BAYES'S THEOREM AND ITS APPLICATIONS (XII, R. S. AGGARWAL)

EXERCISE 30 (Pg.No.: 1301)

1. In a bulb factory, three machines, A, B, C, manufacture 60%, 25% and 15% of the total production respectively. Of their respective outputs, 1%, 2% and 1% are defective. A bulb is drawn at random from the total product and it is found to be defective. Find the probability that it was manufactured by machine C.

Sol. Let E_1, E_2, E_3 be the events of drawing a bulb product by machine A, B and C respectively, let D be the event of drawing a deflective bulb.

We have $P(E_1)=\frac{60}{100}, P(E_2)=\frac{25}{100}, P(E_3)=\frac{15}{100}$;

$P(D/E_1)=\frac{1}{100}, P(D/E_2)=\frac{2}{100}, P(D/E_3)=\frac{1}{100}$

P (defective bulb is product by machine C) $=P(E_3/D)$

$$=\frac{P(E_3).P(D/E_3)}{P(E_1).P(D/E_1)+P(E_2).P(D/E_2)+P(E_3).P(D/E_3)}$$

$$=\frac{\frac{15}{100}\times\frac{1}{100}}{\frac{60}{100}\times\frac{1}{100}+\frac{25}{100}\times\frac{2}{100}+\frac{15}{100}\times\frac{1}{100}}=\frac{15}{60+50+15}=\frac{15}{125}=\frac{3}{25}.$$

2. A company manufactures scooters at two plants. A and B. Plant A produces 80% and plant B produces 20% of the total product. 85% of the scooters produced at plant A and 65% of the scooters produced at plant B are of standard quality. A scooter produced by the company is selected at random and it is found to be of standard quality. What is the probability that it was manufactured at plant A?

Sol. Let E_1 = event that the selected scooter is produced at plant A, and E_2 = event that the selected scooter is produced at plant B.

Then, $P(E_1)=\frac{80}{100}=\frac{4}{5}$ and $P(E_2)=\frac{20}{100}=\frac{1}{5}$

Let E be the event of choosing a scooter which is of standard quantity.

Then, $P(E/E_1)=\frac{85}{100}=\frac{17}{20}$, and $P(E/E_2)=\frac{65}{100}=\frac{13}{20}$

Probability that the selected scooter was produced at plant A, given that it is of standard quality.

$$=P(E_1/E)=\frac{P(E/E_1).P(E_1)}{P(E/E_1).P(E_1)+P(E/E_2).P(E_2)}=\frac{\frac{17}{20}\cdot\frac{4}{5}}{\frac{17}{20}\cdot\frac{4}{5}+\frac{13}{20}\cdot\frac{1}{5}}=\frac{68}{68+13}=\frac{68}{81}$$

3. In a certain college, 4% of boys and 1% of girls are taller than 1.75 metres. Further more, 60% of the students are girls. If a student is selected at random and is taller them 1.75 metres, what is the probability the selected student is a girl?

Sol. Let E_1 and E_2 be the events of selecting a boy and a girl respectively.

Then $P(E_1)=\frac{40}{100}=\frac{2}{5}$, and $P(E_2)=\frac{60}{100}=\frac{3}{5}$

Let E = event that the student selected is taller than 1.75 m.

Then $P(E/E_1)=\frac{4}{100}=\frac{1}{25}$ and $P(E/E_2)=\frac{1}{100}$.

Probability that the selected student is a girl. Given that she is taller than 1.75 m

$$=P(E_1/E)=\frac{P(E/E_2).P(E_2)}{P(E/E_1).P(E_1)+P(E/E_2).P(E_2)}=\frac{\frac{1}{100}.\frac{3}{5}}{\frac{1}{25}.\frac{2}{5}+\frac{1}{100}\times\frac{3}{5}}=\frac{\frac{3}{500}}{\frac{2}{125}+\frac{3}{500}}=\frac{\frac{3}{500}}{\frac{8+3}{500}}=\frac{3}{11}$$

4. In a class, 5% of the boys and 10% of the girls have an IQ of more than 150. In this class, 60% of the students are boys. If a student is selected at random and found to have an IQ of more than 150, find the probability that the student is a boy.

