CROSS, OR VECTOR, PRODUCT OF VECTORS (XII, R. S. AGGARWAL)

EXERCISE 24 [Pg. No.: 1057]

1. Find $(\vec{a}\times\vec{b})$ and $|\vec{a}\times\vec{b}|$, when

(i) $\vec{a}=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}-4\hat{k}$

(ii) $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+5\hat{j}-2\hat{k}$

(iii) $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

(iv) $\vec{a}=4\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=3\hat{i}+\hat{k}$

(v) $\vec{a}=3\hat{i}+4\hat{j}$ and $\vec{b}=\hat{i}+\hat{j}+\hat{k}$

Sol. (i) Let $\vec{a}=(\hat{i}-\hat{j}+2\hat{k})$ and $\vec{b}=(2\hat{i}+3\hat{j}-4\hat{k})$

$$(\vec{a}\times\vec{b})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-1&2\\2&3&-4\end{vmatrix}=\hat{i}\begin{vmatrix}-1&2\\3&-4\end{vmatrix}-\hat{j}\begin{vmatrix}1&2\\2&-4\end{vmatrix}+\hat{k}\begin{vmatrix}1&-1\\2&3\end{vmatrix}$$

$$=\hat{i}(4-6)-\hat{j}(-4-4)+\hat{k}(3+2)=(-2\hat{i}+8\hat{j}+5\hat{k})$$

$$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{(-2)^2+(8)^2+(5)^2}=\sqrt{4+64+25}=\sqrt{93}$$

(ii) Let $a=(2\hat{i}+\hat{j}+3\hat{k})$ and $\vec{b}=(3\hat{i}+5\hat{j}-2\hat{k})$

$$(\vec{a}\times\vec{b})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&1&3\\3&5&-2\end{vmatrix}=\hat{i}\begin{vmatrix}1&3\\5&-2\end{vmatrix}-\hat{j}\begin{vmatrix}2&3\\3&-2\end{vmatrix}+\hat{k}\begin{vmatrix}2&1\\3&5\end{vmatrix}$$

$$=\hat{i}(-2-15)-\hat{j}(-4-9)+\hat{k}(10-3)=(-17\hat{i}+13\hat{j}+7\hat{k})$$

$$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{(-17)^2+(13)^2+(7)^2}=\sqrt{289+169+49}=\sqrt{507}=13\sqrt{3}$$

(iii) Let $\vec{a}=(\hat{i}-7\hat{j}+7\hat{k})$ and $\vec{b}=(3\hat{i}-2\hat{j}+2\hat{k})$, $(\vec{a}\times\vec{b})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-7&7\\3&-2&2\end{vmatrix}=\hat{i}\begin{vmatrix}-7&7\\-2&2\end{vmatrix}-\hat{j}\begin{vmatrix}1&7\\3&2\end{vmatrix}+\hat{k}\begin{vmatrix}1&-7\\3&-2\end{vmatrix}$

$$=\hat{i}(-14+14)-\hat{j}(2-21)+\hat{k}(-2+21)=(19\hat{j}+19\hat{k})$$

$$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{(19)^2+(19)^2}=\sqrt{19^2(1+1)}=19\sqrt{2}$$

(iv) Let $\vec{a}=(4\hat{i}+\hat{j}-2\hat{k})$ and $\vec{b}=(3\hat{i}+\hat{k})$, $(\vec{a}\times\vec{b})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\4&1&-2\\3&0&1\end{vmatrix}=\hat{i}\begin{vmatrix}1&-2\\0&1\end{vmatrix}-\hat{j}\begin{vmatrix}4&-2\\3&1\end{vmatrix}+\hat{k}\begin{vmatrix}4&1\\3&0\end{vmatrix}$

$$=\hat{i}(1-0)-\hat{j}(4+6)+\hat{k}(0-3)=(\hat{i}-10\hat{j}-3\hat{k})$$

$$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{(1)^2+(-10)^2+(-3)^2}=\sqrt{1+100+9}=\sqrt{110}$$

(v) Let $\vec{a}=(3\hat{i}+4\hat{j})$ and $\vec{b}=(\hat{i}+\hat{j}+\hat{k})$, $(\vec{a}\times\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 4 & 0\\ 1 & 1 & 1\end{vmatrix}=\hat{i}\begin{vmatrix}4 & 0\\ 1 & 1\end{vmatrix}-\hat{j}\begin{vmatrix}3 & 0\\ 1 & 1\end{vmatrix}+\hat{k}\begin{vmatrix}3 & 4\\ 1 & 1\end{vmatrix}$

$$=\hat{i}(4-0)-\hat{j}(3-0)+\hat{k}(3-4)=(4\hat{i}-3\hat{j}-\hat{k})$$

$$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{(4)^2+(-3)^2+(-1)^2}=\sqrt{16+9+1}=\sqrt{26}$$

2. Find λ if $(2\hat{i}+6\hat{j}+14\hat{k})\times(\hat{i}-\lambda\hat{j}+7\hat{k})=\vec{0}$

Sol. $\because \left(2\hat{i}+6\hat{j}+14\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+7\hat{k}\right)=\vec{0}$

$$\Rightarrow\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 6 & 14\\ 1 & -\lambda & 7\end{vmatrix}=\vec{0}\Rightarrow\begin{vmatrix}6 & 14\\ 7 & 7\end{vmatrix}\hat{i}-\begin{vmatrix}2 & 14\\ 1 & 7\end{vmatrix}\hat{j}+\begin{vmatrix}2 & 6\\ 1 & -\lambda\end{vmatrix}\hat{k}=\vec{0}$$

$$\Rightarrow(42-98)\hat{i}(7-14)\hat{j}+(-2\lambda-6)\hat{k}=\vec{0}\Rightarrow -56\hat{i}+7\hat{j}-2(\lambda+3)\hat{k}=\vec{0}\Rightarrow -2(\lambda+3)=0\Rightarrow\lambda=\ 3\ \text{Ans.}$$

3. If $\vec{a}=(-3\hat{i}+4\hat{j}-7\hat{k})$ and $\vec{b}=(6\hat{i}+2\hat{j}-3\hat{k})$, find $(\vec{a}\times\vec{b})$

Verify that (i) $\vec{a}$ and $(\vec{a}\times\vec{b})$ are perpendicular to each other

And (ii) $\vec{b}$ and $(\vec{a}\times\vec{b})$ are perpendicular to each other

Sol. $\vec{a}=\left(-3\hat{i}+4\hat{j}-7\hat{k}\right)$

$\vec{b}=\left(6\hat{i}+2\hat{j}-3\hat{k}\right)$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -3 & 4 & -7\\ 6 & 2 & -3\end{vmatrix}$$

$$=\begin{vmatrix}4 & -7\\ 2 & -3\end{vmatrix}\hat{i}-\begin{vmatrix}-3 & -7\\ 6 & -3\end{vmatrix}\hat{j}+\begin{vmatrix}-3 & 4\\ 6 & 2\end{vmatrix}\hat{k}=(-12+14)\hat{i}-(9+42)\hat{j}+(-6-24)\hat{k}=2\hat{i}-51\hat{j}-30\hat{k}$$

