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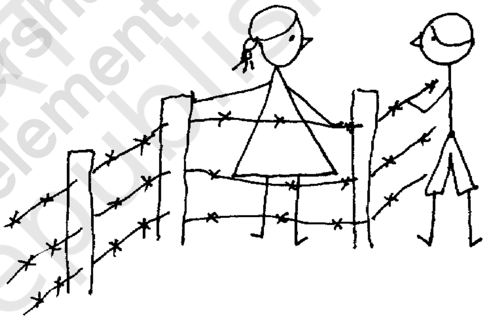
APPENDIX 1

PROOFS IN MATHEMATICS

A1.1 Introduction



Suppose your family owns a plot of land and there is no fencing around it. Your neighbour decides one day to fence off his land. After he has fenced his land, you discover that a part of your family's land has been enclosed by his fence. How will you prove to your neighbour that he has tried to encroach on your land? Your first step may be to seek the help of the village elders to sort out the difference in boundaries. But, suppose opinion is divided among the elders. Some feel you are right and others feel your neighbour is right. What can you do? Your only option is to find a way of establishing your claim for the boundaries of your land that is acceptable to all. For example, a government approved survey map of your village can be used, if necessary in a court of law, to prove (claim) that you are correct and your neighbour is wrong.

Let us look at another situation. Suppose your mother has paid the electricity bill of your house for the month of August, 2005. The bill for September, 2005, however, claims that the bill for August has not been paid. How will you disprove the claim made by the electricity department? You will have to produce a receipt proving that your August bill has been paid.

You have just seen some examples that show that in our daily life we are often called upon to prove that a certain statement or claim is true or false. However, we also accept many statements without bothering to prove them. But, in mathematics we only accept a statement as true or false (except for some axioms) if it has been proved to be so, according to the logic of mathematics.

In fact, proofs in mathematics have been in existence for thousands of years, and they are central to any branch of mathematics. The first known proof is believed to have been given by the Greek philosopher and mathematician Thales. While mathematics was central to many ancient civilisations like Mesopotamia, Egypt, China and India, there is no clear evidence that they used proofs the way we do today.

In this chapter, we will look at what a statement is, what kind of reasoning is involved in mathematics, and what a mathematical proof consists of.

A1.2 Mathematically Acceptable Statements

In this section, we shall try to explain the meaning of a mathematically acceptable statement. A 'statement' is a sentence which is not an order or an exclamatory sentence. And, of course, a statement is not a question! For example,

"What is the colour of your hair?" is not a statement, it is a question.

"Please go and bring me some water." is a request or an order, not a statement.

"What a marvellous sunset!" is an exclamatory remark, not a statement.

However, "The colour of your hair is black" is a statement.

In general, statements can be one of the following:

- *always true*
- *always false*
- *ambiguous*

The word 'ambiguous' needs some explanation. There are two situations which make a statement ambiguous. The first situation is when we cannot decide if the statement is always true or always false. For example, "Tomorrow is Thursday" is ambiguous, since enough of a context is not given to us to decide if the statement is true or false.

The second situation leading to ambiguity is when the statement is subjective, that is, it is true for some people and not true for others. For example, "Dogs are intelligent" is ambiguous because some people believe this is true and others do not.

Example 1 : State whether the following statements are always true, always false or ambiguous. Justify your answers.

(i) There are 8 days in a week.

(ii) It is raining here.

(iii) The sun sets in the west.

(iv) Gauri is a kind girl.

(v) The product of two odd integers is even.

(vi) The product of two even natural numbers is even.

Solution :

(i) This statement is always false, since there are 7 days in a week.

(ii) This statement is ambiguous, since it is not clear where 'here' is.

(iii) This statement is always true. The sun sets in the west no matter where we live.

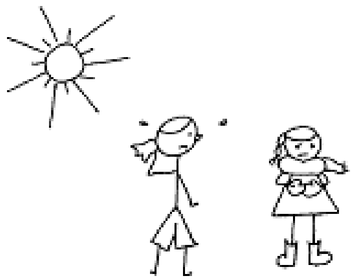
(iv) This statement is ambiguous, since it is subjective–Gauri may be kind to some and not to others.

(v) This statement is always false. The product of two odd integers is always odd.

(vi) This statement is always true. However, to justify that it is true we need to do some work. It will be proved in Section A1.4.

As mentioned before, in our daily life, we are not so careful about the validity of statements. For example, suppose your friend tells you that in July it rains everyday in Manantavadi, Kerala. In all probability, you will believe her, even though it may not have rained for a day or two in July. Unless you are a lawyer, you will not argue with her!

As another example, consider statements we often make to each other like "it is very hot today". We easily accept such statements because we know the context even though these statements are ambiguous. 'It is very hot today' can mean different things to different people because what is very hot for a person from Kumaon may not be hot for a person from Chennai.



But a mathematical statement cannot be ambiguous. *In mathematics, a statement is only acceptable or valid, if it is either true or false.* We say that a statement is true, if it is always true otherwise it is called a false statement.

For example, 5 + 2 = 7 is always true, so '5 + 2 = 7' is a true statement and 5 + 3 = 7 is a false statement.

Example 2 : State whether the following statements are true or false:

(i) The sum of the interior angles of a triangle is 180°.

(ii) Every odd number greater than 1 is prime.

(iii) For any real number x, $4x + x = 5x$.

(iv) For every real number x, $2x > x$.

(v) For every real number x, $x^2 \geq x$.

(vi) If a quadrilateral has all its sides equal, then it is a square.

Solution :

(i) This statement is true. You have already proved this in Chapter 6.

(ii) This statement is false; for example, 9 is not a prime number.

(iii) This statement is true.

(iv) This statement is false; for example, $2 \times (-1) = -2$, and -2 is not greater than -1.

(v) This statement is false; for example, $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, and $\frac{1}{4}$ is not greater than $\frac{1}{2}$.

(vi) This statement is false, since a rhombus has equal sides but need not be a square.

You might have noticed that to establish that a statement is not true according to mathematics, all we need to do is to find one case or example where it breaks down. So in (ii), since 9 is not a prime, it is an example that shows that the statement "Every odd number greater than 1 is prime" is not true. Such an example, that counters a statement, is called a *counter-example*. We shall discuss counter-examples in greater detail in Section A1.5.

You might have also noticed that while Statements (iv), (v) and (vi) are false, they can be restated with some conditions in order to make them true.

Example 3 : Restate the following statements with appropriate conditions, so that they become true statements.

(i) For every real number x, $2x > x$.

(ii) For every real number x, $x^2 \geq x$.

(iii) If you divide a number by itself, you will always get 1.

(iv) The angle subtended by a chord of a circle at a point on the circle is 90°.

(v) If a quadrilateral has all its sides equal, then it is a square.

Solution :

(i) If $x > 0$, then $2x > x$.

(ii) If $x \leq 0$ or $x \geq 1$, then $x^2 \geq x$.

(iii) If you divide a number except zero by itself, you will always get 1.

(iv) The angle subtended by a diameter of a circle at a point on the circle is 90°.

(v) If a quadrilateral has all its sides and interior angles equal, then it is a square.

EXERCISE A1.1

1. State whether the following statements are always true, always false or ambiguous. Justify your answers.

(i) There are 13 months in a year.

(ii) Diwali falls on a Friday.

(iii) The temperature in Magadi is 26^0 C.

(iv) The earth has one moon.

(v) Dogs can fly.

(vi) February has only 28 days.

2. State whether the following statements are true or false. Give reasons for your answers.

(i) The sum of the interior angles of a quadrilateral is 350°.

(ii) For any real number x, $x^2 \geq 0$.

(iii) A rhombus is a parallelogram.

(iv) The sum of two even numbers is even.

(v) The sum of two odd numbers is odd.

3. Restate the following statements with appropriate conditions, so that they become true statements.

(i) All prime numbers are odd.

(ii) Two times a real number is always even.

(iii) For any x, $3x + 1 > 4$.

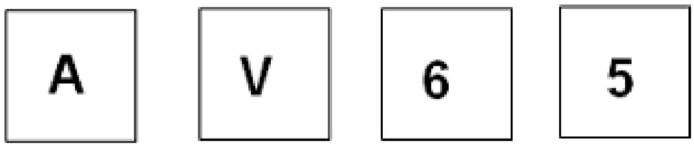
(iv) For any x, $x^3 \geq 0$.

(v) In every triangle, a median is also an angle bisector.

A1.3 Deductive Reasoning

The main logical tool used in establishing the truth of an **unambiguous** statement is *deductive reasoning*. To understand what deductive reasoning is all about, let us begin with a puzzle for you to solve.

You are given four cards. Each card has a number printed on one side and a letter on the other side.



Suppose you are told that these cards follow the rule:

"If a card has an even number on one side, then it has a vowel on the other side."

What is the **smallest number** of cards you need to turn over to check if the rule is true?

Of course, you have the option of turning over all the cards and checking. But can you manage with turning over a fewer number of cards?

Notice that the statement mentions that a card with an even number on one side has a vowel on the other. It does not state that a card with a vowel on one side must have an even number on the other side. That may or may not be so. The rule also does not state that a card with an odd number on one side must have a consonant on the other side. It may or may not.

So, do we need to turn over 'A'? No! Whether there is an even number or an odd number on the other side, the rule still holds.

What about '5'? Again we do not need to turn it over, because whether there is a vowel or a consonant on the other side, the rule still holds.

But you do need to turn over V and 6. If V has an even number on the other side, then the rule has been broken. Similarly, if 6 has a consonant on the other side, then the rule has been broken.

The kind of reasoning we have used to solve this puzzle is called **deductive reasoning**. It is called 'deductive' because we arrive at (i.e., deduce or infer) a result or a statement from a previously established statement using logic. For example, in the puzzle above, by a series of logical arguments we deduced that we need to turn over only V and 6.

Deductive reasoning also helps us to conclude that a particular statement is true, because it is a special case of a more general statement that is known to be true. For example, once we prove that the product of two odd numbers is always odd, we can immediately conclude (without computation) that 70001×134563 is odd simply because 70001 and 134563 are odd.

Deductive reasoning has been a part of human thinking for centuries, and is used all the time in our daily life. For example, suppose the statements "The flower Solaris blooms, only if the maximum temperature is above 28° C on the previous day" and "Solaris bloomed in Imaginary Valley on 15th September, 2005" are true. Then using deductive reasoning, we can conclude that the maximum temperature in Imaginary Valley on 14th September, 2005 was more than 28° C.

Unfortunately we do not always use correct reasoning in our daily life! We often come to many conclusions based on faulty reasoning. For example, if your friend does not smile at you one day, then you may conclude that she is angry with you. While it may be true that "if she is angry with me, she will not smile at me", it may also be true that "if she has a bad headache, she will not smile at me". Why don't you examine some conclusions that you have arrived at in your day-to-day existence, and see if they are based on valid or faulty reasoning?

EXERCISE A1.2

1. Use deductive reasoning to answer the following:

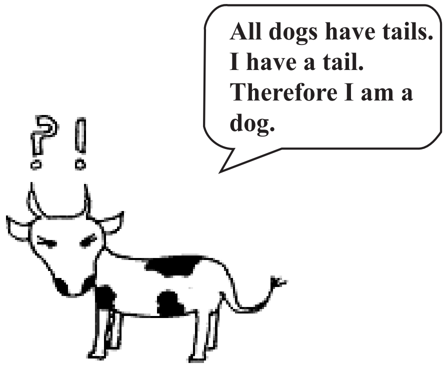
(i) Humans are mammals. All mammals are vertebrates. Based on these two statements, what can you conclude about humans?

(ii) Anthony is a barber. Dinesh had his hair cut. Can you conclude that Antony cut Dinesh's hair?

(iii) Martians have red tongues. Gulag is a Martian. Based on these two statements, what can you conclude about Gulag?

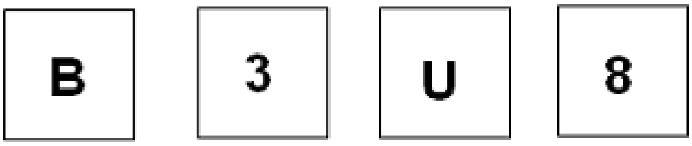
(iv) If it rains for more than four hours on a particular day, the gutters will have to be cleaned the next day. It has rained for 6 hours today. What can we conclude about the condition of the gutters tomorrow?

(v) What is the fallacy in the cow's reasoning in the cartoon below?



2. Once again you are given four cards. Each card has a number printed on one side and a letter on the other side. Which are the only two cards you need to turn over to check whether the following rule holds?

"If a card has a consonant on one side, then it has an odd number on the other side."



A1.4 Theorems, Conjectures and Axioms

So far we have discussed statements and how to check their validity. In this section, you will study how to distinguish between the three different kinds of statements mathematics is built up from, namely, a theorem, a conjecture and an axiom.

You have already come across many theorems before. So, what is a theorem? A mathematical statement whose truth has been established (proved) is called a *theorem*. For example, the following statements are theorems, as you will see in Section A1.5.

Theorem A1.1 : *The sum of the interior angles of a triangle is* 180º.

Theorem A1.2 : *The product of two even natural numbers is even.*

Theorem A1.3 : *The product of any three consecutive even natural numbers is divisible by* 16.

A *conjecture* is a statement which we believe is true, based on our mathematical understanding and experience, that is, our mathematical intuition. The conjecture may turn out to be true or false. If we can prove it, then it becomes a theorem. Mathematicians often come up with conjectures by looking for patterns and making intelligent mathematical guesses. Let us look at some patterns and see what kind of intelligent guesses we can make.

Example 4 : Take any three consecutive even numbers and add them, say,

$2 + 4 + 6 = 12$, $4 + 6 + 8 = 18$, $6 + 8 + 10 = 24$, $8 + 10 + 12 = 30$, $20 + 22 + 24 = 66$.

Is there any pattern you can guess in these sums? What can you conjecture about them?

Solution : One conjecture could be :

(i) the sum of three consecutive even numbers is even.

Another could be :

(ii) the sum of three consecutive even numbers is divisible by 6.

Example 5 : Consider the following pattern of numbers called the Pascal's Triangle:

Line		**Sum of numbers**
1	1	1
2	1 1	2
3	1 2 1	4
4	1 3 3 1	8
5	1 4 6 4 1	16
6	1 5 10 10 5 1	32
7	: :	:
8	: :	:

What can you conjecture about the sum of the numbers in Lines 7 and 8? What about the sum of the numbers in Line 21? Do you see a pattern? Make a guess about a formula for the sum of the numbers in line n.