Sol. Let E_1 and E_2 be the event of selecting a boy and a girl respectively. Then

$P(E_1)=\frac{60}{100}=\frac{3}{5}$ and $P(E_2)=\frac{40}{100}=\frac{2}{5}$

Let E be the event of selecting a student having an IQ of more than 150.

Then, $P(E/E_1)=\frac{5}{100}=\frac{1}{20}$, and $P(E/E_2)=\frac{10}{100}=\frac{1}{10}$.

Required probability $=P(E_1/E)=\frac{P(E/E_1).P(E_1)}{P(E/E_1).P(E_1)+P(E/E_2).P(E_2)}$

$$=\frac{\frac{3}{5}.\frac{1}{20}}{\frac{3}{5}.\frac{1}{20}+\frac{2}{5}.\frac{1}{10}}=\frac{\frac{3}{100}}{\frac{3}{100}+\frac{2}{50}}=\frac{3}{3+4}=\frac{3}{7}$$

5. Suppose 5 men out of 100 and 25 women out of 1000 are good orators. An orator is chosen at random. Find the probability that a male person is selected Assume that there are equal numbers of men and women.

Sol. Let there be 1000 men and 1000 women Let E_1 and E_2 be the event of choosing a man and a women respectively. Then,

$P(E_1)=\frac{1000}{2000}=\frac{1}{2}, P(E_2)=\frac{1000}{2000}=\frac{1}{2}$

Let E be the event of choosing on orator. Then,

$P(E/E_1)=\frac{50}{1000}=\frac{1}{20}$ and $P(E/E_2)=\frac{25}{1000}=\frac{1}{40}$

Probability of selecting a male person, given that the person is a good orator

$$=P(E_1/E)=\frac{P(E/E_1).P(E_1)}{P(E/E_1).P(E_1)+P(E/E_2).P(E_2)}$$

$$=\frac{\frac{1}{20}.\frac{1}{2}}{\frac{1}{20}.\frac{1}{2}+\frac{1}{40}.\frac{1}{2}}=\frac{\frac{1}{40}}{\frac{1}{40}+\frac{1}{80}}=\frac{\frac{1}{40}}{\frac{2+1}{80}}=\frac{1}{40}\times\frac{80}{3}=\frac{2}{3}$$

6. Two groups are competing for the positions on the board of directors of a corporation. The probability that the first and second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and when the second group wins, the corresponding probability is 0.3. Find the probability that the new product introduced was by the second group.

Sol. Let E_1 = event that the first group wins, E_2 = even that the second group wins

and E = even that a new product is introduced. Then $P(E_1)=0.6, P(E_2)=0.4$

$P(E/E_1)=0.7, P(E/E_2)=0.3$

Required probability $=P(E_2/E)=\dfrac{P(E/E_2).P(E_2)}{P(E/E_1).P(E_1)+P(E/E_2).P(E_2)}$

$$=\frac{0.3\times0.4}{0.6\times0.7+0.3\times0.4}=\frac{0.12}{0.42+0.12}=\frac{0.12}{0.54}=\frac{2}{9}$$

7. A bag A contains 1 white and 6 red balls. Another bag contains 4 white and 3 red balls. One of the bags is selected at random and a ball is drawn from it, which is found to be white. Find the probability that the ball drawn is from the bag A.

Sol. Let E_1 = the event of choosing bag A. E_2 = the event of choosing bag B.

E = the event of choosing white ball.

