PROBABILITY DISTRIBUTION (XII, R. S. AGGARWAL)

EXERCISE 31 (Pg.No.: 1315)

1. Find the mean (μ), variance (σ^2) and standard deviation (σ) for each of the following probability distributions

(i)

x	0	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

(ii)

x_i	1	2	3	4
p_i	0.4	0.3	0.2	0.1

(iii)

x_i	-3	-1	0	2
p_i	0.2	0.4	0.3	0.1

(iv)

x_i	-2	-1	0	1	2
p_i	0.1	0.2	0.4	0.2	0.1

Sol. (i) **Calculation of mean and variance**

X	p i.e., $P(X)$	pX	pX^2
0	$\frac{1}{6}=\frac{5}{30}$	**0**	0
1	$\frac{1}{2}=\frac{15}{30}$	$\frac{15}{30}$	$\frac{15}{30}$
2	$\frac{3}{10}=\frac{9}{30}$	$\frac{18}{30}$	$\frac{36}{30}$
3	$\frac{1}{30}$	$\frac{3}{30}$	$\frac{9}{30}$
	$\sum p_i=1$	$\sum p_i x_i=\frac{36}{30}=1.2$	$\sum p_i x_i^2=\frac{60}{30}=2$

$\therefore$ Mean $(\mu)=\sum p_i x_i=1.2$ Variance $(\sigma^2)=\sum p_i x_i^2-\mu^2=2-(1.2)^2=0.56$

standard deviation $(\sigma)=0.74$

(ii)

X	p i.e., $P(X)$	pX	pX^2
1	$0.4=\frac{4}{10}$	$\frac{4}{10}$	$\frac{4}{10}$
2	$0.3=\frac{3}{10}$	$\frac{6}{10}$	$\frac{12}{10}$
3	$0.2=\frac{2}{10}$	$\frac{6}{10}$	$\frac{18}{10}$
4	$0.10=\frac{1}{10}$	$\frac{4}{10}$	$\frac{16}{10}$
	$\sum p_i=1$	$\sum p_i x_i=\frac{20}{10}=2$	$\sum p_i x_i^2=\frac{50}{10}=5$

$\therefore$ Mean $(\mu)=\sum p_i x_i=2$ Variance $(\sigma^2)=\sum p_i x_i^2-\mu^2=5-(2)^2=1$

standard deviation $(\sigma)=1$

(iii)

X	p i.e., $P(X)$	pX	pX^2
-3	$0.2=\frac{2}{10}$	$-\frac{6}{10}$	$\frac{18}{10}$
-1	$0.4=\frac{4}{10}$	$-\frac{4}{10}$	$\frac{4}{10}$
0	$0.3=\frac{3}{10}$	0	0
2	$0.1=\frac{1}{10}$	$\frac{2}{10}$	$\frac{4}{10}$
	$\sum p_i=1$	$\sum p_i x_i=-\frac{8}{10}=-0.8$	$\sum p_i x_i^2=\frac{26}{10}=2.6$

$\therefore$ Mean $(\mu)=\sum p_i x_i=-0.8$ Variance $(\sigma^2)=\sum p_i x_i^2-\mu^2=3.6-(-0.8)^2=2.6-0.64=1.96$

standard deviation $(\sigma)=1.612$

(iv)

X	p i.e., $P(X)$	pX	pX^2
-2	$0.1=\frac{1}{10}$	$-\frac{2}{10}$	$\frac{4}{10}$
-1	$0.2=\frac{2}{10}$	$-\frac{2}{10}$	$\frac{2}{10}$
0	$0.4=\frac{4}{10}$	0	0
1	$0.2=\frac{2}{10}$	$\frac{2}{10}$	$\frac{2}{10}$
2	$0.1=\frac{1}{10}$	$\frac{2}{10}$	$\frac{4}{10}$
	$\sum p_i=1$	$\sum p_i x_i=\frac{0}{10}=0$	$\sum p_i x_i^2=\frac{12}{10}=1.2$

