac{dv}{dx} = v - \sin^2 v$$

$$\Rightarrow x \cdot \frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \operatorname{cosec}^2 v \cdot dv = -\sin^2 v$$

Integrating both sides we have $\int \operatorname{cosec}^2 v \cdot dv = -\int \frac{dx}{x}$

$$\Rightarrow -\cot v = -\log|x| + C$$

$$\Rightarrow \cot \frac{y}{x} = \log|x| - C$$

Putting $x=1$ and $y=\frac{\pi}{4}$ we have $\cot \frac{\pi}{4} = \log|1| - C$

$$\Rightarrow 1 = -C$$

$$\Rightarrow C = -1$$

Putting $C = -1$ in (i) we have $\cot\left(\frac{y}{x}\right) = \log|x| + 1$

This is the required solution of given differential equation

27. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{y(2y-x)}{x(2y+x)}$ given that $y=1$ when $x=1$

Sol. For checking homogeneous putting tx for x and ty for y .

The given equation remains unchanged so it is a homogeneous equation.

Put $y = vx$, we get, $\frac{dy}{dx} = v + x \frac{dv}{dx} \Rightarrow v + x \frac{dv}{dx} = \frac{vx(2vx-x)}{x(2vx+x)} \Rightarrow v + x \frac{dv}{dx} = \frac{2v^2-v}{2v+1}$

$$\Rightarrow x \frac{dv}{dx} = \frac{2v^2-v}{2v+1} - \frac{v}{1} \Rightarrow x \frac{dv}{dx} = \frac{-2v}{2v+1} \Rightarrow \int \frac{2v+1}{2v} dv = -\int \frac{dx}{x}$$

$$\Rightarrow v + \frac{1}{2} \log v = -\log|x| + c \Rightarrow \frac{y}{x} + \frac{1}{2} \log\left(\frac{y}{x}\right) = -\log|x| + c$$

$$\Rightarrow \frac{y}{x} + \frac{1}{2} \log y - \frac{1}{2} \log x + \log|x| = c \Rightarrow \frac{y}{x} + \frac{1}{2} \log y + \frac{1}{2} \log x = c$$

$$\Rightarrow \frac{y}{x} + \frac{1}{2} \log(xy) = c \quad \dots (1)$$

Given, $x = 1, y = 1, \quad \frac{1}{1} + \frac{1}{2} \log(1.1) = c \Rightarrow c = 1$

Putting the value of c in equation (1), $\therefore \frac{y}{x} + \frac{1}{2} \log(xy) = 1$

28. Find the particular solution of the differential equation $(xe^{y/x} + y)dx = xdy$, given that $y(1) = 0$

Sol. Given differential equation is $(x \cdot e^{y/x} + y)dx = xdy$

$$\Rightarrow \frac{dy}{dx} = e^{y/x} + \frac{y}{x} \quad \dots (i)$$

This is of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

This the given differential equation is homogeneous differential equation

Let $\frac{y}{x} = v$

$$\Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Now from (ii) we have $v + x \cdot \frac{dv}{dx} = e^v + v$

$$\Rightarrow x \cdot \frac{dv}{dx} = e^v$$

$$\Rightarrow e^{-v} dv = \frac{dx}{x}$$

Integrating both sides we have

$$\int e^{-v} dv = \int \frac{dx}{x}$$

$$\Rightarrow -e^{-v} = \log|x| + C$$

$$\Rightarrow -e^{-y/x} = \log|x| + C \quad \dots (i)$$

Putting $x = 1$ & $y = 0$ in (i) we have

$$-e^0 = \log(1) + C$$

$$\Rightarrow C = -1$$

Putting $C = -1$ in (ii) we have

$$-e^{y/x} = \log|x| - 1$$

$$\Rightarrow \log|x| + e^{-y/x} = 1$$

This is the required solution of given differential equation

29. Find the particular solution of the differential equation $xe^{y/x} - y + x \frac{dy}{dx} = 0$, given that $y(e) = 0$

Sol. Given differential equation is

$$x \cdot e^{y/x} - y + x \cdot \frac{dy}{dx} = 0$$

HOMOGENEOUS DIFFERENTIAL EQUATIONS (XII, R. S. AGGARWAL)

$$\Rightarrow x \cdot \frac{dy}{dx} = y - x \cdot e^{y/x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - e^{-y/x} \quad \dots (i)$$

This is of the form $\frac{dy}{dx} = f\left\{\frac{y}{x}\right\}$

This the given differential equation is homogeneous differential equation

$$\text{Let } \frac{y}{x} = v$$

$$\Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

$$\text{Now, from (i) } v + x \cdot \frac{dv}{dx} = v - e^v$$

$$\Rightarrow x \cdot \frac{dv}{dx} = -e^v$$

$$\Rightarrow -e^{-v} dv = \frac{dx}{x}$$

$$\text{Integrating both sides we get } -\int e^{-v} dv = \int \frac{dx}{x}$$

$$\Rightarrow e^{-v} = \log|x| + C$$

$$\Rightarrow e^{-y/x} = \log|x| + C \quad \dots (i)$$

$$\because y(e) = 0$$

$$\therefore e^0 = \log e + C$$

$$\Rightarrow 1 = 1 + C$$

$$\Rightarrow C = 0$$

$$\text{Putting } C = 0 \text{ in (i) we have } e^{-y/x} = \log|x|$$

$$\text{Taking log on both sides } -\frac{y}{x} = \log(\log|x|)$$

$$\Rightarrow y = -x \log(\log|x|)$$

This is the required solution of given differential equation

30. The slope of the tangent to a curve at any point (x, y) on it is given by $\frac{y}{x} - \left(\cot \frac{y}{x}\right) \left(\cos \frac{y}{x}\right)$, where $x > 0$ and $y > 0$. If the curve passes through the point $\left(1, \frac{\pi}{4}\right)$, find the equation of the curve

Sol. Since slope of the tangent to a curve at (x, y) on it is $\frac{y}{x} - \left(\cot \frac{y}{x}\right) \cdot \cos\left(\frac{y}{x}\right)$

$$\therefore \frac{dy}{dx} = \frac{y}{x} - \cot\left(\frac{y}{x}\right) \cdot \cos\left(\frac{y}{x}\right) \quad \dots (i)$$

$$\text{This is of the form } \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Thus the differential equation is homogeneous differential equation

$$\text{Let } \frac{y}{x} = v$$

$$\Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

Now from (i) we have

$$v + x \cdot \frac{dv}{dx} = v - \cot(v) \cdot \cos(v)$$

$$\Rightarrow x \cdot \frac{dv}{dx} = -\cot v \cdot \cos v$$

$$\Rightarrow \sec v \cdot \tan v \, dv = -\frac{dx}{x}$$

$$\text{Integrating both sides we have } \int \sec v \cdot \tan v \, dv = -\int \frac{dx}{x}$$

$$\Rightarrow \sec v = -\log|x| + C$$

$$\Rightarrow \sec\left(\frac{y}{x}\right) = -\log|x| + C \quad \dots (ii)$$

Since the curve passes through the point $\left(1, \frac{\pi}{4}\right)$

$$\therefore \sec\left(\frac{\pi}{4}\right) = -\log|1| + C$$

$$\Rightarrow \sqrt{2} = C$$

$$\text{Putting } C = \sqrt{2} \text{ in equation we have } \sec\left(\frac{y}{x}\right) + \log|x| = \sqrt{2}$$

This is the required solution of given differential equation