Since the ball drawn is found to be white, $\therefore$ We have to find $P(E_1/E)$

Since both the bags are likely to be chosen, we have $P(E_1)=\dfrac{1}{2}=P(E_2)$

Also $P(E/E_1)$ = the probability that the ball drawn is white, if it is drawn from bag $A=\dfrac{1}{7}$

Similarly $P(E/E_2)=\dfrac{4}{7}$.

$$\therefore\ P(E)=P(E_1).P(E/E_1)+P(E_2).P(E/E_2)=\frac{1}{2}.\frac{1}{7}+\frac{1}{2}.\frac{4}{7}=\frac{1}{14}+\frac{4}{14}=\frac{5}{14}$$

$$\therefore\ P(E_1/E)=\frac{P(E_1).P(E/E_1)}{P(E)}=\frac{\frac{1}{2}.\frac{1}{7}}{\frac{5}{14}}=\frac{1}{5}$$

8. There are two bags I and II. Bag I contains 3 white and 4 black balls, and bag II contains 5 white and 6 black balls. One ball is drawn at random from one of the bags and is found to be white. Find the probability that it was drawn from bag I.

Sol. Let E_1 be the event of choosing the bag I, E_2 be the event of choosing the bag II and A be the event of choosing a white ball. Then $P(E_1)=P(E_2)=\dfrac{1}{2}$

Again $P(A/E_1)=P$(drawing a white ball from bag I) $=\dfrac{3}{7}$

and $P(A/E_2)=P$(drawing a white ball from bag II) $=\dfrac{5}{11}$

The required probability of drawing a ball from bag I.

$\therefore$ By Baye's theorem

$$P(E_1/A)=\frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1)+P(E_2).P(A/E_2)}$$

$$=\frac{\frac{3}{7}\times\frac{1}{2}}{\frac{3}{7}\times\frac{1}{2}+\frac{1}{2}\times\frac{5}{11}}=\frac{\frac{3}{14}}{\frac{3}{14}+\frac{5}{22}}=\frac{\frac{3}{14}}{\frac{33+35}{154}}=\frac{3}{14}\times\frac{154}{68}=\frac{33}{68}$$

9. A box contains 2 gold and 3 silver coins. Another box contains 3 gold and 3 silver coins. A box is chosen at random, and a coin is drawn from it. If the selected coin is a gold coin, find the probability that it was drawn from the second box.

Sol. Let E_1, E_2 be the events that box I and II are chosen respectively.

Then $P(E_1) = P(E_2) = \frac{1}{2}$. Also, let A be the event that the coin draw is of gold

Then $P(A/E_1) = P$ (a gold coin from box I) $= \frac{2}{5}$

$P(A/E_2) = P$ (a gold coin from box II) $= \frac{3}{6} = \frac{1}{2}$

The required probability of drawing a ball from box II.

$$P(E_2/A) = \frac{P(E_2).P(A/E_2)}{P(E_1).P(A/E_1)+P(E_2).P(A/E_2)} = \frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{2}{5}+\frac{1}{2}\times\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{5}+\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{4+5}{20}} = \frac{1}{4}\times\frac{20}{9} = \frac{5}{9}$$

10. Three urns A, B and C contain 6 red and 4 white; 2 red and 6 white; and 1 red and 5 white balls respectively. An urn is chosen at random and a ball is drawn. If the ball drawn is found to be red, find the probability that the ball was drawn from the urn A.

Sol. Let E_1, E_2 and E_3 be the events of choosing the urns A, B and C respectively and let E be the event of drawing a red ball. Then. $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

$P(E/E_1) = \frac{6}{10}, P(E/E_2) = \frac{2}{8}$ and $P(E/E_3) = \frac{1}{6}$

Required probability $= P(E_1/E) = \frac{P(E_1)\times P(E/E_1)}{P(E_1)\times P(E/E_1)+P(E_2)\times P(E/E_2)+P(E_3)\times P(E/E_3)}$

$$= \frac{\frac{1}{3}\times\frac{6}{10}}{\left(\frac{1}{3}\times\frac{6}{10}\right)+\left(\frac{1}{3}\times\frac{2}{8}\right)+\left(\frac{1}{3}\times\frac{1}{6}\right)} = \frac{\frac{2}{10}}{\frac{2}{10}+\frac{1}{12}+\frac{1}{18}} = \frac{\frac{2}{10}}{\frac{36+15+10}{180}} = \frac{2}{10}\times\frac{180}{61} = \frac{36}{61}$$

11. Three urns contain 2 white and 3 black balls; 3 white and 2 black balls, and 4 white and 1 black ball respectively. One ball is drawn from an urn chosen at random and it was found to be white. Find the probability that it was drawn from the first urn.