$\therefore$ Mean $(\mu)=\sum p_i x_i=0$ Variance $(\sigma^2)=\sum p_i x_i^2-\mu^2=1.2-(0)^2=1.2$

standard deviation $(\sigma)=1.095$

2. Find the mean and variance of the number of heads when two coins are tossed simultaneously

Sol. Given if two coins are tossed once we can write the sample space as follows $S=\{HH,HT,TT\}$

Since all these 4 events are equally likely, $P(HH)=P(HT)=P(TT)=\frac{1}{4}$

Let X be the number of heads in S such that $\rightarrow X(HH)=2, X(HT)=1, X(TH)=1$

And $X(TT)=0$. it follows that

$P(X=0)=P(TT)=\frac{1}{4}$

$P(X=1)=P(TH,HT)=\frac{2}{4}=\frac{1}{2}$

$P(X=2)=P(HH)=\frac{1}{4}$

We can then construct the probability distribution table as follows

X	0	1	2
$P(X)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Mean $(\mu)=$

Variance $(\sigma^2)=$

3. Find the mean and variance of the number of tails when three coins are tossed simultaneously

Sol. If 3 coins are tossed once, we have n(s) = 8

Let X be the random variable denoted the number of tails, then the probability distribution is

X	0	1	2	3
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Now, $\mu=\sum_{i=1}^{n} p_i x_i =\left(\frac{1}{8}\times 0\right)+\left(\frac{3}{8}\times 1\right)+\left(\frac{3}{8}\times 2\right)+\left(\frac{1}{8}\times 3\right) =0+\frac{3}{8}+\frac{6}{8}+\frac{3}{8} =\frac{12}{8}=\frac{3}{2}$

Variance $\left(\sigma^2\right)=\sum_{i=1}^{n} p_i x_i^2-(\mu)^2 =\left(\frac{1}{8}\times 0^2+\frac{3}{8}\times 1^2+\frac{3}{8}\times 2^2+\frac{1}{8}\times 3^2\right)-\left(\frac{3}{2}\right)^2$

$=\left(0+\frac{3}{8}+\frac{3}{2}+\frac{9}{8}\right)-\frac{9}{4}=\frac{6}{8}=\frac{3}{4}$

4. A die is tossed twice. 'Cetting and odd number on at toss' is considered a success. Find the probability distribution of number of successes. Also, find the mean and variance of the number of successes

Sol. Let E be the event of getting a success i.e. of getting an odd number in the toss of die. On a die, Odd numbers are 1, 3, 5.

$\therefore P(E)=\frac{3}{6}=\frac{1}{2}$ and $P(\overline{E})=1-\frac{1}{2}=\frac{1}{2}$. Let x denotes the random variable "number of success".

$\therefore$ The possible values of x are: 0, 1, 2.

$P(X=0)=P(\overline{E}_1\overline{E}_2)=P(\overline{E}_1)P(\overline{E}_2)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

$P(X=1)=P(E_1\overline{E}_2 \text{or} \overline{E}_1 E_2)=P(E_1)P(\overline{E}_2)+P(\overline{E}_1)P(E_2)=\left(\frac{1}{2}\times\frac{1}{2}\right)+\left(\frac{1}{2}\times\frac{1}{2}\right)=\frac{1}{2}$

$P(X=2)=P(E_1E_2)=P(E_1)p(E_2)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$.

Calculation of mean and variance

X	p	pX	pX^2
0	$\frac{1}{4}$	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{4}$	$\frac{1}{2}$	1
	$\sum p=1$	$\sum pX=1$	$\sum pX^2=\frac{3}{2}$

Mean $(\mu)=\sum p_i x_i=1$ Variance $=\sum p_i x_i^2-\mu^2=\frac{3}{2}-(1)^2=0.5$.

