FUNDAMENTAL CONCEPTS OF 3-DIMENSIONAL GEOMETRY (XII, R. S. AGGARWAL)

EXERCISE 26 [Pg.No.: 1103]

1. Find the direction cosines of a line segment whose direction ratios are

(i) $2,-6,3$ (ii) $2,-,1-2$ (iii) $-9,6,-2$

Sol. (i) Here, a = 2, b = -6 and c = 3

Now, $\sqrt{a^2+b^2+c^2}=\sqrt{2^2+(-6)^2+3^2}=7$

Hence, direction cosines are $\frac{2}{7},-\frac{6}{7},\frac{3}{7}$

(ii) Here, a = 2, b = -1 and c = -2

Now, $\sqrt{a^2+b^2+c^2}=\sqrt{2^2+(-1)^2+(-2)^2}=3$ Hence, direction cosines are $\frac{2}{3},-\frac{1}{3},-\frac{2}{3}$.

(iii) Here, a = -9, b = 6 and c = -2

Now, $\sqrt{a^2+b^2+c^2}=\sqrt{(-9)^2+6^2+(-2)^2}=\sqrt{121}=11$ Hence, direction cosines are $-\frac{9}{11},\frac{6}{11},-\frac{2}{11}$.

2 Find the direction ratios and the direction consines of the line segment joining the points

(i) $A(1,0,0)$ and $B(0,1,1)$ (ii) $A(5,6,-3)$ and $B(1,-6,3)$

(iii) $A(-5,7,-9)$ and $B(-3,4,-6)$

Sol. (i) Direction rations of AB

are (0 – 1), (1 – 0), (1 – 0)

i.e. -1, 1, 1

Now, $\sqrt{(-1)^2+1^2+1^2}=\sqrt{3}$ Hence, Direction cosines are $-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}$.

(iii) Direction ratios of AB

are, (1 – 5), (-6 – 6) and (3 + 3)i.e. -4, -12, 6

Now, $\sqrt{(-4)^2+(-12)^2+6^2}=\sqrt{196}=14$

Here, direction cosines are $-\frac{4}{14},-\frac{12}{14},\frac{6}{14}$ i.e, $-\frac{2}{7},-\frac{6}{7},\frac{3}{7}$

(iii) Direction ratios of AB

are (-3 + 5), (4 – 7), (-6 + 9)i.e 2, -3, 3

Now, $\sqrt{2^2+(-3)^2+3^2}=\sqrt{22}$ Hence, direction cosines are $\frac{2}{\sqrt{22}},-\frac{3}{\sqrt{22}},\frac{3}{\sqrt{22}}$.

3. Show that the line joining the points $A(1,-1,2)$ and $B(3,4,-2)$ is perpendicular to the line joining the points $C(0,3,2)$ and $D(3,5,6)$

Sol. Direction ratios of AB

Are (3 – 1), (4 + 1), (-2, -2)

i.e., 2, 5, -4

Direction ratios of CD are, 3 – 0, 5 – 3, 6 – 2 i.e. 3, 2, 4

Now, 2 × 3 + 5 × 2 + (-4) × 4= 6 + 10 – 16= 0

Hence, AB and CD are perpendicular.

4. Show that the line segment joining the orgin to the point $A(2,1,1)$ is perpendicular to the line segment joining the points $B(3,5,-1)$ and $C(4,3,-1)$

Sol. Let, 0 be the origin.

Now, Direction ratio of OA are 2, 1, 1

Direction ratio of BC are $4-3,\ 3-5,\ -1-(-1)$

i.e, 1, -2, 0 Now, $2 \times 1 + 1 \times (-2) + 1 \times 0 = 0$

Hence, $OA \perp BC$.

5. Find the value of p for which the line through the points $A(4,1,2)$ and $B(5,p,0)$ is perpendicular to the line through the points $C(2,1,1)$ and $D(3,3,-1)$

Sol. Direction ratio of AB

are (5 – 4), (P – 1), (0 – 2)

i.e 1, P – 1, -2

Direction ratio of CD are, 3 – 2, 3 – 1, -1 – 1

i.e, 1, 2, -2 $\because AB \perp CD$

$\Rightarrow 1 \times 1 + 2(P-1) + (-2)(-2) = 0 \Rightarrow 1 + 2P - 2 + 4 = 0 \Rightarrow 2P = -3 \Rightarrow P = -\frac{3}{2}$

6. If O be the origin, and $P(2,3,4)$ and $Q(1,-2,1)$ be any two points, show that $OP \perp OQ$.

Sol. The direction ratio of the vector OP are 2, 3, 4.