Sol. Let E_1, E_2 and E_3 are the events of selection of urns.

$A =$ event of getting white

$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$ and $P(A/E_1) = \frac{2}{5}, P(A/E_2) = \frac{3}{5}$, and $P(A/E_3) = \frac{4}{5}$

Required probability $= P(E_1/A) = \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1)+P(E_2).P(A/E_2)+P(E_3).P(A/E_3)}$

$$= \frac{\frac{1}{3}\times\frac{2}{5}}{\left(\frac{1}{3}\times\frac{2}{5}\right)+\left(\frac{1}{3}\times\frac{3}{5}\right)+\left(\frac{1}{3}\times\frac{4}{5}\right)} = \frac{\frac{2}{15}}{\frac{2}{15}+\frac{3}{15}+\frac{4}{15}} = \frac{\frac{2}{15}}{\frac{2+3+4}{15}} = \frac{2}{9}$$

12. There are three boxes, the first one containing 1 white, 2 red and 3 black balls; the second one containing 2 white, 3 red and 1 black ball and the third one containing 3 white, 1 red and 2 black balls. A box is chosen at random and from it two balls are drawn at random. One ball is red and the other, white. What is the probability that they come from the second box?

Sol. Let E_1, E_2 and E_3 be the events that the balls are drawn from box A, box B and box C respectively, and let E be the event the balls drawn are one red and another white then,

$P(E_1)=P(E_2)=P(E_3)=\frac{1}{3}$

$P(E/E_1)=$ probability that the balls drawn are one white and 1 red, given that the balls are from

Box A. $=\frac{{}^1C_1\times{}^2C_1}{{}^6C_2}=\frac{2}{15}$

$P(E/E_2)=$ Probability that the balls drawn are one red and one white given that the balls are from

Box $B=\frac{{}^2C_1\times{}^3C_1}{{}^6C_2}=\frac{2\times3}{15}=\frac{2}{5}$

$P(E/E_3)=$ probability that the balls drawn are one red and 1 white, given that the balls are from

Box C. $=\frac{{}^3C_1\times{}^1C_1}{{}^6C_2}=\frac{3}{15}=\frac{1}{5}$

Probability that the balls drawn are from box B, it is being given that the ball drawn are one red and one white $=P(E_2/E)=\frac{P(E/E_2).P(E_2)}{P(E_1).P(E/E_1)+P(E_2).P(E/E_2)+P(E_3).P(E/E_3)}$

$$=\frac{\frac{1}{3}\times\frac{2}{5}}{\frac{1}{3}\times\frac{2}{15}+\frac{1}{3}\times\frac{2}{5}+\frac{1}{3}\times\frac{1}{5}}=\frac{\frac{2}{15}}{\frac{2}{45}+\frac{2}{15}+\frac{1}{15}}=\frac{\frac{2}{15}}{\frac{2+6+3}{45}}=\frac{2}{15}\times\frac{45}{11}=\frac{6}{11}$$

13. Urn A contains 7 white and 3 black balls; urn B contains 4 white and 6 black balls; urn C contains 2 white and 8 black balls. One of these urns is chosen at random with probabilities 0.2, 0.6 and 0.2 respectively. From the chosen urn, two balls are drawn at random without replacement. Both the balls happen to be white. Find the probability that the balls drawn are from the urn C.