4. Find the value of

(i) $(\hat{i}\times\hat{j})\cdot\hat{k}+\hat{i}\cdot\hat{j}$ (ii) $(\hat{k}\times\hat{j})\cdot\hat{i}+\hat{j}\cdot\hat{k}$ iii) $\hat{i}\times(\hat{j}+\hat{k})+\hat{j}\times(\hat{k}+\hat{i})+\hat{k}\times(\hat{i}+\hat{j})$

Sol. (i) $\left(\hat{i}\times\hat{j}\right)\cdot\hat{k}+\hat{i}\cdot\hat{j}=\hat{k}\cdot\hat{k}+\hat{i}\cdot\hat{j}=1+0=1$

(ii) $\left(\hat{k}\times\hat{j}\right)\cdot\hat{i}+\hat{j}\cdot\hat{k}=-\hat{i}\cdot\hat{i}+\hat{j}\cdot\hat{k}=-1+0=-1$

(iii) $\hat{i}\times\left(\hat{j}+\hat{k}\right)+\hat{j}\times\left(\hat{k}+\hat{i}\right)+\hat{k}\times\left(\hat{i}+\hat{j}\right)$

$$=\left(\hat{i}\times\hat{j}\right)+\left(\hat{i}\times\hat{k}\right)+\left(\hat{j}\times\hat{k}\right)+\left(\hat{j}\times\hat{i}\right)+\left(\hat{k}\times\hat{i}\right)+\left(\hat{k}\times\hat{j}\right)=\hat{k}+\left(-\hat{j}\right)+\hat{i}+\left(-\hat{k}\right)+\hat{j}+\left(-\hat{i}\right)=0$$

5. Find the unit vectors perpendicular to both $\vec{a}$ and $\vec{b}$, when

(i) $\vec{a}=3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}-\hat{k}$ (ii) $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$

(iii) $\vec{a}=\hat{i}+3\hat{j}-2\hat{k}$ and $\vec{b}=-\hat{i}+3\hat{k}$ (iv) $\vec{a}=4\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}+4\hat{j}-\hat{k}$

Sol. **(i)** Let $\vec{a}=\left(3\hat{i}+\hat{j}-2\hat{k}\right)$ and $\vec{b}=\left(2\hat{i}+3\hat{j}-\hat{k}\right)$

$$\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&1&-2\\2&3&-1\end{vmatrix}=\hat{i}\begin{vmatrix}1&-2\\3&-1\end{vmatrix}-\hat{j}\begin{vmatrix}3&-2\\2&-1\end{vmatrix}+\hat{k}\begin{vmatrix}3&1\\2&3\end{vmatrix}$$

$$=\hat{i}(-1+6)-\hat{j}(-3+4)+\hat{k}(9-2)=\left(5\hat{i}-\hat{j}+7\hat{k}\right)$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right|=\sqrt{(5)^2+(-1)^2+(7)^2}=\sqrt{25+1+49}=\sqrt{75}=\pm5\sqrt{3}$$

Hence, the required unit vector $=\dfrac{\left(\vec{a}\times\vec{b}\right)}{\left|\vec{a}\times\vec{b}\right|}=\dfrac{\left(5\hat{i}-\hat{j}+7\hat{k}\right)}{\pm5\sqrt{3}}=\pm\dfrac{1}{5\sqrt{3}}\left(5\hat{i}-\hat{j}+7\hat{k}\right)$

(ii) Let $\vec{a}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)$ and $\vec{b}=\left(\hat{i}+2\hat{j}-\hat{k}\right)$

$$\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-2&3\\1&2&-1\end{vmatrix}=\hat{i}\begin{vmatrix}-2&3\\2&-1\end{vmatrix}-\hat{j}\begin{vmatrix}1&3\\1&-1\end{vmatrix}+\hat{k}\begin{vmatrix}1&-2\\1&2\end{vmatrix}$$

$$=\hat{i}(2-6)-\hat{j}(-1-3)+\hat{k}(2+2)=\left(-4\hat{i}+4\hat{j}+4\hat{k}\right)$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right|=\sqrt{(4)^2+(4)^2+(4)^2}=\pm4\sqrt{3}$$

Hence the required unit vector $=\dfrac{\left(\vec{a}\times\vec{b}\right)}{\left|\vec{a}\times\vec{b}\right|}=\dfrac{4\left(-\hat{i}+\hat{j}+\hat{k}\right)}{4\sqrt{3}}=\dfrac{\left(-\hat{i}+\hat{j}+\hat{k}\right)}{\sqrt{3}}$

(iii) $\vec{a}=\left(\hat{i}+3\hat{j}-2\hat{k}\right)$ and $\vec{b}=\left(-\hat{i}+3\hat{k}\right)$

$$\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&3&-2\\-1&0&3\end{vmatrix}=\hat{i}\begin{vmatrix}3&-2\\0&3\end{vmatrix}-\hat{j}\begin{vmatrix}1&-2\\-1&3\end{vmatrix}+\hat{k}\begin{vmatrix}1&3\\-1&0\end{vmatrix}$$

$$=\hat{i}(9-0)-\hat{j}(3-2)+\hat{k}(0+3)=\left(9\hat{i}-\hat{j}+3\hat{k}\right)$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right|=\sqrt{(9)^2+(-1)^2+(3)^2}=\sqrt{81+1+9}=\pm\sqrt{91}$$

Hence, the required unit vector $=\dfrac{\left(\vec{a}\times\vec{b}\right)}{\left|\vec{a}\times\vec{b}\right|}=\dfrac{\left(9\hat{i}-\hat{j}+3\hat{k}\right)}{\pm\sqrt{91}}$

(iv) Let $\vec{a}=\left(4\hat{i}+2\hat{j}-\hat{k}\right)$ and $\vec{b}=\left(\hat{i}+4\hat{j}-\hat{k}\right)$

$$\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\4&2&-1\\1&4&-1\end{vmatrix}=\hat{i}\begin{vmatrix}2&-1\\4&-1\end{vmatrix}-\hat{j}\begin{vmatrix}4&-1\\1&-1\end{vmatrix}+\hat{k}\begin{vmatrix}4&2\\1&4\end{vmatrix}$$

$$=\hat{i}(-2+4)-\hat{j}(-4+1)+\hat{k}(16-2)=\left(2\hat{i}+3\hat{j}+14\hat{k}\right)$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right|=\sqrt{(2)^2+(3)^2+(14)^2}=\sqrt{4+9+196}=\pm\sqrt{209}$$

Hence, the required unit vector $=\dfrac{\left(\vec{a}\times\vec{b}\right)}{\left|\vec{a}\times\vec{b}\right|}=\dfrac{\left(2\hat{i}+3\hat{j}+14\hat{k}\right)}{\pm\sqrt{209}}$

6. Find the unit vectors perpendicular to the plane of the vectors $\vec{a}=2\hat{i}-6\hat{j}-3\hat{k}$ and $\vec{b}=4\hat{i}+3\hat{j}-2\hat{k}$.

