Chapter 13 Probability

EXERCISE 13.1

Question 1:

Given that E and F are events such that $P(E) = 0.6, P(F) = 0.3$ and $P(E \cap F) = 0.2$, find $P(E|F)$ and $P(F|E)$.

Solution:

Given, $P(E) = 0.6, P(F) = 0.3$ and $P(E \cap F) = 0.2$

$$\Rightarrow P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$= \frac{0.2}{0.3}$$

$$= \frac{2}{3}$$

$$\Rightarrow P(F|E) = \frac{P(E \cap F)}{P(E)}$$

$$= \frac{0.2}{0.6}$$

$$= \frac{1}{3}$$

Question 2:

Compute $P(A|B)$, if $P(B) = 0.5$ and $P(A \cap B) = 0.32$.

Solution:

Given, $P(B) = 0.5$ and $P(A \cap B) = 0.32$

$$\Rightarrow P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{0.32}{0.5}$$

$$= \frac{16}{25}$$

Question 3:

If $P(A) = 0.8, P(B) = 0.5$ and $P(B|A) = 0.4$, find

(i) $P(A \cap B)$ (ii) $P(A|B)$ (iii) $P(A \cup B)$

Solution:

Given, $P(A) = 0.8, P(B) = 0.5$ and $P(B|A) = 0.4$

(i) $P(B|A) = 0.4$

$\therefore \dfrac{P(A \cap B)}{P(A)} = 0.4$

$\Rightarrow \dfrac{P(A \cap B)}{0.8} = 0.4$

$\Rightarrow P(A \cap B) = 0.32$

(ii) $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$\Rightarrow P(A|B) = \dfrac{0.32}{0.5}$

$\qquad\qquad = 0.64$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cup B) = 0.8 + 0.5 - 0.32$

$\qquad\qquad = 0.98$

Question 4:

Evaluate $P(A \cup B)$, if $2P(A) = P(B) = \dfrac{5}{13}$ and $P(A|B) = \dfrac{2}{5}$.

Solution:

Given, $2P(A) = P(B) = \dfrac{5}{13}$

$\Rightarrow P(A) = \dfrac{5}{26}$ and $P(B) = \dfrac{5}{13}$

$\Rightarrow P(A|B) = \dfrac{2}{5}$

$\Rightarrow \dfrac{P(A \cap B)}{P(B)} = \dfrac{2}{5}$

$\Rightarrow P(A \cap B) = \dfrac{2}{5} \times \dfrac{5}{13}$

$\qquad\qquad = \dfrac{2}{13}$

It is known that, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow P(A \cup B) = \frac{5}{26} + \frac{5}{13} - \frac{2}{13}$$

$$\Rightarrow P(A \cup B) = \frac{5 + 10 - 4}{26}$$

$$\Rightarrow P(A \cup B) = \frac{11}{26}$$

Question 5:

If $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and $P(A \cup B) = \frac{7}{11}$, find

(i) $P(A \cap B)$ (ii) $P(A|B)$ (iii) $P(B|A)$

Solution:

Given, $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and $P(A \cup B) = \frac{7}{11}$

(i) $P(A \cup B) = \frac{7}{11}$

$$\therefore P(A) + P(B) - P(A \cap B) = \frac{7}{11}$$

$$\Rightarrow \frac{6}{11} + \frac{5}{11} - P(A \cap B) = \frac{7}{11}$$

$$\Rightarrow P(A \cap B) = \frac{11}{11} - \frac{7}{11}$$

$$= \frac{4}{11}$$

(ii) Since, $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$$\Rightarrow P(A|B) = \frac{\frac{4}{11}}{\frac{5}{11}}$$

$$= \frac{4}{5}$$

(iii) Since, $P(B|A) = \frac{P(A \cap B)}{P(A)}$

$$\Rightarrow P(B|A) = \dfrac{\dfrac{4}{11}}{\dfrac{6}{11}}$$

$$= \dfrac{4}{6}$$

$$= \dfrac{2}{3}$$

Determine $P(E|F)$ in the following.

Question 6:

A coin is tossed three times, where
(i) E: head on third toss, F: heads on first two tosses
(ii) E: at least two heads, F: at most two heads
(iii) E: at most two tails, F: at least one tail.

Solution:

Sample space S is $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

(i) $E = \{HHH, HTH, THH, TTH\}$
$F = \{HHH, HHT\}$
$\therefore E \cap F = \{HHH\}$

$P(F) = \dfrac{2}{8} = \dfrac{1}{4}$ and $P(E \cap F) = \dfrac{1}{8}$

$$P(E|F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{\dfrac{1}{8}}{\dfrac{1}{4}}$$

$$= \dfrac{4}{8}$$

$$= \dfrac{1}{2}$$

(ii) $E = \{HHH, HHT, HTH, THH\}$
$F = \{HHT, HTH, HTT, THH, THT, TTH, TTT\}$
$\therefore E \cap F = \{HHT, HTH, THH\}$

Clearly, $P(E \cap F) = \dfrac{3}{8}$ and $P(F) = \dfrac{7}{8}$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\dfrac{3}{8}}{\dfrac{7}{8}}$$

$$= \frac{3}{7}$$

(iii) $E = \{HHH, HHT, HTH, HTT, THH, THT, TTH\}$

$F = \{HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$\therefore E \cap F = \{HHT, HTH, THH, HTT, THT, TTH\}$

$P(F) = \dfrac{7}{8}$ and $P(E \cap F) = \dfrac{6}{8}$

Thus,

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\dfrac{6}{8}}{\dfrac{7}{8}}$$

$$= \frac{6}{7}$$

Question 7:

Two coins are tossed once, where
(i) E: tail appears on one coin, F: one-coin shows head
(ii) E: not tail appears, F: no head appears

Solution:

Sample space S is $S = \{HH, HT, TH, TT\}$

(i) $E = \{HT, TH\}$

$F = \{HT, TH\}$

$\therefore E \cap F = \{HT, TH\}$

$P(F) = \dfrac{2}{4} = \dfrac{1}{2}$

$P(E \cap F) = \dfrac{2}{4} = \dfrac{1}{2}$

$\therefore P(E|F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{\dfrac{1}{2}}{\dfrac{1}{2}}$

$$= 1$$

(ii) $E = \{HH\}$

$F = \{TT\}$

$\therefore E \cap F = \phi$

$P(F) = 1$ and $P(E \cap F) = 0$

$\therefore P(E|F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{0}{1} = 0$

Question 8:

A die is thrown three times,

E: 4 appears on the third toss, F: 6 and 5 appears respectively on first two tosses.

Solution:

Sample space will be $6 \times 6 \times 6 = 216$

$$E = \begin{Bmatrix} (1,1,4),(1,2,4),.........(1,6,4) \\ (2,1,4),(2,2,4),.......(2,6,4) \\ (3,1,4),(3,2,4),.......(3,6,4) \\ (4,1,4),(4,2,4),.......(4,6,4) \\ (5,1,4),(5,2,4),.........(5,6,4) \\ (6,1,4),(6,2,4),.......(6,6,4) \end{Bmatrix}$$

$F = \{(6,5,1),(6,5,2),(6,5,3),(6,5,4),(6,5,5),(6,5,6)\}$

$\therefore E \cap F = \{(6,5,4)\}$

$P(F) = \dfrac{6}{216}$ and $P(E \cap F) = \dfrac{1}{216}$

$\therefore P(E|F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{\dfrac{1}{216}}{\dfrac{6}{216}}$

$= \dfrac{1}{6}$

Question 9:

Mother, father and son line up at random for a family picture

E: son on one end, F: father in middle.

Solution:

Let mother (M), father (F), and son (S) line up for the family picture, then the sample space will be

$S = \{MFS, MSF, FMS, FSM, SMF, SFM\}$

$\Rightarrow E = \{MFS, FMS, SMF, SFM\}$

$F = \{MFS, SFM\}$

$\therefore E \cap F = \{MFS, SFM\}$

$P(E \cap F) = \dfrac{2}{6} = \dfrac{1}{3}$

$P(F) = \dfrac{2}{6} = \dfrac{1}{3}$

$\therefore P(E|F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{\frac{1}{3}}{\frac{1}{3}} = 1$

Question 10:

A black and a red dice are rolled.

(a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.

(b) Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Solution:

Let the first observation be from the black die and second from the red die.

When two dice (one black and another red) are rolled, the sample space S has $6 \times 6 = 36$ number of elements.

(a) Let A: Obtaining a sum greater than 9

$= \{(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)\}$

B: Black die results in a 5.

$= \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$

$\therefore A \cap B = \{(5,5), (5,6)\}$

The conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5, is given by $P(A|B)$

$\therefore P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{2}{36}}{\frac{6}{36}} = \dfrac{2}{6} = \dfrac{1}{3}$

(b) E: Sum of the observations is 8.

$= \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

F: Red die resulted in a number less than 4.

$$= \left\{\begin{array}{l}(1,1),(1,2),(1,3),(2,1),(2,2),(2,3) \\ (3,1),(3,2),(3,3),(4,1),(4,2),(4,3) \\ (5,1),(5,2),(5,3),(6,1),(6,2),(6,3)\end{array}\right\}$$

$$\therefore E \cap F = \{(5,3),(6,2)\}$$

$$P(F) = \frac{18}{36} \text{ and } P(E \cap F) = \frac{2}{36}$$

The conditional probability of obtaining the sum equal to 8, given that the red die resulted in a number less than 4, is given by $P(E|F)$.

Thus,

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{2}{36}}{\frac{18}{36}} = \frac{2}{18} = \frac{1}{9}$$

Question 11:

A fair die is rolled. Consider events $E = \{1,3,5\}, F = \{2,3\}$ and $G = \{2,3,4,5\}$. Find

(i) $P(E|F)$ and $P(F|E)$ (ii) $P(E|G)$ and $P(G|E)$

(iii) $P((E \cup F)|G)$ and $P((E \cap F)|G)$

Solution:

The sample space S will be $S = \{1,2,3,4,5,6\}$

Given, $E = \{1,3,5\}, F = \{2,3\}$ and $G = \{2,3,4,5\}$

$$\therefore P(E) = \frac{3}{6} = \frac{1}{2}$$

$$P(F) = \frac{2}{6} = \frac{1}{3}$$

$$P(G) = \frac{4}{6} = \frac{2}{3}$$

(i) $E \cap F = \{3\}$

$$\therefore P(E \cap F) = \frac{1}{6}$$

$$\therefore P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$$

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

(ii) $E \cap G = \{3,5\}$

$$\therefore P(E \cap G) = \frac{2}{6} = \frac{1}{3}$$

$$P(E|G) = \frac{P(E \cap G)}{P(G)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$$P(G|E) = \frac{P(E \cap G)}{P(E)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

(iii) $E \cup F = \{1,2,3,5\}$

$(E \cup F) \cap G = \{1,2,3,5\} \cap \{2,3,4,5\} = \{2,3,5\}$

$E \cap F = \{3\}$

$(E \cap F) \cap G = \{3\} \cap \{2,3,4,5\} = \{3\}$

$$\therefore P(E \cap G) = \frac{4}{6} = \frac{2}{3}$$

$$\therefore P(E \cup G) = \frac{4}{6} = \frac{2}{3}$$

$$\therefore P((E \cup F) \cap G) = \frac{3}{6} = \frac{1}{2}$$

$$P(E \cap F) = \frac{1}{6}$$

$$P((E \cap F) \cap G) = \frac{1}{6}$$

$$\therefore P((E \cup F)|G) = \frac{P((E \cup F) \cap G)}{P(G)}$$

$$= \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4}$$

$$P((E \cap F)|G) = \frac{P((E \cap F) \cap G)}{P(G)} = \frac{\frac{1}{6}}{\frac{2}{3}} = \frac{1}{4}$$

Question 12:

Assume that each born child is equally like to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that (i) the youngest is a girl, (ii) at least one is a girl?

Solution:

Let B and G represent the boy and girl child respectively. If a family has two children, the sample space will be

$$S = \{(B,B), (B,G), (G,B), (G,G)\}$$

Let E be the event that both children are girls.

$$\therefore E = \{(G,G)\}$$

(i) Let E' be the event that the youngest child is a girl.

$$\therefore E' = \{(B,G), (G,G)\}$$

$$\Rightarrow E \cap E' = \{(G,G)\}$$

$$\therefore P(E') = \frac{2}{4} = \frac{1}{2}$$

$$P(E \cap E') = \frac{1}{4}$$

The conditional probability that both are girls, given that the youngest child is a girl, is given by $P(E|E')$.

$$P(E|E') = \frac{P(E \cap E')}{P(E')} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Thus, the required probability is $\frac{1}{2}$.

(ii) Let A be the event that at least one child is a girl.

$$\therefore A = \{(B,G), (G,B), (G,G)\}$$

$$\Rightarrow E \cap A = \{G,G\}$$

$$\Rightarrow P(A) = \frac{3}{4}$$

$$P(E \cap A) = \frac{1}{4}$$

The conditional probability that both are girls, given that at least one child is a girl, is given by $P(E|A)$.

$$P(E|A) = \frac{P(E \cap A)}{P(A)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Thus,

Question 13:

An instructor has a question bank consisting of 300 easy True/False questions, 200 difficult True/False questions, 500 easy multiple-choice questions and 400 difficult multiple-choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple-choice question?

Solution:

The given data can be represented as

	True/False	Multiple Choice	Total
Easy	300	500	800
Difficult	200	400	600
Total	500	900	1400

Let us denote E=Easy questions, M=Multiple choice questions, D=difficult questions, and T=True/False questions.

Total number of questions=1400
Total number of Multiple-choice questions=900
Thus, probability of selecting an easy multiple-choice question is

$$P(E \cap M) = \frac{500}{1400} = \frac{5}{14}$$

Probability of selecting a multiple-choice question, P(M), is $\frac{900}{1400} = \frac{9}{14}$

$P(E|M)$ represents the probability that a randomly selected question will be an easy question, given that it is a multiple-choice question.

$$\therefore P(E|M) = \frac{P(E \cap M)}{P(M)} = \frac{\frac{5}{14}}{\frac{9}{14}} = \frac{5}{9}$$

Thus, the required probability is $\frac{5}{9}$.

Question 14:

Given that the two numbers appearing on throwing the two dice are different. Find the probability of the event 'the sum of numbers on the dice is 4'.

Solution:

Number of observations in the sample space $S = 6 \times 6 = 36$.

Let A be the event that the sum of the numbers on the dice is 4 and B be the event that the two numbers appearing on throwing the two dice are different.

$\therefore A = \{(1,3),(2,2),(3,1)\}$

$$B = \begin{cases} (1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5) \end{cases}$$

$A \cap B = \{(1,3),(3,1)\}$

$\therefore P(B) = \dfrac{30}{36} = \dfrac{5}{6}$ and $P(A \cap B) = \dfrac{2}{36} = \dfrac{1}{18}$

Let $P(A|B)$ represents the probability that the sum of the numbers on the dice is 4, given that the two numbers appearing on throwing the two dice are different.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{18}}{\frac{5}{6}} = \frac{1}{15}$$

Thus, the required probability is $\dfrac{1}{15}$.