Sol. Let E_1, E_2, E_3 be the events that the balls are drawn from urn A, urn B and urn C respectively. and

Then $P(E_1)=0.2, P(E_2)=0.6$ and $P(E_3)=0.2$

Let E be the event that 2 white ball are drawn,

Then, $P(E/E_1)=\frac{{}^7C_2}{{}^{10}C_2}=\frac{21}{45}=\frac{7}{15}$; $P(E/E_2)=\frac{{}^4C_2}{{}^{10}C_2}=\frac{6}{45}$; $P(E/E_3)=\frac{2C_2}{10C_2}=\frac{1}{45}$

$\therefore$ required probability $=P(E_3/E)=\frac{P(E/E_3).P(E_3)}{P(E_1).P(E/E_1)+P(E_2).P(E/E_2)+P(E_3).P(E/E_3)}$

$$=\frac{\frac{1}{45}\times0.2}{0.2\times\frac{7}{15}+0.6\times\frac{6}{45}+0.2\times\frac{1}{45}}=\frac{\frac{2}{450}}{\frac{14}{150}+\frac{36}{450}+\frac{2}{450}}=\frac{\frac{2}{450}}{\frac{42+36+2}{450}}=\frac{2}{80}=\frac{1}{40}$$

14. There are 3 bags, each containing 5 white and 3 black balls. Also, there are 2 bags, each containing 2 white and 4 black balls. A white ball is drawn at random. Find the probability that this ball is from a bag of the first group.

Sol. Let E_1 = event of selecting a bag from the first group and,

E_2 = event of selecting a bag from the second group.

Then, $P(E_1)=\frac{3}{5}$ and $P(E_2)=\frac{2}{5}$

Let E = event that the ball drawn is white, then $P(E/E_1)=\frac{5}{8}, P(E/E_2)=\frac{2}{6}=\frac{1}{3}$

$$\therefore\ P(E_1/E)=\frac{P(E/E_1).P(E_1)}{P(E/E_1).P(E_1)+P(E/E_2).P(E_2)}$$

$$=\frac{\frac{3}{5}\times\frac{5}{8}}{\frac{3}{5}\times\frac{5}{8}+\frac{2}{5}\times\frac{1}{3}}=\frac{\frac{3}{8}}{\frac{3}{8}+\frac{2}{15}}=\frac{\frac{3}{8}}{\frac{45+16}{120}}=\frac{3}{8}\times\frac{120}{61}=\frac{45}{61}$$

15. There are four boxes, A, B, C and D, containing marbles. A contains 1 red, 6 white and 3 black marbles; B contains 6 red, 2 white and 2 black marbles; C contains 8 red, 1 white and 1 black marbles; and D contains 6 white and 4 black marbles. One of the boxes is selected at random and a single marble is drawn from it. If the marble is red, what is the probability that it was drawn from the box A?

Sol. Let E_1, E_2, E_3, E_4 be the events of selecting boxes A, B, C, D respectively. Then,

$P(E_1)=P(E_2)=P(E_3)=P(E_4)=\frac{1}{4}$. Let E = event that the marble drawn is red. Then

$$P(E/E_1)=\frac{1}{10}, P(E/E_2)=\frac{6}{10}=\frac{3}{5};\ P(E/E_3)=\frac{8}{10}=\frac{4}{5}, P(E/E_4)=0$$

$$\therefore\ P(E_1/E)=\frac{P(E/E_1).P(E_1)}{P(E/E_1).P(E_1)+P(E/E_2).P(E_2)+P(E/E_3).P(E_3)+P(E/E_4).P(E_4)}$$

$$=\frac{\frac{1}{4}\times\frac{1}{10}}{\frac{1}{4}\times\frac{1}{10}+\frac{1}{4}\times\frac{3}{5}+\frac{1}{4}\times\frac{4}{5}+\frac{1}{4}\times 0}=\frac{\frac{1}{40}}{\frac{1}{40}+\frac{3}{20}+\frac{4}{20}}=\frac{\frac{1}{40}}{\frac{1+6+8}{40}}=\frac{1}{15}$$

16. A car manufacturing factory has two plants X and Y. Plant X manufactures 70% of the cars and plant Y manufactures 30%. At plant X, 80% of the cars are rated of standard quality and at plant Y, 90% are rated of standard quality. A car is picked up at random and is found to be of standard quality. Find the probability that it has come from plats X.