Solution : Sum of the numbers in Line 7 = $2 \times 32 = 64 = 2^6$

Sum of the numbers in Line 8 = $2 \times 64 = 128 = 2^7$

Sum of the numbers in Line 21 = 2^{20}

Sum of the numbers in Line n = 2^{n-1}

Example 6 : Consider the so-called triangular numbers T_n:

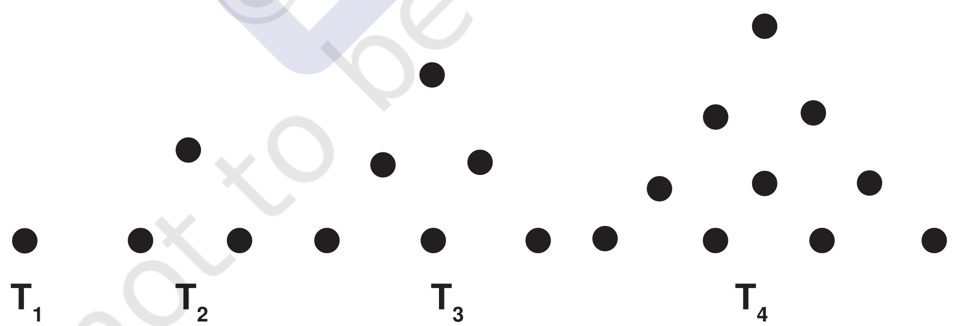


Fig. A1.1

The dots here are arranged in such a way that they form a triangle. Here $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$, and so on. Can you guess what T_5 is? What about T_6? What about T_n?

Make a conjecture about T_n.

It might help if you redraw them in the following way.

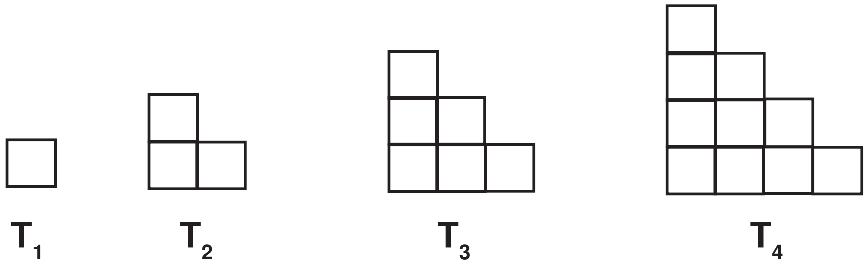


Fig. A1.2

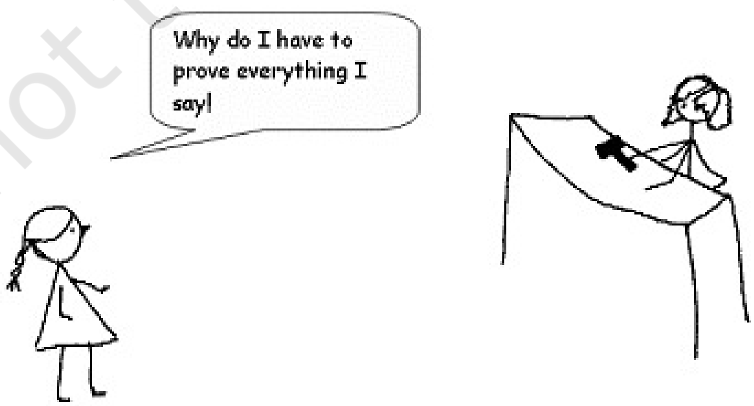
Solution : $T_5 = 1 + 2 + 3 + 4 + 5 = 15 = \dfrac{5 \times 6}{2}$

$$T_6 = 1 + 2 + 3 + 4 + 5 + 6 = 21 = \frac{6 \times 7}{2}$$

$$T_n = \frac{n \times (n+1)}{2}$$

A favourite example of a conjecture that has been open (that is, it has not been proved to be true or false) is the Goldbach conjecture named after the mathematician Christian Goldbach (1690 – 1764). This conjecture states that "*every even integer greater than* 4 *can be expressed as the sum of two odd primes*." Perhaps you will prove that this result is either true or false, and will become famous!

You might have wondered – do we need to prove everything we encounter in mathematics, and if not, why not?



The fact is that every area in mathematics is based on some statements which are assumed to be true and are not proved. These are 'self-evident truths' which we take to be true without proof. These statements are called *axioms*. In Chapter 5, you would have studied the axioms and postulates of Euclid. (We do not distinguish between axioms and postulates these days.)

For example, the first postulate of Euclid states:

A straight line may be drawn from any point to any other point.

And the third postulate states:

A circle may be drawn with any centre and any radius.

These statements appear to be perfectly true and Euclid assumed them to be true. Why? This is because we cannot prove everything and we need to start somewhere. We need some statements which we accept as true and then we can build up our knowledge using the rules of logic based on these axioms.

You might then wonder why we don't just accept all statements to be true when they appear self-evident. There are many reasons for this. Very often our intuition can be wrong, pictures or patterns can deceive and the only way to be sure that something is true is to prove it. For example, many of us believe that if a number is multiplied by another, the result will be larger than both the numbers. But we know that this is not always true: for example, $5 \times 0.2 = 1$, which is less than 5.

Also, look at the Fig. A1.3. Which line segment is longer, AB or CD?

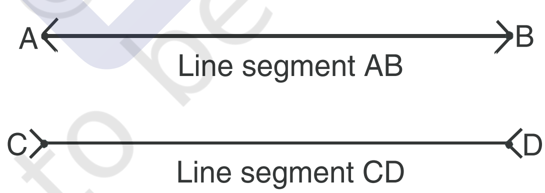


Fig. A1.3

It turns out that both are of exactly the same length, even though AB appears shorter!

You might wonder then, about the validity of axioms. Axioms have been chosen based on our intuition and what appears to be self-evident. Therefore, we expect them to be true. However, it is possible that later on we discover that a particular axiom is not true. What is a safeguard against this possibility? We take the following steps:

(i) Keep the axioms to the bare minimum. For instance, based on only axioms and five postulates of Euclid, we can derive hundreds of theorems.

(ii) Make sure the axioms are consistent.

We say a collection of axioms is *inconsistent,* if we can use one axiom to show that another axiom is not true. For example, consider the following two statements. We will show that they are inconsistent.

Statement1: No whole number is equal to its successor.

Statement 2: A whole number divided by zero is a whole number.

(Remember, **division by zero is not defined**. But just for the moment, we assume that it is possible, and see what happens.)

From Statement 2, we get $\frac{1}{0} = a$, where a is some whole number. This implies that, $1 = 0$. But this disproves Statement 1, which states that no whole number is equal to its successor.

(iii) A false axiom will, sooner or later, result in a contradiction. We say that *there is a contradiction, when we find a statement such that, both the statement and its negation are true*. For example, consider Statement 1 and Statement 2 above once again.

From Statement 1, we can derive the result that $2 \neq 1$.

Now look at $x^2 - x^2$. We will factorise it in two different ways as follows:

(i) $x^2 - x^2 = x(x - x)$ and

(ii) $x^2 - x^2 = (x + x)(x - x)$

So, $x(x - x) = (x + x)(x - x)$.

From Statement 2, we can cancel $(x - x)$ from both sides.

We get $x = 2x$, which in turn implies $2 = 1$.

So we have both the statement $2 \neq 1$ and its negation, $2 = 1$, true. This is a contradiction. The contradiction arose because of the false axiom, that a whole number divided by zero is a whole number.

So, the statements we choose as axioms require a lot of thought and insight. We must make sure they do not lead to inconsistencies or logical contradictions. Moreover, the choice of axioms themselves, sometimes leads us to new discoveries. From Chapter 5, you are familiar with Euclid's fifth postulate and the discoveries of non-Euclidean geometries. You saw that mathematicians believed that the fifth postulate need not be a postulate and is actually a theorem that can be proved using just the first four postulates. Amazingly these attempts led to the discovery of non-Euclidean geometries.

We end the section by recalling the differences between an axiom, a theorem and a conjecture. An **axiom** is a mathematical statement which is assumed to be true without proof; a **conjecture** is a mathematical statement whose truth or falsity is yet to be established; and a **theorem** is a mathematical statement whose truth has been logically established.

EXERCISE A1.3

1. Take any three consecutive even numbers and find their product; for example, $2 \times 4 \times 6 = 48$, $4 \times 6 \times 8 = 192$, and so on. Make three conjectures about these products.

2. Go back to Pascal's triangle.

   Line 1 : $1 = 11^0$

   Line 2 : 1 $1 = 11^1$

   Line 3 : 1 2 $1 = 11^2$

   Make a conjecture about Line 4 and Line 5. Does your conjecture hold? Does your conjecture hold for Line 6 too?

3. Let us look at the triangular numbers (see Fig.A1.2) again. Add two consecutive triangular numbers. For example, $T_1 + T_2 = 4$, $T_2 + T_3 = 9$, $T_3 + T_4 = 16$.

   What about $T_4 + T_5$? Make a conjecture about $T_{n-1} + T_n$.

4. Look at the following pattern:

   $1^2 = 1$

   $11^2 = 121$

   $111^2 = 12321$

   $1111^2 = 1234321$

   $11111^2 = 123454321$

   Make a conjecture about each of the following:

   $111111^2 =$

   $1111111^2 =$

   Check if your conjecture is true.

5. List five axioms (postulates) used in this book.

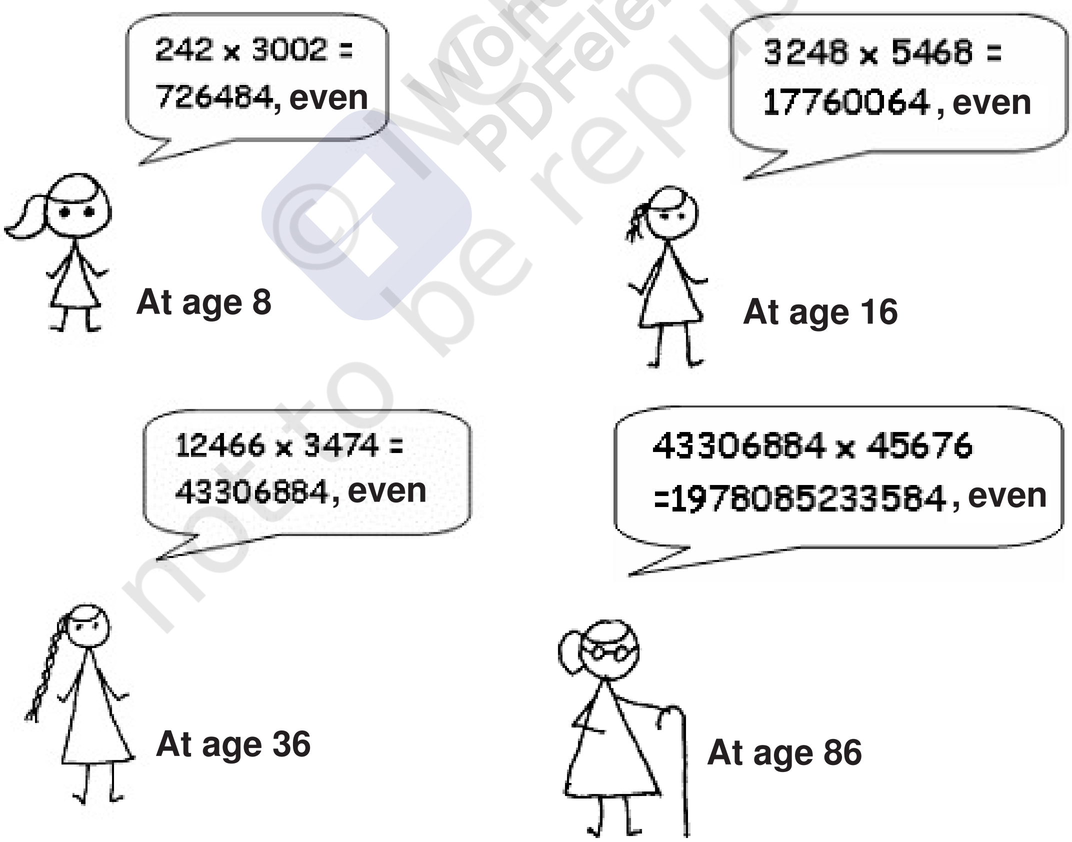
A1.5 What is a Mathematical Proof?

Let us now look at various aspects of proofs. We start with understanding the difference between verification and proof. Before you studied proofs in mathematics, you were mainly asked to verify statements.

For example, you might have been asked to verify with examples that "the product of two even numbers is even". So you might have picked up two random even numbers, say 24 and 2006, and checked that 24 × 2006 = 48144 is even. You might have done so for many more examples.

Also, you might have been asked as an activity to draw several triangles in the class and compute the sum of their interior angles. Apart from errors due to measurement, you would have found that the interior angles of a triangle add up to 180°.

What is the flaw in this method? There are several problems with the process of verification. While it may help you to make a statement you believe is true, you cannot be *sure* that it is true in *all* cases. For example, the multiplication of several pairs of even numbers may lead us to guess that the product of two even numbers is even. However, it does not ensure that the product of all pairs of even numbers is even. You cannot physically check the products of all possible pairs of even numbers. If you did, then like the girl in the cartoon, you will be calculating the products of even numbers for the rest of your life. Similarly, there may be some triangles which you have not yet drawn whose interior angles do not add up to 180°. We cannot measure the interior angles of all possible triangles.



Moreover, verification can often be misleading. For example, we might be tempted to conclude from Pascal's triangle (Q.2 of Exercise A1.3), based on earlier verifications, that 11^5 = 15101051. But in fact 11^5 =161051.

So, you need another approach that does not depend upon verification for some cases only. There is another approach, namely 'proving a statement'. A process which can establish the truth of a mathematical statement based purely on logical arguments is called a *mathematical proof*.

In Example 2 of Section A1.2, you saw that to establish that a mathematical statement is false, it is enough to produce a single counter-example. So while it is not enough to establish the validity of a mathematical statement by checking or verifying it for thousands of cases, it is enough to produce one counter-example to *disprove* a statement (i.e., to show that something is false). This point is worth emphasising.

To show that a mathematical statement is false, it is enough to find a single counter-example.

So, 7 + 5 = 12 is a counter-example to the statement that the sum of two odd numbers is odd.

Let us now look at the list of basic ingredients in a proof:

(i) To prove a theorem, we should have a rough idea as to how to proceed.