Sol. Let $\vec{a}=\left(2\hat{i}-6\hat{j}-3\hat{k}\right)$, $\vec{b}=\left(4\hat{i}+3\hat{j}-\hat{k}\right)$

$$\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -6 & -3\\ 4 & 3 & -1\end{vmatrix}=\hat{i}\begin{vmatrix}-6 & -3\\ 3 & -1\end{vmatrix}-\hat{j}\begin{vmatrix}2 & -3\\ 4 & -1\end{vmatrix}+\hat{k}\begin{vmatrix}2 & -6\\ 4 & 3\end{vmatrix}$$

$$=\hat{i}(6+9)-\hat{j}(-2+12)+\hat{k}(6+24)=\left(15\hat{i}-10\hat{j}+30\hat{k}\right)$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right|=\sqrt{(15)^2+(-10)^2+(30)^2}=\sqrt{225+100+900}=\sqrt{1225}=35$$

Hence, the required unit vector $=\dfrac{\left(\vec{a}\times\vec{b}\right)}{\left|\vec{a}\times\vec{b}\right|}=\dfrac{5\left(3\hat{i}-2\hat{j}+6\hat{k}\right)}{35}=\pm\dfrac{1}{7}\left(3\hat{i}-2\hat{j}+6\hat{k}\right)$

7. Find a vector of magnitude 6 which is perpendicular to both the vectors $\vec{a}=4\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=-2\hat{i}+\hat{j}-2\hat{k}$.

Sol. Let $\vec{a}=\left(4\hat{i}-\hat{j}+3\hat{k}\right)$ and $\vec{b}=\left(-2\hat{i}+\hat{j}-2\hat{k}\right)$

$$\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 3\\ -2 & 1 & -2\end{vmatrix}=\hat{i}\begin{vmatrix}1 & 3\\ 1 & -2\end{vmatrix}-\hat{j}\begin{vmatrix}4 & 3\\ -2 & -2\end{vmatrix}+\hat{k}\begin{vmatrix}4 & -1\\ -2 & 1\end{vmatrix}$$

$$=\hat{i}(2-3)-\hat{j}(-8+6)+\hat{k}(4-2)=\left(-\hat{i}+2\hat{j}+2\hat{k}\right)$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right|=\sqrt{(-1)^2+(2)^2+(2)^2}=\sqrt{1+4+4}=\sqrt{9}=3$$

So, a unit vector $\perp$ to both $\vec{a}\,\&\,\vec{b}=\dfrac{\left(\vec{a}\times\vec{b}\right)}{\left|\vec{a}\times\vec{b}\right|}=\dfrac{\left(-\hat{i}+2\hat{j}+2\hat{k}\right)}{3}$

The required unit vector $=\dfrac{6\left(-\hat{i}+2\hat{j}+2\hat{k}\right)}{3}=\pm2\left(-\hat{i}+2\hat{j}+2\hat{k}\right)$

8. Find a vector of magnitude 5 units, perpendicular to each of $\left(\vec{a}+\vec{b}\right)$ and $\left(\vec{a}-\vec{b}\right)$, where $\vec{a}=\left(\hat{i}+\hat{j}+\hat{k}\right)$ and $\vec{b}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)$.

Sol. Let $\vec{a}=\left(\hat{i}+\hat{j}+\hat{k}\right)$ and $\vec{b}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)$

$$\left(\vec{a}+\vec{b}\right)=\left(\hat{i}+\hat{j}+\hat{k}\right)+\left(\hat{i}+2\hat{j}+3\hat{k}\right)=\left(2\hat{i}+3\hat{j}+4\hat{k}\right)$$

$$\left(\vec{a}-\vec{b}\right)=\left(\hat{i}+\hat{j}+\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)=\left(-\hat{j}-2\hat{k}\right)$$

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 0 & -1 & -2\end{vmatrix}=\hat{i}\begin{vmatrix}3 & 4\\ -1 & -2\end{vmatrix}-\hat{j}\begin{vmatrix}2 & 4\\ 0 & -2\end{vmatrix}+\hat{k}\begin{vmatrix}2 & 3\\ 0 & -1\end{vmatrix}$$

$$=\hat{i}(-6+4)-\hat{j}(-4-0)+\hat{k}(-2-0)=(-2\hat{i}+4\hat{j}-2\hat{k})=2(-\hat{i}+2\hat{j}-\hat{k})$$

$$\Rightarrow \left|(\vec{a}\times\vec{b})\times(\vec{a}-\vec{b})\right|=\sqrt{(-2)^2+(4)^2+(-2)^2}=\sqrt{4+16+4}=\sqrt{24}=2\sqrt{16}$$

So, a unit vector $\perp$ to both $\vec{a}\,\&\,\vec{b}=\dfrac{\left[(\vec{a}\times\vec{b})\times(\vec{a}-\vec{b})\right]}{\left|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})\right|}$

The required vector $=\dfrac{5.2(-\hat{i}+2\hat{j}-\hat{k})}{2\sqrt{6}}=\dfrac{5}{\sqrt{6}}(-\hat{i}+2j-\hat{k})=\dfrac{5\sqrt{6}}{6}(-\hat{i}+2\hat{j}-\hat{k})$

9. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and $|\vec{a}\times\vec{b}|=\sqrt{3}$.

Sol. Let, θ = Angle between $\vec{a}$ and $\vec{b}$.

Given:- $|\vec{a}|=1, |\vec{b}|=2$ and $|\vec{a}\times\vec{b}|=\sqrt{3}$

We have, $|\vec{a}\times\vec{b}|=|\vec{a}|\,|\vec{b}|\cdot\sin\theta \Rightarrow \sqrt{3}=1\times 2\times\sin\theta \Rightarrow \sin\theta=\dfrac{\sqrt{3}}{2}\Rightarrow\theta=\dfrac{\pi}{3}$ Ans.

10. Let $\vec{a}=(\hat{i}-\hat{j}), \vec{b}=(3\hat{j}-\hat{k})$ and $\vec{c}=(7\hat{i}-\hat{k})$. Find a vector $\vec{d}$ such that it is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c}.\vec{d}=1$.

Sol. Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$\vec{d}\perp\vec{a}$, $\vec{d}.\vec{a}=0 \quad\Rightarrow(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})(\hat{i}-\hat{j})=0$

$\Rightarrow a_1-a_2=0 \quad$ ...(i)

$\vec{d}\perp\vec{b}$, $\vec{a}.\vec{b}=0 \quad\Rightarrow(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})(3j-\hat{k})=0$

$\Rightarrow 3a_2-a_3=0 \quad$ ...(ii)

$\vec{d}.\vec{c}=1 \quad\Rightarrow(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})(7\hat{i}-\hat{k})=1$

$\Rightarrow 7a_1-a_3=1 \quad$ ...(iii)

Solving equation (i) and (ii) we get $3a_1-a_3=0$ ...(iv)

Again solving equation (iii) & (iv) we get $a_1=\dfrac{1}{4}$

From equation (i), $a_1-a_2=0$ or $a_1=a_2=\dfrac{1}{4}$

From equation (ii), $3a_2-a_3=0 \Rightarrow 3.\dfrac{1}{4}=a_3 \Rightarrow a_3=\dfrac{3}{4}$. Hence $\vec{d}=\dfrac{1}{4}\hat{i}+\dfrac{1}{4}\hat{j}+\dfrac{3}{4}\hat{k}$.