$\therefore$ its direction cosines are $\frac{2}{\sqrt{(2)^2+(3)^2(4)^2}}, \frac{3}{\sqrt{(2)^2+(3)^2+(4)^2}}, \frac{4}{\sqrt{(2)^2+(3)^2+(4)^2}}$

i.e., $\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}} \Rightarrow l_1 = \frac{2}{\sqrt{29}}, m_1 = \frac{3}{\sqrt{29}}, n_1 = \frac{4}{\sqrt{29}}$

The direction ratio of the OQ are 1, –2, 1.

$\therefore$ its direction cosines are $\frac{1}{\sqrt{(1)^2+(2)^2+(1)^2}}, \frac{-2}{\sqrt{(1)^2+(-2)^2+(1)^2}}, \frac{1}{\sqrt{(1)^2+(-2)^2+(1)^2}}$

i.e., $\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \Rightarrow l_2 = \frac{1}{\sqrt{6}}, m_2 = \frac{-2}{\sqrt{6}}, n_2 = \frac{1}{\sqrt{6}}$

$OP \perp OQ$ then $l_1 l_2 + m_1 m_2 + n_1 . n_2 = 0$

$\Rightarrow \frac{2}{\sqrt{29}} \times \frac{1}{\sqrt{6}} + \frac{3}{\sqrt{29}} \times \left(\frac{-2}{\sqrt{6}}\right) + \frac{4}{\sqrt{29}} \times \frac{1}{\sqrt{6}} = 0 \Rightarrow \frac{2}{\sqrt{174}} - \frac{6}{\sqrt{174}} + \frac{4}{\sqrt{174}} = 0$

Hence $OP \perp OQ$ proved.

7. Show that the line segment joining the points $A(1,2,3)$ and $B(4,5,7)$ is parallel to the line segment joining the points $C(-4,3,-6)$ and $D(2,9,2)$.

Sol. Let, $\vec{A} = (\hat{i} + 2\hat{j} + 3\hat{k})$, $\vec{B} = (4\hat{i} + 5\hat{j} + 7\hat{k})$, $\vec{C} = (-4\hat{i} + 3\hat{j} - 6\hat{k})$ & $\vec{D} = (2\hat{i} + 9\hat{j} + 2\hat{k})$

$\overrightarrow{AB} =$ (Position vector of B – position vector of A)

$= (4\hat{i} + 5\hat{j} + 7\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = (3\hat{i} + 3\hat{j} + 4\hat{k})$

$\overrightarrow{AB} = (3\hat{i} + 3\hat{j} + 4\hat{k}) \Rightarrow |\overrightarrow{AB}| = \sqrt{(3)^2 + (3)^2 + (4)^2} = \sqrt{9+9+16} = \sqrt{34}$

$\Rightarrow$ Direction ratios of the vector $\overrightarrow{AB}$ are $(3,3,4)$.

Its direction cosines are $\left(\frac{3}{\sqrt{34}},\frac{3}{\sqrt{34}},\frac{4}{\sqrt{34}}\right)$ i.e., $l_1=\frac{3}{\sqrt{34}}, m_1=\frac{3}{\sqrt{34}}, n_1=\frac{4}{\sqrt{34}}$

$\overrightarrow{CD}$ = Position vector of $D-$ Position vector of C

$=\left(2\hat{i}+9\hat{j}+2\hat{k}\right)-\left(-4\hat{i}+3\hat{j}-6\hat{k}\right)=\left(6\hat{i}+6\hat{j}+8\hat{k}\right)$

$\overrightarrow{CD}=\left(6\hat{i}+6\hat{j}+8\hat{k}\right)\Rightarrow\left|\overrightarrow{CD}\right|=\sqrt{(6)^2+(6)^2+(8)^2}=\sqrt{36+36+64}=\sqrt{136}=2\sqrt{34}$

The direction ratios of the vector $\overrightarrow{CD}$ are $(6,6,8)$.