(ii) The information already given to us in a theorem (i.e., the hypothesis) has to be clearly understood and used.

For example, in Theorem A1.2, which states that the product of two even numbers is even, we are given two even natural numbers. So, we should use their properties. In the Factor Theorem (in Chapter 2), you are given a polynomial $p(x)$ and are told that $p(a) = 0$. You have to use this to show that $(x - a)$ is a factor of $p(x)$. Similarly, for the converse of the Factor Theorem, you are given that $(x - a)$ is a factor of $p(x)$, and you have to use this hypothesis to prove that $p(a) =0$.

You can also use constructions during the process of proving a theorem. For example, to prove that the sum of the angles of a triangle is 180°, we draw a line parallel to one of the sides through the vertex opposite to the side, and use properties of parallel lines.

(iii) A proof is made up of a successive sequence of mathematical statements. Each statement in a proof is logically deduced from a previous statement in the proof, or from a theorem proved earlier, or an axiom, or our hypothesis.

(iv) The conclusion of a sequence of mathematically true statements laid out in a logically correct order should be what we wanted to prove, that is, what the theorem claims.

To understand these ingredients, we will analyse Theorem A1.1 and its proof. You have already studied this theorem in Chapter 6. But first, a few comments on proofs in geometry. We often resort to diagrams to help us prove theorems, and this is very important. However, each statement in the proof has to be established **using only logic**. Very often, we hear students make statements like "Those two angles are equal because in the drawing they look equal" or "that angle must be 90^0, because the two lines look as if they are perpendicular to each other". Beware of being deceived by what you see (remember Fig A1.3)! .

So now let us go to Theorem A1.1.

Theorem A1.1 : *The sum of the interior angles of a triangle is* 180°.

Proof : Consider a triangle ABC (see Fig. A1.4).

We have to prove that $\angle ABC + \angle BCA + \angle CAB = 180°$ (1)

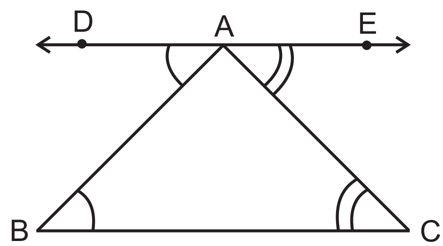


Fig A 1.4

Construct a line DE parallel to BC passing through A. (2)

DE is parallel to BC and AB is a transversal.

So, ∠ DAB and ∠ ABC are alternate angles. Therefore, by Theorem 6.2, Chapter 6, they are equal, i.e. ∠ DAB = ∠ ABC (3)

Similarly, ∠ CAE = ∠ ACB (4)

Therefore, ∠ ABC + ∠ BAC+ ∠ ACB = ∠ DAB +∠ BAC + ∠ CAE (5)

But ∠ DAB +∠ BAC + ∠ CAE = 180°, since they form a straight angle. (6)

Hence, ∠ ABC + ∠ BAC+ ∠ ACB = 180°. (7)

Now, we comment on each step of the proof.

Step 1 : Our theorem is concerned with a property of triangles, so we begin with a triangle.

Step 2 : This is the key idea – the intuitive leap or understanding of how to proceed so as to be able to prove the theorem. Very often geometric proofs require a construction.

Steps 3 and 4 : Here we conclude that ∠ DAE = ∠ ABC and ∠ CAE = ∠ ACB, by using the fact that DE is parallel to BC (our construction), and the previously proved Theorem 6.2, which states that if two parallel lines are intersected by a transversal, then the alternate angles are equal.

Step 5 : Here we use Euclid's axiom (see Chapter 5) which states that: "If equals are added to equals, the wholes are equal" to deduce

∠ ABC + ∠ BAC+ ∠ ACB = ∠ DAB +∠ BAC + ∠ CAE.

That is, the sum of the interior angles of the triangle are equal to the sum of the angles on a straight line.

Step 6 : Here we use the Linear pair axiom of Chapter 6, which states that the angles on a straight line add up to 180°, to show that ∠ DAB +∠ BAC + ∠ CAE = 180°.

Step 7 : We use Euclid's axiom which states that "things which are equal to the same thing are equal to each other" to conclude that ∠ ABC + ∠ BAC + ∠ ACB = ∠ DAB +∠ BAC + ∠ CAE = 180°. Notice that Step 7 is the claim made in the theorem we set out to prove.

We now prove Theorems A1.2 and A1.3 without analysing them.

Theorem A1.2 : *The product of two even natural numbers is even.*

Proof : Let x and y be any two even natural numbers.

We want to prove that xy is even.

Since x and y are even, they are divisible by 2 and can be expressed in the form $x = 2m$, for some natural number m and $y = 2n$, for some natural number n.

Then $xy = 4\,mn$. Since $4\,mn$ is divisible by 2, so is xy.

Therefore, xy is even.

Theorem A1.3 : *The product of any three consecutive even natural numbers is divisible by* 16.

Proof : Any three consecutive even numbers will be of the form $2n$, $2n + 2$ and $2n + 4$, for some natural number n. We need to prove that their product $2n(2n + 2)(2n + 4)$ is divisible by 16.

Now, $2n(2n + 2)(2n + 4) = 2n \times 2(n + 1) \times 2(n + 2)$

$= 2 \times 2 \times 2n(n + 1)(n + 2) = 8n(n + 1)(n + 2)$.

Now we have two cases. Either n is even or odd. Let us examine each case.

Suppose n is even : Then we can write $n = 2m$, for some natural number m.

And, then $2n(2n + 2)(2n + 4) = 8n(n + 1)(n + 2) = 16m(2m + 1)(2m + 2)$.

Therefore, $2n(2n + 2)(2n + 4)$ is divisible by 16.

Next, suppose n is odd. Then $n + 1$ is even and we can write $n + 1 = 2r$, for some natural number r.

We then have :

$$\begin{aligned} 2n(2n + 2)(2n + 4) &= 8n(n + 1)(n + 2) \\ &= 8(2r - 1) \times 2r \times (2r + 1) \\ &= 16r(2r - 1)(2r + 1) \end{aligned}$$

Therefore, $2n(2n + 2)(2n + 4)$ is divisible by 16.

So, in both cases we have shown that the product of any three consecutive even numbers is divisible by 16.

We conclude this chapter with a few remarks on the difference between how mathematicians discover results and how formal rigorous proofs are written down. As mentioned above, each proof has a key intuitive idea (sometimes more than one). Intuition is central to a mathematician's way of thinking and discovering results. Very often the proof of a theorem comes to a mathematician all jumbled up. A mathematician will often experiment with several routes of thought, and logic, and examples, before she/he can hit upon the correct solution or proof. It is only after the creative phase subsides that all the arguments are gathered together to form a proper proof.

It is worth mentioning here that the great Indian mathematician Srinivasa Ramanujan used very high levels of intuition to arrive at many of his statements, which he claimed were true. Many of these have turned out be true and are well known theorems. However, even to this day mathematicians all over the world are struggling to prove (or disprove) some of his claims (conjectures).

Srinivasa Ramanujan (1887–1920)
Fig. A1.5

EXERCISE A1.4

1. Find counter-examples to disprove the following statements:
   (i) If the corresponding angles in two triangles are equal, then the triangles are congruent.
   (ii) A quadrilateral with all sides equal is a square.
   (iii) A quadrilateral with all angles equal is a square.
   (iv) For integers a and b, $\sqrt{a^2 + b^2} = a + b$
   (v) $2n^2 + 11$ is a prime for all whole numbers n.
   (vi) $n^2 - n + 41$ is a prime for all positive integers n.
2. Take your favourite proof and analyse it step-by-step along the lines discussed in Section A1.5 (what is given, what has been proved, what theorems and axioms have been used, and so on).
3. Prove that the sum of two odd numbers is even.
4. Prove that the product of two odd numbers is odd.
5. Prove that the sum of three consecutive even numbers is divisible by 6.
6. Prove that infinitely many points lie on the line whose equation is $y = 2x$.
   (*Hint* : Consider the point $(n, 2n)$ for any integer n.)
7. You must have had a friend who must have told you to think of a number and do various things to it, and then without knowing your original number, telling you what number you ended up with. Here are two examples. Examine why they work.
   (i) Choose a number. Double it. Add nine. Add your original number. Divide by three. Add four. Subtract your original number. Your result is seven.
   (ii) Write down any three-digit number (for example, 425). Make a six-digit number by repeating these digits in the same order (425425). Your new number is divisible by 7, 11 and 13.

A1.6 Summary

In this Appendix, you have studied the following points:

1. In mathematics, a statement is only acceptable if it is either always true or always false.
2. To show that a mathematical statement is false, it is enough to find a single counter-example.
3. Axioms are statements which are assumed to be true without proof.
4. A conjecture is a statement we believe is true based on our mathematical intuition, but which we are yet to prove.
5. A mathematical statement whose truth has been established (or proved) is called a theorem.
6. The main logical tool in proving mathematical statements is deductive reasoning.
7. A proof is made up of a successive sequence of mathematical statements. Each statement in a proof is logically deduced from a previouly known statement, or from a theorem proved earlier, or an axiom, or the hypothesis.

APPENDIX 2

INTRODUCTION TO MATHEMATICAL MODELLING

A 2.1 Introduction

Right from your earlier classes, you have been solving problems related to the real-world around you. For example, you have solved problems in simple interest using the formula for finding it. The formula (or equation) is a relation between the interest and the other three quantities that are related to it, the principal, the rate of interest and the period. This formula is an example of a **mathematical model**. A **mathematical model** is a mathematical relation that describes some real-life situation.

Mathematical models are used to solve many real-life situations like:

- launching a satellite.
- predicting the arrival of the monsoon.
- controlling pollution due to vehicles.
- reducing traffic jams in big cities.

In this chapter, we will introduce you to the process of constructing mathematical models, which is called **mathematical modelling**. In mathematical modelling, we take a real-world problem and write it as an equivalent mathematical problem. We then solve the mathematical problem, and interpret its solution in terms of the real-world problem. After this we see to what extent the solution is valid in the context of the real-world problem. So, the *stages* involved in mathematical modelling are formulation, solution, interpretation and validation.

We will start by looking at the process you undertake when solving word problems, in Section A2.2. Here, we will discuss some word problems that are similar to the ones you have solved in your earlier classes. We will see later that the steps that are used for solving word problems are some of those used in mathematical modelling also.

In the next section, that is Section A2.3, we will discuss some simple models.

In Section A2.4, we will discuss the overall process of modelling, its advantages and some of its limitations.

A2.2 Review of Word Problems

In this section, we will discuss some word problems that are similar to the ones that you have solved in your earlier classes. Let us start with a problem on direct variation.

Example 1 : I travelled 432 kilometres on 48 litres of petrol in my car. I have to go by my car to a place which is 180 km away. How much petrol do I need?

Solution : We will list the steps involved in solving the problem.

Step 1 : Formulation : You know that farther we travel, the more petrol we require, that is, the amount of petrol we need varies directly with the distance we travel.

Petrol needed for travelling 432 km = 48 litres

Petrol needed for travelling 180 km = ?

Mathematical Description : Let

x = distance I travel

y = petrol I need

y varies directly with x.

So, $y = kx$, where k is a constant.

I can travel 432 kilometres with 48 litres of petrol.

So, $y = 48, x = 432.$

Therefore, $$k = \frac{y}{x} = \frac{48}{432} = \frac{1}{9}.$$

Since $y = kx$,

therefore, $$y = \frac{1}{9}x \qquad (1)$$

Equation or Formula (1) describes the relationship between the petrol needed and distance travelled.

Step 2 : Solution : We want to find the petrol we need to travel 180 kilometres; so, we have to find the value of y when $x = 180$. Putting $x = 180$ in (1), we have

$$y = \frac{180}{9} = 20.$$

Step 3 : Interpretation : Since $y = 20$, we need 20 litres of petrol to travel 180 kilometres.

Did it occur to you that you may not be able to use the formula (1) in all situations? For example, suppose the 432 kilometres route is through mountains and the 180 kilometres route is through flat plains. The car will use up petrol at a faster rate in the first route, so we cannot use the same rate for the 180 kilometres route, where the petrol will be used up at a slower rate. So the formula works if all such conditions that affect the rate at which petrol is used are the same in both the trips. Or, if there is a difference in conditions, the effect of the difference on the amount of petrol needed for the car should be very small. The petrol used will vary directly with the distance travelled only in such a situation. We assumed this while solving the problem.

Example 2 : Suppose Sudhir has invested ₹ 15,000 at 8% simple interest per year. With the return from the investment, he wants to buy a washing machine that costs ₹ 19,000. For what period should he invest ₹ 15,000 so that he has enough money to buy a washing machine?

Solution : Step 1 : Formulation of the problem : Here, we know the principal and the rate of interest. The interest is the amount Sudhir needs in addition to 15,000 to buy the washing machine. We have to find the number of years.

Mathematical Description : The formula for simple interest is $I = \frac{Pnr}{100}$,

where P = Principal,

n = Number of years,

r % = Rate of interest

I = Interest earned

Here, the principal = ₹ 15,000

The money required by Sudhir for buying a washing machine = ₹ 19,000

So, the interest to be earned = ₹ (19,000 – 15,000)

= ₹ 4,000

The number of years for which ₹ 15,000 is deposited = n

The interest on ₹ 15,000 for n years at the rate of 8% = I

Then,
$$I = \frac{15000 \times n \times 8}{100}$$

So, $$\text{I} = 1200n \qquad (1)$$

gives the relationship between the number of years and interest, if ₹ 15000 is invested at an annual interest rate of 8%.

We have to find the period in which the interest earned is ₹ 4000. Putting I = 4000 in (1), we have

$$4000 = 1200n \qquad (2)$$

Step 2 : Solution of the problem : Solving Equation (2), we get

$$n = \frac{4000}{1200} = 3\frac{1}{3}.$$

Step 3 : Interpretation : Since $n = 3\frac{1}{3}$ and one third of a year is 4 months, Sudhir can buy a washing machine after 3 years and 4 months.

Can you guess the assumptions that you have to make in the example above? We have to assume that the interest rate remains the same for the period for which we calculate the interest. Otherwise, the formula $\text{I} = \frac{\text{P}nr}{100}$ will not be valid. We have also assumed that the price of the washing machine does not increase by the time Sudhir has gathered the money.

Example 3 : A motorboat goes upstream on a river and covers the distance between two towns on the riverbank in six hours. It covers this distance downstream in five hours. If the speed of the stream is 2 km/h, find the speed of the boat in still water.