Question 15:

Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows as 3'.

Solution:

The sample space of the experiment is,

$$S = \begin{cases} (1,H),(1,T),(2,H),(2,T),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6), \\ (4,H),(4,T),(5,H),(5,T),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{cases}$$

Let A be the event that the coin shows a tail and B be the event that at least one die shows 3.

$\therefore A = \{(1,T),(2,T),(4,T),(5,T)\}$

$B = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(6,3)\}$

$\Rightarrow A \cap B = \phi$

$\therefore P(A \cap B) = 0$

Then, $P(B) = P\{(3,1)\} + P\{(3,2)\} + P\{(3,3)\} + P\{(3,4)\} + P\{(3,5)\} + P\{(3,6)\} + P\{(6,3)\}$

$= \dfrac{1}{36} + \dfrac{1}{36} + \dfrac{1}{36} + \dfrac{1}{36} + \dfrac{1}{36} + \dfrac{1}{36} + \dfrac{1}{36}$

$= \dfrac{7}{36}$

Probability of the event that the coin shows a tail, given that at least one die shows 3, is given by $P(A|B)$.

$$P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0}{\dfrac{7}{36}} = 0$$

Question 16:

If $P(A) = \dfrac{1}{2}, P(B) = 0$ then $P(A|B)$ is

(A) 0   (B) $\dfrac{1}{2}$ (C) Not defined   (D) 1

Solution:

Given, $P(A) = \dfrac{1}{2}$ and $P(B) = 0$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A \cap B)}{0}$

Thus, $P(A|B)$ is not defined.

The correct answer is C.

Question 17:

If A and B are events such that $P(A|B) = P(B|A)$, then

(A) $A \subset$ but $A \neq B$ (B) $A = B$ (C) $A \cap B = \phi$ (D) $P(A) = P(B)$

Solution:

Given, $P(A|B) = P(B|A)$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(A) = P(B)$$

Thus, the correct answer is D.

EXERCISE 13.2

Question 1:

If $P(A) = \dfrac{3}{5}$ and $P(B) = \dfrac{1}{5}$, find $P(A \cap B)$ if A and B are independent events.

Solution:

Given, $P(A) = \dfrac{3}{5}$ and $P(B) = \dfrac{1}{5}$

A and B are independent events.

Thus, $P(A \cap B) = P(A) \times P(B) = \dfrac{3}{5} \times \dfrac{1}{5} = \dfrac{3}{25}$

Question 2:

Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Solution:

Since there are 26 black cards in a deck of 52 cards.

Let $P(A)$ be the probability of getting a black card in the first draw.

$\therefore P(A) = \dfrac{26}{52} = \dfrac{1}{2}$

Let $P(B)$ be the probability of getting a black card on second draw.
Since the card is not replaced,

$\therefore P(B) = \dfrac{25}{51}$

Therefore, probability of getting both the cards black $= \dfrac{1}{2} \times \dfrac{25}{51} = \dfrac{25}{102}$

Question 3:

A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

Solution:

Let A, B, and C be the respective events that the first, second, and the third drawn orange is good.

Thus, probability that first drawn orange is good, $P(A) = \dfrac{12}{15}$

The oranges are not replaced.

Thus, probability of getting second orange good, $P(B) = \dfrac{11}{14}$

Similarly, probability of getting third orange good, $P(C) = \dfrac{10}{13}$

Since the box is approved for sale, if all the three oranges are good.

Thus, probability of getting all the oranges good $= \dfrac{12}{15} \times \dfrac{11}{14} \times \dfrac{10}{13} = \dfrac{44}{91}$

Thus, the probability that the box is approved for sale is $= \dfrac{44}{91}$

Question 4:

A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.

Solution:

The sample space is given by, $S = \begin{Bmatrix} (H,1),(H,2),(H,3),(H,4),(H,5),(H,6) \\ (T,1),(T,2),(T,3),(T,4),(T,5),(T,6) \end{Bmatrix}$

Let A: Head appears on the coin

$A = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6)\}$

$\Rightarrow P(A) = \dfrac{6}{12} = \dfrac{1}{2}$

B: 3 on die $= \{(H,3),(T,3)\}$

$\Rightarrow P(B) = \dfrac{2}{12} = \dfrac{1}{6}$

$A \cap B = \{(H,3)\}$

$P(A \cap B) = \dfrac{1}{12}$

$P(A) \times P(B) = \dfrac{1}{2} \times \dfrac{2}{6} = P(A \cap B)$

Therefore, A and B are independent events.

Question 5:

A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the events, 'the number is even,' and B be the event, 'the number is red'. Are A and B independent?

Solution:

The sample space (S) is $S = \{1, 2, 3, 4, 5, 6\}$

Let A: the number is even $= \{2, 4, 6\}$

$$\Rightarrow P(A) = \frac{3}{6} = \frac{1}{2}$$

B: the number is red $= \{1, 2, 3\}$

$$\Rightarrow P(B) = \frac{3}{6} = \frac{1}{2}$$

$$\therefore A \cap B = \{2\}$$

$$P(AB) = P(A \cap B) = \frac{1}{6}$$

$$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \neq \frac{1}{6}$$

$$P(A) \times P(B) \neq P(AB)$$

Therefore, A and B are not independent events.

Question 6:

Let E and F be the events with $P(E) = \frac{3}{5}, P(F) = \frac{3}{10}$ and $P(E \cap F) = \frac{1}{5}$. Are E and F are independent?

Solution:

Given $P(E) = \frac{3}{5}, P(F) = \frac{3}{10}$ and $P(EF) = P(E \cap F) = \frac{1}{5}$

$$P(E) \times P(F) = \frac{3}{5} \times \frac{3}{10} = \frac{9}{50} \neq \frac{1}{5}$$

$$P(E) \times P(F) \neq P(E \cap F)$$

Therefore, E and F are not independent.

Question 7:

Given that the events A and B are such that $P(A) = \frac{1}{2}, P(A \cap B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are (i) mutually exclusive (ii) independent.

Solution:

Given, $P(A) = \frac{1}{2}, P(A \cap B) = \frac{3}{5}$ and $P(B) = p$

(i) When A and B are mutually exclusive, $A \cap B = \phi$

$\therefore \ P(A \cap B) = 0$

Since, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \dfrac{3}{5} = \dfrac{1}{2} + p - 0$

$\Rightarrow p = \dfrac{3}{5} - \dfrac{1}{2} = \dfrac{1}{10}$

(ii) When A and B are independent, $P(A \cap B) = P(A) \times P(B) = \dfrac{1}{2}p$

Since, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \dfrac{3}{5} = \dfrac{1}{2} + p - \dfrac{1}{2}p$

$\Rightarrow \dfrac{3}{5} = \dfrac{1}{2} + \dfrac{p}{2}$

$\Rightarrow \dfrac{p}{2} = \dfrac{3}{5} - \dfrac{1}{2} = \dfrac{1}{10}$

$\Rightarrow \dfrac{p}{2} = \dfrac{2}{10} = \dfrac{1}{5}$

Question 8:

Let A and B be independent events with $P(A) = 0.3$ and $P(B) = 0.4$. Find

(i) $P(A \cap B)$ (ii) $P(A \cup B)$

(iii) $P(A|B)$ (iv) $P(B|A)$

Solution:

Given, $P(A) = 0.3$ and $P(B) = 0.4$.

(i)   If A and B are independent events, then

     $P(A \cap B) = P(A) \times P(B)$

           $= 0.3 \times 0.4$

           $= 0.12$

(ii)   Since, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

     $\Rightarrow P(A \cup B) = 0.3 + 0.4 - 0.12 = 0.58$

(iii)   Since, $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

     $\Rightarrow P(A|B) = \dfrac{0.12}{0.4} = 0.3$

(iv) Since,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(B|A) = \frac{0.12}{0.3} = 0.4$$

Question 9:

If A and B are two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{8}$, find P (not A and not B).

Solution:

Given, $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{8}$.

P (not on A and not on B) = $P(A' \cap B')$

P (not on A and not on B) = $P((A \cup B)')$ $[A' \cap B' = (A \cup B)']$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - \left[\frac{1}{4} + \frac{1}{2} - \frac{1}{8}\right]$$

$$= 1 - \frac{5}{8}$$

$$= \frac{3}{8}$$

Question 10:

Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and P (not A or not B) $= \frac{1}{4}$. State whether A and B are independent?

Solution:

Given, $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and P (not A or not B) $= \frac{1}{4}$

$$\Rightarrow P(A' \cup B') = \frac{1}{4}$$

$$\Rightarrow P((A \cap B)') = \frac{1}{4} \qquad [A' \cup B' = (A \cap B)']$$

$$\Rightarrow 1 - P(A \cap B) = \frac{1}{4}$$

$$\Rightarrow P(A \cap B) = \frac{3}{4} \qquad \ldots (1)$$

But, $P(A) \times P(B) = \frac{1}{2} \times \frac{7}{12} = \frac{7}{24} \qquad \ldots (2)$

Here, $\frac{3}{4} \neq \frac{7}{24}$

$\therefore P(A \cap B) \neq P(A) \times P(B)$

Therefore, A and B are not independent events.

Question 11:

Given two independent events A and B such $P(A) = 0.3, P(B) = 0.6$. Find

(i) $P(A \text{ and } B)$ (ii) $P(A \text{ and not } B)$

(iii) $P(A \text{ or } B)$ (iv) $P(\text{neither } A \text{ nor } B)$

Solution:

Given, $P(A) = 0.3$ and $P(B) = 0.6$

A and B are independent events.

(i) $P(A \text{ and } B) = P(A) \times P(B)$

$\Rightarrow P(A \cap B) = 0.3 \times 0.6 = 0.18$

(ii) $P(A \text{ and not } B) = P(A \cap B')$

$= P(A) - P(A \cap B)$

$= 0.3 - 0.18$

$= 0.12$

(iii) $P(A \text{ or } B) = P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= 0.3 + 0.6 - 0.18$

$= 0.72$

(iv) $P(\text{neither } A \text{ nor } B) = P(A' \cap B')$

$$= P\left((A \cup B)'\right) \qquad \left[A' \cap B' = (A \cup B)'\right]$$
$$= 1 - P(A \cup B)$$
$$= 1 - 0.72$$
$$= 0.28$$

Question 12:

A die tossed thrice. Find the probability of getting an odd number at least once.

Solution:

Probability of getting an odd number in a single throw of a die $= \dfrac{3}{6} = \dfrac{1}{2}$

Similarly, probability of getting an even number $= \dfrac{3}{6} = \dfrac{1}{2}$

Probability of getting an even number three times $= \dfrac{1}{2} \times \dfrac{1}{2} \times \dfrac{1}{2} = \dfrac{1}{8}$

= 1 − Probability of getting an odd number in none of the throws

= 1 − Probability of getting an even number thrice

$$= 1 - \dfrac{1}{8}$$

$$= \dfrac{7}{8}$$

Question 13:

Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that
(i) both balls are red.
(ii) first ball is black and second is red.
(iii) one of them is black and other is red.

Solution:

Given,
Total number of balls = 18
Number of red balls = 8
Number of black balls = 10

(i) Probability of getting a red ball in the first draw $= \dfrac{8}{18} = \dfrac{4}{9}$
The ball is replaced after the first draw.

$\therefore$ Probability of getting a red ball in the second draw $= \dfrac{8}{18} = \dfrac{4}{9}$

Therefore, probability of getting both the balls red $= \dfrac{4}{9} \times \dfrac{4}{9} = \dfrac{16}{81}$

(ii) Probability of getting first ball black $= \dfrac{10}{18} = \dfrac{5}{9}$
The ball is replaced after the first draw.

Probability of getting second ball as red $= \dfrac{8}{18} = \dfrac{4}{9}$

Therefore, probability of getting first ball as black and second ball as red $= \dfrac{5}{9} \times \dfrac{4}{9} = \dfrac{20}{81}$

(iii) Probability of getting first ball as red $= \dfrac{8}{18} = \dfrac{4}{9}$
The ball is replaced after the first draw.

Probability of getting second ball as black $= \dfrac{10}{18} = \dfrac{5}{9}$

Therefore, probability of getting first ball as black and second ball as red $= \dfrac{4}{9} \times \dfrac{5}{9} = \dfrac{20}{81}$
Therefore, probability that one of them is black and other is red = Probability of getting first ball black and second as red + Probability of getting first ball red and second ball black

$= \dfrac{20}{81} + \dfrac{20}{81} = \dfrac{40}{81}$

Question 14:

Probability of solving specific problem independently by A and B are $\dfrac{1}{2}$ and $\dfrac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that
(i) the problem is solved
(ii) exactly one of them solves the problem.

Solution:

Probability of solving the problem by A, $P(A) = \dfrac{1}{2}$

Probability of solving the problem by B, $P(B) = \dfrac{1}{3}$
Since the problem is solved independently by A and B,

$$P(AB) = P(A) \times P(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

$$P(A') = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(B') = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$$

(i) Probability that the problem is solved $= P(A \cup B)$

$$= P(A) + P(B) - P(AB)$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6}$$

$$= \frac{4}{6} = \frac{2}{3}$$

(ii) Probability that exactly one of them solves the problem $= P(A) \cdot P(B') + P(B) \cdot P(A')$

$$= \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{3}$$

$$= \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

Question 15:

One card is drawn at random from a well shuffled deck of 52 cards. In which of the following cases are the events E and F independent?

(i) E: 'the card drawn is a spade'
   F: 'the card drawn is an ace'

(ii) E: 'the card drawn is black'
    F: 'the card drawn is a king'

(iii) E: 'the card drawn is a king and queen'
     F: 'the card drawn is a queen or jack'

Solution:

(i) In a deck of 52 cards, 13 cards are spades and 4 cards are aces.

$$\therefore \ P(E) = P(\text{the card drawn is a spade}) = \frac{13}{52} = \frac{1}{4}$$

$$\therefore \ P(F) = P(\text{the card drawn in an ace}) = \frac{4}{52} = \frac{1}{13}$$

In the deck of cards, only 1 card is an ace of spades.

$$P(EF) = P(\text{the card drawn is spade and an ace}) = \frac{1}{52}$$

$$P(E) \times P(F) = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52} = P(EF)$$

$$\Rightarrow P(E) \times P(F) = P(EF)$$

Therefore, the events E and F are independent.

(ii) In a deck of 52 cards, 26 cards are black and 4 cards are kings.

$$\therefore \ P(E) = P(\text{the card drawn is a black}) = \frac{26}{52} = \frac{1}{2}$$

$$\therefore \ P(F) = P(\text{the card drawn in an ace}) = \frac{4}{52} = \frac{1}{13}$$

In the pack of 52 cards, 2 cards are black as well as kings.

$$P(EF) = P(\text{the card drawn is black king}) = \frac{2}{52} = \frac{1}{26}$$

$$P(E) \times P(F) = \frac{1}{2} \times \frac{1}{13} = \frac{1}{26} = P(EF)$$

Therefore, the given events E and F are independent.