Its direction cosine $\left(\frac{6}{2\sqrt{34}},\frac{6}{2\sqrt{34}},\frac{8}{2\sqrt{34}}\right)=\left(\frac{3}{\sqrt{34}},\frac{3}{\sqrt{34}},\frac{4}{\sqrt{34}}\right)$ i.e., $l_2=\frac{3}{\sqrt{34}}, m_2=\frac{3}{\sqrt{34}}, n_2=\frac{4}{\sqrt{34}}$

Now, the line l_1 & l_2 are parallel

$\Leftrightarrow \frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2} \Rightarrow \frac{4}{\sqrt{34}}\times\frac{\sqrt{34}}{4}=\frac{3}{\sqrt{34}}\times\frac{\sqrt{34}}{3}=\frac{3}{\sqrt{34}}\times\frac{\sqrt{34}}{3}=1$ Hence, $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

8. if the line segment joining the points $A(7,p,2)$ and $B(q,-2,5)$ be parallel to the line segment joining the points $C(2,-2,5)$ and $D(-6,-15,11)$ find the values of p and q

Sol. Direction ratios of AB are q – 7, -2-P, 5 – 2

i.e, q – 7, -2-P, 3

Direction rations of CD are -6-2, -15 + 3, 11 – 5

i.e, - 8, - 12, 6

$\because$ AB || CD

$\Rightarrow \frac{q-7}{-8}=\frac{-2-P}{-12}=\frac{3}{6}\Rightarrow\frac{q-7}{-8}=\frac{1}{2}$ or, $\frac{-2-P}{12}=\frac{1}{2}$

$\Rightarrow$ 2q – 14 = - 8 or, -2 – P = -6

$\Rightarrow$ 2q = 6 or, -P = -4

$\Rightarrow$ q = 3 or P = 4

Hence, P = 4 and q = 3 Ans.

9. Show that the points $A(2,3,4), B(-1,-2,1)$ and $C(5,8,7)$ are collinear

Sol. Direction ratios of AB are -1 – 2, -2 -3, 1 – 4

i.e. -3, -5, -3

Direction rations of BC are 5 – (-1), 8 – (-2), 7 – 1

i.e, 6, 10, 6

Now, $\frac{-3}{6}=\frac{-5}{10}=\frac{-3}{6}$ $\therefore$ AB || BC

Here, B is common Hence, A, B and C are collinear

10. Show that the points $A(-2,4,7)$, $B(3,-6,-8)$ and $C(1,-2,-2)$ are collinear

Sol. Direction ratios of AB are 3 – (-2), -6 – 4, - 8 – 7

i.e. 5, -10, -15

Direction ratios of BC are 1 – 3, -2 –(-6), -2 – (-8)

i.e. -2, 4, 6

$\because \frac{5}{-2}=\frac{-10}{4}=\frac{-15}{6}$ $\therefore$ AB||BC

Here, B is common Hence, A, B and C are collinear.

11. Find the value of p for which the points $A(-1,3,2), B(-4,2,-2)$ and $C(5,5,p)$ are collinear

Sol. Direction rations of AB are -4 – (-1), 2 – 3, -2 – 2

i.e. -3, -1, -4

Direction ratios of BC are 5 – (-4), 5 – 2, P – (-2)

i.e. 9, 3, (P + 2)

$\because$ A, B & C are collinear

$\Rightarrow \frac{-3}{9} = \frac{-1}{3} = \frac{-4}{P+2} \Rightarrow \frac{-1}{3} = \frac{-4}{P+2} \Rightarrow \frac{1}{3} = \frac{4}{P+2} \quad \Rightarrow P + 2 = 12 \Rightarrow P = 10$ Ans.

12. Find the angle between the two lines whose direction cosines are $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$ and $\frac{3}{7}, \frac{2}{7}, \frac{6}{7}$

Sol. Angle between two lines having direction cosines ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 is gives by, $\theta = \cos^{-1}(\ell_1\ell_2 + m_1m_2, + n_1n_2)$

Here, $\theta = \cos^{-1}\left(\frac{2}{3}\times\frac{3}{7} + \frac{-1}{3}\times\frac{2}{7} + \frac{-2}{3}\times\frac{6}{7}\right) = \cos^{-1}\left(\frac{2}{7} - \frac{2}{21} - \frac{4}{7}\right) = \cos^{-1}\left(\frac{-8}{21}\right)$ Ans.

13. Find the angle between the two lines whose direction rations are a, b, c and $(b-c), (c-a)(a-b)$

Sol. Let, θ be the angle b/ω the two lines

we have, $\theta = \cos^{-1}\left\{\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right\}$

$$\Rightarrow \theta = \cos^{-1}\left\{\frac{a(b-c) + b(c-a) + c(a-b)}{\sqrt{a^2+b^2+c^2}\sqrt{(b-c)^2 + (c-a)^2 + (a-b)^2}}\right\}$$

$$\Rightarrow \theta = \cos^{-1}\left\{\frac{ab - ac + bc - ab + ac - bc}{\sqrt{a^2+b^2+c^2}\sqrt{(b-c)^2 + (c-a)^2 + (a-b)^2}}\right\}$$

$$\Rightarrow \theta = \cos^{-1}\left\{\frac{0}{\sqrt{a^2+b^2+c^2}\sqrt{(b-c)^2 + (c-a)^2 + (a-b)^2}}\right\}$$

$\Rightarrow \theta = \cos^{-1}\{0\} \Rightarrow \theta = \frac{\pi^c}{2}$ Ans.