5. A die is tossed twice. 'Getting a number greater than 4' is considered a success. Find the probability distribution of number of sucesses. Also, find the mean and variance of the number of sucesses

Sol. When a die tossed, sample space $=\{1,2,3,4,5,6\}$ it has six equally likely outcomes

Let E be the event ' a number greater than 4' then

$E=\{5,6\}$, so $n(E)=2$ $p=P(E)=\frac{2}{6}=\frac{1}{3}$, so $q=1-\frac{1}{3}=\frac{2}{3}$

As the die is tossed twice, so there are 2 Burnoullian trials i.e., $n=2$

Let X denote the number of successes, then X can take value $0,1,2$

$P(0)={}^2C_0q^2=1.\left(\frac{2}{3}\right)^2=\frac{4}{9},\ P(1)={}^2C_1pq=2.\frac{1}{3}.\frac{2}{3}=\frac{4}{9},\ P(2)={}^2C_2p^2=1.\left(\frac{1}{3}\right)^2=\frac{1}{9}$

$\therefore$ Probability distribution of number of success is $\begin{pmatrix}0 & 1 & 2\\ \frac{4}{9} & \frac{4}{9} & \frac{1}{9}\end{pmatrix}$

Mean $=np=2\times\frac{1}{3}=\frac{2}{3}$ variance $=npq=2\times\frac{1}{3}\times\frac{2}{3}=\frac{4}{9}$. Standard deviation $=\sqrt{\text{Variance}}=\sqrt{\frac{4}{9}}=\frac{2}{3}$

6. A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability distribution of number of sucesses . Also find the mean and variance of number of sucesses

Sol. When a pair of dice is thrown, the sample space has 36 equally likely outcomes, out of which six are doubles $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$, $(5,5)$ and $(6,6)$.

If $p=$ Probability of getting a doublet, then $p=\frac{6}{36}=\frac{1}{6}$. So, $q=1-\frac{1}{6}=\frac{5}{6}$. Here, $n=4$.

Thus, we have a binomial distribution with $p=\frac{1}{6}$, $q=\frac{5}{6}$ and $n=4$

If X denoted the number of doubles obtained, then X takes the values $0,1,2,3,4$

$P(0)={}^4C_0q^4=\left(\frac{5}{6}\right)^4=\frac{625}{1296}$ $P(1)={}^4C_1pq^3=4.\frac{1}{6}.\left(\frac{5}{6}\right)^3=\frac{500}{1296}$,

$P(2)={}^4C_2p^2q^2=6\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2=\frac{150}{1296}$, $P(3)={}^4C_3p^3q=4\left(\frac{1}{6}\right)^3\frac{5}{6}=\frac{20}{1296}$

and $P(4)={}^4C_4p^4=\left(\frac{1}{6}\right)^4=\frac{1}{1296}$

$\therefore$ The required probability distribution is $\begin{pmatrix}0 & 1 & 2 & 3 & 4\\ \frac{625}{1296} & \frac{500}{1296} & \frac{150}{1296} & \frac{20}{1296} & \frac{1}{1296}\end{pmatrix}$

Mean $(\mu)=n\cdot p=4\times\frac{1}{6}=\frac{2}{3}$

Variance $(\sigma^2)=n\cdot p\cdot q$

$=4\times\frac{1}{6}\times\frac{5}{6}=\frac{5}{9}$

7. A coin is tossed 4 times. Let X denote the number of heads. Find the probability distribution of X. Also the mean and variance of X

Sol. When a fair coin is tossed once, $p=$ probability of head $=\frac{1}{2}$, so $q=1-\frac{1}{2}=\frac{1}{2}$

As the coin is tossed 4 times, $n=4$. Thus, we get a binomial distribution with $p=\frac{1}{2},q=\frac{1}{2},n=2$

$\therefore$ mean $=np=4\times\frac{1}{2}=2$; Variance $=npq=4\times\frac{1}{2}\times\frac{1}{2}=1$

8. Let X denote the number of times' a total of 9' appears in two thrown of a pair of dice. Find the probability distribution of X. Also find the mean variance and standard deviation of X

Sol. Let p denote probability of the event E: "a total of 9" in a single throw of two dice, then

$p=\frac{4}{36}=\frac{1}{9}$ $\quad (\because (3,6),(6,3),(4,5)$ and $(5,4)$ are the only favourable outcomes$)$

Let q denote possibility of 'not getting a total of 9', then $q=1-p=1-\frac{1}{9}=\frac{8}{9}$, i.e. $P(E^c)=\frac{8}{9}$.