(iii) In a deck of 52 cards, 4 cards are kings, 4 cards are queens, and 4 cards are jacks.

$$\therefore \ P(E) = P(\text{the card drawn is a king or a queen}) = \frac{8}{52} = \frac{2}{13}$$

$$\therefore \ P(F) = P(\text{the card drawn in a queen or a jack}) = \frac{8}{52} = \frac{2}{13}$$

There are 4 cards which are king or queen and queen or jack.

$$P(EF) = P(\text{the card drawn is king or a queen, or queen or a jack}) = \frac{4}{52} = \frac{1}{13}$$

$$P(E) \times P(F) = \frac{2}{13} \times \frac{2}{13} = \frac{4}{169} \neq \frac{1}{13}$$

$$P(E) \times P(F) \neq P(EF)$$

Therefore, the given events E and F are not independent.

Question 16:

In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.

(a) Find the probability that she reads neither Hindi nor English newspapers.

(b) If she reads Hindi newspaper, find the probability that she reads English newspaper.

(c) If she reads English newspaper, find the probability that she reads Hindi newspaper.

Solution:

Let H denote the students who read Hindi newspaper and E denote the students who read English newspaper.
Given,

$$P(H) = 60\% = \frac{60}{100} = \frac{3}{5}$$

$$P(E) = 40\% = \frac{40}{100} = \frac{2}{5}$$

$$P(H \cap E) = 20\% = \frac{20}{100} = \frac{1}{5}$$

(i) Probability that a student reads Hindi or English newspaper

$$P(H \cup E') = 1 - P(H \cup E)$$
$$= 1 - \{P(H) + P(E) - P(H \cap E)\}$$
$$= 1 - \left(\frac{3}{5} + \frac{2}{5} - \frac{1}{5}\right)$$
$$= 1 - \frac{4}{5} = \frac{1}{5}$$

(ii) Probability that a randomly chosen student reads English newspaper, if she reads Hindi newspaper, is given by $P(E \mid H)$.

$$P(E|H) = \frac{P(E \cap H)}{P(H)} = \frac{\frac{1}{5}}{\frac{3}{5}} = \frac{1}{3}$$

(iii) Probability that a randomly chosen student reads Hindi newspaper, if she reads English newspaper, is given by P (H|E).

$$P\left(H|E\right) = \frac{P\left(H \cap E\right)}{P\left(E\right)} = \frac{\frac{1}{5}}{\frac{2}{5}} = \frac{1}{2}$$

Question 17:

The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

(A) 0   (B) $\frac{1}{3}$ (C) $\frac{1}{12}$ (D) $\frac{1}{36}$

Solution:

When two dice are rolled, the number of outcomes is 36.
The only even prime number is 2.
Let E be the event of getting an even prime number on each die.

$$\therefore E = \left\{(2,2)\right\}$$

$$\Rightarrow P\left(E\right) = \frac{1}{36}$$

Therefore, the correct answer is D.

Question 18:

Two events A and B will be independent, if
(A) A and B are mutually exclusive
(B) $P\left(A'B'\right) = \left[1 - P\left(A\right)\right]\left[1 - P\left(B\right)\right]$
(C) $P\left(A\right) = P\left(B\right)$
(D) $P\left(A\right) + P\left(B\right) = 1$

Solution:

Two events A and B are said to be independent, if $P\left(AB\right) = P\left(A\right) \times P\left(B\right)$
Let's take option B.

$$P\left(A'B'\right) = \left[1 - P\left(A\right)\right]\left[1 - P\left(B\right)\right]$$

$$\Rightarrow P\left(A' \cap B'\right) = 1 - P\left(A\right) - P\left(B\right) + P\left(A\right) \times P\left(B\right)$$

$$\Rightarrow 1 - P\left(A \cup B\right) = 1 - P\left(A\right) - P\left(B\right) + P\left(A\right) \times P\left(B\right)$$

$$\Rightarrow P\left(A \cup B\right) = P\left(A\right) + P\left(B\right) - P\left(A\right) \times P\left(B\right)$$

$$\Rightarrow P\left(A\right) + P\left(B\right) - P\left(AB\right) = P\left(A\right) + P\left(B\right) - P\left(A\right) \times P\left(B\right)$$

$$\Rightarrow P\left(AB\right) = P\left(A\right) \times P\left(B\right)$$

This implies that A and B are independent, if $P(A'B') = [1 - P(A)][1 - P(B)]$

A. Let $P(A) = m, P(B) = n, 0 < m, n < 1$

A and B are mutually exclusive.

$\therefore A \cap B = \phi$

$\Rightarrow P(AB) = 0$

However, $P(A) \times P(B) = mn \neq 0$

$\therefore P(A) \times P(B) \neq P(AB)$

C. Let A: Event of getting an odd number on throw of a die $= \{1, 3, 5\}$

$\Rightarrow P(A) = \dfrac{3}{6} = \dfrac{1}{2}$

Let B: Event of getting an even number on throw of a die $= \{2, 4, 6\}$

$\Rightarrow P(B) = \dfrac{3}{6} = \dfrac{1}{2}$

Here, $A \cap B = \phi$

$\therefore P(AB) = 0$

$P(A) \times P(B) = \dfrac{1}{4} \neq 0$

$P(A) \times P(B) \neq P(AB)$

D. From the above example, it can be seen that,

$P(A) + P(B) = \dfrac{1}{2} + \dfrac{1}{2} = 1$

However, it cannot be inferred that A and B are independent.

Thus, the correct answer is B.

EXERCISE 13.3

Question 1:

An urn contains 5 red and 5 black balls. A ball is drawn at random; its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. What is the probability that the second ball is red?

Solution:

The urn contains 5 red and 5 black balls.

Let a red ball be drawn in the first attempt.

$$\therefore P\left(\text{drawing a red ball}\right) = \frac{5}{10} = \frac{1}{2}$$

If two red balls are added to the urn, then the urn contains 7 red and 5 black balls.

$$P\left(\text{drawing a red ball}\right) = \frac{7}{12}$$

Let a black ball be drawn in the first attempt.

$$\therefore P\left(\text{drawing a black ball in the first attempt}\right) = \frac{5}{10} = \frac{1}{2}$$

If two black balls are added to the urn, then the urn contains 5 red and 7 black balls.

$$P\left(\text{drawing a red ball}\right) = \frac{5}{12}$$

Thus, probability of drawing second ball as red is

$$= \frac{1}{2} \times \frac{7}{12} + \frac{1}{2} \times \frac{5}{12}$$

$$= \frac{1}{2}\left(\frac{7}{12} + \frac{5}{12}\right)$$

$$= \frac{1}{2} \times 1$$

$$= \frac{1}{2}$$

Question 2:

A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

Solution:

Let E_1 and E_2 be the events of selecting first bag and second bag respectively.

$$P(E_1) = P(E_2) = \frac{1}{2}$$

Let A be the event of getting a red ball.

$$\Rightarrow P(A|E_1) = P(\text{drawing a red ball from first bag}) = \frac{4}{8} = \frac{1}{2}$$

$$\Rightarrow P(A|E_2) = P(\text{drawing a red ball from second bag}) = \frac{2}{8} = \frac{1}{4}$$

The probability of drawing a ball from the first bag, given that it is red, is given by $P(E_1|A)$.

By using Bayes' theorem, we obtain

$$P(E_1|A) = \frac{P(E_1) \cdot P(A|E_1)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4}}$$

$$= \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{8}} = \frac{\frac{1}{4}}{\frac{3}{8}} = \frac{2}{3}$$

Question 3:

Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is hostlier?

Solution:

Let E_1 and E_2 and be the events that the student is a hostlier and a day scholar respectively and A be the event that the chosen student gets grade A.

$$P(E_1) = 60\% = \frac{60}{100} = 0.6$$

$$P(E_2) = 40\% = \frac{40}{100} = 0.4$$

$$P(A|E_1) = P(\text{student getting an A grade is a hostler}) = 30\% = 0.3$$

$$P(A|E_2) = P(\text{student getting an A grade is a day scholar}) = 20\% = 0.2$$

The probability that a randomly chosen student is a hostler, given that he has an A grade, is given by $P(E_1|A)$.

By using Bayes' theorem, we obtain

$$P(E_1|A) = \frac{P(E_1) \cdot P(A|E_1)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)}$$

$$= \frac{0.6 \times 0.3}{0.6 \times 0.3 + 0.4 \times 0.2}$$

$$= \frac{0.18}{0.18 + 0.08} = \frac{18}{26} = \frac{9}{13}$$

Question 4:

In answering a question on a multiple-choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?

Solution:

Let E_1 and E_2 and be the respective events that the student knows the answer and he guesses the answer.

Let A be the event that the answer is correct.

$$P(E_1) = \frac{3}{4}$$

$$P(E_2) = \frac{1}{4}$$

The probability that the students answered correctly, given that he knows the answer, is 1.

$$\therefore P(A|E_1) = 1$$

Probability that the student answered correctly, given that he guessed, is $\frac{1}{4}$.

$$\therefore P(A|E_2) = \frac{1}{4}$$

The probability that the student knows the answer, given that he answered it correctly, is given by $P(E_1|A)$.

By using Bayes' theorem, we obtain

$$P(E_1|A) = \frac{P(E_1) \cdot P(A|E_1)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)}$$

$$= \frac{\dfrac{3}{4} \times 1}{\dfrac{3}{4} \times 1 + \dfrac{1}{4} \times \dfrac{1}{4}}$$

$$= \frac{\dfrac{3}{4}}{\dfrac{3}{4} + \dfrac{1}{16}} = \frac{\dfrac{3}{4}}{\dfrac{13}{16}} = \frac{12}{13}$$

Question 5:

A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (that is, if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive?

Solution:

Let E_1 and E_2 be the respective events that a person has a disease and a person has no disease.

Since E_1 and E_2 are events complimentary to each other,

$$\therefore P(E_1) + P(E_2) = 1$$
$$\Rightarrow P(E_2) = 1 - P(E_1) = 1 - 0.001 = 0.999$$

Let A be the event that the blood test result is positive.

$$P(E_1) = 0.1\% = \frac{0.1}{100} = 0.001$$

$$P(A|E_1) = P(\text{result is positive given the person has disease}) = 99\% = 0.99$$

$$P(A|E_2) = P(\text{result is positive given the person has no disease}) = 0.5\% = 0.005$$

Probability that a person has a disease has a disease, given that his test result is positive, is given by $P(E_1|A)$.

By using Bayes' theorem, we get

$$P(E_1|A) = \frac{P(E_1) \cdot P(A|E_1)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)}$$

$$= \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.005}$$

$$= \frac{0.00099}{0.00099 + 0.004995}$$

$$= \frac{0.00099}{0.005985}$$

$$= \frac{990}{5985} = \frac{110}{665} = \frac{22}{133}$$

Question 6:

There are three coins. One is two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin?

Solution:

Let E_1, E_2 and E_3 be the respective events of choosing a two headed coin, a biased coin, and an unbiased coin.

$$\therefore P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Let A be the event that the coin shows heads.

A two-headed coin will always show heads.

$$\therefore P(A|E_1) = P(\text{coin showing heads, given that it is a two headed coin}) = 1$$

Probability of heads coming up, given that it is a biased coin = 75%

$$\therefore P(A|E_2) = P(\text{coin showing heads, given that it is a biased coin}) = \frac{75}{100} = \frac{3}{4}$$

Since the third coin is unbiased, the probability that it shows heads is always $\frac{1}{2}$.

$$\therefore P(A|E_3) = P(\text{coin showing heads, given that it is an unbiased coin}) = \frac{1}{2}$$

The probability that the coin is two – headed, given that it shows heads, is given by $P(E_1|A)$.
By using Bayes' theorem, we get

$$P\left(E_1|A\right) = \frac{P\left(E_1\right) \cdot P\left(A|E_1\right)}{P\left(E_1\right) \cdot P\left(A|E_1\right) + P\left(E_2\right) \cdot P\left(A|E_2\right) + P\left(E_3\right) \cdot P\left(A|E_3\right)}$$

$$= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times \frac{3}{4} + \frac{1}{3} \times \frac{1}{2}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3}\left(1 + \frac{3}{4} + \frac{1}{2}\right)}$$

$$= \frac{1}{\frac{9}{4}} = \frac{4}{9}$$

Question 7:

An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

Solution:

Let E_1, E_2 and E_3 be the respective events that the driver is a scooter driver, a car driver, and a truck driver.

Let A be the event that the person meets with an accident.

There are 2000 scooter drivers, 4000 car drivers, and 6000 truck drivers.

Total number of drivers $= 2000 + 4000 + 6000 = 12000$

$$P\left(E_1\right) = P\left(\text{driver is a scooter driver}\right) = \frac{2000}{12000} = \frac{1}{6}$$

$$P\left(E_2\right) = P\left(\text{driver is a car driver}\right) = \frac{4000}{12000} = \frac{1}{3}$$

$$P\left(E_3\right) = P\left(\text{driver is a truck driver}\right) = \frac{6000}{12000} = \frac{1}{2}$$

$$P\left(A|E_1\right) = P\left(\text{scooter driver met with an accident}\right) = 0.01 = \frac{1}{100}$$

$$P\left(A|E_2\right) = P\left(\text{car driver met with an accident}\right) = 0.03 = \frac{3}{100}$$

$$P\left(A|E_3\right) = P\left(\text{truck driver met with an accident}\right) = 0.15 = \frac{15}{100}$$

The probability that the driver is a scooter driver, given that he met with an accident, is given by $P(E_1|A)$.

By using Bayes' theorem, we get

$$P(E_1|A) = \frac{P(E_1) \cdot P(A|E_1)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + P(E_3) \cdot P(A|E_3)}$$

$$= \frac{\dfrac{1}{6} \times \dfrac{1}{100}}{\dfrac{1}{6} \times \dfrac{1}{100} + \dfrac{1}{3} \times \dfrac{3}{100} + \dfrac{1}{2} \times \dfrac{15}{100}}$$

$$= \frac{\dfrac{1}{6} \times \dfrac{1}{100}}{\dfrac{1}{100}\left(\dfrac{1}{6} + 1 + \dfrac{15}{2}\right)}$$

$$= \frac{\dfrac{1}{6}}{\dfrac{104}{12}} = \frac{1}{52}$$

Question 8:

A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Future, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen random from this and is found to be defective. What is the probability that was produced by machine B?

Solution:

Let E_1 and E_2 be the respective events of items produced by machines A and B. Let X be the event that the produced items was found to be defective.

∴ Probability of items produced by machine A, $P(E_1) = 60\% = \dfrac{3}{5}$

Probability of items produced by machine B, $P(E_2) = 40\% = \dfrac{2}{5}$

Probability that machine A produced defective items, $P(X|E_1) = 2\% = \dfrac{2}{100}$

Probability that machine B produced defective items, $P(X|E_2) = 1\% = \dfrac{1}{100}$

The probability that the randomly selected items was from machine B, given that it is defective, is given by $P(E_2|X)$.