14. Find the angle between the lines whose direction ratios are $2, -3, 4$ and 1, 2, 1

Sol. Let, θ be the angle between the two lines.

we have, $\theta = \cos^{-1}\left\{\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right\}$

$\Rightarrow \theta = \cos^{-1}\left\{\frac{2\times1 + (-3)\times2 + 4\times1}{\sqrt{2^2 + (-3)^2 + 4^2}\sqrt{1^2 + 2^2 + 1^2}}\right\} \Rightarrow \theta = \cos^{-1}\left\{\frac{2-6+4}{\sqrt{29}\sqrt{6}}\right\} \Rightarrow \theta = \cos^{-1}\{0\} \Rightarrow \theta = \frac{\pi^c}{2}$ Ans.

15. Find the angle between the lines whose direction ratios are 1, 1, 2 and $(\sqrt{3}-1), (-\sqrt{3}-1), 4$

Sol. Let, θ be the angle between the two lines,

we have, $\theta = \cos^{-1}\left\{\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right\}$

$$\Rightarrow \theta = \cos^{-1}\left\{\frac{1\times(\sqrt{3}-1)+1\times1(-\sqrt{3}-1)+2\times4}{\sqrt{1^2+1^2+2^2}\sqrt{(\sqrt{3}-1)^2+(-\sqrt{3}-1)^2+4^2}}\right\}$$

$$\Rightarrow \theta = \cos^{-1}\left\{\frac{\sqrt{3}-1-\sqrt{3}-1+8}{\sqrt{6}\sqrt{3+1-2\sqrt{3}+3+1+2\sqrt{3}+16}}\right\} \Rightarrow \theta = \cos^{-1}\left\{\frac{6}{\sqrt{6}\sqrt{24}}\right\}$$

$$\Rightarrow \theta = \cos^{-1}\left\{\frac{6}{6\times2}\right\} \qquad \Rightarrow \theta = \cos^{-1}\left\{\frac{1}{2}\right\} \qquad \Rightarrow \theta = \frac{\pi^c}{3} \text{ Ans.}$$

16. Find the angle between the vectors $\vec{r}_1 = (3\hat{i} - 2\hat{j} + \hat{k})$ and $\vec{r}_2 = (4\hat{i} + 5\hat{j} + 7\hat{k})$

Sol. Let, θ be the angle between $\vec{r}_1$ and $\vec{r}_2$

$$\theta = \cos^{-1}\left\{\frac{\vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1||\vec{r}_2|}\right\}$$

$$\Rightarrow \theta = \cos^{-1}\left\{\frac{(3\hat{i}-2\hat{j}+\hat{k})\cdot(4\hat{i}+5\hat{j}+7\hat{k})}{\sqrt{3^2+(-2)^2+1^2}\sqrt{4^2+5^2+7^2}}\right\} \Rightarrow \theta = \cos^{-1}\left\{\frac{12-10+7}{\sqrt{14}\sqrt{90}}\right\}$$

$$\Rightarrow \theta = \cos^{-1}\left\{\frac{9}{\sqrt{2\times7\times2\times3\times3\times5}}\right\} \Rightarrow \theta = \cos^{-1}\left\{\frac{9}{6\sqrt{35}}\right\} \Rightarrow \theta = \cos^{-1}\left\{\frac{3}{2\sqrt{35}}\right\} \text{ Ans.}$$

17. Find the angles made by the following vectors with the coordinates axes

(i) $(\hat{i} - \hat{j} + \hat{k})$ (ii) $(\hat{j} - \hat{k})$ (iii) $(\hat{i} - 4\hat{j} + 9\hat{k})$

Sol. (i) Let, $\vec{r} = \hat{i} - \hat{j} + \hat{k}$

Direction cosines of $\vec{r}$ are, $\frac{1}{\sqrt{1^2+(-1)^2+1^2}}, \frac{-1}{\sqrt{1^2+(-1)^2+1^2}}, \frac{1}{\sqrt{1^2+(-1)^2+1^2}}$

i.e, $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}},$

Thus, Angle between $\vec{r}$ and x – axis, $\beta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Angle between $\vec{r}$ and y – axis, $\beta = \cos^{1}\left(\frac{-1}{\sqrt{3}}\right)$