11. If $\vec{a}=(4\hat{i}+5\hat{j}-\hat{k}), \vec{b}=(\hat{i}-4\hat{j}+5\hat{k})$, and $\vec{c}=(3\hat{i}+\hat{j}-\hat{k})$, find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and for which $\vec{c}\cdot\vec{d}=21$

Sol. Given:- $\vec{a}=4\hat{i}+5\hat{j}-\hat{k}$

$\vec{b}=\hat{i}-4\hat{j}+5\hat{k}$

and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$

$\because \vec{d} \perp \vec{a}$ and $\vec{b}$

$\Rightarrow \vec{d} \parallel \left(\vec{a} \times \vec{b}\right) \Rightarrow \vec{d} = k\left(\vec{a} \times \vec{b}\right)$ (Let)

$$\Rightarrow \vec{d} = k\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & -1 \\ 1 & -4 & 5 \end{vmatrix} \Rightarrow \vec{d} = k\left\{\begin{vmatrix} 5 & -1 \\ -4 & 5 \end{vmatrix}\hat{i} - \begin{vmatrix} 4 & -1 \\ -1 & 5 \end{vmatrix}\hat{j} + \begin{vmatrix} 4 & 5 \\ 1 & -4 \end{vmatrix}\hat{k}\right\}$$

$\Rightarrow \vec{d} = k\{(25-4)\vec{i} - (20+1)\vec{j} + (-16-15)\vec{k}\} \Rightarrow \vec{d} = 21k\vec{i} - 21k\vec{j} - 21k\vec{k}\}$

Now, $\vec{c} \cdot \vec{d} = 21$

$\Rightarrow (3\hat{i} + \hat{j} - \hat{k}) \cdot (21k\hat{i} - 21k\hat{j} - 21k\hat{k}) = 21 \Rightarrow 63k - 21k + 21k = 21 \Rightarrow 63k = 21 \Rightarrow k = \frac{1}{3}$

$\therefore \vec{d} = 21 \times \frac{1}{3}\hat{i} - 21 \times \frac{1}{3}\hat{j} - 21 \times \frac{1}{3}\hat{k} = \vec{d} = \left(7\hat{i} - 7\hat{j} - 7\hat{k}\right)$ Ans.

12. Prove that $\left|\vec{a} \times \vec{b}\right| = \left(\vec{a}.\vec{b}\right)\tan\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

Sol. L.H.S $\left|\vec{a} \times \vec{b}\right| = |\vec{a}|\left|\vec{b}\right|\sin\theta$ ...(i)

We know that $\vec{a}.\vec{b} = |\vec{a}|\left|\vec{b}\right|\cos\theta \quad \Rightarrow |\vec{a}|\left|\vec{b}\right| = \frac{\vec{a}.\vec{b}}{\cos\theta}$

From equation (i), $\left|\vec{a} \times \vec{b}\right| = \frac{\vec{a}.\vec{b}}{\cos\theta}\sin\theta \quad \Rightarrow \left|\vec{a} \times \vec{b}\right| = \left(\vec{a}.\vec{b}\right)\tan\theta$

13. Write the value of p for which $\vec{a} = \left(3\hat{i} + 2\hat{j} + 9\hat{k}\right)$ and $\vec{b} = \left(\hat{i} + p\hat{j} + 3\hat{k}\right)$ are parallel vectors

Sol. $\because \vec{a} \parallel \vec{b}$

$\Rightarrow \vec{a} \times \vec{b} = \vec{0}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 9 \\ 1 & p & 3 \end{vmatrix} = \vec{0}$$

$$\begin{vmatrix} 2 & 9 \\ P & 3 \end{vmatrix}\hat{i} - \begin{vmatrix} 3 & 9 \\ 1 & 3 \end{vmatrix}\hat{j} + \begin{vmatrix} 3 & 2 \\ 1 & P \end{vmatrix}\hat{k} = \vec{0}$$

$\Rightarrow (6-9P)\hat{i} - (9-9)\hat{j} + (3P-2)\hat{k} = \vec{0} \Rightarrow 3(2-3P)\hat{i} + (3P-2)\hat{k} = \vec{0} \Rightarrow 2-3P = 0$ and $3P-2 = 0$

$\Rightarrow P = \frac{2}{3}$ Ans.

14. Verify that $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) + \left(\vec{a} \times \vec{c}\right)$, when

(i) $\vec{a} = \hat{i} - \hat{j} - 3\hat{k}, \vec{b} = 4\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 2\hat{k}$

(ii) $\vec{a} = 4\hat{i} - \hat{j} + \hat{k}, \vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} + \hat{k}$

Sol. **(i)** $\vec{a} = \left(\hat{i} - \hat{j} - 3\hat{k}\right)$, $\vec{b} = \left(4\hat{i} - 3\hat{j} + \hat{k}\right)$ & $\vec{c} = \left(2\hat{i} - \hat{j} + 2\hat{k}\right)$

$\Rightarrow (\vec{b}+\vec{c}) = (4\hat{i}-3\hat{j}+\hat{k})+(2\hat{i}-\hat{j}+2\hat{k}) = (6\hat{i}-4\hat{j}+3\hat{k})$

$$\text{L.H.S} = \{\vec{a}\times(\vec{b}+\vec{c})\} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & -3\\ 6 & -4 & 3\end{vmatrix} = \hat{i}\begin{vmatrix}-1 & -3\\ -4 & 3\end{vmatrix} - \hat{j}\begin{vmatrix}1 & -3\\ 6 & 3\end{vmatrix} + \hat{k}\begin{vmatrix}1 & -1\\ 6 & -4\end{vmatrix}$$

$$= \hat{i}(-3-12)-\hat{j}(3+18)+\hat{k}(-4+6) = (-15\hat{i}-21\hat{j}+2\hat{k})$$

$$\text{R.H.S} = (\vec{a}\times\vec{b})+(\vec{a}\times\vec{c}) = \{(\hat{i}-\hat{j}-3\hat{k})\times(4\hat{i}-3\hat{j}+\hat{k})\}+\{(\hat{i}-\hat{j}-3\hat{k})\times(2\hat{i}-\hat{j}+2\hat{k})\}$$