By using Bayes' theorem, we get

$$P(E_2|X) = \frac{P(E_2) \cdot P(X|E_2)}{P(E_1) \cdot P(X|E_1) + P(E_2) \cdot P(X|E_2)}$$

$$= \frac{\dfrac{2}{5} \times \dfrac{1}{100}}{\dfrac{3}{5} \times \dfrac{2}{100} + \dfrac{2}{5} \times \dfrac{1}{100}}$$

$$= \frac{\dfrac{2}{500}}{\dfrac{6}{500} + \dfrac{2}{500}} = \frac{2}{8} = \frac{1}{4}$$

Question 9:

Two groups are competing for the position on board of directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

Solution:

Let E_1 and E_2 be the respective events that the first group and the second group win the competition. Let A be the event of introducing a new product.

Probability that the first group wins the competition, $P(E_1) = 0.6$

Probability that the second group wins the competition, $P(E_2) = 0.4$

Probability of introducing a new product if the first group wins, $P(A|E_1) = 0.7$

Probability of introducing a new product if the second group wins, $P(A|E_2) = 0.3$

The probability that the new product is introduced by the second group is given by $P(E_2|A)$.

By using Bayes' theorem, we get

$$P\left(E_2 | A\right) = \frac{P\left(E_2\right) \cdot P\left(A | E_2\right)}{P\left(E_1\right) \cdot P\left(A | E_1\right) + P\left(E_2\right) \cdot P\left(A | E_2\right)}$$

$$= \frac{0.4 \times 0.3}{0.6 \times 0.7 + 0.4 \times 0.3}$$

$$= \frac{0.12}{0.42 + 0.12}$$

$$= \frac{0.12}{0.54}$$

$$= \frac{12}{54} = \frac{2}{9}$$

Question 10:

Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?

Solution:

Let E_1 be the event that the outcome on the die is 5 or 6 and E_2 be the event that the outcome on the die is 1, 2, 3, or 4.

$$P\left(E_1\right) = \frac{2}{6} = \frac{1}{3} \text{ and } P\left(E_2\right) = \frac{4}{6} = \frac{2}{3}$$

Let A be the event of getting exactly one head.

Probability of getting exactly one head by tossing the coin three times if she gets 5 or 6,

$$P\left(A | E_1\right) = \frac{3}{8}$$

Probability of getting exactly one head in a single throw of coin if she gets 1, 2, 3, or 4,

$$P\left(A | E_2\right) = \frac{1}{2}$$

The probability that the girl threw 1, 2, 3, or 4 with die, if she obtained exactly one head, is given by $P\left(E_2 | A\right)$.

By using Bayes' theorem, we get

$$P(E_2|A) = \frac{P(E_2) \cdot P(A|E_2)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)}$$

$$= \frac{\dfrac{2}{3} \times \dfrac{1}{2}}{\dfrac{1}{3} \times \dfrac{3}{8} + \dfrac{2}{3} \times \dfrac{1}{2}}$$

$$= \frac{\dfrac{1}{3}}{\dfrac{1}{3}\left(\dfrac{3}{8} + 1\right)}$$

$$= \frac{1}{\dfrac{11}{8}} = \frac{8}{11}$$

Question 11:

A manufacture has three machine operators A, B and C. The first operator A produces 1% defective items, whereas the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that was produced by A?

Solution:

Let E_1, E_2 and E_3 be the respective events that the time consumed by machine A, B, and C for the job.

$$P(E_1) = 50\% = \frac{50}{100} = \frac{1}{2}$$

$$P(E_2) = 30\% = \frac{30}{100} = \frac{3}{10}$$

$$P(E_3) = 20\% = \frac{20}{100} = \frac{1}{5}$$

Let X be the event of producing defective items.

$$P(X|E_1) = 1\% = \frac{1}{100}$$

$$P(X|E_2) = 5\% = \frac{5}{100}$$

$$P(X|E_3) = 7\% = \frac{7}{100}$$

The probability that the defective item was produced by A is given by $P(E_1|X)$.

By using Bayes' theorem, we get

$$P(E_1|X) = \frac{P(E_1) \cdot P(X|E_1)}{P(E_1) \cdot P(X|E_1) + P(E_2) \cdot P(X|E_2) + P(E_3) \cdot P(X|E_3)}$$

$$= \frac{\dfrac{1}{2} \times \dfrac{1}{100}}{\dfrac{1}{2} \times \dfrac{1}{100} + \dfrac{3}{10} \times \dfrac{5}{100} + \dfrac{1}{5} \times \dfrac{7}{100}}$$

$$= \frac{\dfrac{1}{2} \times \dfrac{1}{100}}{\dfrac{1}{100}\left(\dfrac{1}{2} + \dfrac{3}{2} + \dfrac{7}{5}\right)}$$

$$= \frac{\dfrac{1}{2}}{\dfrac{17}{5}} = \frac{5}{34}$$

Question 12:

A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.

Solution:

Let E_1 and E_2 be the respective events of choosing a diamond cards and a card which is not diamond.

Let A denote the lost card.

Out of 52 cards, 13 cards are diamonds and 39 cards are not diamonds.

$$P(E_1) = \frac{13}{52} = \frac{1}{4}$$

$$P(E_2) = \frac{39}{52} = \frac{3}{4}$$

When one diamond card is lost, there are 12 diamonds cards out of 51 cards.

Two cards can be drawn out of 12 diamonds cards in $^{12}C_2$ ways.

Similarly, 2 diamonds cards can be drawn out of 51 cards in $^{51}C_2$ ways.

The probability of getting two cards, when one diamond card is lost, is given by $P(A|E_1)$.

$$P(A|E_1) = \frac{^{12}C_2}{^{51}C_2} = \frac{12!}{2! \times 10!} \times \frac{2! \times 49!}{51!} = \frac{11 \times 12}{51 \times 50} = \frac{22}{425}$$

When the lost card is not a diamond, there are 13 diamonds cards out of 51 cards.

Two cards can be drawn out of 13 diamonds cards in $^{13}C_2$ ways whereas 2 cards can be drawn out of 51 cards in $^{51}C_2$ ways.

The probability of getting two cards, when one card is lost which is not diamond, is given by $P(A|E_2)$.

$$P(A|E_2) = \frac{^{13}C_2}{^{51}C_2} = \frac{13!}{2! \times 11!} \times \frac{2! \times 49!}{51!} = \frac{13 \times 12}{51 \times 50} = \frac{26}{425}$$

The probability that the lost card is diamond is given by $P(E_1|A)$.

By using Bayes' theorem, we get

$$P(E_1|A) = \frac{P(E_1) \cdot P(A|E_1)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)}$$

$$= \frac{\dfrac{1}{4} \times \dfrac{22}{425}}{\dfrac{1}{4} \times \dfrac{22}{425} + \dfrac{3}{4} \times \dfrac{26}{425}}$$

$$= \frac{\dfrac{22}{1700}}{\dfrac{100}{1700}}$$

$$= \frac{22}{100} = \frac{11}{50}$$

Question 13:

Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A report that a head appears. The probability that actually there was head is

A. $\frac{4}{5}$　　　B. $\frac{1}{2}$　　　C. $\frac{1}{5}$　　　D. $\frac{2}{5}$

Solution:

Let E_1 and E_2 be the events such that

E_1 : A speaks truth

E_2 : A speaks false

Let X be the event that a head appears.

$$P(E_1) = \frac{4}{5}$$

$$P(E_2) = 1 - P(E_1) = 1 - \frac{4}{5} = \frac{1}{5}$$

If a coin is tossed, then it may result in either head (H) or tail (T).

The probability of getting a head is $\frac{1}{2}$ whether A speaks truth or not.

$$\therefore P(X|E_1) = P(X|E_2) = \frac{1}{2}$$

The probability that there is actually a head is given by $P(E_1|X)$.

$$P(E_1|X) = \frac{P(E_1) \cdot P(X|E_1)}{P(E_1) \cdot P(X|E_1) + P(E_2) \cdot P(X|E_2)}$$

$$= \frac{\frac{4}{5} \times \frac{1}{2}}{\frac{4}{5} \times \frac{1}{2} + \frac{1}{5} \times \frac{1}{2}}$$

$$= \frac{\frac{4}{5} \times \frac{1}{2}}{\frac{1}{2}\left(\frac{4}{5} + \frac{1}{5}\right)}$$

$$= \frac{\frac{4}{5}}{1} = \frac{4}{5}$$

Therefore, the correct answer is A.

Question 14:

If A and B are two events such that $A \subset B$ and $P(B) \neq 0$, then which of the following is correct?

A. $P(A|B) = \frac{P(B)}{P(A)}$ B. $P(A|B) < P(A)$ C. $P(A|B) \geq P(A)$ D. None of these

Solution:

If $A \subset B$, then $A \cap B = A$

$\Rightarrow A \cap B = P(A)$

Also, $P(A) < P(B)$

Consider $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)}{P(B)} \neq \dfrac{P(B)}{P(A)}$ ....(1)

Consider $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)}{P(B)}$ ....(2)

Since, $P(B) \leq 1$

$\Rightarrow \dfrac{1}{P(B)} \geq 1$

$\Rightarrow \dfrac{P(A)}{P(B)} \geq P(A)$

From (2), we get

$\Rightarrow P(A|B) \geq P(A)$ ....(3)

Thus, option (C) is correct.

EXERCISE 13.4

Question 1:

State which of the following are not the probability distribution of a random variable. Give reasons for your answer.

(i)

X	0	1	2
P (X)	0.4	0.4	0.2

(ii)

X	0	1	2	3	4
P (X)	0.1	0.5	0.2	-0.1	0.3

(iii)

Y	-1	0	1
P (Y)	0.6	0.1	0.2

(iv)

Z	3	2	1	0	-1
P (Z)	0.3	0.2	0.4	0.1	0.05

Solution:

Since the sum of all the probabilities in a probability distribution is one.

(i) Sum of the probabilities $= 0.4 + 0.4 + 0.2 = 1$

Thus, the given table is a probability distribution of random variable.

(ii) For $X = 3, \ P(X) = -0.1$

Since probability of any observation is not negative. Therefore, the given table is not a probability distribution of random variables.

(iii) Sum of the probabilities $= 0.6 + 0.1 + 0.2 = 0.9 \neq 1$

Thus, the given table is not a probability distribution of random variables.

(iv) Sum of the probabilities $= 0.3 + 0.2 + 0.4 + 0.1 + 0.05 = 1.05 \neq 1$

Thus, the given table is not a probability distribution of random variable.

Question 2:

An urn contains 5 red and 2 black balls. Two balls are randomly drawn. Let X represents the number of black balls. What are the possible values of X? Is X a random variable?

Solution:

Let B represents a black ball and R represents a red ball.

The two balls selected can be represented as BB, BR, RB, RR.

$X(BB)=2$

$X(BR)=1$

$X(RB)=1$

$X(RR)=0$

Thus, the possible values of X are 0, 1, and 2.

Yes, X is a random variable.

Question 3:

Let X represents the difference between the number of heads and the number of tails obtained when a coin is tossed 6 times. What are possible values of X?

Solution:

A coin is tossed six times and X represents the difference between the number of heads and the number of tails.

$X(6H, 0T) = |6 - 0| = 6$

$X(5H, 1T) = |5 - 1| = 4$

$X(4H, 2T) = |4 - 2| = 2$

$X(3H, 3T) = |3 - 3| = 0$

$X(2H, 4T) = |2 - 4| = 2$

$X(1H, 5T) = |1 - 5| = 4$

$X(0H, 6T) = |0 - 6| = 6$

Therefore, the possible values of X are 6, 4, 2, and 0.

Question 4:

Find the probability distribution of
(i) number of heads in two tosses of a coin
(ii) number of tails in the simultaneous tosses of three coins
(iii) number of heads in four tosses of a coin

Solution:

(i) When one coin is tossed twice. the sample space is $\{HH, HT, TH, TT\}$

Let X represent the number of heads.

$$\therefore X(HH) = 2, X(HT) = 1, X(TH) = 1, X(TT) = 0$$

Thus, X can take the value of 0, 1, or 2.

Since,

$$P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$$

$$P(X = 0) = P(TT) = \frac{1}{4}$$

$$P(X = 1) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(X = 2) = P(HH) = \frac{1}{4}$$

Therefore, the required probability distribution is as follows.

X	0	1	2
P (X)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(ii) The sample space is $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
Let X represents the number of tails.

Since, X can take the value of 0, 1, 2, or 3.

$$P(X = 0) = P(HHH) = \frac{1}{8}$$

$$P(X = 1) = P(HHT) + P(HTH) + P(THH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

$$P(X = 2) = P(HTT) + P(THT) + P(TTH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

$$P(X = 3) = P(TTT) = \frac{1}{8}$$

Therefore, the probability distribution is as follows.

X	0	1	2	3
P (X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(iii) The sample space is $S = \left\{ \begin{array}{l} HHHH, HHHT, HHTH, HHTT, HTHT, HTHH, HTTH, HTTT \\ THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT \end{array} \right\}$

Let X be the random variable, which represents the number of heads.

Since, X can take the value of 0, 1, 2, 3, or 4.

$$P(X=0) = P(TTTT) = \frac{1}{16}$$

$$P(X=1) = P(TTTH) + P(TTHT) + P(THTT) + P(HTTT)$$

$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{4}{16} = \frac{1}{4}$$

$$P(X=2) = P(HHTT) + P(THHT) + P(TTHH) + P(HTTH) + P(HTHT) + P(THTH)$$

$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{6}{16} = \frac{3}{8}$$

$$P(X=3) = P(HHHT) + P(HHTH) + P(HTHH) + P(THHH)$$

$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{4}{16} = \frac{1}{4}$$

$$P(X=4) = P(HHHH) = \frac{1}{16}$$

Therefore, the probability distribution is as follows.

X	0	1	2	3	4
P (X)	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

Question 5:
Find the probability distribution of the number of success in two tosses of die, where a success is defined as
(i) number greater than 4
(ii) six appears on at least one die

Solution:
When a die is tossed two times, we obtain $(6 \times 6) = 36$ number of observations.
Let X be the random variable, which represents the number of success.

(i) Here, success refers to the number greater than 4.

$P(X=0) =$ P (number less than or equal to 4 on both the tosses) $= \frac{4}{6} \times \frac{4}{6} = \frac{4}{9}$

$P(X = 1) = P$ (number less than or equal to 4 on first toss and greater than 4 on second toss) + P (number greater than 4 on first toss and less than or equal to 4 on second toss)

$$= \frac{4}{6} \times \frac{2}{6} + \frac{4}{6} \times \frac{2}{6} = \frac{4}{9}$$

$P(X = 2) = P$ (number greater than 4 on both the tosses) $= \frac{2}{6} \times \frac{2}{6} = \frac{1}{9}$

Therefore, the probability distribution is as follows.

X	0	1	2
P (X)	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

(ii) Here, success means six appears on at least one die.

$P(Y = 0) = P$ (six appears on none of the dice) $= \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$

$P(Y = 1) = P$ (six appears on at least one of the dice) $= \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} = \frac{5}{36} + \frac{5}{36} = \frac{10}{36}$

Therefore, the required probability distribution is as follows.