Since, X denotes the number of 'a total of 9' is obtained in two throws, therefore, X can take values $0,1,2$:

$P(X=0)=P(E^c\text{in the first throw and }E^c\text{in second throw})$

$=P(E^c)P(E^c)=qq=\left(\frac{8}{9}\right)^2=\frac{64}{81}$,

$P(X=1)=P((E^c\text{ in the first throw and E in second throw})$

$\text{or}(\text{E in first throw and }E^c\text{ in second throw}))$

$P(E^c)P(E)+P(E)P(E^c) \Rightarrow qp+pq=2pq=2\times\frac{1}{9}\times\frac{8}{9}=\frac{16}{81}$ and

$P(X=2)=P(\text{E in first throw and E in second throw}) \quad \Rightarrow pp=p^2=\left(\frac{1}{9}\right)^2=\frac{1}{81}$

Hence, the probability distribution of X is:

X	0	1	2
P(X)	$\frac{64}{81}$	$\frac{16}{81}$	$\frac{1}{81}$

Mean $\mu=\sum XP(X)=0\times\frac{64}{81}+1\times\frac{16}{81}+2\times\frac{1}{81}=\frac{18}{81}=\frac{2}{9}$.

Variance $\sigma^2=\sum X^2P(X)-\mu^2=0^2\times\frac{64}{81}+1^2\times\frac{16}{81}+2^2\times\frac{1}{81}-\left(\frac{2}{9}\right)^2 \Rightarrow \frac{20}{81}-\frac{4}{81}=\frac{16}{81}$.

9. There are 5 cards, numbered 1 to 5, one number on each card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two cards drawn. Find the mean and variance of X

Sol. In this case the sample space contains $5\times4=20$ sample points of the type $(x,y), x\neq y,\ x,y\in\{1,2,3,4,5\}$.

$\therefore$ Sum of the numbers on the two card can take values from 3 to 9.

$P(X=3)=P(\{(1,2),(2,1)\})=\frac{2}{20}=\frac{1}{10}$, $\quad \Rightarrow P(X=4)=P(\{(1,3),(3,1)\})=\frac{2}{20}=\frac{1}{10}$,

$P(X=5)=P(\{(1,4),(4,1),(2,3),(3,2)\})=\frac{4}{20}=\frac{1}{5}$

$P(X=6)=(\{(1,5),(2,4),(4,2),(5,1)\})=\frac{4}{20}=\frac{1}{5}$

$P(X=7)=P(\{(2,5),(5,2),(3,4)(4,3)\})=\frac{4}{20}=\frac{1}{5}$

$P(X=8)=P(\{(3,5),(5,3)\})=\frac{2}{20}=\frac{1}{10} \quad \Rightarrow P(X=9)=P(\{(4,5),(5,4)\})=\frac{2}{20}=\frac{1}{10}$

Thus, the probability distribution of X is as follows:

X	3	4	5	6	7	8	9
$P(X)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

Hence mean $= \sum p_i x_i \quad \Rightarrow 3\times\frac{1}{10}+4\times\frac{1}{10}+5\times\frac{1}{5}+6\times\frac{1}{5}+7\times\frac{1}{5}+8\times\frac{1}{10}+9\times\frac{1}{10}$

$\Rightarrow \frac{3+4+10+12+14+8+9}{10}=\frac{60}{10}=6$ and variance $= \sum p_i x_i^2-(\text{mean})^2$

$\Rightarrow 3^2\times\frac{1}{10}+4^2\times\frac{1}{10}+5^2\times\frac{1}{5}+6^2\times\frac{1}{5}+7^2\times\frac{1}{5}+8^2\times\frac{1}{10}+9^2\times\frac{1}{10}-(6)^2$

$\Rightarrow \frac{9+16+50+72+98+64+81}{10}-36 \quad \Rightarrow \frac{390}{10}-36=39-36=3$

10. Tow cards are drawn from a well-shuffled pack of 52 cards. Find the probability distribution of number of kings. Also compute the variance for the number of kings

Sol. Let k_I be the event of drawing a king and K_i the event of not drawing a king from a pack of 52 cards in i^{th} th draw