Solution : Step 1 : Formulation : We know the speed of the river and the time taken to cover the distance between two places. We have to find the speed of the boat in still water.

Mathematical Description : Let us write x for the speed of the boat, t for the time taken and y for the distance travelled. Then

$$y = tx \qquad (1)$$

Let d be the distance between the two places.

While going upstream, the actual speed of the boat

= speed of the boat – speed of the river,

because the boat is travelling against the flow of the river.

So, the speed of the boat upstream = $(x - 2)$ km/h

It takes 6 hours to cover the distance between the towns upstream. So, from (1), we get $$d = 6(x - 2) \qquad (2)$$

When going downstream, the speed of the river has to be *added* to the speed of the boat.

So, the speed of the boat downstream = $(x + 2)$ km/h

The boat takes 5 hours to cover the same distance downstream. So,

$$d = 5(x + 2) \quad (3)$$

From (2) and (3), we have

$$5(x + 2) = 6(x - 2) \quad (4)$$

Step 2 : Finding the Solution

Solving for x in Equation (4), we get $x = 22$.

Step 3 : Interpretation

Since $x = 22$, therefore the speed of the motorboat in still water is 22 km/h.

In the example above, we know that the speed of the river is not the same everywhere. It flows slowly near the shore and faster at the middle. The boat starts at the shore and moves to the middle of the river. When it is close to the destination, it will slow down and move closer to the shore. So, there is a small difference between the speed of the boat at the middle and the speed at the shore. Since it will be close to the shore for a small amount of time, this difference in speed of the river will affect the speed only for a small period. So, we can ignore this difference in the speed of the river. We can also ignore the small variations in speed of the boat. Also, apart from the speed of the river, the friction between the water and surface of the boat will also affect the actual speed of the boat. We also assume that this effect is very small.

So, we have assumed that

1. The speed of the river and the boat remains constant all the time.
2. The effect of friction between the boat and water and the friction due to air is negligible.

We have found the speed of the boat in still water with the *assumptions* (*hypotheses*) above.

As we have seen in the word problems above, there are 3 *steps in solving a word problem. These are*

1. **Formulation :** We analyse the problem and see which factors have a major influence on the solution to the problem. These are the **relevant factors**. In our first example, the relevant factors are the distance travelled and petrol consumed. We ignored the other factors like the nature of the route, driving speed, etc. Otherwise, the problem would have been more difficult to solve. The factors that we ignore are the **irrelevant factors**.

We then describe the problem mathematically, in the form of one or more mathematical equations.

2. **Solution :** We find the solution of the problem by solving the mathematical equations obtained in Step 1 using some suitable method.
3. **Interpretation :** We see what the solution obtained in Step 2 means in the context of the original word problem.

Here are some exercises for you. You may like to check your understanding of the steps involved in solving word problems by carrying out the three steps above for the following problems.

EXERCISE A 2.1

In each of the following problems, clearly state what the relevant and irrelevant factors are while going through Steps 1, 2 and 3 given above.

1. Suppose a company needs a computer for some period of time. The company can either hire a computer for ₹ 2,000 per month or buy one for ₹ 25,000. If the company has to use the computer for a long period, the company will pay such a high rent, that buying a computer will be cheaper. On the other hand, if the company has to use the computer for say, just one month, then hiring a computer will be cheaper. Find the number of months beyond which it will be cheaper to buy a computer.
2. Suppose a car starts from a place A and travels at a speed of 40 km/h towards another place B. At the same instance, another car starts from B and travels towards A at a speed of 30 km/h. If the distance between A and B is 100 km, after how much time will the cars meet?
3. The moon is about 3,84,000 km from the earth, and its path around the earth is nearly circular. Find the speed at which it orbits the earth, assuming that it orbits the earth in 24 hours. (Use $\pi = 3.14$)
4. A family pays ₹ 1000 for electricity on an average in those months in which it does not use a water heater. In the months in which it uses a water heater, the average electricity bill is ₹ 1240. The cost of using the water heater is ₹ 8.00 per hour. Find the average number of hours the water heater is used in a day.

A2.3 Some Mathematical Models

So far, nothing was new in our discussion. In this section, we are going to add another step to the three steps that we have discussed earlier. This step is called *validation.* What does validation mean? Let us see. In a real-life situation, we cannot accept a model that gives us an answer that does not match the reality. This process of checking the answer against reality, and modifying the mathematical description if necessary, is called *validation*. This is a very important step in modelling. We will introduce you to this step in this section.

First, let us look at an example, where we do not have to modify our model after validation.

Example 4 : Suppose you have a room of length 6 m and breadth 5 m. You want to cover the floor of the room with square mosaic tiles of side 30 cm. How many tiles will you need? Solve this by constructing a mathematical model.

Solution : Formulation : We have to consider the area of the room and the area of a tile for solving the problem. The side of the tile is 0.3 m. Since the length is 6 m, we can fit in $\frac{6}{0.3} = 20$ tiles along the length of the room in one row (see Fig. A2.1.).

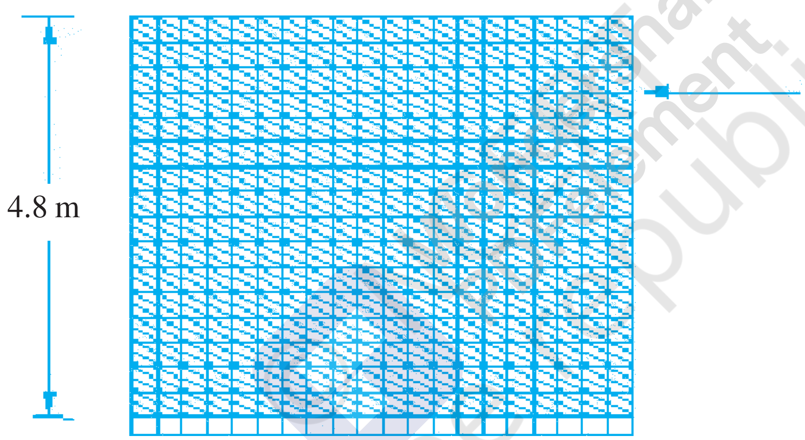


Fig. A2.1

Since the breadth of the room is 5 metres, we have $\frac{5}{0.3} = 16.67$. So, we can fit in 16 tiles in a column. Since $16 \times 0.3 = 4.8$, $5 - 4.8 = 0.2$ metres along the breadth will not be covered by tiles. This part will have to be covered by cutting the other tiles. The breadth of the floor left uncovered, 0.2 metres, is more than half the length of a tile, which is 0.3 m. So we cannot break a tile into two equal halves and use both the halves to cover the remaining portion.

Mathematical Description : We have:

Total number of tiles required = (Number of tiles along the length

× Number of tiles along the breadth) + Number of tiles along the uncovered area (1)

Solution : As we said above, the number of tiles along the length is 20 and the number of tiles along the breadth is 16. We need 20 more tiles for the last row. Substituting these values in (1), we get (20 × 16) + 20 = 320 + 20 = 340.

Interpretation : We need 340 tiles to cover the floor.

Validation : In real-life, your mason may ask you to buy some extra tiles to replace those that get damaged while cutting them to size. This number will of course depend upon the skill of your mason! But, we need not modify Equation (1) for this. This gives you a rough idea of the number of tiles required. So, we can stop here.

Let us now look at another situation now.

Example 5 : In the year 2000, 191 member countries of the U.N. signed a declaration. In this declaration, the countries agreed to achieve certain development goals by the year 2015. These are called the *millennium development goals*. One of these goals is to promote gender equality. One indicator for deciding whether this goal has been achieved is the ratio of girls to boys in primary, secondary and tertiary education. India, as a signatory to the declaration, is committed to improve this ratio. The data for the percentage of girls who are enrolled in primary schools is given in Table A2.1.

Table A2.1

Year	Enrolment (in %)
1991-92	41.9
1992-93	42.6
1993-94	42.7
1994-95	42.9
1995-96	43.1
1996-97	43.2
1997-98	43.5
1998-99	43.5
1999-2000	43.6*
2000-01	43.7*
2001-02	44.1*

Source : *Educational statistics, webpage of Department of Education, GOI.*

* *indicates that the data is provisional.*

Using this data, mathematically describe the rate at which the proportion of girls enrolled in primary schools grew. Also, estimate the year by which the enrolment of girls will reach 50%.

Solution : Let us first convert the problem into a mathematical problem.

Step 1 : Formulation : Table A2.1 gives the enrolment for the years 1991-92, 1992-93, etc. Since the students join at the beginning of an academic year, we can take the years as 1991, 1992, etc. Let us assume that the percentage of girls who join primary schools will continue to grow at the same rate as the rate in Table A2.1. So, the number of years is important, not the specific years. (To give a similar situation, when we find the simple interest for, say, ₹ 1500 at the rate of 8% for three years, it does not matter whether the three-year period is from 1999 to 2002 or from 2001 to 2004. What is important is the interest rate in the years being considered). Here also, we will see how the enrolment grows after 1991 by comparing the number of years that has passed after 1991 and the enrolment. Let us take 1991 as the 0th year, and write 1 for 1992 since 1 year has passed in 1992 after 1991. Similarly, we will write 2 for 1993, 3 for 1994, etc. So, Table A2.1 will now look like as Table A2.2.

Table A2.2

Year	Enrolment (in %)
0	41.9
1	42.6
2	42.7
3	42.9
4	43.1
5	43.2
6	43.5
7	43.5
8	43.6
9	43.7
10	44.1

The increase in enrolment is given in the following table :

Table A2.3

Year	Enrolment (in %)	Increase
0	41.9	0
1	42.6	0.7
2	42.7	0.1
3	42.9	0.2
4	43.1	0.2
5	43.2	0.1
6	43.5	0.3
7	43.5	0
8	43.6	0.1
9	43.7	0.1
10	44.1	0.4

At the end of the one-year period from 1991 to 1992, the enrolment has increased by 0.7% from 41.9% to 42.6%. At the end of the second year, this has increased by 0.1%, from 42.6% to 42.7%. From the table above, we cannot find a definite relationship between the number of years and percentage. But the increase is fairly steady. Only in the first year and in the 10th year there is a jump. The mean of the values is

$$\frac{0.7 + 0.1 + 0.2 + 0.2 + 0.1 + 0.3 + 0 + 0.1 + 0.1 + 0.4}{10} = 0.22$$

Let us assume that the enrolment steadily increases at the rate of 0.22 per cent.

Mathematical Description : We have assumed that the enrolment increases steadily at the rate of 0.22% per year.

So, the Enrolment Percentage (EP) in the first year = 41.9 + 0.22

EP in the second year = $41.9 + 0.22 + 0.22 = 41.9 + 2 \times 0.22$

EP in the third year = $41.9 + 0.22 + 0.22 + 0.22 = 41.9 + 3 \times 0.22$

So, the enrolment percentage in the nth year = $41.9 + 0.22n$, for $n \geq 1$. (1)

Now, we also have to find the number of years by which the enrolment will reach 50%. So, we have to find the value of n in the equation or formula

$$50 = 41.9 + 0.22n \qquad (2)$$

Step 2 : Solution : Solving (2) for n, we get

$$n = \frac{50 - 41.9}{0.22} = \frac{8.1}{0.22} = 36.8$$

Step 3 : Interpretation : Since the number of years is an integral value, we will take the next higher integer, 37. So, the enrolment percentage will reach 50% in 1991 + 37 = 2028.

In a word problem, we generally stop here. But, since we are dealing with a real-life situation, we have to see to what extent this value matches the real situation.

Step 4 : Validation: Let us check if Formula (2) is in agreement with the reality. Let us find the values for the years we already know, using Formula (2), and compare it with the known values by finding the difference. The values are given in Table A2.4.

Table A2.4

Year	Enrolment (in %)	Values given by (2) (in %)	Difference (in %)
0	41.9	41.90	0
1	42.6	42.12	0.48
2	42.7	42.34	0.36
3	42.9	42.56	0.34
4	43.1	42.78	0.32
5	43.2	43.00	0.20
6	43.5	43.22	0.28
7	43.5	43.44	0.06
8	43.6	43.66	–0.06
9	43.7	43.88	–0.18
10	44.1	44.10	0.00

As you can see, some of the values given by Formula (2) are less than the actual values by about 0.3% or even by 0.5%. This can give rise to a difference of about 3 to 5 years since the increase per year is actually 1% to 2%. We may decide that this much of a difference is acceptable and stop here. In this case, (2) is our mathematical model.

Suppose we decide that this error is quite large, and we have to improve this model. Then we have to go back to Step 1, the formulation, and change Equation (2). Let us do so.

Step 1 : Reformulation : We still assume that the values increase steadily by 0.22%, but we will now introduce a correction factor to reduce the error. For this, we find the mean of all the errors. This is

$$\frac{0+0.48+0.36+0.34+0.32+0.2+0.28+0.06-0.06-0.18+0}{10}=0.18$$

We take the mean of the errors, and correct our formula by this value.

Revised Mathematical Description : Let us now add the mean of the errors to our formula for enrolment percentage given in (2). So, our corrected formula is:

Enrolment percentage in the nth year = $41.9 + 0.22n + 0.18 = 42.08 + 0.22n$, for $n \geq 1$ (3)

We will also modify Equation (2) appropriately. The new equation for n is:

$$50 = 42.08 + 0.22n \quad (4)$$

Step 2 : Altered Solution : Solving Equation (4) for n, we get

$$n = \frac{50-42.08}{0.22} = \frac{7.92}{0.22} = 36$$

Step 3 : Interpretation: Since $n = 36$, the enrolment of girls in primary schools will reach 50% in the year 1991 + 36 = 2027.

Step 4 : Validation: Once again, let us compare the values got by using Formula (4) with the actual values. Table A2.5 gives the comparison.

Table A2.5

Year	Enrolment (in %)	Values given by (2)	Difference between values	Values given by (4)	Difference between values
0	41.9	41.90	0	41.9	0
1	42.6	42.12	0.48	42.3	0.3
2	42.7	42.34	0.36	42.52	0.18
3	42.9	42.56	0.34	42.74	0.16
4	43.1	42.78	0.32	42.96	0.14
5	43.2	43.00	0.2	43.18	0.02
6	43.5	43.22	0.28	43.4	0.1
7	43.5	43.44	0.06	43.62	– 0.12
8	43.6	43.66	– 0.06	43.84	– 0.24
9	43.7	43.88	– 0.18	44.06	– 0.36
10	44.1	44.10	0	44.28	– 0.18

As you can see, many of the values that (4) gives are closer to the actual value than the values that (2) gives. The mean of the errors is 0 in this case.