Y	0	1
P (Y)	$\frac{25}{36}$	$\frac{10}{36}$

Question 6:

From a lot of 30 bulbs which includes 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

Solution:

Given, out of 30 bulbs, 6 are defective.

$\therefore$ Number of non-defective bulbs $= 30 - 6 = 24$

4 bulbs are drawn from the lot with replacement.

Let X be the random variable that denotes the number of defective bulbs in the selected bulbs.

$$P(X=0) = P(4 \text{ non-defective and } 0 \text{ defective}) = {}^4C_0 \times \left(\frac{4}{5}\right)^4 = \frac{256}{625}$$

$$P(X=1) = P(3 \text{ non-defective and } 1 \text{ defective}) = {}^4C_1 \times \left(\frac{1}{5}\right) \times \left(\frac{4}{5}\right)^3 = \frac{256}{625}$$

$$P(X=2) = P(2 \text{ non-defective and } 2 \text{ defective}) = {}^4C_2 \times \left(\frac{1}{5}\right)^2 \times \left(\frac{4}{5}\right)^2 = \frac{96}{625}$$

$$P(X=3) = P(1 \text{ non-defective and } 3 \text{ defective}) = {}^4C_3 \times \left(\frac{1}{5}\right)^3 \times \left(\frac{4}{5}\right) = \frac{16}{625}$$

$$P(X=4) = P(0 \text{ non-defective and } 4 \text{ defective}) = {}^4C_4 \times \left(\frac{1}{5}\right)^4 = \frac{1}{625}$$

Thus, the required probability distribution is as follows.

X	0	1	2	3	4
P (X)	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

Question 7:

A coin is biased so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution of number of tails.

Solution:

Let the probability of getting a tail in the biased coin be x.

$$\therefore P(T) = x$$
$$\Rightarrow P(H) = 3x$$

For a biased coin, $P(T) + P(H) = 1$
$$\Rightarrow x + 3x = 1$$
$$\Rightarrow 4x = 1$$
$$\Rightarrow x = \frac{1}{4}$$
$$\therefore P(T) = \frac{1}{4} \text{ and } P(H) = \frac{3}{4}$$

When the coin is tossed twice, the sample space is $\{HH, TT, HT, TH\}$.

Let X be the random variable representing the number of tails.

$$P(X=0) = P(\text{no tail}) = P(H) \times P(H) = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$$

$$P(X=1) = P(\text{one tail}) = P(HT) + P(TH)$$

$$= \frac{3}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{3}{4}$$

$$= \frac{3}{16} + \frac{3}{16}$$

$$= \frac{3}{8}$$

$$P(X=2) = P(\text{two tails}) = P(TT) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

Thus, the required probability distribution is as follows.

X	0	1	2
P (X)	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$

Question 8:

A random variable X has the following probability distribution.

X	0	1	2	3	4	5	6	7
P (X)	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Determine

(i) k

(ii) $P(X < 3)$

(iii) $P(X > 6)$

(iv) $P(0 < X < 3)$

Solution:

(i) Since, the sum of probabilities of a probability distribution of random variable is one.

$$\therefore 0 + k + 2k + 3k + k^2 + 2k^2 + (7k^2 + k) = 1$$

$$\Rightarrow 10k^2 + 9k - 1 = 0$$

$$\Rightarrow (10k - 1)(k + 1) = 0$$

$$\Rightarrow k = -1, \frac{1}{10}$$

(ii) $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$

$= 0 + k + 2k$

$= 3k$

$= 3 \times \dfrac{1}{10}$

$= \dfrac{3}{10}$

(iii) $P(X > 6) = P(X = 7)$

$= 7k^2 + k$

$= 7 \times \left(\dfrac{1}{10}\right)^2 + \dfrac{1}{10}$

$= \dfrac{7}{100} + \dfrac{1}{10}$

$= \dfrac{17}{100}$

(iv) $P(0 < X < 3) = P(X = 1) + P(X = 2)$

$= k + 2k$

$= 3k$

$= 3 \times \dfrac{1}{10}$

$= \dfrac{3}{10}$

Question 9:

The random variable X has probability $P(X)$ of the following form, where *k* is some number:

$$P(X) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Determine the final value of *k*.

(b) Find $P(X < 2), P(X \le 2), P(X \ge 2)$.

Solution:

(a) Since, the sum of probabilities of a probability distribution of random variable is one.

$$\therefore k + 2k + 3k + 0 = 1$$

$$\Rightarrow 6k = 1$$

$$\Rightarrow k = \frac{1}{6}$$

(b) $P(X < 2) = P(X = 0) + P(X = 1)$

$$\therefore k + 2k$$

$$\Rightarrow 3k = \frac{3}{6} = \frac{1}{2}$$

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$\therefore k + 2k + 3k$$

$$\Rightarrow 6k = \frac{6}{6} = 1$$

$$P(X \geq 2) = P(X = 2) + P(X > 2)$$

$$= 3k + 0$$

$$\Rightarrow 3k = \frac{3}{6} = \frac{1}{2}$$

Question 10:

Find the mean number of heads in three tosses of a fair coin.

Solution:

Let X denote the success of getting heads.

Thus, the sample space is $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Here, X can take the value of 0, 1, 2, or 3.

$$\therefore P(X = 0) = P(TTT)$$

$$= P(T) \times P(T) \times P(T)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

$$\therefore P(X = 1) = P(HTT) + P(THT) + P(TTH)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{3}{8}$$

$$\therefore P(X=2) = P(HHT) + P(HTH) + P(THH)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{3}{8}$$

$$\therefore P(X=3) = P(HHH)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{8}$$

Thus, the required probability is as follows.

X	0	1	2	3
P (X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Mean $E(X) = \Sigma X_i P(X_i)$

$$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$$

$$= \frac{3}{8} + \frac{3}{4} + \frac{3}{8}$$

$$= \frac{12}{8} = \frac{3}{2} = 1.5$$

Question 11:

Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X.

Solution:

Here, X represents the number of sixes obtained when two dice are thrown simultaneously.
Thus, X can take the value of 0, 1 or 2.

$$\therefore P(X=0) = P(\text{not getting six on any of the dice}) = \frac{25}{36}$$

$$P(X=1) = P(\text{six on first die and no six on second die}) + P(\text{no six on first die and six on second die})$$

$$= 2 \times \left(\frac{1}{6} \times \frac{5}{6} \right) = \frac{10}{36}$$

$$P(X=2) = P(\text{six on both the dice}) = \frac{1}{36}$$

Thus, the required probability distribution is as follows.

X	0	1	2
P (X)	$\dfrac{25}{36}$	$\dfrac{10}{36}$	$\dfrac{1}{36}$

Then, expectation of $X = E(X) = \Sigma X_i P(X_i)$

$$= 0 \times \frac{25}{36} + 1 \times \frac{10}{36} + 2 \times \frac{1}{36}$$

$$= \frac{1}{3}$$

Question 12:

Two numbers are selected at random (without replacement) from the first six positive integers.

Let X denotes the larger of two numbers obtained. Find $E(X)$.

Solution:

The two positive integers can be selected from the first six positive integers without

replacement in $6 \times 5 = 30 \text{ ways}$.

X represents the larger of the two numbers obtained.

Thus, X can take the value of 2, 3, 4, 5 or 6.

For $X = 2$, the possible observations are $(1,2) \text{ and } (2,1)$.

$$\therefore P(X = 2) = \frac{2}{30} = \frac{1}{15}$$

For $X = 3$, the possible observations are $(1,3), (2,3), (3,1) \text{ and} (3,2)$.

$$\therefore P(X = 3) = \frac{4}{30} = \frac{2}{15}$$

For $X = 4$, the possible observations are $(1,4), (2,4), (3,4), (4,3), (4,2) \text{ and} (4,1)$.

$$\therefore P(X = 4) = \frac{6}{30} = \frac{1}{5}$$

For $X = 5$, the possible observations are $(1,5), (2,5), (3,5), (4,5), (5,4), (5,3), (5,2) \text{ and } (5,1)$

$$\therefore P(X = 5) = \frac{8}{30} = \frac{4}{15}$$

For $X = 6$, the possible observations are

$(1,6),(2,6),(3,6),(4,6),(5,6),(6,4),(6,3),(6,2),(6,1)$.

$\therefore P(X = 6) = \dfrac{10}{30} = \dfrac{1}{3}$

Thus, the required probability distribution is as follows.

X	2	3	4	5	6
P (X)	$\dfrac{1}{15}$	$\dfrac{2}{15}$	$\dfrac{1}{5}$	$\dfrac{4}{15}$	$\dfrac{1}{3}$

Then, $E(X) = \Sigma X_i P(X_i)$

$= 2 \times \dfrac{1}{15} + 3 \times \dfrac{2}{15} + 4 \times \dfrac{1}{5} + 5 \times \dfrac{4}{15} + 6 \times \dfrac{1}{3}$

$= \dfrac{2}{15} + \dfrac{2}{5} + \dfrac{4}{5} + \dfrac{4}{3} + 2$

$= \dfrac{70}{15} = \dfrac{14}{3}$

Question 13:

Let X denotes the sum of the number obtained when two fair dice are rolled. Find the variance and standard deviation of X.

Solution:

Here, X can take values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12.

$$P(X=2) = P(1,1) = \frac{1}{36}$$

$$P(X=3) = P(1,2) + P(2,1) = \frac{2}{36} = \frac{1}{18}$$

$$P(X=4) = P(1,3) + P(2,2) + P(3,1) = \frac{3}{36} = \frac{1}{12}$$

$$P(X=5) = P(1,4) + P(2,3) + P(3,2) + P(4,1) = \frac{4}{36} = \frac{1}{9}$$

$$P(X=6) = P(1,5) + P(2,4) + P(3,3) + P(4,2) + P(5,1) = \frac{5}{36}$$

$$P(X=7) = P(1,6) + P(2,5) + P(3,4) + P(4,3) + P(5,2) + P(6,1) = \frac{6}{36} = \frac{1}{6}$$

$$P(X=8) = P(2,6) + P(3,5) + P(4,4) + P(5,3) + P(6,2) = \frac{5}{36}$$

$$P(X=9) = P(3,6) + P(4,5) + P(5,4) + P(6,3) = \frac{4}{36} = \frac{1}{9}$$

$$P(X=10) = P(4,6) + P(5,5) + P(6,4) = \frac{3}{36} = \frac{1}{12}$$

$$P(X=11) = P(5,6) + P(6,5) = \frac{2}{36} = \frac{1}{18}$$

$$P(X=12) = P(6,6) = \frac{1}{36}$$

Thus, the required probability distribution is as follows.

X	2	3	4	5	6	7	8	9	10	11	12
P (X)	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

Then, $E(X) = \Sigma X_i P(X_i)$

$$E(X) = \Sigma X_i P(X_i)$$

$$= 2 \times \frac{1}{36} + 3 \times \frac{1}{18} + 4 \times \frac{1}{12} + 5 \times \frac{1}{9} + 6 \times \frac{5}{36} + 7 \times \frac{1}{6} + 8 \times \frac{5}{36} + 9 \times \frac{1}{9} + 10 \times \frac{1}{12} + 11 \times \frac{1}{18} + 12 \times \frac{1}{36}$$

$$= \frac{1}{18} + \frac{1}{6} + \frac{1}{3} + \frac{5}{9} + \frac{5}{6} + \frac{7}{6} + \frac{10}{9} + 1 + \frac{5}{6} + \frac{11}{18} + \frac{1}{3}$$

$$= 7$$

$$E(X^2) = \Sigma X_i^2 P(X_i)$$

$$= 4 \times \frac{1}{36} + 9 \times \frac{1}{18} + 16 \times \frac{1}{12} + 25 \times \frac{1}{9} + 36 \times \frac{5}{36} + 49 \times \frac{1}{6} + 64 \times \frac{5}{36} + 81 \times \frac{1}{9} + 100 \times \frac{1}{12} + 121 \times \frac{1}{18} + 144 \times \frac{1}{36}$$

$$= \frac{1}{9} + \frac{1}{2} + \frac{4}{3} + \frac{25}{9} + 5 + \frac{49}{6} + \frac{80}{9} + 9 + \frac{25}{3} + \frac{121}{18} + 4$$

$$= \frac{987}{18} = \frac{329}{6} = 54.833$$

Then, $\text{Var}(X) = E(X^2) - \left[E(X)\right]^2$

$$= 54.833 - 7^2$$
$$= 54.833 - 49$$
$$= 5.833$$

$\therefore$ Standard Variation $= \sqrt{\text{Var}(X)}$

$$= \sqrt{5.833}$$
$$= 2.415$$

Question 14:

A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. What is the probability distribution of the random variable X? Find mean, variance and standard deviation of X.

Solution:

There are 15 students in the class. Each student has the same chance to be chosen.

Thus, the probability of each student to be selected is $\frac{1}{15}$.

The given information can be shown in the frequency table as follows.

X	14	15	16	17	18	19	20	21
f	2	1	2	3	1	2	3	1

$$P(X = 14) = \frac{2}{15}, P(X = 15) = \frac{1}{15}, P(X = 16) = \frac{2}{15}, P(X = 17) = \frac{3}{15}$$

$$P(X = 18) = \frac{1}{15}, P(X = 19) = \frac{2}{15}, P(X = 20) = \frac{3}{15}, P(X = 21) = \frac{1}{15}$$

Thus, the probability distribution of random variable X is as follows.

X	14	15	16	17	18	19	20	21
P (X)	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

Then, Mean $X = E(X)$

$= \Sigma X_i P(X_i)$

$= 14 \times \dfrac{2}{15} + 15 \times \dfrac{1}{15} + 16 \times \dfrac{2}{15} + 17 \times \dfrac{3}{15} + 18 \times \dfrac{1}{15} + 19 \times \dfrac{2}{15} + 20 \times \dfrac{3}{15} + 21 \times \dfrac{1}{15}$

$= \dfrac{1}{15}(28 + 15 + 32 + 51 + 18 + 38 + 60 + 21)$

$= \dfrac{263}{15} = 17.53$

$E(X^2) = \Sigma X_i^2 P(X_i)$

$= (14)^2 \times \dfrac{2}{15} + (15)^2 \times \dfrac{1}{15} + (16)^2 \times \dfrac{2}{15} + (17)^2 \times \dfrac{3}{15} + (18)^2 \times \dfrac{1}{15} + (19)^2 \times \dfrac{2}{15} + (20)^2 \times \dfrac{3}{15} + (21)^2 \times \dfrac{1}{15}$

$= \dfrac{1}{15}(392 + 225 + 512 + 867 + 324 + 722 + 1200 + 441)$

$= \dfrac{4683}{15} = 312.2$

$\therefore$ Variance$(X) = E(X^2) - [E(X)]^2$

$= 312.2 - \left(\dfrac{263}{15}\right)^2$

$= 312.2 - 307.4177$

$= 4.7823$

$= 4.78$

$\therefore$ Standard Variation $= \sqrt{\text{Variance}(X)}$

$= \sqrt{4.78}$

$= 2.186 \approx 2.19$

Question 15:

In a meeting, 70% of the members favour and 30% oppose a certain proposal. A member is selected at random and we take $X = 0$ if he opposed, and $X = 1$ if he is in favour. Find $E(X)$ and $var(X)$.