Angle between $\vec{r}$ and z – axis, $y = \cos^{-1}\frac{1}{\sqrt{3}}$

(ii) Let, $\vec{r} = \hat{j} - \hat{k}$

Direction cosines of $\vec{r}$ are $\frac{0}{\sqrt{(-1)^2+12}}, \frac{1}{\sqrt{(-1)^2+1^2}}, \frac{-1}{\sqrt{(-1)^2+1^2}}$

i.e, $0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$

thus, Angle between $\vec{r}$ and x –axis, $\alpha = \cos^{-1}(0) = \frac{\pi^c}{2}$

Angle between $\vec{r}$ and y – axis, $\beta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi^C}{4}$

Angle between $\vec{r}$ and z-axis $y = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

$\Rightarrow y = \pi - \cos^{-1}\frac{1}{\sqrt{2}} \Rightarrow y = \pi - \frac{\pi}{4} \Rightarrow y = \frac{3\pi}{4}$

(iii) Let, $\vec{r} = \hat{i} - 4\hat{j} + 8\hat{k}$

Direction cosines of

$\vec{r}$ are $\frac{1}{\sqrt{1^2+(-4)^2+8^2}}, \frac{-4}{\sqrt{1^2+(-4)^2+8^2}}, \frac{8}{\sqrt{1^2+(-4)^2+8^2}}$

i.e. $\frac{1}{\sqrt{81}}, \frac{-4}{\sqrt{81}}, \frac{8}{\sqrt{81}}$ i.e. $\frac{1}{9}, \frac{-4}{9}, \frac{8}{9}$

thus, Angle b/ω x-axis and $\vec{r}$, $\alpha = \cos^{-1}\frac{1}{9}$

Angle between y-axis and $\vec{r}$, $\beta = \cos^{-1}\left(\frac{-4}{9}\right)$ Angle between z-axis and $\vec{r}$, $y = \cos^{-1}\left(\frac{8}{9}\right)$

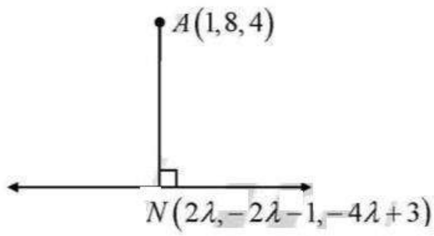
18. Find the coordinates of the foot of the perpendicular drawn from the point $A(1,8,4)$ to the line joining the points $B(0,-1,3)$ and $C(2,-3,-1)$

Sol. Find the coordinates of the foot of the perpendicular drawn from the point $A(1,8,4)$ to the line joining the points $B(0,-1,3)$ and $C(2,-3,-1)$.

Sol. The given line BC is

$\frac{x-0}{2-0} = \frac{y-(-1)}{-3-(-1)} = \frac{z-3}{-1-3} = \lambda$

$\frac{x}{2} = \frac{y+1}{-2} = \frac{z-3}{-4} = \lambda\text{(say)}$...(i)



The general point on this line is $(2\lambda, -2\lambda-1, -4\lambda+3)$.

Let N be the foot of the perpendicular drawn from the point $A(1,8,4)$ to the given line.

Then, this point is $N(2\lambda, -2\lambda-1, -4\lambda+3)$ for some value of λ.

Direction ratio of AN are $(2\lambda-1, -2\lambda-1-8, -4\lambda+3-4) \Rightarrow (2\lambda-1), (-2\lambda-9), (-4\lambda-1)$

Direction ratio of given line (i) are $(2, -2, -4)$

Since $AN \perp$ given line (i) we have, $2(2\lambda-1) - 2(-2\lambda-9) - 4(-4\lambda-1) = 0$

$\Rightarrow 4\lambda - 2 + 4\lambda + 18 + 16\lambda + 4 = 0 \quad \Rightarrow 24\lambda + 20 = 0 \quad \Rightarrow \lambda = \frac{-20}{24} = \frac{-5}{6}$

So, the required point of $N\left(\frac{-5}{3}, \frac{2}{3}, \frac{19}{3}\right)$.

Hence the required co–ordinate foot of the perpendicular is $\left(\frac{-5}{3}, \frac{2}{3}, \frac{19}{3}\right)$.