We will stop our process here. So, Equation (4) is our mathematical description that gives a mathematical relationship between years and the percentage of enrolment of girls of the total enrolment. We have constructed a mathematical model that describes the growth.

The process that we have followed in the situation above is called mathematical modelling.

We have tried to construct a mathematical model with the mathematical tools that we already have. There are better mathematical tools for making predictions from the data we have. But, they are beyond the scope of this course. Our aim in constructing this model is to explain the process of modelling to you, not to make accurate predictions at this stage.

You may now like to model some real-life situations to check your understanding of our discussion so far. Here is an Exercise for you to try.

EXERCISE A2.2

1. We have given the timings of the gold medalists in the 400-metre race from the time the event was included in the Olympics, in the table below. Construct a mathematical model relating the years and timings. Use it to estimate the timing in the next Olympics.

Table A2.6

Year	Timing (in seconds)
1964	52.01
1968	52.03
1972	51.08
1976	49.28
1980	48.88
1984	48.83
1988	48.65
1992	48.83
1996	48.25
2000	49.11
2004	49.41

A2.4 The Process of Modelling, its Advantages and Limitations

Let us now conclude our discussion by drawing out aspects of mathematical modelling that show up in the examples we have discussed. With the background of the earlier sections, we are now in a position to give a brief overview of the steps involved in modelling.

Step 1 : Formulation : You would have noticed the difference between the formulation part of Example 1 in Section A2.2 and the formulation part of the model we discussed in A2.3. In Example 1, all the information is in a readily usable form. But, in the model given in A2.3 this is not so. Further, it took us some time to find a mathematical description. We tested our first formula, but found that it was not as good as the second one we got. This is usually true in general, i.e. when trying to model real-life situations; the first model usually needs to be revised. When we are solving a real-life problem, formulation can require a lot of time. For example, Newton's three laws of motion, which are mathematical descriptions of motion, are simple enough to state. But, Newton arrived at these laws after studying a large amount of data and the work the scientists before him had done.

Formulation involves the following three steps :

(i) Stating the problem : Often, the problem is stated vaguely. For example, the broad goal is to ensure that the enrolment of boys and girls are equal. This may mean that 50% of the total number of boys of the school-going age and 50% of the girls of the school-going age should be enrolled. The other way is to ensure that 50% of the school-going children are girls. In our problem, we have used the second approach.

(ii) Identifying relevant factors : Decide which quantities and relationships are important for our problem and which are unimportant and can be neglected. For example, in our problem regarding primary schools enrolment, the percentage of girls enrolled in the previous year can influence the number of girls enrolled this year. This is because, as more and more girls enrol in schools, many more parents will feel they also have to put their daughters in schools. But, we have ignored this factor because this may become important only after the enrolment crosses a certain percentage. Also, adding this factor may make our model more complicated.

(iii) Mathematical Description : Now suppose we are clear about what the problem is and what aspects of it are more relevant than the others. Then we have to find a relationship between the aspects involved in the form of an equation, a graph or any other suitable mathematical description. If it is an equation, then every important aspect should be represented by a variable in our mathematical equation.

Step 2 : Finding the solution : The mathematical formulation does not give the solution. We have to solve this mathematical equivalent of the problem. This is where your mathematical knowledge comes in useful.

Step 3 : Interpreting the solution : The mathematical solution is some value or values of the variables in the model. We have to go back to the real-life problem and see what these values mean in the problem.

Step 4 : Validating the solution : As we saw in A2.3, after finding the solution we will have to check whether the solution matches the reality. If it matches, then the mathematical model is acceptable. If the mathematical solution does not match, *we go back to the formulation step* again and try to improve our model.

This step in the process is one major difference between solving word problems and mathematical modelling. This is one of the most important step in modelling that is missing in word problems. Of course, it is possible that in some real-life situations, we do not need to validate our answer because the problem is simple and we get the correct solution right away. This was so in the first model we considered in A2.3.

We have given a summary of the order in which the steps in mathematical modelling are carried out in Fig. A2.2 below. Movement from the validation step to the formulation step is shown using a **dotted arrow**. This is because it may not be necessary to carry out this step again.

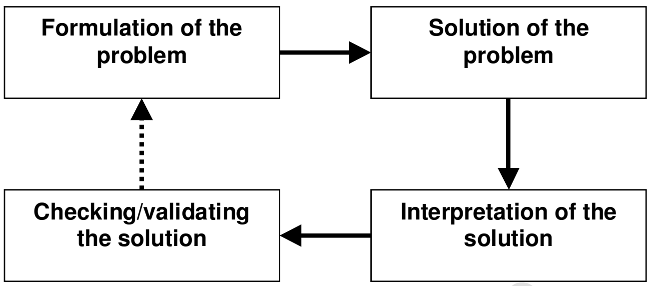


Fig.A2.2

Now that you have studied the stages involved in mathematical modelling, let us discuss some of its aspects.

The *aim* of mathematical modelling is to get some useful information about a real-world problem by converting it into a mathematical problem. This is especially useful when it is not possible or very expensive to get information by other means such as direct observation or by conducting experiments.

You may also wonder why we should undertake mathematical modelling? Let us look at some **advantages of modelling**. Suppose we want to study the corrosive effect of the discharge of the Mathura refinery on the Taj Mahal. We would not like to carry out experiments on the Taj Mahal directly since it may not be safe to do so. Of course, we can use a scaled down physical model, but we may need special facilities for this, which may be expensive. Here is where mathematical modelling can be of great use.

Again, suppose we want to know how many primary schools we will need after 5 years. Then, we can only solve this problem by using a mathematical model. Similarly, it is only through modelling that scientists have been able to explain the existence of so many phenomena.

You saw in Section A2.3, that we could have tried to improve the answer in the second example with better methods. But we stopped because we do not have the mathematical tools. This can happen in real-life also. Often, we have to be satisfied with very approximate answers, because mathematical tools are not available. For example, the model equations used in modelling weather are so complex that mathematical tools to find exact solutions are not available.

You may wonder to what extent we should try to improve our model. Usually, to improve it, we need to take into account more factors. When we do this, we add more variables to our mathematical equations. We may then have a very complicated model that is difficult to use. A model must be simple enough to use. A good model balances two factors:

1. Accuracy, i.e., how close it is to reality.
2. Ease of use.

For example, Newton's laws of motion are very simple, but powerful enough to model many physical situations.

So, is mathematical modelling the answer to all our problems? Not quite! It has its limitations.

Thus, we should keep in mind that a model is *only a simplification* of a real-world problem, and the two are not the same. It is something like the difference between a map that gives the physical features of a country, and the country itself. We can find the height of a place above the sea level from this map, but we cannot find the characteristics of the people from it. So, we should use a model only for the purpose it is supposed to serve, remembering all the factors we have neglected while constructing it. We should apply the model only within the limits where it is applicable. In the later classes, we shall discuss this aspect a little more.

EXERCISE A2.3

1. How are the solving of word problems that you come across in textbooks different from the process of mathematical modelling?

2. Suppose you want to minimise the waiting time of vehicles at a traffic junction of four roads. Which of these factors are important and which are not?

(i) Price of petrol.

(ii) The rate at which the vehicles arrive in the four different roads.

(iii) The proportion of slow-moving vehicles like cycles and rickshaws and fast moving vehicles like cars and motorcycles.

A2.5 Summary

In this Appendix, you have studied the following points :

1. The steps involved in solving word problems.

2. Construction of some mathematical models.

3. The steps involved in mathematical modelling given in the box below.

1. **Formulation :**
   (i) Stating the question
   (ii) Identifying the relevant factors
   (iii) Mathematical description
2. **Finding the solution.**
3. **Interpretation of the solution in the context of the real-world problem.**
4. **Checking/validating to what extent the model is a good representation of the problem being studied.**

4. The aims, advantages and limitations of mathematical modelling.

ANSWERS/HINTS

EXERCISE 1.1

1. Yes. $0 = \frac{0}{1} = \frac{0}{2} = \frac{0}{3}$ etc., denominator q can also be taken as negative integer.

2. There can be infinitely many rationals betwen numbers 3 and 4, one way is to take them

$3 = \frac{21}{6+1}, 4 = \frac{28}{6+1}$. Then the six numbers are $\frac{22}{7}, \frac{23}{7}, \frac{24}{7}, \frac{25}{7}, \frac{26}{7}, \frac{27}{7}$.

3. $\frac{3}{5} = \frac{30}{50}, \frac{4}{5} = \frac{40}{50}$. Therefore, five rationals are : $\frac{31}{50}, \frac{32}{50}, \frac{33}{50}, \frac{34}{50}, \frac{35}{50}$

4. (i) True, since the collection of whole numbers contains all the natural numbers.

(ii) False, for example -2 is not a whole number.

(iii) False, for example $\frac{1}{2}$ is a rational number but not a whole number.

EXERCISE 1.2

1. (i) True, since collection of real numbers is made up of rational and irrational numbers.

(ii) False, no negative number can be the square root of any natural number.

(iii) False, for example 2 is real but not irrational.

2. No. For example, $\sqrt{4} = 2$ is a rational number.

3. Repeat the procedure as in Fig. 1.8 several times. First obtain $\sqrt{4}$ and then $\sqrt{5}$.

EXERCISE 1.3

1. (i) 0.36, terminating. (ii) $0.\overline{09}$, non-terminating repeating.

(iii) 4.125, terminating. (iv) $0.\overline{230769}$, non-terminating repeating.

(v) $0.\overline{18}$ non-terminating repeating. (vi) 0.8225 terminating.

2. $\frac{2}{7} = 2 \times \frac{1}{7} = 0.\overline{285714}$, $\frac{3}{7} = 3 \times \frac{1}{7} = 0.\overline{428571}$, $\frac{4}{7} = 4 \times \frac{1}{7} = 0.\overline{571428}$,

$\frac{5}{7} = 5 \times \frac{1}{7} = 0.\overline{714285}$, $\frac{6}{7} = 6 \times \frac{1}{7} = 0.\overline{857142}$

3. (i) $\frac{2}{3}$ [Let $x = 0.666\ldots$ So $10x = 6.666\ldots$ or, $10x = 6 + x$ or, $x = \frac{6}{9} = \frac{2}{3}$]

(ii) $\frac{43}{90}$ (iii) $\frac{1}{999}$

4. 1 [Let $x = 0.9999\ldots$ So $10\,x = 9.999\ldots$ or, $10\,x = 9 + x$ or, $x = 1$]

5. $0.\overline{0588235294117647}$

6. The prime factorisation of q has only powers of 2 or powers of 5 or both.

7. 0.01001000100001. . ., 0.202002000200002. . ., 0.003000300003. . .

8. 0.75075007500075000075. . ., 0.767076700767000767. . ., 0.808008000800008. . .

9. (i) and (v) irrational; (ii), (iii) and (iv) rational.

EXERCISE 1.4

1. Proceed as in Section 1.4 for 2.665.

2. Proceed as in Example 11.

EXERCISE 1.5

1. (i) Irrational (ii) Rational (iii) Rational (iv) Irrational

(v) Irrational

2. (i) $6 + 3\sqrt{2} + 2\sqrt{3} + \sqrt{6}$ (ii) 6 (iii) $7 + 2\sqrt{10}$ (iv) 3

3. There is no contradiction. Remember that when you measure a length with a scale or any other device, you only get an approximate rational value. So, you may not realise that either c or d is irrational.

4. Refer Fig. 1.17.

5. (i) $\frac{\sqrt{7}}{7}$ (ii) $\sqrt{7}+\sqrt{6}$ (iii) $\frac{\sqrt{5}-\sqrt{2}}{3}$ (iv) $\frac{\sqrt{7}+2}{3}$

EXERCISE 1.6

1. (i) 8 (ii) 2 (iii) 5 **2.** (i) 27 (ii) 4 (iii) 8 (iv) $\frac{1}{5}\left[(125)^{-\frac{1}{3}} = \left(5^3\right)^{-\frac{1}{3}} = 5^{-1}\right]$

3. (i) $2^{\frac{13}{15}}$ (ii) 3^{-21} (iii) $11^{\frac{1}{4}}$ (iv) $56^{\frac{1}{2}}$

EXERCISE 2.1

1. (i) and (ii) are polynomials in one variable, (v) is a polynomial in three variables, (iii), (iv) are not polynomials, because in each of these exponent of the variable is not a whole number.

2. (i) 1 (ii) –1 (iii) $\frac{\pi}{2}$ (iv) 0

3. $3x^{35}-4$; $\sqrt{2}\,y^{100}$ (You can write some more polynomials with different coefficients.)

4. (i) 3 (ii) 2 (iii) 1 (iv) 0

5. (i) quadratic (ii) cubic (iii) quadratic (iv) linear
(v) linear (vi) quadratic (vii) cubic

EXERCISE 2.2

1. (i) 3 (ii) –6 (iii) –3

2. (i) 1, 1, 3 (ii) 2, 4, 4 (iii) 0, 1, 8 (iv) –1, 0, 3

3. (i) Yes (ii) No (iii) Yes (iv) Yes
(v) Yes (vi) Yes
(vii) $-\frac{1}{\sqrt{3}}$ is a zero, but $\frac{2}{\sqrt{3}}$ is not a zero of the polynomial (viii) No

4. (i) –5 (ii) 5 (iii) $\frac{-5}{2}$ (iv) $\frac{2}{3}$
(v) 0 (vi) 0 (vii) $-\frac{d}{c}$

EXERCISE 2.3

1. (i) 0 (ii) $\frac{27}{8}$ (iii) 1 (iv) $-\pi^3 + 3\pi^2 - 3\pi + 1$ (v) $-\frac{27}{8}$

2. $5a$ **3.** No, since remainder is not zero.

EXERCISE 2.4

1. $(x + 1)$ is a factor of (i), but not the factor of (ii), (iii) and (iv).