Solution:

Given, $P(X = 0) = 30\% = \dfrac{30}{100} = 0.3$

$P(X = 1) = 70\% = \dfrac{70}{100} = 0.7$

Thus, the probability distribution is as follows.

X	0	1
P (X)	0.3	0.7

$$E(X) = \Sigma X_i P(X_i)$$
$$= 0 \times 0.3 + 1 \times 0.7$$
$$= 0.7$$

$$E(X^2) = \Sigma X_i^2 P(X_i)$$
$$= 0^2 \times 0.3 + 1^2 \times 0.7$$
$$= 0.7$$

$$Var(X) = E(X^2) - \left[E(X)\right]^2$$
$$= 0.7 - (0.7)^2$$
$$= 0.7 - 0.49$$
$$= 0.21$$

Question 16:

The mean of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face is

(A) 1   (B) 2   (C) 5   (D) $\dfrac{8}{3}$

Solution:

Let X be the random variable representing a number on the die.
The total number of observations is six.

$$\therefore P(X=1) = \frac{3}{6} = \frac{1}{2}$$
$$P(X=2) = \frac{2}{6} = \frac{1}{3}$$
$$P(X=5) = \frac{1}{6}$$

Thus, the probability distribution is as follows.

X	1	2	5
P (X)	$\dfrac{1}{2}$	$\dfrac{1}{3}$	$\dfrac{1}{6}$

$$\text{Mean} = E(X) = \Sigma X_i P(X_i)$$

$$= \frac{1}{2} \times 1 + \frac{1}{3} \times 2 + \frac{1}{6} \times 5$$

$$= \frac{1}{2} + \frac{2}{3} + \frac{5}{6}$$

$$= \frac{12}{6} = 2$$

The correct answer is B.

Question 17:

Suppose that two cards are drawn at random from a deck of cards. Let X be the number of aces obtained. Then the value of $E(X)$ is

(A) $\dfrac{37}{221}$ (B) $\dfrac{5}{13}$ (C) $\dfrac{1}{13}$ (D) $\dfrac{2}{13}$

Solution:

Let X denote the number of aces obtained.

Thus, X can take any of the values of 0, 1, or 2.

Since, in a deck of 52 cards, 4 cards are aces. Thus, there are 48 non-ace cards.

$$\therefore P(X=0) = P(0 \text{ ace and 2 non-ace cards}) = \frac{{}^{4}C_{0} \times {}^{48}C_{2}}{{}^{52}C_{2}} = \frac{1128}{1326}$$

$$P(X=1) = P(1 \text{ ace and 1 non-ace cards}) = \frac{{}^{4}C_{1} \times {}^{48}C_{1}}{{}^{52}C_{2}} = \frac{192}{1326}$$

$$P(X=2) = P(2 \text{ ace and 0 non-ace cards}) = \frac{{}^{4}C_{2} \times {}^{48}C_{0}}{{}^{52}C_{2}} = \frac{6}{1326}$$

Thus, the probability distribution is as follows.

X	0	1	2
P (X)	$\dfrac{1128}{1326}$	$\dfrac{192}{1326}$	$\dfrac{6}{1326}$

$$E(X) = \Sigma X_i P(X_i)$$

$$= 0 \times \frac{1128}{1326} + 1 \times \frac{192}{1326} + 2 \times \frac{6}{1326}$$

$$= \frac{204}{1326} = \frac{2}{13}$$

Thus, the correct answer is D.

EXERCISE 13.5

Question 1:

A die is thrown 6 times. If 'getting an odd number' is a success, what is the probability of
(i) 5 successes?   (ii) at least 5 success?   (iii) at most 5 successes?

Solution:

Let X denote the number of success of getting odd numbers in an experiment of 6 trials.

Probability of getting an odd number in a single throw of a die is, $p = \dfrac{3}{6} = \dfrac{1}{2}$

$\therefore q = 1 - p = \dfrac{1}{2}$

X has a binominal distribution.

Thus, $P(X = x) = {}^nC_x q^{n-x} p^x$, where n = 0, 1, 2 ....n

$= {}^6C_x \left(\dfrac{1}{2}\right)^{6-x} \times \left(\dfrac{1}{2}\right)^x$

$= {}^6C_x \left(\dfrac{1}{2}\right)^6$

(i) $P(5\ success) = P(X = 5)$

$= {}^6C_5 \left(\dfrac{1}{2}\right)^6$

$= 6 \times \dfrac{1}{64} = \dfrac{3}{32}$

(ii) $P(at\ least\ 5\ success) = P(X \ge 5)$

$= P(X = 5) + P(X = 6)$

$= {}^6C_5 \left(\dfrac{1}{2}\right)^6 + {}^6C_6 \left(\dfrac{1}{2}\right)^6$

$= 6 \times \dfrac{1}{64} + 1 \times \dfrac{1}{64} = \dfrac{7}{64}$

(iii) $P(at\ most\ 5\ success) = P(X \le 5)$

$= 1 - P(X > 5)$

$= 1 - P(X = 6)$

$= 1 - {}^6C_6 \left(\dfrac{1}{2}\right)^6$

$= 1 - \dfrac{1}{64} = \dfrac{63}{64}$

Question 2:

A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of two success.

Solution:

Let X denote the number of times of getting doublets in an experiment of throwing two dice simultaneously four times.

Probability of getting doublets in a single throw of the pair of dice is

$$p = \frac{6}{36} = \frac{1}{6}$$

$$\therefore \quad q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

Clearly, X has the binomial distribution with $n = 4,\ p = \frac{1}{6}$ and $q = \frac{5}{6}$

$$P(X = x) = {}^{n}C_{x}q^{n-x}p^{x}, \text{where } x = 0, 1, 2, 3 \ldots n$$

$$= {}^{4}C_{x}\left(\frac{5}{6}\right)^{4-x} \times \left(\frac{1}{6}\right)^{x}$$

$$= {}^{4}C_{x} \times \frac{5^{4-x}}{6^{4}}$$

$$\therefore \quad P(2 \text{ successes}) = P(X = 2)$$

$$= {}^{4}C_{2} \times \frac{5^{4-2}}{6^{4}}$$

$$= 6 \times \frac{25}{1296} = \frac{25}{216}$$

Question 3:

There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Solution:

Let X denote the number of defective items in a sample of 10 items drawn successively.

$$\Rightarrow p = \frac{5}{100} = \frac{1}{20}$$

$$\therefore \quad q = 1 - \frac{1}{20} = \frac{19}{20}$$

X has a binomial distribution with $n = 10$, $p = \dfrac{1}{20}$ and $q = \dfrac{19}{20}$

$P(X = x) = {}^{n}C_{x} q^{n-x} p^{x}$, where $x = 0, 1, 2 \ldots n$

$= {}^{10}C_{x} \left(\dfrac{19}{20}\right)^{10-x} \times \left(\dfrac{1}{20}\right)^{x}$

$P(\text{not more than 1 defective item}) = P(X \leq 1)$

$= P(X = 0) + P(X = 1)$

$= {}^{10}C_{0} \left(\dfrac{19}{20}\right)^{10} \times \left(\dfrac{1}{20}\right)^{0} + {}^{10}C_{1} \left(\dfrac{19}{20}\right)^{9} \times \left(\dfrac{1}{20}\right)^{1}$

$= \left(\dfrac{19}{20}\right)^{10} + 10 \times \left(\dfrac{19}{20}\right)^{9} \times \left(\dfrac{1}{20}\right)$

$= \left(\dfrac{19}{20}\right)^{9} \left(\dfrac{19}{20} + \dfrac{10}{20}\right)$

$= \left(\dfrac{29}{20}\right) \times \left(\dfrac{19}{20}\right)^{9}$

Question 4:

Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that
(i) all the five cards are spades?
(ii) only 3 cards are spades?
(iii) none is a spade?

Solution:

Let X represent the number of spade cards among the five cards drawn.

In a well shuffled deck of 52 cards, there are 13 spades cards.

$p = \dfrac{13}{52} = \dfrac{1}{4}$

$\therefore\ q = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

X has a binomial distribution with $n = 5$, $p = \dfrac{1}{4}$ and $q = \dfrac{3}{4}$

$P(X = x) = {}^{n}C_{x} q^{n-x} p^{x}$, where $x = 0, 1, 2 \ldots n$

$= {}^{5}C_{x} \left(\dfrac{3}{4}\right)^{5-x} \times \left(\dfrac{1}{4}\right)^{x}$

(i) $P(\text{all five cards are spades}) = P(X = 5)$

$$= {}^5C_5 \left(\frac{3}{4}\right)^0 \times \left(\frac{1}{4}\right)^5$$

$$= 1 \times \frac{1}{1024} = \frac{1}{1024}$$

(ii) $P(\text{only 3 cards are spades}) = P(X = 3)$

$$= {}^5C_3 \left(\frac{3}{4}\right)^2 \times \left(\frac{1}{4}\right)^3$$

$$= 10 \times \frac{9}{16} \times \frac{1}{64}$$

$$= \frac{45}{512}$$

(iii) $P(\text{none is a spade}) = P(X = 0)$

$$= {}^5C_0 \left(\frac{3}{4}\right)^5 \times \left(\frac{1}{4}\right)^0$$

$$= 1 \times \frac{243}{1024} = \frac{243}{1024}$$

Question 5:

The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. What is the probability that out of 5 such bulbs
(i) none
(ii) not more than one
(iii) more than one
(iv) at least one
Will fuse after 150 days of use.

Solution:

Let X represents the number of bulbs that will fuse after 150 days of use in an experiment of 5 trials.

Given, $p = 0.05$

$\therefore q = 1 - p = 1 - 0.05 = 0.95$

X has a binomial distribution with $n = 5$, $p = 0.05$ and $q = 0.95$

$$P(X = x) = {}^nC_x q^{n-x} p^x, \text{where } x = 0, 1, 2 \ldots n$$

$$= {}^5C_x \times (0.95)^{5-x} \times (0.05)^x$$

(i) $P(\text{none}) = P(X = 0)$

$= {}^5C_0 \times (0.95)^5 \times (0.05)^0$

$= 1 \times (0.95)^5$

$= (0.95)^5$

$= 0.7737$

(ii) $P(\text{not more than one}) = P(X \leq 1)$

$= P(X = 0) + P(X = 1)$

$= {}^5C_0 \times (0.95)^5 \times (0.05)^0 + {}^5C_1 \times (0.95)^4 \times (0.05)^1$

$= 1 \times (0.95)^5 + 5 \times (0.95)^4 \times (0.05)$

$= (0.95)^5 + (0.25) \times (0.95)^4$

$= (0.95)^4 \times [0.95 + 0.25]$

$= (0.95)^4 \times 1.2$

$= 0.977$

(iii) $P(\text{more than 1}) = P(X > 1)$

$= 1 - P(X \leq 1)$

$= 1 - P(\text{not more than 1})$

$= 1 - (0.95)^4 \times 1.2$

$= 0.02$

(iv) $P(\text{at least one}) = P(X \geq 1)$

$= 1 - P(X < 1)$

$= 1 - P(X = 0)$

$= 1 - {}^5C_0 \times (0.95)^5 \times (0.05)^0$

$= 1 - 1 \times (0.95)^5$

$= 1 - (0.95)^5$

$= 0.2263$

Question 6:

A bag consists of 10 balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?

Solution:

Let X denote the number of balls marked with the digit 0 among the 4 balls drawn.

X has a binominal distribution with $n = 4$ and $p = \dfrac{1}{10}$

$\therefore q = 1 - p = 1 - \dfrac{1}{10} = \dfrac{9}{10}$

$P(X = x) = {}^{n}C_{x}\, q^{n-x} p^{x}, \text{where } x = 1, 2 \dots n$

$= {}^{4}C_{x} \left(\dfrac{9}{10}\right)^{4-x} \times \left(\dfrac{1}{10}\right)^{x}$

$P(\text{none marked with } 0) = P(X = 0)$

$= {}^{4}C_{0} \left(\dfrac{9}{10}\right)^{4} \times \left(\dfrac{1}{4}\right)^{0}$

$= 1 \times \left(\dfrac{9}{10}\right)^{4} = \left(\dfrac{9}{10}\right)^{4}$

Question 7:

In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answers 'true'; if it falls tail, he answers 'false'. Find the probability that he answers at least 12 questions correctly.

Solution:

Let X represent the number of correctly answered questions out of 20 questions.

Since "head" on a coin represents the true answer and "tail" represents the false answer, the correctly answered questions are Bernoulli trials.

$\therefore p = \dfrac{1}{2}$

$\therefore q = 1 - p = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

X has a binominal distribution with $n = 20$ and $p = \dfrac{1}{2}$

$P(X = x) = {}^{n}C_{x}\, q^{n-x} p^{x}, \text{where } x = 0, 1, 2 \dots n$

$= {}^{20}C_{x} \left(\dfrac{1}{2}\right)^{20-x} \times \left(\dfrac{1}{2}\right)^{x}$

$= {}^{20}C_{x} \left(\dfrac{1}{2}\right)^{20}$

$P(\text{at least 12 questions answered correctly}) = P(X \geq 12)$

$= P(X = 12) + P(X = 13) + ... + P(X = 20)$

$= {}^{20}C_{12}\left(\dfrac{1}{2}\right)^{20} + {}^{20}C_{13}\left(\dfrac{1}{2}\right)^{20} + ... + {}^{20}C_{20}\left(\dfrac{1}{2}\right)^{20}$

$= \left(\dfrac{1}{2}\right)^{20} \times \left[{}^{20}C_{12} + {}^{20}C_{13} + ... + {}^{20}C_{20} \right]$

Question 8:

Suppose X has a binominal distribution $B\left(6 \times \dfrac{1}{2}\right)$. Show that $X = 3$ is the most likely outcome.

(Hint: $P(X) = 3$ is the maximum among all $P(x_i), \; x_i = 0, 1, 2, 3, 4, 5, 6$)

Solution:

X is the random variable whose binomial distribution is $B\left(6, \dfrac{1}{2}\right)$

Thus, $n = 6$ and $p = \dfrac{1}{2}$

$\therefore \; q = 1 - p = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

Then,

$P(X = x) = {}^{n}C_x q^{n-x} p^{x}$

$= {}^{6}C_x \left(\dfrac{1}{2}\right)^{6-x} \times \left(\dfrac{1}{2}\right)^{x}$

$= {}^{6}C_x \left(\dfrac{1}{2}\right)^{6}$

It can be seen that $P(X = x)$ will be maximum, if ${}^{6}C_x$ will be maximum.

Then, ${}^{6}C_0 = {}^{6}C_6 = \dfrac{6!}{0! \times 6!} = 1$

${}^{6}C_1 = {}^{6}C_5 = \dfrac{6!}{1! \times 5!} = 6$

${}^{6}C_2 = {}^{6}C_4 = \dfrac{6!}{2! \times 4!} = 15$

${}^{6}C_3 = \dfrac{6!}{3! \times 3!} = 20$

The value of ${}^{6}C_3$ is maximum. Therefore, for $x = 3$, $P(X=x)$ is maximum.