$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & -3\\ 4 & -3 & 1\end{vmatrix} + \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & -3\\ 2 & -1 & 2\end{vmatrix}$$

$$= \left\{\hat{i}\begin{vmatrix}-1 & -3\\ -3 & 1\end{vmatrix} - \hat{j}\begin{vmatrix}1 & -3\\ 4 & 1\end{vmatrix} + \hat{k}\begin{vmatrix}1 & -1\\ 4 & -3\end{vmatrix}\right\} + \left\{\hat{i}\begin{vmatrix}-1 & -3\\ -1 & 2\end{vmatrix} - \hat{j}\begin{vmatrix}1 & -3\\ 2 & 2\end{vmatrix} + \hat{k}\begin{vmatrix}1 & -1\\ 2 & -1\end{vmatrix}\right\}$$

$$= \{\hat{i}(-1-9)-\hat{j}(1+12)+\hat{k}(-3+4)\}+\{\hat{i}(-2-3)-\hat{j}(2+6)+\hat{k}(-1+2)\}$$

$$= (-10\hat{i}-13\hat{j}+\hat{k})+(-5\hat{i}-8\hat{j}+\hat{k}) = (-15\hat{i}-21\hat{j}+2\hat{k})$$

Hence. $\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$

(ii) $\vec{a} = (4\hat{i}-\hat{j}+\hat{k})$, $\vec{b} = (\hat{i}+\hat{j}+\hat{k})$ & $\vec{c} = (\hat{i}-\hat{j}+\hat{k})$

$\Rightarrow (\vec{b}+\vec{c}) = (\hat{i}+\hat{j}+\hat{k})+(\hat{i}-\hat{j}+\hat{k}) = (2\hat{i}+2\hat{k})$

$$\Rightarrow (\vec{a}\times\vec{b}) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 1\\ 1 & 1 & 1\end{vmatrix} = \hat{i}\begin{vmatrix}-1 & 1\\ 1 & 1\end{vmatrix} - \hat{j}\begin{vmatrix}4 & 1\\ 1 & 1\end{vmatrix} + \hat{k}\begin{vmatrix}4 & -1\\ 1 & 1\end{vmatrix}$$

$$= \hat{i}(-1-1)-\hat{j}(4-1)+\hat{k}(4+1) = (-2\hat{i}-3\hat{j}+5\hat{k})$$

$$\Rightarrow (\vec{a}\times\vec{c}) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 1\\ 1 & -1 & 1\end{vmatrix} = \hat{i}\begin{vmatrix}-1 & 1\\ -1 & 1\end{vmatrix} - \hat{j}\begin{vmatrix}4 & 1\\ 1 & 1\end{vmatrix} + \hat{k}\begin{vmatrix}4 & -1\\ 1 & -1\end{vmatrix}$$

$$= \hat{i}(-1+1)-\hat{j}(4-1)+\hat{k}(-4+1) = (-3\hat{j}-3\hat{k})$$

$$\text{L.H.S} = \vec{a}\times(\vec{b}+\vec{c}) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 1\\ 2 & 0 & 2\end{vmatrix} = \hat{i}\begin{vmatrix}-1 & 1\\ 0 & 2\end{vmatrix} - \hat{j}\begin{vmatrix}4 & 1\\ 2 & 2\end{vmatrix} + \hat{k}\begin{vmatrix}4 & -1\\ 2 & 0\end{vmatrix}$$

$$= \hat{i}(-2-0)-\hat{j}(8-2)+\hat{k}(0+2) = (-2\hat{i}-6\hat{j}+2\hat{k})$$

$$\text{R.H.S} = (\vec{a}\times\vec{b})+(\vec{a}\times\vec{c}) = (-2\hat{i}-3\hat{j}+5\hat{k})+(-3\hat{j}-3\hat{k}) = (-2\hat{i}-6\hat{j}+2\hat{k})$$

Hence, $\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$

15. Find the area of the parallelogram whose adjacent sides are represented by the vectors

(i) $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=-3\hat{i}-2\hat{j}+\hat{k}$ (ii) $\vec{a}=(3\hat{i}+\hat{j}+4\hat{k})$ and $\vec{b}=(\hat{i}-\hat{j}+\hat{k})$

(iii) $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}$ (iv) $\vec{a}=2\hat{i}$ and $\vec{b}=3\hat{j}$

Sol. **(i)** Let $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=-3\hat{i}-2\hat{j}+\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&2&3\\-3&-2&1\end{vmatrix}=\hat{i}\begin{vmatrix}2&3\\-2&1\end{vmatrix}-\hat{j}\begin{vmatrix}1&3\\-3&1\end{vmatrix}+\hat{k}\begin{vmatrix}1&2\\-3&-2\end{vmatrix}$$

$$=\hat{i}(2+6)-\hat{j}(1+9)+\hat{k}(-2+6)=8\hat{i}-10\hat{j}+4\hat{k}$$

Required area $=|\vec{a}\times\vec{b}|=\sqrt{(8)^2+(-10)^2+(4)^2}=\sqrt{64+100+16}=\sqrt{180}=6\sqrt{5}$ sq. units.

(ii) Let $\vec{a}=(3\hat{i}+\hat{j}+4\hat{k})$ and $\vec{b}=(\hat{i}-\hat{j}+\hat{k})$

$$(\vec{a}\times\vec{b})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&1&4\\1&-1&1\end{vmatrix}=\hat{i}\begin{vmatrix}1&4\\-1&1\end{vmatrix}-\hat{j}\begin{vmatrix}3&4\\1&1\end{vmatrix}+\hat{k}\begin{vmatrix}3&1\\1&-1\end{vmatrix}$$

$$=\hat{i}(1+4)-\hat{j}(3-4)+\hat{k}(-3-1)=(5\hat{i}+\hat{j}-4\hat{k})$$

Required area $=|\vec{a}\times\vec{b}|=\sqrt{(5)^2+(1)^2+(-4)^2}=\sqrt{25+1+16}=\sqrt{42}$ sq units.

(iii) Let $\vec{a}=(2\hat{i}+\hat{j}+3\hat{k})$, $\vec{b}=(\hat{i}-\hat{j})$

$$(\vec{a}\times\vec{b})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&1&3\\1&-1&0\end{vmatrix}=\hat{i}\begin{vmatrix}1&3\\-1&0\end{vmatrix}-\hat{j}\begin{vmatrix}2&3\\1&0\end{vmatrix}+\hat{k}\begin{vmatrix}2&1\\1&-1\end{vmatrix}$$

$$=\hat{i}(0+3)-\hat{j}(0-3)+\hat{k}(-2-1)=(3\hat{i}+3\hat{j}-3\hat{k})$$

Required area $=|\vec{a}\times\vec{b}|-\sqrt{(3)^2+(3)^2+(-3)^2}=\sqrt{3^2(1^2+1^2+1^2)}=3\sqrt{3}$ sq. units

(iv) Let $\vec{a}=2\hat{i}$, $\vec{b}=3\hat{j}$

$$(\vec{a}\times\vec{b})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&0&0\\0&3&0\end{vmatrix}=\hat{i}\begin{vmatrix}0&0\\3&0\end{vmatrix}-\hat{j}\begin{vmatrix}2&0\\0&0\end{vmatrix}+\hat{k}\begin{vmatrix}2&0\\0&3\end{vmatrix}=6\hat{k}$$

Required area $=|\vec{a}\times\vec{b}|=\sqrt{(6)^2}=6$ sq. units.