2. (i) Yes (ii) No (iii) Yes

3. (i) –2 (ii) $-\left(2+\sqrt{2}\right)$ (iii) $\sqrt{2}-1$ (iv) $\frac{3}{2}$

4. (i) $(3x - 1)(4x - 1)$ (ii) $(x + 3)(2x + 1)$ (iii) $(2x + 3)(3x - 2)$ (iv) $(x + 1)(3x - 4)$

5. (i) $(x - 2)(x - 1)(x + 1)$ (ii) $(x + 1)(x + 1)(x - 5)$

(iii) $(x + 1)(x + 2)(x + 10)$ (iv) $(y - 1)(y + 1)(2y + 1)$

EXERCISE 2.5

1. (i) $x^2 + 14x + 40$ (ii) $x^2 - 2x - 80$ (iii) $9x^2 - 3x - 20$

(iv) $y^4 - \frac{9}{4}$ (v) $9 - 4x^2$

2. (i) 11021 (ii) 9120 (iii) 9984

3. (i) $(3x + y)(3x + y)$ (ii) $(2y - 1)(2y - 1)$ (iii) $\left(x+\frac{y}{10}\right)\left(x-\frac{y}{10}\right)$

4. (i) $x^2 + 4y^2 + 16z^2 + 4xy + 16yz + 8xz$

(ii) $4x^2 + y^2 + z^2 - 4xy - 2yz + 4xz$

(iii) $4x^2 + 9y^2 + 4z^2 - 12xy + 12yz - 8xz$

(iv) $9a^2 + 49b^2 + c^2 - 42ab + 14bc - 6ac$

(v) $4x^2 + 25y^2 + 9z^2 - 20xy - 30yz + 12xz$

(vi) $\frac{a^2}{16}+\frac{b^2}{4}+1-\frac{ab}{4}-b+\frac{a}{2}$

5. (i) $(2x + 3y - 4z)(2x + 3y - 4z)$ (ii) $\left(-\sqrt{2}\,x+y+2\sqrt{2}\,z\right)\left(-\sqrt{2}\,x+y+2\sqrt{2}\,z\right)$

6. (i) $8x^3 + 12x^2 + 6x + 1$ (ii) $8a^3 - 27b^3 - 36a^2b + 54ab^2$

(iii) $\frac{27}{8}x^3 + \frac{27}{4}x^2 + \frac{9}{2}x + 1$ (iv) $x^3 - \frac{8}{27}y^3 - 2x^2y + \frac{4xy^2}{3}$

7. (i) 970299 (ii) 1061208 (iii) 994011992

8. (i) $(2a+b)(2a+b)(2a+b)$ (ii) $(2a-b)(2a-b)(2a-b)$

(iii) $(3-5a)(3-5a)(3-5a)$ (iv) $(4a-3b)(4a-3b)(4a-3b)$

(v) $\left(3p-\frac{1}{6}\right)\left(3p-\frac{1}{6}\right)\left(3p-\frac{1}{6}\right)$

10. (i) $(3y+5z)(9y^2+25z^2-15yz)$ (ii) $(4m-7n)(16m^2+49n^2+28mn)$

11. $(3x+y+z)(9x^2+y^2+z^2-3xy-yz-3xz)$

12. Simiplify RHS.

13. Put $x+y+z=0$ in Identity VIII.

14. (i) -1260. Let $a=-12$, $b=7$, $c=5$. Here $a+b+c=0$. Use the result given in Q13.

(ii) 16380

15. (i) One possible answer is : Length $= 5a-3$, Breadth $= 5a-4$

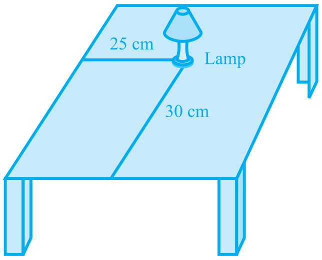
(ii) One possible answer is : Length $= 7y-3$, Breadth $= 5y+4$

16. (i) One possible answer is : 3, x and $x-4$.

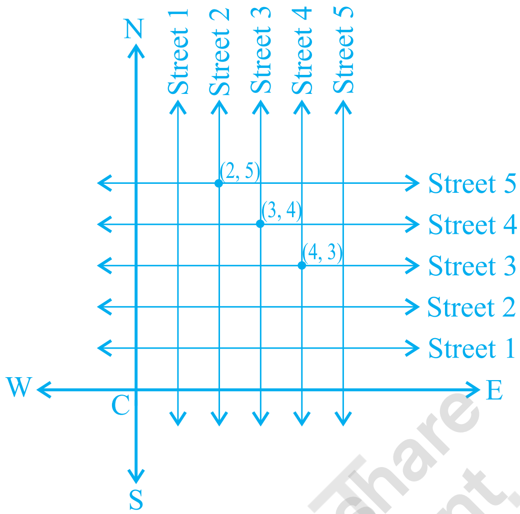
(ii) One possible answer is : $4k$, $3y+5$ and $y-1$.

EXERCISE 3.1

1. Consider the lamp as a point and table as a plane. Choose any two perpendicular edges of the table. Measure the distance of the lamp from the longer edge, suppose it is 25 cm. Again, measure the distance of the lamp from the shorter edge, and suppose it is 30 cm. You can write the position of the lamp as (30, 25) or (25, 30), depending on the order you fix.



2. The Street plan is shown in figure given below.



Both the cross-streets are marked in the figure above. They are *uniquely* found because of the two reference lines we have used for locating them.

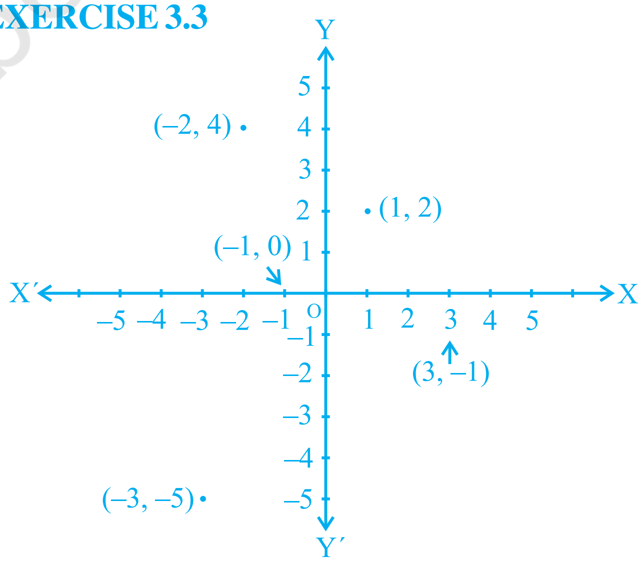
EXERCISE 3.2

1. (i) The x - axis and the y - axis (ii) Quadrants (iii) The origin

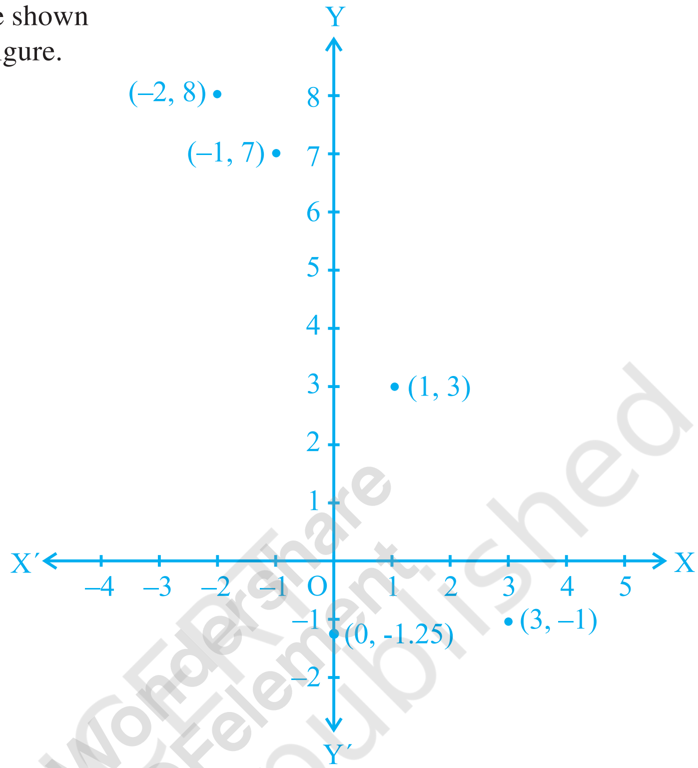
2. (i) (– 5, 2) (ii) (5, –5) (iii) E (iv) G (v) 6 (vi) – 3 (vii) (0, 5) (viii) (–3, 0)

EXERCISE 3.3

1. The point (–2, 4) lies in quadrant II, the point (3, –1) lies in the quadrant IV, the point (– 1, 0) lies on the negative x - axis, the point (1, 2) lies in the quadrant I and the point (–3, –5) lies in the quadrant III. Locations of the points are shown in the adjoining figure.



2. Positions of the points are shown by dots in the adjoining figure.



EXERCISE 4.1

1. $x - 2y = 0$

2. (i) $2x + 3y - 9.3\overline{5} = 0; a = 2, b = 3, c = - 9.3\overline{5}$

(ii) $x - \frac{y}{5} - 10 = 0; a = 1, b = \frac{-1}{5}, c = -10$

(iii) $-2x + 3y - 6 = 0; a = -2, b = 3, c = -6$

(iv) $1.x - 3y + 0 = 0; a = 1, b = -3, c = 0$

(v) $2x + 5y + 0 = 0; a = 2, b = 5, c = 0$

(vi) $3x + 0.y + 2 = 0; a = 3, b = 0, c = 2$

(vii) $0.x + 1.y - 2 = 0; a = 0, b = 1, c = -2$

(viii) $-2x + 0.y + 5 = 0; a = -2, b = 0, c = 5$

EXERCISE 4.2

1. (iii), because for every value of x, there is a corresponding value of y and vice-versa.

2. (i) (0, 7), (1, 5), (2, 3), (4, –1)

(ii) $(1, 9-\pi), (0, 9), (-1, 9+\pi), \left(\frac{9}{\pi}, 0\right)$

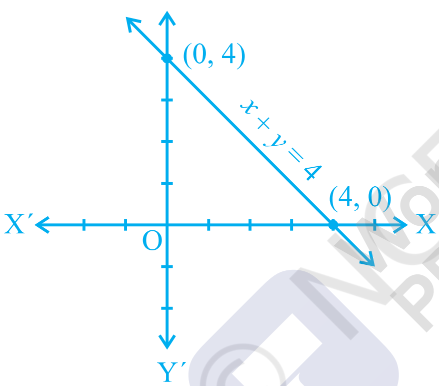
(iii) $(0, 0), (4, 1), (-4, 1), \left(2, \frac{1}{2}\right)$

3. (i) No (ii) No (iii) Yes (iv) No (v) No

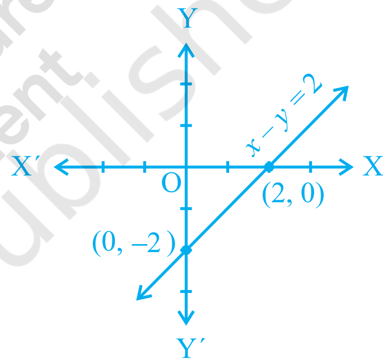
4. 7

EXERCISE 4.3

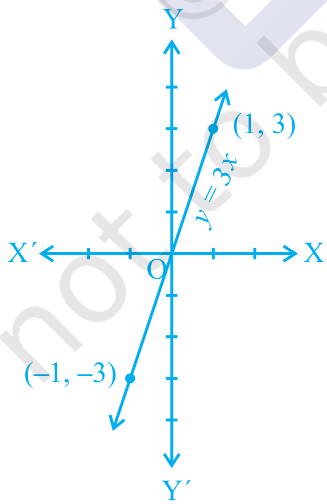
1. (i)



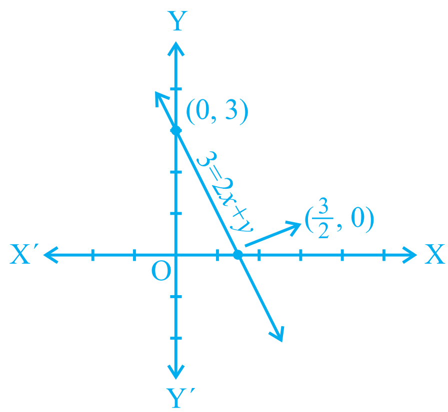
(ii)



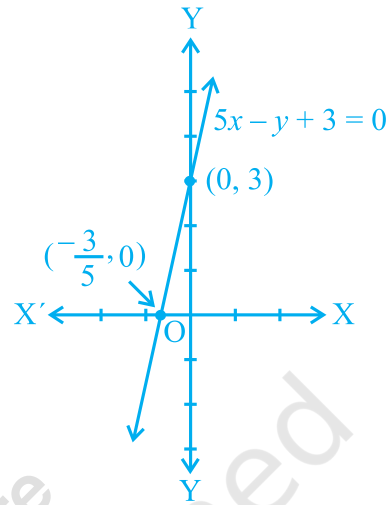
(iii)



(iv)



2. $7x - y = 0$ and $x + y = 16$; infintely many [Through a point infinitely many lines can be drawn]



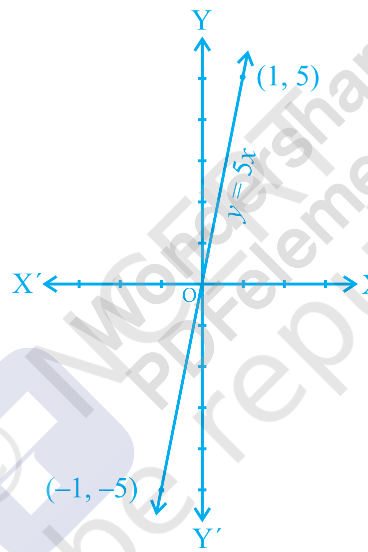
3. $\frac{5}{3}$

4. $5x - y + 3 = 0$

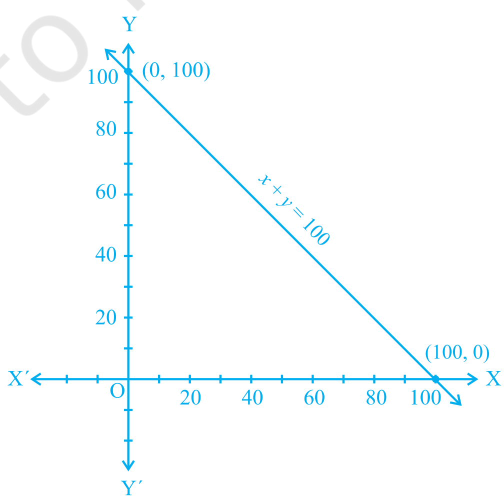
5. For Fig. 4.6, $x + y = 0$ and for Fig. 4.7, $y = -x + 2$.