Let X denote the discrete random variable "number of kings" in two draws. Here, the possible values of X are 0, 1 and 2. Now,

$P(X=0)=P(K_1'K_2')=P(K_1')P(K_2')=\left(\frac{48}{42}\right)\left(\frac{48}{52}\right)=\frac{144}{169}$

$P(X=1)=P(K_1K_2' \text{ or } K_1'K_2)$

$P(K_1K_2')+P(K_1'K_2)=P(K_1)P(K_2')+P(K_1')P(K_2)=\left(\frac{4}{52}\right)\left(\frac{48}{52}\right)+\left(\frac{48}{52}\right)\left(\frac{4}{52}\right)=\frac{24}{169}$

$P(X=2)=P(K_1K_2)=P(K_1)P(K_2)=\left(\frac{4}{52}\right)\left(\frac{4}{52}\right)=\frac{1}{169}$

$\therefore$ The required probability distribution is

X	0	1	2
$P(X=r)$	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

Mean $(\mu)=\sum X\times P(X)$

$=0\times\frac{144}{169}+1\times\frac{24}{169}+2\times\frac{1}{169}=\frac{26}{169}=\frac{2}{13}$

Variance $(\sigma^2)=\sum X^2\times P(X)-(\mu^2)$

$=0^2\times\frac{144}{169}+1^2\times\frac{24}{169}+2^2\times\frac{1}{169}-\left(\frac{2}{13}\right)^2$

$=\frac{28}{169}-\frac{4}{169}=\frac{24}{169}$

11. A box contains 16 bulbs, out of which 4 bulbs are defective. Three bulbs are drawn at random from the box. Let X be the number of defective bulbs drawn. Find the mean and variance of X

Sol. No. of bulbs = 60 , no. of defective bulbs = 4

X = no. of defective bulbs

Since 3 bulbs are drawn ,

$\therefore$ X can take values 0 or 1 or 2 or 3

$P(X=0) = 12/16 \times 11/15 \times 10/14 = 11/28$

$P(X=1) = 3 \times 12/16 \times 11/15 \times 4/14 = 33/70$

$P(X=2) = 3 \times 12/16 \times 4/15 \times 3/14 = 9/70$

$P(X=3) = 4/16 \times 3/15 \times 2/14 = 1/140$

Probability distribution is

X	0	1	2	3
P(X)	11/28	33/70	9/70	1/140

Mean $(\mu) = \sum X \times P(X) = 0 \times \frac{11}{28} + 1 \times \frac{33}{70} + 2 \times \frac{9}{70} + 3 \times \frac{1}{140} = \frac{66+36+3}{140} = \frac{105}{140} = \frac{3}{4}$

Variance $(\sigma)^2 = \sum X^2 \times P(X) - (\mu)^2 = 0^2 \times \frac{11}{28} + 1^2 \times \frac{33}{70} + 2^2 \times \frac{9}{70} + 3^2 \times \frac{1}{140} - \left(\frac{3}{4}\right)^2$

$= \frac{66+72+9}{140} - \frac{9}{16} = \frac{147}{140} - \frac{9}{16} = \frac{39}{80}$

12. 20% of the bulbs produced by a machine are defective. Find the probability distribution of the number of defective bulbs in a sample of 4 bulbs chosen at random

Sol. Let $P(S)$ be the probability of getting a defective bulb

$P(S) = \frac{20}{100} = \frac{1}{5}$

Let $P(F)$ be the probability of getting a good bulb

$P(F) = \frac{80}{100} = \frac{4}{5}$

Clearly X can take values 0,1,2,3,4

Step 2 :

$P(X=0) = P(FFFF)$

$= \frac{4}{5} \times \frac{4}{5} \times \frac{4}{5} \times \frac{4}{5} = \frac{256}{625}$

$P(X=0) = P(SFFF) + P(FSFF) + P(FFSF) + P(FFFS)$

$= 4 \times \frac{1}{5} \times \left(\frac{4}{5}\right)^3 = \frac{256}{625}$

$P(X=2) = P(SSFF) + P(SFSF) + P(FFSS) + P(FSFS)$

$= 4 \times \left(\frac{1}{5}\right)^2 \times \left(\frac{4}{5}\right)^2 = \frac{64}{625}$