16. Find the area of the parallelogram whose diagonals are represented by the vectors

(i) $\vec{d_1}=3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{d_2}=\hat{i}-3\hat{j}+4\hat{k}$ (ii) $\vec{d_1}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{d_2}=3\hat{i}+4\hat{j}-\hat{k}$

(iii) $\vec{d_1}=\hat{i}-3\hat{j}+2\hat{k}$ and $\vec{d_2}=-\hat{i}+2\hat{j}$

Sol. **(i)** Let $\vec{d_1}=(3\hat{i}+\hat{j}-2\hat{k})$, $\vec{d_2}=(\hat{i}-3\hat{j}+4\hat{k})$

$$\left(\vec{d}_1\times\vec{d}_2\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & -2\\ 1 & -3 & 4\end{vmatrix}=\hat{i}\begin{vmatrix}1 & -2\\ -3 & 4\end{vmatrix}-\hat{j}\begin{vmatrix}3 & -2\\ 1 & 4\end{vmatrix}+\hat{k}\begin{vmatrix}3 & 1\\ 1 & -3\end{vmatrix}$$

$$=\hat{i}(4-6)-\hat{j}(12+2)+\hat{k}(-9-1)=\left(-2\hat{i}-14\hat{j}-10\hat{k}\right)$$

$$\Rightarrow \left|\vec{d}_1\times\vec{d}_2\right|=\sqrt{(-2)^2+(-14)^2+(-10)^2}=\sqrt{4+196+100}=\sqrt{300}=10\sqrt{3}$$

$\therefore$ Required area $=\frac{1}{2}\left|\vec{d}_1\times\vec{d}_2\right|=\frac{1}{2}\times 10\sqrt{3}=5\sqrt{3}$ sq. units

(ii) Let $\vec{d}_1=\left(2\hat{i}-\hat{j}+\hat{k}\right)$, $\vec{d}_2=\left(3\hat{i}+4\hat{j}-\hat{k}\right)$

$$\left(\vec{d}_1\times\vec{d}_2\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -1 & 1\\ 3 & 4 & -1\end{vmatrix}=\hat{i}\begin{vmatrix}-1 & 1\\ 4 & -1\end{vmatrix}-\hat{j}\begin{vmatrix}2 & 1\\ 3 & -1\end{vmatrix}+\hat{k}\begin{vmatrix}2 & -1\\ 3 & 4\end{vmatrix}$$

$$=\hat{i}(1-4)-\hat{j}(-2-3)+\hat{k}(8+3)=\left(-3\hat{i}+5\hat{j}+11\hat{k}\right)$$

$$\Rightarrow \left|\vec{d}_1\times\vec{d}_2\right|=\sqrt{(-3)^2+(5)^2+(11)^2}=\sqrt{9+25+121}=\sqrt{155}$$

$\therefore$ Required area $=\frac{1}{2}\left|\vec{d}_1\times\vec{d}_2\right|=\frac{1}{2}\sqrt{155}$ sq. units

(iii) Let $\vec{d}_1=\left(\hat{i}-3\hat{j}+2\hat{k}\right)$, $\vec{d}_2=\left(-\hat{i}+2\hat{j}\right)$,

$$\left(\vec{d}_1\times\vec{d}_2\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 2\\ -1 & 2 & 0\end{vmatrix}=\hat{i}\begin{vmatrix}-3 & 2\\ 2 & 0\end{vmatrix}-\hat{j}\begin{vmatrix}1 & 2\\ -1 & 0\end{vmatrix}+\hat{k}\begin{vmatrix}1 & -3\\ -1 & 2\end{vmatrix}$$

$$=\hat{i}(0-4)-\hat{j}(0+2)+\hat{k}(2-3)=\left(-4\hat{i}-2\hat{j}-\hat{k}\right)$$

$$\Rightarrow \left|\vec{d}_1\times\vec{d}_2\right|=\sqrt{(-4)^2+(-2)^2+(-1)^2}=\sqrt{16+4+1}=\sqrt{21}$$

$\therefore$ Required area $=\frac{1}{2}\left|\vec{d}_1\times\vec{d}_2\right|=\frac{1}{2}\sqrt{21}$ sq. units.

17. Find the area of the triangle whose two adjacent sides are determined by the vectors

(i) $\vec{a}=-2\hat{i}-5\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}-\hat{k}$ (ii) $\vec{a}=3\hat{i}+4\hat{j}$ and $\vec{b}=-5\hat{i}+7\hat{j}$

Sol. (i) Let $\vec{a}=\left(-2\hat{i}-5\hat{k}\right)$, $\vec{b}=\left(\hat{i}-2\hat{j}-\hat{k}\right)$

$$\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 0 & -5\\ 1 & -2 & -1\end{vmatrix}=\hat{i}\begin{vmatrix}0 & -5\\ -2 & -1\end{vmatrix}-\hat{j}\begin{vmatrix}-2 & -5\\ 1 & -1\end{vmatrix}+\hat{k}\begin{vmatrix}-2 & 0\\ 1 & -2\end{vmatrix}$$

$$=\hat{i}(0-10)-\hat{j}(2+5)+\hat{k}(4-0)=\left(-10\hat{i}-7\hat{j}+4\hat{k}\right)$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right|=\sqrt{(-10)^2+(-7)^2+(4)^2}=\sqrt{100+49+16}=\sqrt{165}$$

$\therefore$ Required area $=\frac{1}{2}\left|\vec{a}\times\vec{b}\right|=\frac{1}{2}\sqrt{165}$ sq. units.

(ii) Let $\vec{a}=\left(3\hat{i}+4\hat{j}\right)$, $\vec{b}=\left(-5\hat{i}+7\hat{j}\right)$

$$\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 4 & 0\\ -5 & 7 & 0\end{vmatrix}=\hat{i}\begin{vmatrix}4 & 0\\ 7 & 0\end{vmatrix}-\hat{j}\begin{vmatrix}3 & 0\\ -5 & 0\end{vmatrix}+\hat{k}\begin{vmatrix}3 & 4\\ -5 & 7\end{vmatrix}$$

$$=\hat{k}(21+20)=41\hat{k}\Rightarrow\left|\vec{a}\times\vec{b}\right|=\sqrt{(41)^2}=41$$

$\therefore$ Required area $=\frac{1}{2}\left|\vec{a}\times\vec{b}\right|=\frac{1}{2}\times 41=\frac{41}{2}$ sq. units.

18. Using vectors, find the area of ΔABC whose vertices are

(i) $A(1,1,2), B(2,3,5)$ and $C(1,5,5)$ (ii) $A(1,2,3), B(2,-1,4)$ and $C(4,5,-1)$

(iii) $A(3,-1,2), B(1,-1,-3), C(4,-3,1)$ (iv) $A(1,-1,2), B(2,1,-1)$ and $C(3,-1,2)$

Sol. (i) Given vertices are A(1, 1, 2), B (2, 3, 5) and C(1, 5, 5).