6. Supposing x is the distance and y is the work done. Therefore according to the problem the equation will be $y = 5x$.

(i) 10 units (ii) 0 unit



7. $x + y = 100$



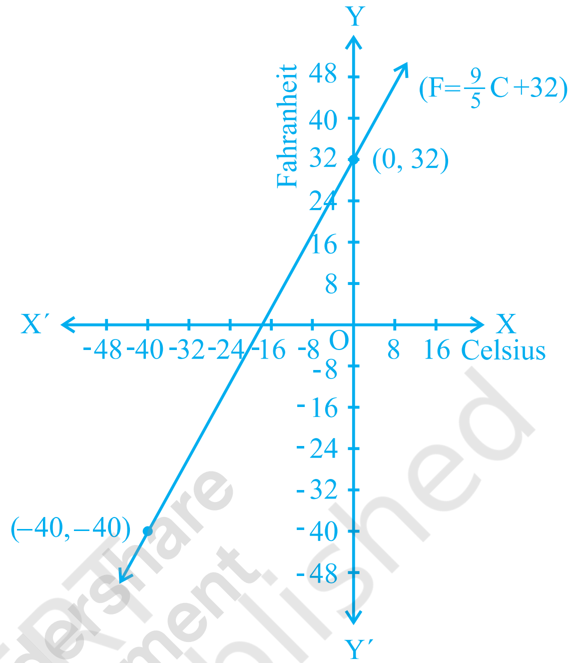
8. (i) See adjacent figure.

(ii) 86° F

(iii) 35° C

(iv) 32° F, –17.8° C (approximately)

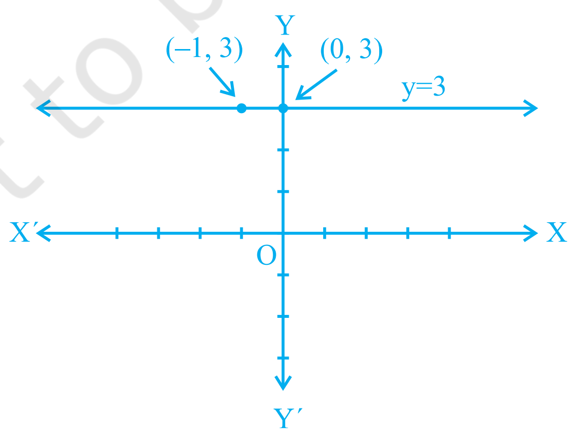
(v) Yes, – 40° (both in F and C)



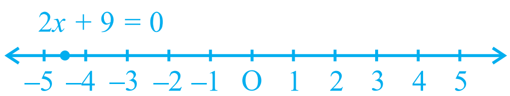
EXERCISE 4.4

1. (i)

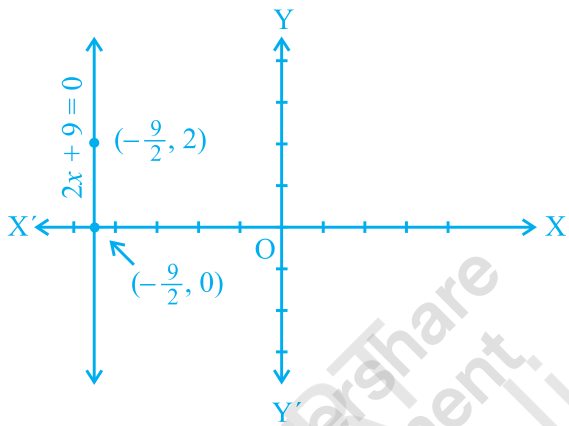
(ii)



2. (i)



(ii)



EXERCISE 5.1

1. (i) False. This can be seen visually by the student.

(ii) False. This contradicts Axiom 5.1.

(iii) True. (Postulate 2)

(iv) True. If you superimpose the region bounded by one circle on the other, then they coincide. So, their centres and boundaries coincide. Therefore, their radii will coincide.

(v) True. The first axiom of Euclid.

3. There are several undefined terms which the student should list. They are consistent, because they deal with two different situations — (i) says that given two points A and B, there is a point C lying on the line in between them; (ii) says that given A and B, you can take C not lying on the line through A and B.

These 'postulates' do not follow from Euclid's postulates. However, they follow from Axiom 5.1.

4. $\text{AC} = \text{BC}$

So, $\text{AC} + \text{AC} = \text{BC} + \text{AC}$ (Equals are added to equals)

i.e., $2\text{AC} = \text{AB}$ (BC + AC coincides with AB)

Therefore, $\text{AC} = \frac{1}{2}\text{AB}$

5. Make a temporary assumption that different points C and D are two mid-points of AB. Now, you show that points C and D are not two different points.

6.
$$AC = BD \qquad \text{(Given)} \quad (1)$$
$$AC = AB + BC \qquad \text{(Point B lies between A and C)} \quad (2)$$
$$BD = BC + CD \qquad \text{(Point C lies between B and D)} \quad (3)$$

Substituting (2) and (3) in (1), you get

$$AB + BC = BC + CD$$

So, $AB = CD$ (Subtracting equals from equals)

7. Since this is true for any thing in any part of the world, this is a universal truth.

EXERCISE 5.2

1. Any formulation the student gives should be discussed in the class for its validity.

2. If a straight line l falls on two straight lines m and n such that sum of the interior angles on one side of l is two right angles, then by Euclid's fifth postulate the line will not meet on this side of l. Next, you know that the sum of the interior angles on the other side of line l will also be two right angles. Therefore, they will not meet on the other side also. So, the lines m and n never meet and are, therefore, parallel.

EXERCISE 6.1

1. 30°, 250° **2.** 126° **4.** Sum of all the angles at a point = 360°

5. $\angle QOS = \angle SOR + \angle ROQ$ and $\angle POS = \angle POR - \angle SOR$. **6.** 122°, 302°

EXERCISE 6.2

1. 130°, 130° **2.** 126° **3.** 126°, 36°, 54° **4.** 60° **5.** 50°, 77°

6. Angle of incidence = Angle of reflection. At point B, draw $BE \perp PQ$ and at point C, draw $CF \perp RS$.

EXERCISE 6.3

1. 65° **2.** 32°, 121° **3.** 92° **4.** 60° **5.** 37°, 53°

6. Sum of the angles of ΔPQR = Sum of the angles of ΔQTR and $\angle PRS = \angle QPR + \angle PQR$.

EXERCISE 7.1

1. They are equal. **6.** $\angle BAC = \angle DAE$

EXERCISE 7.2

6. $\angle$ BCD = $\angle$ BCA + $\angle$ DCA = $\angle$ B + $\angle$ D **7.** each is of 45°

EXERCISE 7.3

3. **(ii)** From (i), $\angle$ ABM = $\angle$ PQN

EXERCISE 7.4

4. Join BD and show $\angle$ B > $\angle$ D. Join AC and show $\angle$ A > $\angle$ C.

5. $\angle$ Q + $\angle$ QPS > $\angle$ R + $\angle$ RPS, etc.

EXERCISE 8.1

1. 36°, 60°, 108° and 156°.

6. (i) From Δ DAC and Δ BCA, show $\angle$ DAC = $\angle$ BCA and $\angle$ ACD = $\angle$ CAB, etc.

(ii) Show $\angle$ BAC = $\angle$ BCA, using Theorem 8.4.

EXERCISE 8.2

2. Show PQRS is a parallelogram. Also show PQ || AC and PS || BD. So, $\angle$ P = 90°.

5. AECF is a parallelogram. So, AF || CE, etc.

EXERCISE 9.1

1. (i) Base DC, parallels DC and AB; (iii) Base QR, parallels QR and PS;

(v) Base AD, parallels AD and BQ

EXERCISE 9.2

1. 12.8 cm. **2.** Join EG; Use result of Example 2.

6. Wheat in Δ APQ and pulses in other two triangles or pulses in Δ APQ and wheat in other two triangles.

EXERCISE 9.3

4. Draw CM ⊥ AB and DN ⊥ AB. Show CM = DN. **12.** See Example 4.

EXERCISE 9.4 (Optional)

7. Use result of Example 3 repeatedly.

EXERCISE 10.1

1. (i) Interior (ii) Exterior (iii) Diameter
   (iv) Semicircle (v) The chord (vi) Three
2. (i) True (ii) False (iii) False
   (iv) True (v) False (vi) True

EXERCISE 10.2

1. Prove exactly as Theorem 10.1 by considering chords of congruent circles.
2. Use SAS axiom of congruence to show the congruence of the two triangles.

EXERCISE 10.3

1. 0, 1, 2. Two
2. Proceed as in Example 1.
3. Join the centres O, O′ of the circles to the mid-point M of the common chord AB. Then, show $\angle$ OMA = 90° and $\angle$ O′MA = 90°.

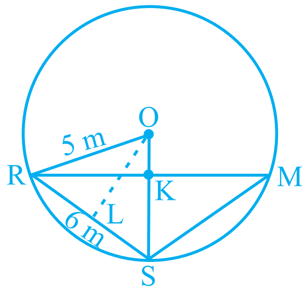
EXERCISE 10.4

1. 6 cm. First show that the line joining centres is perpendicular to the radius of the smaller circle and then that common chord is the diameter of the smaller circle.
2. If AB, CD are equal chords of a circle with centre O intersecting at E, draw perpendiculars OM on AB and ON on CD and join OE. Show that right triangles OME and ONE are congruent.
3. Proceed as in Example 2.
4. Draw perpendicular OM on AD.
5. Represent Reshma, Salma and Mandip by R, S and M respectively. Let KR = x m (see figure).

   Area of Δ ORS $= \frac{1}{2} x \times 5$. Also, area of Δ ORS =

   $\frac{1}{2}$ RS × OL $= \frac{1}{2} \times 6 \times 4$.

   Find x and hence RM.



6. Use the properties of an equilateral triangle and also Pythagoras Theorem.

EXERCISE 10.5

1. 45°
2. 150°, 30°
3. 10°
4. 80°
5. 110°
6. $\angle BCD = 80°$ and $\angle ECD = 50°$
8. Draw perpendiculars AM and BN on CD (AB || CD and AB < CD). Show $\Delta AMD \cong \Delta BNC$. This gives $\angle C = \angle D$ and, therefore, $\angle A + \angle C = 180°$.

EXERCISE 10.6 (Optional)

2. Let O be the centre of the circle. Then perpendicular bisector of both the chords will be same and passes through O. Let r be the radius, then $r^2 = \left(\frac{11}{2}\right)^2 + x^2$ $= \left(\frac{5}{2}\right)^2 + (6 - x)^2$, where x is length of the perpendicular from O on the chord of length 11 cm. This gives $x = 1$. So, $r = \frac{5\sqrt{5}}{2}$ cm.
3. 3 cm.
4. Let $\angle AOC = x$ and $\angle DOE = y$. Let $\angle AOD = z$. Then $\angle EOC = z$ and $x + y + 2z = 360°$.

   $\angle ODB = \angle OAD + \angle DOA = 90° - \frac{1}{2}z + z = 90° + \frac{1}{2}z$. Also $\angle OEB = 90° + \frac{1}{2}z$

8. $\angle ABE = \angle ADE$, $\angle ADF = \angle ACF = \frac{1}{2}\angle C$.

   Therefore, $\angle EDF = \angle ABE + \angle ADF = \frac{1}{2}(\angle B + \angle C) = \frac{1}{2}(180° - \angle A) = 90° - \frac{1}{2}\angle A$.

9. Use Q. 1, Ex. 10.2 and Theorem 10.8.
10. Let angle-bisector of $\angle A$ intersect circumcircle of ΔABC at D. Join DC and DB. Then $\angle BCD = \angle BAD = \frac{1}{2}\angle A$ and $\angle DBC = \angle DAC = \frac{1}{2}\angle A$. Therefore, $\angle BCD = \angle DBC$ or, DB = DC. So, D lies on the perpendicular bisector of BC.

EXERCISE 12.1

1. $\frac{\sqrt{3}}{4}a^2$, 900, 3 cm^2
2. ₹ 1650000
3. $20\sqrt{2}\ m^2$
4. $21\sqrt{11}\ cm^2$
5. 9000 cm^2
6. $9\sqrt{15}\ cm^2$

EXERCISE 12.2

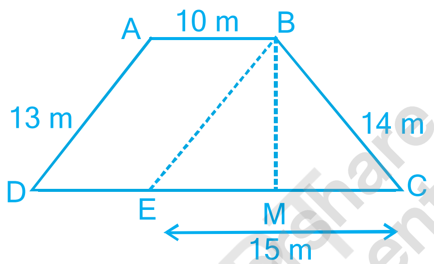
1. 65.5 m^2 (approx.) **2.** 15.2 cm^2 (approx.) **3.** 19.4 cm^2 (approx.)

4. 12 cm **5.** 48 m^2 **6.** $1000\sqrt{6}$ cm^2, $1000\sqrt{6}$ cm^2

7. Area of shade I = Area of shade II = 256 cm^2 and area of shade III = 17.92 cm^2

8. ₹ 705.60 **9.** 196 m^2

[See the figure. Find area of Δ BEC = 84 m^2, then find the height BM.]



EXERCISE 13.1

1. (i) 5.45 m^2 (ii) ₹ 109 **2.** ₹ 555 **3.** 6 m **4.** 100 bricks.

5. (i) Lateral surface area of cubical box is greater by 40 cm^2.

(ii) Total surface area of cuboidal box is greater by 10 cm^2.

6. (i) 4250 cm^2 of glass (ii) 320 cm of tape. [Calculate the sum of all the edges (The 12 edges consist of 4 lengths, 4 breadths and 4 heights)].

7. ₹ 2184 **8.** 47 m^2

EXERCISE 13.2

1. 2 cm **2.** 7.48 m^2 **3.** (i) 968 cm^2 (ii) 1064.8 cm^2 (iii) 2038.08 cm^2

[Total surface area of a pipe is (inner curved surface area + outer curved surface area + areas of the two bases). Each base is a ring of area given by π ($R^2 - r^2$), where R = outer radius and r = inner radius].