$P(X=3) = P(SSSF) + P(SFSS) + P(FSSS) + P(SSFS)$

$= 4 \times \left(\frac{4}{5}\right) \times \left(\frac{1}{5}\right)^3 = \frac{16}{625}$

$P(X=4) = P(SSSS) = \left(\frac{1}{5}\right)^4 = \frac{1}{625}$

X	0	1	2	3	4
P(X)	256/625	256/625	64/625	16/625	1/625

13. Four bad eggs are mixed with 10 good ones. Three eggs are drawn one by one without replacement. Let X be the number of bad egs drawn. Find the mean and variance of X

Sol. Since the eggs are drawn one by one with replacement, the events are independent.

Total number of eggs $=4+10=14$ out of which 4 are bad.

If $p=$ probability of drawing a bad egg, then $p=\frac{4}{14}=\frac{2}{7}$, so, $q=1-\frac{2}{7}=\frac{5}{7}$

Thus, events are independent, we have a binomial distribution with $p=\frac{2}{7}, q=\frac{5}{7}$ and $n=3$

If X denoted the number of bad eggs obtained, then X can take the values $0,1,2,3$.

$$P(0)={}^3C_0q^3=\left(\frac{5}{7}\right)^3=\frac{125}{343},\ P(1)={}^3C_1pq^2=3.\frac{2}{7}.\left(\frac{5}{7}\right)^2=\frac{150}{343},$$

$$P(2)={}^3C_2p^2q=3.\left(\frac{2}{7}\right)^2.\frac{5}{7}=\frac{60}{343} \text{ and } P(3)={}^3C_3p^3=\left(\frac{2}{7}\right)^3=\frac{8}{343}$$

$\therefore$ The required probability distribution is $\begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{125}{343} & \frac{150}{343} & \frac{60}{343} & \frac{8}{343} \end{pmatrix}$

Mean $(\mu)=n\times p=3\times\frac{2}{7}=\frac{6}{7}$

Variance $(\sigma^2)=n\times p\times q=3\times\frac{2}{7}\times\frac{5}{7}=\frac{30}{49}$

14. Four rotten oranges are accidentally mixed with 16 good ones. Three oranges are drawn at random from the mixed lot. Let X be the number of rotten oranges drawn. Find the mean and variance of X

Sol. Let the random variable (number of rotten oranges in a draw of two oranges) be X.

Clearly, X can take values 0, 1, or 2.

Now, $P(X=0)=P(\text{getting no rotten oranges})\Rightarrow P(\text{getting two good oranges})=\frac{{}^{16}C_2}{{}^{20}C_2}=\frac{12}{19}$

$P(X=1)=P(\text{getting one rotten orange and one good orange})=\frac{{}^4C_1\cdot{}^{16}C_1}{{}^{20}C_2}=\frac{32}{95}$

and , $P(X=2)=P(\text{getting two rotten oranges})=\frac{{}^4C_2}{{}^{20}C_2}=\frac{3}{95}$

Thus, the required probability distribution of X is:

X	0	1	2
P(X)	$\frac{12}{19}$	$\frac{32}{95}$	$\frac{3}{95}$

Mean $(\mu)=\sum X\cdot P(X)=0\times\frac{12}{19}+1\times\frac{32}{95}+2\times\frac{3}{95}$

$$\frac{32+6}{95}=\frac{38}{95}$$

Variance $(\sigma^2)=\sum X^2\times P(X)-(\mu^2)=0^2\times\frac{12}{19}+1^2\times\frac{32}{95}+2^2\times\frac{3}{95}-\left(\frac{38}{95}\right)^2$

$= \frac{32+12}{95} - \left(\frac{38}{95}\right)^2 = \frac{44\times 95 - 38\times 38}{95\times 95} = \frac{2736}{9025}$

15. Three balls are drawn simultaneously from a bag containing 5 white and 4 red balls. Let X be the number of red balls drawn. Find the mean and variance of X

Sol. Let the random variable (number of red balls drawn) be X.