$\vec{AB}=(2-1)\hat{i}+(3-1)\hat{j}+(5-2)\hat{k}=\hat{i}+2\hat{j}+3\hat{k}$

$\vec{AC}=(1-1)\hat{i}+(5-1)\hat{j}+(5-2)\hat{k}=4\hat{j}+3\hat{k}$

Now, $\vec{AB}\times\vec{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ 0 & 4 & 3\end{vmatrix}=\begin{vmatrix}2 & 3\\ 4 & 3\end{vmatrix}\hat{i}-\begin{vmatrix}1 & 3\\ 0 & 3\end{vmatrix}\hat{j}+\begin{vmatrix}1 & 2\\ 0 & 4\end{vmatrix}\hat{k}=-6\hat{i}-3\hat{j}+4\hat{k}$

$\left|\vec{AB}\times\vec{AC}\right|=\sqrt{(-6)^2+(-3)^2+4^2}=\sqrt{36+9+16}=\sqrt{61}$

Area of $\Delta ABC=\frac{1}{2}|\vec{AB}\times\vec{AC}|=\frac{1}{2}\sqrt{61}$ square units.

(ii). Given vertices are, A(1, 2, 3), B(2, -1, 4) and C(4, 5, -1)

Now, $\vec{AB}=(2-1)\hat{i}+(-1-2)\hat{j}+(4-3)\hat{k}$

$\Rightarrow\vec{AB}=\hat{i}-3\hat{j}+\hat{k}$

again, $\vec{AC}=(4-2)\hat{i}+(5+1)\hat{j}+(-1-4)\hat{k}$

$\vec{AC}=2\hat{i}+6\hat{j}-5\hat{k}$

Now, $\vec{AB}\times\vec{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 2 & 6 & -5\end{vmatrix}$

$\Rightarrow\vec{AB}\times\vec{AC}=\begin{vmatrix}-3 & 1\\ 6 & -5\end{vmatrix}\hat{i}-\begin{vmatrix}1 & 1\\ 2 & -5\end{vmatrix}\hat{j}+\begin{vmatrix}1 & -3\\ 6 & 6\end{vmatrix}\hat{k}=9\hat{i}+7\hat{j}+12\hat{k}$

$|\vec{AB}\times\vec{AC}|=\sqrt{9^2+7^2+12^2}=\sqrt{81+49+144}=\sqrt{274}$

Area of $\Delta ABC=\frac{1}{2}|\vec{AB}\times\vec{AC}|$

$\frac{1}{2}\sqrt{274}$ Sq. Units.

(iii). $A(3,-1,2), B(1,-1,-3)$ and $C(4,-3,1)$

$$\Rightarrow (\overrightarrow{AB}\times\overrightarrow{AC})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -4 & 5 & 7\\ 4 & -5 & -7\end{vmatrix}=\hat{i}\begin{vmatrix}5 & 7\\ -5 & -7\end{vmatrix}-\hat{j}\begin{vmatrix}-4 & 7\\ 4 & -7\end{vmatrix}+\hat{k}\begin{vmatrix}-4 & 5\\ 4 & -5\end{vmatrix}=0$$

Hence, A, B and C area collinear.

(ii) Let $\vec{A}=(6\hat{i}-7\hat{j}-\hat{k}), \vec{B}=(2\hat{i}-3\hat{j}+\hat{k}), \vec{C}=(4\hat{i}-5\hat{j})$

$\Rightarrow \overrightarrow{AB}=$ Position vector of B – position vector of A

$=(2\hat{i}-3\hat{j}+\hat{k})-(6\hat{i}-7\hat{j}-\hat{k})=(-4\hat{i}+4\hat{j}+2\hat{k})$

$\Rightarrow \overrightarrow{AC}=$ Position vector of C – Position vector of A

$=(4\hat{i}-5\hat{j})-(6\hat{i}-7\hat{j}-\hat{k})=(-2\hat{i}+2\hat{j}+\hat{k})$

$$\Rightarrow (\overrightarrow{AB}\times\overrightarrow{AC})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -4 & 4 & 2\\ -2 & 2 & 1\end{vmatrix}=\hat{i}\begin{vmatrix}4 & 2\\ 2 & 1\end{vmatrix}-\hat{j}\begin{vmatrix}-4 & 2\\ -2 & 1\end{vmatrix}+\hat{k}\begin{vmatrix}-4 & 4\\ -2 & 2\end{vmatrix}$$

$=\hat{i}\ (4-4)-\hat{j}(-4+4)+\hat{k}(-8+8)=0$. Hence, A, B and C are collinear.

20. Show that the points A,B,C with position vectors $(3\hat{i}-2\hat{j}+4\hat{k})$, $(\hat{i}+\hat{j}+\hat{k})$ and $(-\hat{i}+4\hat{j}-2\hat{k})$ respectively are collinear.

Sol. Let $\vec{A}=(3\hat{i}-2\hat{j}+4\hat{k}), \vec{B}=(\hat{i}+\hat{j}+\hat{k}), \vec{C}=(\ \hat{i}+4\hat{j}-2\hat{k})$

$\Rightarrow \overrightarrow{AB}=$ Position vector of B – position vector of A

$=(\hat{i}+\hat{j}+\hat{k})-(3\hat{i}-2\hat{j}+4\hat{k})=(-2\hat{i}+3\hat{j}-3\hat{k})$

$\Rightarrow \overrightarrow{AC}=$ Position vector of C – position vector of A

$=(-\hat{i}+4\hat{j}-2\hat{k})-(3\hat{i}-2\hat{j}+4\hat{k})=(-4\hat{i}+6\hat{j}-6\hat{k})$

$$\Rightarrow (\overrightarrow{AB}\times\overrightarrow{AC})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 3 & -3\\ -4 & 6 & -6\end{vmatrix}=\hat{i}\begin{vmatrix}3 & -3\\ 6 & -6\end{vmatrix}-\hat{j}\begin{vmatrix}-2 & 3\\ -4 & 6\end{vmatrix}+\hat{k}\begin{vmatrix}-2 & 3\\ -4 & 6\end{vmatrix}$$

$=\hat{i}(-18+18)-\hat{j}(-12+12)+\hat{k}(-12+12)=0$

Hence, proved that A,B and C are collinear.

21. Show that the points having position vectors $\vec{a},\vec{b},(3\vec{a}-2\vec{b})$ are collinear, whatever be $\vec{a},\vec{b},\vec{c}$.

Sol. Let $\vec{A}=\vec{a}$, $\vec{B}=\vec{b}$, $\vec{C}=(3\vec{a}-2\vec{b})$

$\Rightarrow \overrightarrow{AB}=$ position vector of B – position vector of $A=(\vec{b}-\vec{a})$

$\Rightarrow \overrightarrow{AC}=$ position vector of C – position vector of A $=(3\vec{a}-2\vec{b})-\vec{a}=(2\vec{a}-2\vec{b})$

$$\Rightarrow (\overrightarrow{AB}\times\overrightarrow{AC})=\begin{vmatrix}\vec{a} & \vec{b} & \vec{c}\\ -1 & 1 & 0\\ 2 & -2 & 0\end{vmatrix}=\vec{a}\begin{vmatrix}1 & 0\\ -2 & 0\end{vmatrix}-\vec{b}\begin{vmatrix}-1 & 0\\ 2 & 0\end{vmatrix}+\vec{c}\begin{vmatrix}-1 & 1\\ 2 & -2\end{vmatrix}=\vec{c}(2-2)=0$$

Hence, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear.