Sol. Let E_1 and E_2 be the events that the car is manufactured by plant X and Y respectively.

Let E be the event that the car is of standard quality. Then, $P(E_1)=\frac{70}{100}=\frac{7}{10}, P(E_2)=\frac{30}{100}=\frac{3}{10}$

$$\Rightarrow P(E/E_1)=\frac{80}{100}=\frac{4}{5},\ P(E/E_2)=\frac{90}{100}=\frac{9}{10}$$

$$P(E_1/E)=\frac{P(E_1)\times P(E/E_1)}{P(E_1)\times P(E/E_1)+P(E_2)\times P(E/E_2)}$$

$$=\frac{\frac{7}{10}\times\frac{4}{5}}{\left(\frac{7}{10}\times\frac{4}{5}\right)+\left(\frac{3}{10}\times\frac{9}{10}\right)}=\frac{\frac{28}{50}}{\frac{28}{50}+\frac{27}{100}}=\frac{\frac{28}{50}}{\frac{56+27}{100}}=\frac{28}{50}\times\frac{100}{83}=\frac{56}{83}$$

17. An insurance company insured 2000 scooters and 3000 motorcycles. The probability of an accident involving a scooter is 0.01 and that of a motorcycle is 0.02. An insured vehicle met with an accident. Find the probability that the accidented vehicle was a motorcycle.

Sol. Let E_1 and E_2 be the events that an insured vehicle is a scooter and a motorcycle respectively.

Let E be the event that the insured vehicle meets an accident.

$$P(E_1)=\frac{2000}{2000+3000}=\frac{2000}{5000}=\frac{2}{5},\quad P(E_2)=\frac{3000}{5000}=\frac{3}{5}$$

$$P(E/E_1)=0.01 \text{ and } P(E/E_2)=0.02$$

Required probability $= P(E_2/E) = \dfrac{P(E_2).P(E/E_2)}{P(E_1).P(E/E_1)+P(E_2).P(E/E_2)}$

$$= \frac{\frac{3}{5}\times 0.02}{\frac{2}{5}\times 0.01+\frac{3}{5}\times 0.02} = \frac{\frac{6}{500}}{\frac{2}{500}+\frac{6}{500}} = \frac{6}{2+6} = \frac{6}{8} = \frac{3}{4}$$

18. In a bulb factory, machines A, B and C manufacture 60%, 30% and 10% bulbs respectively. Out of these bulbs 1%, 2% and 3% of the bulbs produced respectively by A, B and C are found to the defective. A bulb is picked up at random from the total production and found to be defective. Find the probability that this bulb was produced by the machine A.

Sol. Let E_1, E_2, E_3 be the events of drawing a bulb produced by machine A, B, C respectively. Let D be the event of drawing a defective ball.

We have $P(E_1) = \dfrac{60}{100}, P(E_2) = \dfrac{30}{100}, P(E_3) = \dfrac{10}{100}$

$P(D/E_1) = \dfrac{1}{100}, P(D/E_2) = \dfrac{2}{100}, P(D/E_3) = \dfrac{3}{100}$

The events E_1, E_2 and E_3 are mutually exclusive and exhaustive by Bayes' theorem.

P (defective bulb is produced by machine A)

$$= P(E_1/D) = \frac{P(E_1).P(D/E_1)}{P(E_1).P(D/E_1)+P(E_2).P(D/E_2)+P(E_3).P(D/E_3)}$$

$$= \frac{\frac{60}{100}\times\frac{1}{100}}{\frac{60}{100}\times\frac{1}{100}+\frac{30}{100}\times\frac{2}{100}+\frac{10}{100}\times\frac{3}{100}} = \frac{60}{60+60+30} = \frac{60}{150} = \frac{6}{15} = \frac{2}{5}$$