Thus, $X = 3$ is the most likely outcome.

Question 9:

On a multiple-choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

Solution:

Let X represent the number of correct answers by guessing in the set of 5 multiple choice questions.

Probability of getting a correct answer is, $p = \dfrac{1}{3}$

$\therefore \; q = 1 - p = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

Clearly, X has a binomial distribution with $n = 5$ and $p = \dfrac{1}{3}$

$$P(X = x) = {}^nC_x q^{n-x} p^x$$
$$= {}^5C_x \left(\dfrac{2}{3}\right)^{5-x} \times \left(\dfrac{1}{3}\right)^x$$

$$P(\text{guessing more than 4 correct answers}) = P(X \geq 4)$$
$$= P(X = 4) + P(X = 5)$$
$$= {}^5C_4 \left(\dfrac{2}{3}\right) \times \left(\dfrac{1}{3}\right)^4 + {}^5C_5 \times \left(\dfrac{1}{3}\right)^5$$
$$= 5 \times \dfrac{2}{3} \times \dfrac{1}{81} + 1 \times \dfrac{1}{243}$$
$$= \dfrac{10}{243} + \dfrac{1}{243} = \dfrac{11}{243}$$

Question 10:

A person buys a lottery ticket in 50 lotteries, in each of which his change of winning a prize is $\dfrac{1}{100}$. What is the probability that he will in a prize
(A) at least once   (B) exactly once   (C) at least twice?

Solution:

Let X represents the number of winning prizes in 50 lotteries.

Clearly, X has a binominal distribution with $n = 50$ and $p = \dfrac{1}{100}$

$$\therefore \; q = 1 - p = 1 - \frac{1}{100} = \frac{99}{100}$$

$$P(X = x) = {}^{n}C_{x}\, q^{n-x}\, p^{x}$$

$$= {}^{50}C_{x} \left(\frac{99}{100}\right)^{50-x} \times \left(\frac{1}{100}\right)^{x}$$

(a) $P(\text{winning at least once}) = P(X \geq 1)$

$$= 1 - P(X < 1)$$

$$= 1 - P(X = 0)$$

$$= 1 - {}^{50}C_{0} \times \left(\frac{99}{100}\right)^{50}$$

$$= 1 - 1 \times \left(\frac{99}{100}\right)^{50}$$

$$= 1 - \left(\frac{99}{100}\right)^{50}$$

(b) $P(\text{Winning exactly once}) = P(X = 1)$

$$= {}^{50}C_{1} \times \left(\frac{99}{100}\right)^{49} \times \left(\frac{1}{100}\right)^{1}$$

$$= 50 \times \left(\frac{1}{100}\right) \times \left(\frac{99}{100}\right)^{49}$$

$$= \frac{1}{2} \times \left(\frac{99}{100}\right)^{49}$$

(c) $P(\text{winning at least twice}) = P(X \geq 2)$

$$= 1 - P(X < 2)$$

$$= 1 - P(X \leq 1)$$

$$= 1 - \big[P(X = 0) + P(X = 1) \big]$$

$$= \big[1 - P(X = 0) \big] - P(X = 1)$$

$$= 1 - \left(\frac{99}{100} \right)^{50} - \frac{1}{2} \times \left(\frac{99}{100} \right)^{49}$$

$$= 1 - \left(\frac{99}{100} \right)^{49} \times \left[\frac{99}{100} + \frac{1}{2} \right]$$

$$= 1 - \left(\frac{99}{100} \right)^{49} \times \left(\frac{149}{100} \right)$$

$$= 1 - \left(\frac{149}{100} \right) \times \left(\frac{99}{100} \right)^{49}$$

Question 11:

Find the probability of getting 5 exactly twice in 7 throws of a die.

Solution:

Let X represent the number of times of getting 5 in 7 throws of the die.

Probability of getting 5 in a single throw of the die, $p = \dfrac{1}{6}$

$$\therefore q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

Clearly, X has the probability distribution with $n = 7$ and $p = \dfrac{1}{6}$

$$\therefore P(X = x) = {}^{n}C_{x} q^{n-x} p^{x} = {}^{7}C_{x} \left(\frac{5}{6} \right)^{7-x} \times \left(\frac{1}{6} \right)^{x}$$

P(getting 5 exactly twice) $= P(X = 2)$

$$= {}^{7}C_{2} \times \left(\frac{5}{6} \right)^{5} \times \left(\frac{1}{6} \right)^{2}$$

$$= 21 \times \left(\frac{5}{6} \right)^{5} \times \frac{1}{36}$$

$$= \frac{7}{12} \times \left(\frac{5}{6} \right)^{5}$$

Question 12:

Find the probability of throwing at most 2 sixes in 6 throws of a single die.

Solution:

Let X represent the number of times of getting sixes in 6 throws of the die.

Probability of getting six in a single throw of die, $p = \dfrac{1}{6}$

$$\therefore \; q = 1 - p = 1 - \dfrac{1}{6} = \dfrac{5}{6}$$

Clearly, X has a binomial distribution with $n = 6$

$$\therefore \; P(X = x) = {}^{n}C_{x}\, q^{n-x}\, p^{x} = {}^{6}C_{x}\left(\dfrac{5}{6}\right)^{6-x} \times \left(\dfrac{1}{6}\right)^{x}$$

$$P(\text{at most 2 sixes}) = P(X \le 2)$$
$$= P(X = 0) + P(X = 1) + P(X = 2)$$
$$= {}^{6}C_{0}\left(\dfrac{5}{6}\right)^{6} + {}^{6}C_{1}\left(\dfrac{5}{6}\right)^{5} \times \left(\dfrac{1}{6}\right) + {}^{6}C_{2}\left(\dfrac{5}{6}\right)^{4} \times \left(\dfrac{1}{6}\right)^{2}$$
$$= 1 \times \left(\dfrac{5}{6}\right)^{6} + 6 \times \dfrac{1}{6} \times \left(\dfrac{5}{6}\right)^{5} + 15 \times \dfrac{1}{36} \times \left(\dfrac{5}{6}\right)^{4}$$
$$= \left(\dfrac{5}{6}\right)^{6} + \left(\dfrac{5}{6}\right)^{5} + \dfrac{5}{12} \times \left(\dfrac{5}{6}\right)^{4}$$
$$= \left(\dfrac{5}{6}\right)^{4}\left[\left(\dfrac{5}{6}\right)^{2} + \left(\dfrac{5}{6}\right) + \left(\dfrac{5}{12}\right)\right]$$
$$= \left(\dfrac{5}{6}\right)^{4}\left[\dfrac{25}{36} + \dfrac{5}{6} + \dfrac{5}{12}\right]$$
$$= \left(\dfrac{5}{6}\right)^{4}\left[\dfrac{25 + 30 + 15}{36}\right]$$

$$= \dfrac{70}{36} \times \left(\dfrac{5}{6}\right)^{4} = \dfrac{35}{18} \times \left(\dfrac{5}{6}\right)^{4}$$
$$= 0.9377$$

Question 13:

It is known that 10% of certain articles manufactured are defective. What is the probability that in a random sample of 12 such articles, 9 are defective?

Solution:

Let X denote the number of times of selecting defective articles in a random sample space of 12 articles.

Clearly, X has a binominal distribution with $n = 12$ and $p = 10\% = \dfrac{10}{100} = \dfrac{1}{10}$

$\therefore\ q = 1 - p = 1 - \dfrac{1}{10} = \dfrac{9}{10}$

$\therefore\ P(X = x) = {}^nC_x q^{n-x} p^x = {}^{12}C_x \left(\dfrac{9}{10}\right)^{12-x} \times \left(\dfrac{1}{10}\right)^x$

$P(\text{Selecting 9 defective articles}) = {}^{12}C_9 \times \left(\dfrac{9}{10}\right)^3 \times \left(\dfrac{1}{10}\right)^9$

$= 220 \times \dfrac{9^3}{10^3} \times \dfrac{1}{10^9}$

$= \dfrac{22 \times 9^3}{10^{11}}$

Question 14:

In a box containing 100 bulbs, 10 are defective. The probability that out of a sample of 5 bulbs, none is defective is

(A) 10^{-1}

(B) $\left(\dfrac{1}{2}\right)^5$

(C) $\left(\dfrac{9}{10}\right)^5$

(D) $\dfrac{9}{10}$

Solution:

Let X denote the number of defective bulbs out of a sample of 5 bulbs.

Probability of getting a defective bulb $p = \dfrac{10}{100} = \dfrac{1}{10}$

$\therefore\ q = 1 - p = 1 - \dfrac{1}{10} = \dfrac{9}{10}$

Clearly, X has binominal distribution with $n = 5$ and $p = \dfrac{1}{10}$

$\therefore\ P(X = x) = {}^nC_x q^{n-x} p^x = {}^5C_x \left(\dfrac{9}{10}\right)^{5-x} \times \left(\dfrac{1}{10}\right)^x$

$$P(\text{none of the bulbs is defective}) = P(X = 0)$$

$$= {}^5C_0 \times \left(\frac{9}{10}\right)^5$$

$$= 1 \times \left(\frac{9}{10}\right)^5$$

$$= \left(\frac{9}{10}\right)^5$$

The correct answer is C.

Question 15:

The probability that a student is not a swimming is $\frac{1}{5}$. The probability that out of five students, four are swimmers is

(A) ${}^5C_4 \left(\frac{4}{5}\right)^4 \frac{1}{5}$

(B) $\left(\frac{4}{5}\right)^4 \frac{1}{5}$

(C) ${}^5C_1 \frac{1}{5}\left(\frac{4}{5}\right)^4$

(D) None of these

Solution:

Let X denote the number of students, out of 5 students, who are swimmers.

Probability of students who are not swimmers, $q = \frac{1}{5}$

$\therefore \; p = 1 - q = 1 - \frac{1}{5} = \frac{4}{5}$

Clearly, X has a binomial distribution with $n = 5$ and $p = \frac{4}{5}$

$\therefore \; P(X = x) = {}^nC_x q^{n-x} p^x = {}^5C_x \times \left(\frac{1}{5}\right)^{5-x} \times \left(\frac{4}{5}\right)^x$

$P(\text{four students are swimmers}) = P(X = 4)$

$$= {}^5C_4 \times \left(\frac{1}{5}\right) \times \left(\frac{4}{5}\right)^4$$

Therefore, the correct answer is A.

MISCELLANEOUS EXERCISE

Question 1:

A and B are two events such that $P(A) \neq 0$, find $P(B|A)$, if
(i) A is a subset of B   (ii) $A \cap B = \phi$.

Solution:

Given, $P(A) \neq 0$
(i) A is a subset of B.
$\Rightarrow A \cap B = A$
$\therefore P(A \cap B) = P(B \cap A) = P(A)$
$\therefore P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{P(A)}{P(A)} = 1$

(ii) $A \cap B = \phi$
$\Rightarrow P(A \cap B) = 0$
$\therefore P(B|A) = \dfrac{P(A \cap B)}{P(A)} = 0$

Question 2:

A couple has two children,
(i) Find the probability that both children are males, if it is known that at least one of the children is male.
(ii) Find the probability that both children are females, if it is known that the elder child is a female.

Solution:

If a couple has two children, then the sample space is $S = \{(B, B), (B, G), (G, B), (G, G)\}$

(i) Let E and F respectively denote the events that both children are male and atleast one children is a male.

$E \cap F = \{(G, G)\} \Rightarrow P(E \cap F) = \dfrac{1}{4}$

$P(E) = \dfrac{1}{4}$

$P(F) = \dfrac{3}{4}$

$$\Rightarrow P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

(ii) Let C and D respectively denote the events that both children are females and the elder child is a female.

$$C = \{(G,G)\} \Rightarrow P(C) = \frac{1}{4}$$

$$D = \{(G,B),(G,G)\} \Rightarrow P(D) = \frac{2}{4}$$

$$C \cap D = \{(G,G)\} \Rightarrow P(C \cap D) = \frac{1}{4}$$

$$\therefore P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

Question 3:

Suppose that 5% of men and 0.25% of women have grey hair. A haired person is selected at random. What is the probability of this person being male?
Assume that there are equal numbers of males and females.

Solution:

Given, 5% of men and 0.25% of women have grey hair.

Thus, percentage of people with grey hair $= (5 + 0.25)\% = 5.25\%$

Probability that the selected haired person is a male $= \frac{5}{5.25} = \frac{20}{21}$

Question 4:

Suppose that 90% of people are right-handed. What is the probability that at most 6 of a random sample of 10 people are right-handed?

Solution:

A person can be either right-handed or left-handed.

Given, 90% of the people are right-handed.

$$\therefore \quad p = \text{P}(\text{right-handed}) = \frac{9}{10}$$

$$\therefore \quad q = \text{P}(\text{left-handed}) = 1 - p = 1 - \frac{9}{10} = \frac{1}{10}$$

Using binominal distribution, the probability that more than 6 people are right-handed is given by

$$\sum_{r=7}^{10} {}^{10}C_r p^r q^{n-r} = \sum_{r=7}^{10} {}^{10}C_r \left(\frac{9}{10}\right)^r \times \left(\frac{1}{10}\right)^{10-r}$$

Therefore, the probability that at most 6 people are right-handed

$$= 1 - \text{P}(\text{more than 6 are right-handed})$$

$$= 1 - \sum_{r=7}^{10} {}^{10}C_r (0.9)^r \times (0.1)^{10-r}$$

Question 5:

An urn contains 25 balls of which 10 balls bear a mark 'X' and the remaining 15 bear a mark 'Y'. A ball is drawn at random from the urn, its mark is noted down, and it is replaced. If 6 balls are drawn in this way, find the probability that

(i) all will bear 'X' mark.
(ii) not more than 2 will bear 'Y' mark.
(iii) at least one ball will bear 'Y' mark.
(iv) the number of balls with 'X' mark and 'Y' mark will be equal.

Solution:

Total number of balls in the urn $= 25$

Balls bearing mark 'X' $= 10$

Balls bearing mark 'Y' $= 15$

$$p = \text{P}(\text{ball bearing mark 'X'}) = \frac{10}{25} = \frac{2}{5}$$

$$q = \text{P}(\text{Balls bearing mark 'Y'}) = \frac{15}{25} = \frac{3}{5}$$

Let Z be the random variable that represents the number of balls with 'Y' mark on them in the trials.