22. Show that the points having position vectors $\left(-2\vec{a}+3\vec{b}+5\vec{c}\right)$, $\left(\vec{a}+2\vec{b}+3\vec{c}\right)$ and $\left(7\vec{a}-\vec{c}\right)$ are collinear, whatever be $\vec{a}, \vec{b}, \vec{c}$.

Sol. Let, $\vec{A}=\left(-2\vec{a}+3\vec{b}+5\vec{c}\right)$, $\vec{B}=\left(\vec{a}+2\vec{b}+3\vec{c}\right)$, $\vec{C}=\left(7\vec{a}-\vec{c}\right)$

$\Rightarrow \overrightarrow{AB}=$ Position vector of $B-$ position vector of A

$=\left(\vec{a}+2\vec{b}+3\vec{c}\right)-\left(-2\vec{a}+3\vec{b}+5\vec{c}\right)=\left(3\vec{a}-\vec{b}-2\vec{c}\right)$

$\Rightarrow \overrightarrow{AC}=$ Position vector of $C-$ position vector of A

$=\left(7\vec{a}-\vec{c}\right)-\left(-2\vec{a}+3\vec{b}+5\vec{C}\right)=\left(9\vec{a}-3\vec{b}-6\vec{c}\right)$

$$\Rightarrow \left(\overrightarrow{AB}\times\overrightarrow{AC}\right)=\begin{vmatrix}\vec{a} & \vec{b} & \vec{c}\\ 3 & -1 & -2\\ 9 & -3 & -6\end{vmatrix}=\vec{a}\begin{vmatrix}-1 & -2\\ -3 & -6\end{vmatrix}-\vec{b}\begin{vmatrix}3 & -2\\ 9 & -6\end{vmatrix}+\vec{c}\begin{vmatrix}3 & -1\\ 9 & -3\end{vmatrix}$$

$$=\vec{a}(6-6)-\vec{b}(-18+18)+\vec{c}(-9+9)=0.$$

Hence, $\vec{A}$, $\vec{B}$ and $\vec{C}$ are collinear.

23. Find a unit vector perpendicular to the plane ABC, where the points A, B, C are $(3,-1,2)$, $(1,-1,-3)$ and $(4,-3,1)$ respectively.

Sol. Let $\vec{A}=\left(3\hat{i}-\hat{j}+2\hat{k}\right)$, $\vec{B}=\left(\hat{i}-\hat{j}-3\hat{k}\right)$, $\vec{C}=\left(4\hat{i}-3\hat{j}+\hat{k}\right)$

$\Rightarrow \overrightarrow{AB}=$ Position vector of $B-$ position vector of A

$=\left(\hat{i}-\hat{j}-3\hat{k}\right)-\left(3\hat{i}-\hat{j}+2\hat{k}\right)=\left(-2\hat{i}-5\hat{k}\right)$

$\Rightarrow \overrightarrow{AC}=$ Position vector of $C-$ position vector of A

$=\left(4\hat{i}-3\hat{j}+\hat{k}\right)-\left(3\hat{i}-\hat{j}+2\hat{k}\right)=\left(\hat{i}-2\hat{j}-\hat{k}\right)$

$$\Rightarrow \left(\overrightarrow{AB}\times\overrightarrow{AC}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 0 & -5\\ 1 & -2 & -1\end{vmatrix}=\hat{i}\begin{vmatrix}0 & -5\\ -2 & -1\end{vmatrix}-\hat{j}\begin{vmatrix}-2 & -5\\ 1 & -1\end{vmatrix}+\hat{k}\begin{vmatrix}-2 & 0\\ 1 & -2\end{vmatrix}$$

$$=\hat{i}(0-10)-\hat{j}(2+5)+\hat{k}(4-0)=\left(-10\hat{i}-7\hat{j}+4\hat{k}\right)$$

$$\Rightarrow \left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{(-10)^2+(-7)^2+(4)^2}=\sqrt{100+49+16}=\sqrt{165}$$

$\therefore$ Required unit vector is $=\dfrac{\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)}{\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|}=\dfrac{\left(-10\hat{i}-7\hat{j}+4\hat{k}\right)}{\sqrt{165}}$

24. If $\vec{a}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)$ and $\vec{b}=\left(\hat{i}-3\hat{k}\right)$ then find $\left|\vec{b}\times 2\vec{a}\right|$

Sol. Given : - $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$

and, $\vec{b}=\hat{i}-3\hat{k}$

Now, $2\vec{a}=2\hat{i}-4\hat{j}+6\hat{k}$

and, $\vec{b}\times 2\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -3 \\ 1 & 0 & -3 \end{vmatrix}$

$= \begin{vmatrix} -2 & -3 \\ 0 & -3 \end{vmatrix}\hat{i} - \begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix}\hat{j} + \begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix}\hat{k} = 6\hat{i} - 6\hat{j} + 2\hat{k} \Rightarrow |\vec{b}\times 2\vec{a}| = \sqrt{6^2+6^2+2^2} = \sqrt{76} = 2\sqrt{19}$

25. if $|\vec{a}| = 2, |\vec{b}| = 5$ and $|\vec{a}\times\vec{b}| = 8$, find $\vec{a}\cdot\vec{b}$

Sol. Given:- $|\vec{a}| = 2,\ |\vec{b}| = 5$ and $|\vec{a}\times\vec{b}| = 8$

$\because |\vec{a}\times\vec{b}| = 8$

$\Rightarrow |\vec{a}||\vec{b}|\cdot\sin\theta = 8 \Rightarrow 2\times 5\times\sin\theta = 8 \Rightarrow \sin\theta = \frac{4}{5}$

Now, $\cos\theta = \sqrt{1-\sin^2\theta}$

$\Rightarrow \cos\theta = \sqrt{1-\frac{16}{25}} \Rightarrow \cos\theta = \frac{3}{5}$

Now, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = 2\times 5\times\frac{3}{5} = 6$ Ans.

26. If $|\vec{a}| = 2, |\vec{b}| = 7$ and $(\vec{a}\times\vec{b}) = (3\hat{i}+2\hat{j}+6\hat{k})$, find the angle between $\vec{a}$ and $\vec{b}$

Sol. Given:- $|\vec{a}| = 2,\ |\vec{b}| = 7$ & $\vec{a}\times\vec{b} = 3\hat{i}+2\hat{j}+6\hat{k}$

Let, θ = Angle between $\vec{a}$ and $\vec{b}$

Now, $|\vec{a}\times\vec{b}| = \sqrt{3^2+2^2+6^2} = 7$

Now, $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\cdot\sin\theta \Rightarrow 7 = 2\times 7\times\sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi^c}{6}$ Ans