4. 1584 m^2 **5.** ₹ 68.75 **6.** 1 m

7. (i) 110 m^2 (ii) ₹ 4400 **8.** 4.4 m^2

9. (i) 59.4 m^2 (ii) 95.04 m^2

[Let the actual area of steel used be x m^2. Since $\frac{1}{12}$ of the actual steel used was wasted, the area of steel which has gone into the tank = $\frac{11}{12}$ of x. This means that the actual area of steel used = $\frac{12}{11} \times 87.12\,\text{m}^2$]

10. 2200 cm²; Height of the cylinder should be treated as (30 + 2.5 + 2.5) cm

11. 7920 cm²

EXERCISE 13.3

1. 165 cm²
2. 1244.57 m²
3. (i) 7 cm (ii) 462 cm²
4. (i) 26 m (ii) ₹ 137280
5. 63 m
6. ₹ 1155
7. 5500 cm²
8. ₹ 384.34 (approx.)

EXERCISE 13.4

1. (i) 1386 cm² (ii) 394.24 cm² (iii) 2464 cm²
2. (i) 616 cm² (ii) 1386 cm² (iii) 38.5 m²
3. 942 cm²
4. 1 : 4
5. ₹ 27.72
6. 3.5 cm
7. 1 : 16
8. 173.25 cm²
9. (i) $4\pi r^2$ (ii) $4\pi r^2$ (iii) 1 : 1

EXERCISE 13.5

1. 180 cm³
2. 135000 litres
3. 4.75 m
4. ₹ 4320
5. 2 m
6. 3 days
7. 16000
8. 6 cm, 4 : 1
9. 4000 m³

EXERCISE 13.6

1. 34.65 litres

2. 3.432 kg [Volume of a pipe = $\pi h \times (R^2 - r^2)$, where R is the outer radius and r is the inner radius].

3. The cylinder has the greater capacity by 85 cm³.

4. (i) 3 cm (ii) 141.3 cm³

5. (i) 110 m² (ii) 1.75 m (iii) 96.25 *kl*

6. 0.4708 m²

7. Volume of wood = 5.28 cm³, Volume of graphite = 0.11 cm³.

8. 38500 cm³ or 38.5 *l* of soup

EXERCISE 13.7

1. (i) 264 cm^3 (ii) 154 cm^3 **2.** (i) 1.232 *l* (ii) $\frac{11}{35}$ *l*

3. 10 cm **4.** 8 cm **5.** 38.5 *kl*

6. (i) 48 cm (ii) 50 cm (iii) 2200 cm^2 **7.** 100π cm^3 **8.** 240π cm^3; 5 : 12

9. 86.625x m^3, 99.825 m^2

EXERCISE 13.8

1. (i) $1437\frac{1}{3}$ cm^3 (ii) 1.05 m^3 (approx.)

2. (i) $11498\frac{2}{3}$ cm^3 (ii) 0.004851 m^3 **3.** 345.39 g (approx.) **4.** $\frac{1}{64}$

5. 0.303*l* (approx.) **6.** 0.06348 m^3 (approx.)

7. $179\frac{2}{3}$ cm^3 **8.** (i) 249.48 m^2 (ii) 523.9 m^3 (approx.) **9.** (i) 3*r* (ii) 1 : 9

10. 22.46 mm^3 (approx.)

EXERCISE 13.9 (Optional)

1. ₹ 6275

2. ₹ 2784.32 (approx.) [Rememeber to subtract the part of the sphere that is resting on the support while calculating the cost of silver paint]. **3.** 43.75%

EXERCISE 14.1

1. Five examples of data that we can gather from our day-to-day life are :

(i) Number of students in our class.

(ii) Number of fans in our school.

(iii) Electricity bills of our house for last two years.

(iv) Election results obtained from television or newspapers.

(v) Literacy rate figures obtained from Educational Survey.

Of course, remember that there can be many more different answers.

2. Primary data; (i), (ii) and (iii)

Secondary data; (iv) and (v)

EXERCISE 14.2

1.

Blood group	Number of students
A	9
B	6
O	12
AB	3
Total	30

Most common – O , Rarest – AB

2.

Distances (in km)	Tally Marks	Frequency
0 - 5	𝍸	5
5 - 10	𝍸 𝍸 |	11
10 - 15	𝍸 𝍸 |	11
15 - 20	𝍸 ||||	9
20 - 25	|	1
25 - 30	|	1
30 - 35	||	2
Total		40

3. (i)

Relative humidity (in %)	Frequency
84 - 86	1
86 - 88	1
88 - 90	2
90 - 92	2
92 - 94	7
94 - 96	6
96 - 98	7
98 - 100	4
Total	30

(ii) The data appears to be taken in the rainy season as the relative humidity is high.

(iii) Range = 99.2 – 84.9 = 14.3

4. (i)

Heights (in cm)	Frequency
150 - 155	12
155 - 160	9
160 - 165	14
165 - 170	10
170 - 175	5
Total	50

(ii) One conclusion that we can draw from the above table is that more than 50% of students are shorter than 165 cm.

5. (i)

Concentration of Sulphur dioxide (in ppm)	Frequency
0.00 - 0.04	4
0.04 - 0.08	9
0.08 - 0.12	9
0.12 - 0.16	2
0.16 - 0.20	4
0.20 - 0.24	2
Total	30

(ii) The concentration of sulphur dioxide was more than 0.11 ppm for 8 days.

6.

Number of heads	Frequency
0	6
1	10
2	9
3	5
Total	30

7. (i)

Digits	Frequency
0	2
1	5
2	5
3	8
4	4
5	5
6	4
7	4
8	5
9	8
Total	50

(ii) The most frequently occurring digits are 3 and 9. The least occurring is 0.

8. (i)

Number of hours	Frequency
0 - 5	10
5 - 10	13
10 - 15	5
15 - 20	2
Total	30

(ii) 2 children.

9.

Life of batteries (in years)	Frequency
2.0 - 2.5	2
2.5 - 3.0	6
3.0 - 3.5	14
3.5 - 4.0	11
4.0 - 4.5	4
4.5 - 5.0	3
Total	40

EXERCISE 14.3

1. (ii) Reproductive health conditions.

3. (ii) Party A **4.** (ii) Frequency polygon (iii) No **5.** (ii) 184

8.

Age (in years)	Frequency	Width	Length of the rectangle
1 - 2	5	1	$\frac{5}{1} \times 1 = 5$
2 - 3	3	1	$\frac{3}{1} \times 1 = 3$
3 - 5	6	2	$\frac{6}{2} \times 1 = 3$
5 - 7	12	2	$\frac{12}{2} \times 1 = 6$
7 - 10	9	3	$\frac{9}{3} \times 1 = 3$
10 - 15	10	5	$\frac{10}{5} \times 1 = 2$
15 - 17	4	2	$\frac{4}{2} \times 1 = 2$

Now, you can draw the histogram, using these lengths.

9. (i)

Number of letters	Frequency	Width of interval	Length of rectangle
1 - 4	6	3	$\frac{6}{3} \times 2 = 4$
4 - 6	30	2	$\frac{30}{2} \times 2 = 30$
6 - 8	44	2	$\frac{44}{2} \times 2 = 44$
8 - 12	16	4	$\frac{16}{4} \times 2 = 8$
12 - 20	4	8	$\frac{4}{8} \times 2 = 1$

Now, draw the histogram.

(ii) 6 - 8

EXERCISE 14.4

1. Mean = 2.8; Median = 3; Mode = 3
2. Mean = 54.8; Median = 52; Mode = 52
3. $x = 62$ **4.** 14
5. Mean salary of 60 workers is ₹ 5083.33.

EXERCISE 15.1

1. $\frac{24}{30}$, i.e., $\frac{4}{5}$ **2.** (i) $\frac{19}{60}$ (ii) $\frac{407}{750}$ (iii) $\frac{211}{1500}$ **3.** $\frac{3}{20}$ **4.** $\frac{9}{25}$

5. (i) $\frac{29}{2400}$ (ii) $\frac{579}{2400}$ (iii) $\frac{1}{240}$ (iv) $\frac{1}{96}$ (v) $\frac{1031}{1200}$ **6.** (i) $\frac{7}{90}$ (ii) $\frac{23}{90}$

7. (i) $\frac{27}{40}$ (ii) $\frac{13}{40}$ **8.** (i) $\frac{9}{40}$ (ii) $\frac{31}{40}$ (iii) 0 **11.** $\frac{7}{11}$ **12.** $\frac{1}{15}$ **13.** $\frac{1}{10}$

EXERCISE A1.1

1. (i) False. There are 12 months in a year.
(ii) Ambiguous. In a given year, Diwali may or may not fall on a Friday.
(iii) Ambiguous. At some time in the year, the temperature in Magadi, may be 26° C.
(iv) Always true.
(v) False. Dogs cannot fly.
(vi) Ambiguous. In a leap year, February has 29 days.

2. (i) False. The sum of the interior angles of a quadrilateral is 360°.
(ii) True (iii) True (iv) True
(v) False, for example, $7 + 5 = 12$, which is not an odd number.

3. (i) All prime numbers greater than 2 are odd. (ii) Two times a natural number is always even. (iii) For any $x > 1$, $3x + 1 > 4$. (iv) For any $x \geq 0$, $x^3 \geq 0$.
(v) In an equilateral triangle, a median is also an angle bisector.

EXERCISE A1.2

1. (i) Humans are vertebrates. (ii) No. Dinesh could have got his hair cut by anybody else. (iii) Gulag has a red tongue. (iv) We conclude that the gutters will have to be cleaned tomorrow. (v) All animals having tails need not be dogs. For example, animals such as buffaloes, monkeys, cats, etc. have tails but are not dogs.

2. You need to turn over B and 8. If B has an even number on the other side, then the rule has been broken. Similarly, if 8 has a consonant on the other side, then the rule has been broken.

EXERCISE A1.3

1. Three possible conjectures are:

(i) The product of any three consecutive even numbers is even. (ii) The product of any three consecutive even numbers is divisible by 4. (iii) The product of any three consecutive even numbers is divisible by 6.

2. Line 4: $1\ 3\ 3\ 1 = 11^3$; Line 5: $1\ 4\ 6\ 4\ 1 = 11^4$; the conjecture holds for Line 4 and Line 5; No, because $11^5 \neq 15101051$.

3. $T_4 + T_5 = 25 = 5^2$; $T_{n-1} + T_n = n^2$.

4. $111111^2 = 12345654321$; $1111111^2 = 1234567654321$

5. Student's own answer. For example, Euclid's postulates.

EXERCISE A1.4

1. (i) You can give any two triangles with the same angles but of different sides.

(ii) A rhombus has equal sides but may not be a square.

(iii) A rectangle has equal angles but may not be a square.

(iv) For $a = 3$ and $b = 4$, the statement is not true.

(v) For $n = 11$, $2n^2 + 11 = 253$ which is not a prime.

(vi) For $n = 41$, $n^2 - n + 41$ is not a prime.

2. Student's own answer.

3. Let x and y be two odd numbers. Then $x = 2m + 1$ for some natural number m and $y = 2n + 1$ for some natural number n.

$x + y = 2(m + n + 1)$. Therefore, $x + y$ is divisible by 2 and is even.

4. See Q.3. $xy = (2m + 1)(2n + 1) = 2(2mn + m + n) + 1$.

Therefore, $x\,y$ is not divisible by 2, and so it is odd.

5. Let $2n$, $2n + 2$ and $2n + 4$ be three consecutive even numbers. Then their sum is $6(n + 1)$, which is divisible by 6.

7. (i) Let your original number be n. Then we are doing the following operations:

$$n \to 2n \to 2n + 9 \to 2n + 9 + n = 3n + 9 \to \frac{3n + 9}{3} = n + 3 \to n + 3 + 4 = n + 7 \to n + 7 - n = 7.$$

(ii) Note that $7 \times 11 \times 13 = 1001$. Take any three digit number say, abc. Then $abc \times 1001 = abcabc$. Therefore, the six digit number $abcabc$ is divisible by 7, 11 and 13.

EXERCISE A2.1

1. **Step 1: Formulation :**

The relevant factors are the time period for hiring a computer, and the two costs given to us. We assume that there is no significant change in the cost of purchasing or hiring the computer. So, we treat any such change as irrelevant. We also treat all brands and generations of computers as the same, i.e. these differences are also irrelevant.

The expense of hiring the computer for x months is ₹ $2000x$. If this becomes more than the cost of purchasing a computer, we will be better off buying a computer. So, the equation is

$$2000x = 25000 \quad (1)$$

Step 2 : Solution : Solving (1), $x = \frac{25000}{2000} = 12.5$

Step 3 : Interpretation : Since the cost of hiring a computer becomes more **after** 12.5 months, it is cheaper to buy a computer, if you have to use it for more than 12 months.

2. **Step1 : Formulation :** We will assume that cars travel at a constant speed. So, any change of speed will be treated as irrelevant. If the cars meet after x hours, the first car would have travelled a distance of $40x$ km from A and the second car would have travelled $30x$ km, so that it will be at a distance of $(100 - 30x)$ km from A. So the equation will be $40x = 100 - 30x$, i.e., $70x = 100$.

Step 2 : Solution : Solving the equation, we get $x = \frac{100}{70}$.

Step 3 : Interpretation : $\frac{100}{70}$ is approximately 1.4 hours. So, the cars will meet after 1.4 hours.

3. **Step1 : Formulation :** The speed at which the moon orbits the earth is $\frac{\text{Length of the orbit}}{\text{Time taken}}$.

Step 2 : Solution : Since the orbit is nearly circular, the length is $2 \times \pi \times 384000$ km $= 2411520$ km

The moon takes 24 hours to complete one orbit.

So, speed $= \frac{2411520}{24} = 100480$ km/hour.

Step 3 : Interpretation : The speed is 100480 km/h.

4. **Formulation :** An assumption is that the difference in the bill is only because of using the water heater.

Let the average number of hours for which the water heater is used = x

Difference per month due to using water heater = ₹ 1240 – ₹ 1000 = ₹ 240

Cost of using water heater for one hour = ₹ 8

So, the cost of using the water heater for 30 days = $8 \times 30 \times x$

Also, the cost of using the water heater for 30 days = Difference in bill due to using water heater

So, $$240x = 240$$

Solution : From this equation, we get $x = 1$.

Interpretation : Since $x = 1$, the water heater is used for an average of 1 hour in a day.

EXERCISE A2.2

1. We will not discuss any particular solution here. You can use the same method as we used in last example, or any other method you think is suitable.

EXERCISE A2.3

1. We have already mentioned that the formulation part could be very detailed in real-life situations. Also, we do not validate the answer in word problems. Apart from this word problem have a 'correct answer'. This need not be the case in real-life situations.

2. The important factors are (ii) and (iii). Here (i) is not an important factor although it can have an effect on the number of vehicles sold.