Since there are 4 red balls, therefore X can have values $0, 1, 2, 3$

Let R_1 = The event of drawing a red ball in the ith draw.

Now, $P(X=0) = \frac{{}^5C_3}{{}^9C_3} = \frac{5}{42}$

$P(X=1) = \frac{\left({}^4C_1 \times {}^5C_2\right)}{{}^9C_3} = \frac{10}{21}$

$P(X=2) = \frac{\left({}^4C_2 \times {}^5C_1\right)}{{}^9C_3} = \frac{5}{14}$

$P(X=3) = \frac{{}^4C_3}{{}^9C_3} = \frac{1}{21}$

Therefore required probability distribution of X is :

X	0	1	2	3
P(X)	$\frac{5}{42}$	$\frac{10}{21}$	$\frac{5}{14}$	$\frac{1}{21}$

Mean $(\mu) = \sum X \cdot P(X) = 0\times\frac{5}{42} + 1\times\frac{10}{21} + 2\times\frac{5}{14} + 3\times\frac{1}{21} = \frac{10}{21} + \frac{15}{21} + \frac{3}{21} = \frac{28}{21} = \frac{4}{3}$

Variance $(\sigma^2) = \sum X^2 \times P(X) - (\mu^2)$

$= 0^2\times\frac{5}{42} + 1^2\times\frac{10}{21} + 2^2\times\frac{5}{14} + 3^2\times\frac{1}{21} - \left(\frac{4}{3}\right)^2 = \frac{10+30+9}{21} - \frac{16}{9} = \frac{49}{21} - \frac{16}{9} = \frac{147-112}{63} = \frac{35}{63} = \frac{5}{9}$

16. Two cards are drawn without replacement from a well-shuffled deck of 52 cards. Let X be the number of face cards drawn. Find the mean and variance of X

Sol. There are 12 face cards (4 kings, 4 queens and 4 jacks)

Clearly, X = 0 or 1 or 2

$P(X=0) = P(\text{no face card})$

= P(drawing 2 cards out of 40 non-face cards)

$= \frac{{}^{40}C_2}{{}^{52}C_2} = \left(\frac{40\times 39}{2\times 1} \times \frac{2\times 1}{52\times 51}\right) = \frac{10}{17}$

$P(X=1) = P(1 \text{ face card and } 1 \text{ non-face card})$

$= \frac{\left({}^{12}C_1 \times {}^{40}C_1\right)}{{}^{52}C_2} = \left(12\times 40\times\frac{2\times 1}{52\times 51}\right) = \frac{80}{221}$

P (X = 2) = P(2 face cards)

$= \frac{{}^{40}C_2}{{}^{52}C_2} = \left(\frac{40\times 39}{2\times 1} \times \frac{2\times 1}{52\times 51}\right) = \frac{10}{17}$

Thus we have

$X = x_i$	0	1	2
p_i	$\frac{10}{17}$	$\frac{80}{221}$	$\frac{10}{17}$

Now find the mean and variance

Mean $(\mu) = \sum X \cdot P(X) = 0 \times \frac{10}{17} + 1 \times \frac{80}{221} + 2 \times \frac{10}{17} = \frac{80+260}{221} = \frac{340}{221} = \frac{20}{13}$

Variance $(\sigma^2) = \sum X^2 \times P(X) - (\mu^2) = 0^2 \times \frac{10}{17} + 1^2 \times \frac{80}{221} + 2^2 \times \frac{10}{17} - \left(\frac{20}{13}\right)^2 = \frac{1000}{2873}$

17. Two cards are drawn one by one with replacement from a well-shuffled deck of 52 cards. Find the mean and variance of the number of aces

Sol. We may draw no ace, one ace or two aces. So, X can take values 0, 1, 2.

$P(X=0) = P$ (no ace is drawn) $= P$(two non-ace cards are drawn)

$\Rightarrow \frac{48}{52} \times \frac{47}{51} = \frac{188}{221}$, note that the first card is not replaced before second is drawn.