Clearly, Z has a binomial distribution with $n = 6$ and $p = \frac{2}{5}$.

$$\therefore \quad \text{P}(Z = z) = {}^n C_z p^{n-z} q^z$$

(i) $\quad \text{P}(\text{all will bear 'X' mark}) = \text{P}(Z = 0) = {}^6 C_0 \left(\frac{2}{5}\right)^6 = \left(\frac{2}{5}\right)^6$

(ii) $P(\text{not more than 2 bear 'Y' mark}) = P(Z \le 2)$

$= P(Z = 0) + P(Z = 1) + P(Z = 2)$

$= {}^6C_0 \times (p)^6 \times (q)^0 + {}^6C_1 \times (p)^5 \times (q)^1 + {}^6C_2 \times (p)^4 \times (q)^2$

$= \left(\dfrac{2}{5}\right)^6 + 6 \times \left(\dfrac{2}{5}\right)^5 \times \left(\dfrac{3}{5}\right) + 15 \times \left(\dfrac{2}{5}\right)^4 \times \left(\dfrac{3}{5}\right)^2$

$= \left(\dfrac{2}{5}\right)^4 \times \left[\left(\dfrac{2}{5}\right)^2 + 6 \times \left(\dfrac{2}{5}\right) \times \left(\dfrac{3}{5}\right) + 15 \times \left(\dfrac{3}{5}\right)^2 \right]$

$= \left(\dfrac{2}{5}\right)^4 \times \left[\dfrac{4}{25} + \dfrac{36}{25} + \dfrac{135}{25} \right]$

$= \left(\dfrac{2}{5}\right)^4 \times \left[\dfrac{175}{25} \right]$

$= 7 \times \left(\dfrac{2}{5}\right)^4 = 0.1792$

(iii) $P(\text{at least one ball bear 'Y' mark}) = P(Z \ge 1) = 1 - P(Z = 0)$

$= 1 - \left(\dfrac{2}{5}\right)^6$

(iv) $P(\text{equal number of balls with 'X' mark and 'Y' mark}) = P(Z = 3)$

$= {}^6C_3 \times \left(\dfrac{2}{5}\right)^3 \times \left(\dfrac{3}{5}\right)^3$

$= \dfrac{20 \times 8 \times 27}{15625}$

$= \dfrac{864}{3125}$

Question 6:

In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\dfrac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?

Solution:

Let p and q respectively be the probability that the player will clear and knock down the hurdle.

$\therefore p = \dfrac{5}{6}$

$\Rightarrow q = 1 - p = 1 - \dfrac{5}{6} = \dfrac{1}{6}$

Let X be the random variable that represents the number of times the player will knock down the hurdle.

Thus, by binomial distribution, we get

$$\therefore P(X = x) = {}^nC_x p^{n-x} q^x$$

$$P(\text{Player knocking down less than 2 hurdles}) = P(X < 2)$$

$$= P(X = 0) + P(X = 1)$$

$$= {}^{10}C_0 \times (q)^0 \times (p)^{10} + {}^{10}C_1 \times (q)^1 \times (p)^9$$

$$= \left(\frac{5}{6}\right)^{10} + 10 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^9$$

$$= \left(\frac{5}{6}\right)^9 + \left[\frac{5}{6} + \frac{10}{6}\right]$$

$$= \frac{5}{2} \times \left(\frac{5}{6}\right)^9$$

$$= \frac{(5)^{10}}{2 \times (6)^9}$$

Question 7:

A die is thrown again and again until three sixes are obtained. Find the probability of obtaining the third six in the sixth throw of the die.

Solution:

The probability of getting a six in a throw of die is $\frac{1}{6}$ and not getting a six is $\frac{5}{6}$.

Let $p = \frac{1}{6}$ and $q = \frac{5}{6}$

The probability that the 2 sixes come in the first five throws of the die is

$${}^5C_2 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^3 = \frac{10 \times (5)^3}{(6)^5}$$

Probability that third six comes in the sixth throw $= \frac{10 \times (5)^3}{(6)^5} \times \frac{1}{6}$

$$= \frac{10 \times 125}{(6)^6}$$

$$= \frac{10 \times 125}{46656}$$

$$= \frac{625}{23328}$$

Question 8:

If a leap year is selected at random, what is the change that it will contain 53 Tuesdays?

Solution:

In a leap year, there are 366 days i.e., 52 weeks and 2 days.

In 52 weeks, there are 52 Tuesdays.

Therefore, the probability that the leap year will contain 53 Tuesdays is equal to the probability that the remaining 2 days will be Tuesdays.

The remaining 2 days can be any of the following:
Monday and Tuesday, Tuesday and Wednesday, Wednesday and Thursday, Thursday and Friday, Friday and Saturday, Saturday and Sunday and Sunday and Monday.

Total number of cases $= 7$
Favourable cases $= 2$

Probability that a leap year will have 53 Tuesdays $= \frac{2}{7}$

Question 9:

An experiment succeeds twice as often as it fails. Find the probability that in the next six trials, there will be at least 4 successes.

Solution:

Given, the probability of success is twice the probability of failure.
Let the probability of failure be x.
Probability of success $= 2x$

$$x + 2x = 1$$

$$\Rightarrow 3x = 1$$

$$\Rightarrow x = \frac{1}{3}$$

$$\therefore 2x = \frac{2}{3}$$

Let $p = \dfrac{1}{3}$ and $q = \dfrac{2}{3}$

Let X be the random variable that represents the number of successes in six trials.

By binomial distribution, we obtain

$$P(X = x) = {}^{n}C_{x}\, p^{n-x}\, q^{x}$$

Probability of at least 4 successes $= P(X \geq 4)$

$$= P(X = 4) + P(X = 5) + P(X = 6)$$

$$= {}^{6}C_{4} \times \left(\frac{2}{3}\right)^{4} \times \left(\frac{1}{3}\right)^{2} + {}^{6}C_{5} \times \left(\frac{2}{3}\right)^{5} \times \left(\frac{1}{3}\right) + {}^{6}C_{6} \times \left(\frac{2}{3}\right)^{6}$$

$$= \frac{15 \times (2)^{4}}{3^{6}} + \frac{6 \times (2)^{5}}{3^{6}} + \frac{(2)^{6}}{3^{6}}$$

$$= \frac{(2)^{4}}{(3)^{6}} \times [15 + 12 + 4]$$

$$= \frac{31 \times (2)^{4}}{(3)^{6}}$$

$$= \frac{31}{9} \times \left(\frac{2}{3}\right)^{4}$$

Question 10:

How many times must a man toss a fair coin so that the probability of having at least one head is more than 90%?

Solution:

Let the man toss the coin n times. The n tosses are n Bernoulli trials.

Probability (p) of getting a head at the toss of a coin is $\dfrac{1}{2}$

$$p = \frac{1}{2}, \, q = \frac{1}{2}$$

$$P(X = x) = {}^{n}C_{x}\, p^{n-x}\, q^{x} = {}^{n}C_{x}\left(\frac{1}{2}\right)^{n-x}\left(\frac{1}{2}\right)^{x} = {}^{n}C_{x}\left(\frac{1}{2}\right)^{n}$$

$$P\left(\text{getting at least one head}\right) > \frac{90}{100}$$

$$P\left(x \geq 1\right) > 0.9$$

$$1 - P\left(x = 0\right) > 0.9$$

$$1 - {}^{n}C_{0} \times \frac{1}{2^{n}} > 0.9$$

$${}^{n}C_{0} \times \frac{1}{2^{n}} < 0.1$$

$$\frac{1}{2^{n}} < 0.1$$

$$2^{n} > \frac{1}{0.1}$$

$$2^{n} > 10 \qquad \qquad \ldots(1)$$

The minimum value of *n* that satisfies the given inequality is 4.

Thus, the man should toss the coin 4 or more than 4 times.

Question 11:

In a game, a man wins a rupee for a six and loss a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins/loses.

Solution:

Here, the probability of getting a six is $\frac{1}{6}$ and the probability of not getting a 6 is $\frac{5}{6}$.
Three cases can occur.

(a) If he gets a six in the first throw, then the required probability is $\frac{1}{6}$
Amount he will receive $= \text{Re } 1$.

(b) If he does not get a six in the first throw and gets a six in the second throw, then probability $= \left(\frac{5}{6} \times \frac{1}{6}\right) = \frac{5}{36}$
Amount he will receive $= -\text{Re } 1 + \text{Re } 1 = 0$

(c) If he does not get a six in the first two throws and gets a six in the third throw, then probability $= \left(\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}\right) = \frac{25}{216}$

Amount he will receive $= -\text{Re } 1 - \text{Re } 1 + \text{Re } 1 = -1$

Probability that he does not get a six in any of the three throws $= \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$

Expected value he can win $= \frac{1}{6} \times (1) + \left(\frac{5}{6} \times \frac{1}{6} \right) \times (0) + \left(\left(\frac{5}{6} \right)^2 \times \frac{1}{6} \right) \times (-1)$

$$= \frac{1}{6} - \frac{25}{216}$$

$$= \frac{36 - 25}{216} = \frac{11}{216}$$

Question 12:

Suppose we have four boxes. A, B, C and D containing coloured marbles as given below.

Box	Marble Colour		
	Red	**White**	**Black**
A	1	6	3
B	6	2	2
C	8	1	1
D	0	6	4

One of the boxes has been selected at random and a single marble is drawn from it. If the marble is red, what is the probability that it was drawn from box A?, box B?, box C?

Solution:

Let R be the event of drawing the red marble.

Let E_A, E_B and E_C respectively denote the events of selecting the box A, B, and C.

Total number of marbles = 40
Number of red marbles = 15

$$\therefore \; P(R) = \frac{15}{40} = \frac{3}{8}$$

Probability of drawing the red marble from box A is given by $P(E_A|R)$.

$$\therefore \; P(E_A|R) = \frac{P(E_A \cap R)}{P(R)} = \frac{\frac{1}{40}}{\frac{3}{8}} = \frac{1}{15}$$

Probability that the red marble is from box B is $P(E_B|R)$.

$$\therefore P(E_B|R) = \frac{P(E_B \cap R)}{P(R)} = \frac{\frac{6}{40}}{\frac{3}{8}} = \frac{2}{5}$$

Probability that the red marble is from box C is $P(E_c|R)$.

$$\therefore P(E_C|R) = \frac{P(E_C \cap R)}{P(R)} = \frac{\frac{8}{40}}{\frac{3}{8}} = \frac{8}{15}$$

Question 13:

Assume that the changes of the patient having a heart attack are 40%. It is also assumed that a meditation and yoga course reduce the risk of heart attack by 30% and prescription of certain drug reduces its changes by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga?

Solution:

Let A, E_1 and E_2 respectively denote the events that a person has a heart attack, the selected person followed the course of yoga and meditation, and the person adopted the drug prescription.

$$\therefore P(A) = 0.40$$

$$P(E_1) = P(E_2) = \frac{1}{2}$$

$$\Rightarrow P(A|E_1) = 0.40 \times 0.70 = 0.28$$

$$\Rightarrow P(A|E_2) = 0.40 \times 0.75 = 0.30$$

Probability that the patient suffering a heart attack followed a course of meditation and yoga is given by $P(E_1|A)$.

$$P(E_1|A) = \frac{P(E_1) \times P(A|E_1)}{P(E_1) \times P(A|E_1) + P(E_2) \times P(A|E_2)}$$

$$= \frac{\frac{1}{2} \times 0.28}{\frac{1}{2} \times 0.28 + \frac{1}{2} \times 0.30} = \frac{14}{29}$$

Question 14:

If each element of a second order determinant is either zero or one, what is the probability that the value of the determinant is positive? (Assume that the individual entries of the determinant are chosen independently, each value being assumed with probability $\frac{1}{2}$).

Solution:

The total number of determinants of second order with each element being 0 or 1 is $(2)^4 = 16$.

The value of determinant is positive in the following cases. $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$

Therefore, Required probability $= \frac{3}{16}$

Question 15:

An electronic assembly consists of two subsystems, say, A and B. From previous testing procedures, the following probabilities are assumed to be known:

$P(A \text{ fails}) = 0.2$

$P(B \text{ fails alone}) = 0.15$

$P(A \text{ and } B \text{ fails}) = 0.15$

Evaluate the following probabilities

(i) $P(A \text{ fails} | B \text{ has failed})$ (ii) $P(A \text{ fails alone})$

Solution:

Let the event in which A fails and B fails be denote by E_A and E_B

$P(E_A) = 0.2$

$P(E_A \cap E_B) = 0.15$

$P(B \text{ fails alone}) = P(E_B) - P(E_A \cap E_B)$

$\therefore 0.15 = P(E_B) - 0.15$

$\therefore P(E_B) = 0.3$

(i) $P(E_A | E_B) = \frac{P(E_A \cap E_B)}{P(E_B)} = \frac{0.15}{0.3} = 0.5$

(ii) $P(A \text{ fails alone}) = P(E_A) - P(E_A \text{ and } E_B)$

$= 0.2 - 0.15 = 0.05$

Question 16:

Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

Solution:

Let E_1 and E_2 respectively denote the event that a red ball is transferred from bag I to II and a black ball is transferred from bag I to II.

$$P(E_1) = \frac{3}{7} \text{ and } P(E_2) = \frac{4}{7}$$

Let A be the event that the ball drawn is red.
When a red ball is transferred from bag I to II,

$$P(A|E_1) = \frac{5}{10} = \frac{1}{2}$$

When a black ball is transferred from bag I to II,

$$P(A|E_2) = \frac{4}{10} = \frac{2}{5}$$

$$P(E_2|A) = \frac{P(E_2) \times P(A|E_2)}{P(E_1) \times P(A|E_1) + P(E_2) \times P(A|E_2)}$$

$$= \frac{\frac{4}{7} \times \frac{2}{5}}{\frac{3}{7} \times \frac{1}{2} + \frac{4}{7} \times \frac{2}{5}} = \frac{16}{31}$$

Question 17:

If A and B are two events such that P (A) ≠ 0 and P (B|A) = 1, then
(A) $A \subset B$
(B) $B \subset A$
(C) $B = \phi$
(D) $A = \phi$

Solution:

Given, $P(A) \neq 0$ and $P(A|B) = 1$.

$$\therefore P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$1 = \frac{P(B \cap A)}{P(A)}$$

$$P(A) = P(B \cap A)$$

$$\Rightarrow A \subset B$$

Thus, the correct answer is A.

Question 18:

If and $P(A|B) > P(A)$, then which of the following is correct:

(A) $P(B|A) < P(B)$ (B) $P(A \cap B) < P(A) \times P(B)$

(C) $P(B|A) > P(B)$ (D) $P(B|A) = P(B)$

Solution:

Given, $P(A|B) > P(A)$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} > P(A)$$

$$\Rightarrow P(A \cap B) > P(A) \times P(B)$$

$$\Rightarrow \frac{P(A \cap B)}{P(A)} > P(B)$$

$$\Rightarrow P(B|A) > P(B)$$

Thus, the correct answer is C.

Question 19:

If A and B are any two events such that $P(A) + P(B) - P(A \text{ and } B) = P(A)$, then

(A) $P(B|A) = 1$ (B) $P(A|B) = 1$

(C) $P(B|A) = 0$ (D) $P(A|B) = 0$

Solution:

Given, $P(A) + P(B) - P(A \text{ and } B) = P(A)$

$$\Rightarrow P(A) + P(B) - P(A \cap B) = P(A)$$

$$\Rightarrow P(B) - P(A \cap B) = 0$$

$$\Rightarrow P(A \cap B) = P(B)$$

$$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

Thus, the correct answer is B.