$P(X=1) = P$(one ace is drawn) $= P$(one ace and one non-ace is drawn)

$\Rightarrow \frac{4}{52} \times \frac{48}{51} + \frac{48}{52} \times \frac{4}{51} = \frac{32}{221}$ (ace may drawn either at first or at second draw)

and $P(X=2) = P$(two aces are drawn) $= \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$

Thus, the probability distribution of X is as shown below:

X	0	1	2
$P(X)$	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

$\therefore$ Mean $= \sum p_i x_i = 0 \times \frac{188}{221} + 1 \times \frac{32}{221} + 2 \times \frac{1}{221} = \frac{34}{221} = \frac{2}{13}$ and

Variance $= \sum x_i^2 p_i - (\text{mean})^2$

$\Rightarrow 0^2 \times \frac{188}{221} + 1^2 \times \frac{32}{221} + 2^2 \times \frac{1}{221} - \left(\frac{2}{13}\right)^2 = 0 + \frac{36}{221} - \frac{4}{169} = \frac{400}{2873}$

18. Three cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Find the mean and variance of the number of aces

Sol. Let X be the random variable denoting the number of aces when 2 cards are drawn,with replacement from well-shuffled pack of 52 cards.

X can take the values 0 or 1 or 2

$P(X=0) =$ 3 probability of no aces

$= \frac{48}{52} \times \frac{48}{52} = \frac{144}{169}$

P(X=1)= probability of 1 ace

$= 2 \times \frac{4}{52} \times \frac{48}{52} = 2 \times \frac{1}{13} \times \frac{12}{13} = \frac{24}{169}$

P(X=2) = Probability of 2 aces

$= \frac{4}{52} \times \frac{4}{52} = \frac{1}{169}$

The probability distribution of X is given by

X = x	0	1	2
$P(X=x)$	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

Mean $= E(X) = \sum x_i P_i = 0\times\frac{144}{169} + 1\times\frac{24}{169} + 2\times\frac{1}{169} = \frac{26}{169} = \frac{2}{13}$

Var $(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \sum x_i^2 P_i = 0\times\frac{144}{169} + 1\times\frac{24}{169} + 4\times\frac{1}{169} = \frac{28}{169}$

$\therefore Var(X) = \frac{28}{169} - \frac{4}{169} = \frac{24}{169}$

19. Five defective bulbs are accidentally mixed with 20 good. It is not possible to just look at a bulb and tell whether or not a bulb is defective. Four bulbs are drawn at random from this lot. Find the probability distribution from this lot

Sol. Let us denote by X, the number of defective bulbs. Clearly X can take the values 0,1,2,3,4

P(X = 0) = (no defective bulb) = P (all 4 goods ones)

$$= \frac{{}^{20}C_1 \times {}^{20}C_3}{{}^{25}C_4} = \frac{\frac{5}{1}\times\frac{20\times19\times18}{1\times2\times3}}{\frac{25\times24\times23\times22}{1\times2\times3\times4}} = \frac{1140}{2530}$$

P(X = 2) = P (2 defective and 2 good ones)

$$= \frac{{}^{5}C_2 \times {}^{20}C_2}{{}^{25}C_4} = \frac{\frac{5\times4}{1\times2}\times\frac{20\times19}{1\times2}}{\frac{25\times24\times23\times22}{1\times2\times3\times4}} = \frac{380}{2530}$$

P(X = 3) = P(3 defective and one good one)

$$= \frac{{}^{5}C_3 \times {}^{20}C_1}{{}^{25}C_1} = \frac{\frac{5\times4}{1\times2}\times\frac{20}{1}}{\frac{25\times24\times23\times22}{1\times2\times3\times4}} = \frac{40}{2530}$$

P(X = 4)= P(all 4 defective)

$$= \frac{{}^{5}C_4}{{}^{25}C_4} = \frac{\frac{5}{1}}{\frac{25\times24\times23\times22}{1\times2\times3\times4}} = \frac{1}{2530}$$

$\therefore$ probability distribution table is ,

X	0	1	2	3	4
$P(X)$	$\frac{969}{2530}$	$\frac{1140}{2530}$	$\frac{380}{2530}$	$\frac{40}{2530}$	$\frac{1}{2530}$