Differential Equations and their Applications

Workbook

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https://mathofcourse.com/

Contents

1 Introduction and Equilibrium Solutions

An *ordinary differential equation* (ODE) is an equation involving rates of change with ordinary derivatives e.g. $\frac{dy}{dt}$ rather than partial derivatives e.g. $\frac{\partial y}{\partial t}$. Often, problems in Mathematics, Science or Engineering are described by a differential equation such as

$$R\frac{dQ}{dt}+\frac{Q}{C}=E(t),$$

which describes the rate at which the charge Q in a circuit changes with respect to time $\frac{dQ}{dt}$ and how that is related to the resistance R, charge Q, capacitance C, and voltage $E(t)$. To model the relationship between these variables, we must solve the differential equation to give an equation relating the variable in the equation. If $R = 5$ ohms, $C = 0.05$ farads, and $E(t) = 60$ volts, then solving the ODE would give the equation

$$Q(t) = 3-3e^{-4t},$$

showing that the charge approaches 3 with time, see Figure 1.

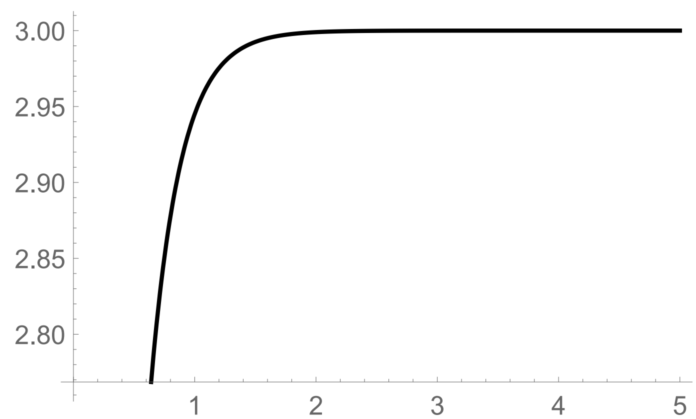


Figure 1: A plot of $Q(t) = 3-3e^{-4t}$.

The differential equation above is an example of a 1st order linear ODE. In this course you will learn how to solve these differential equations and several other varieties of differential equations.

1.1 Constant absolute rate of change

$$\frac{dy}{dt} = a \qquad \text{or} \qquad y'(t) = a,$$

where a is constant and represents the absolute rate of change.

When $a > 0$, $y(t)$ is increasing (growth).
When $a < 0$, $y(t)$ is decreasing (decay).

The general solution to the differential equation $y'(t) = a$ is

$$y(t) = at + b,$$

where b is an arbitrary constant.

1.2 Constant relative rate of change

$$\frac{1}{y}\frac{dy}{dt} = k \qquad \text{or} \qquad y'(t) = ky(t),$$

where k is a constant and represents the relative rate of change.

When $k > 0$, $y(t)$ is increasing (k is the relative growth rate).
When $k < 0$, $y(t)$ is decreasing ($|k|$ is the relative decay rate).

The general solution to the differential equation $y'(t) = ky(t)$ is

$$y(t) = Ae^{kt}.$$

To find the doubling time when $k > 0$, we seek the time t such that

$$y(t) = 2y(0).$$

To find the half life when $k < 0$, we seek the time t such that

$$y(t) = \frac{1}{2}y(0).$$

For example, it has been said that after drinking a cup of coffee, it takes approximately 2 to 3 hours for your body to eliminate half of the caffeine from your system.

We have

- Most drug concentrations decay exponentially (nicotine half life ≈ 2 hours, caffeine ≈ 2 to 3 hours, cannabis ≈ 1 to 3 days)
- blood alcohol concentration decays linearly (health note ≈ 10 g/hour - 1 standard drink)

1.3 Equilibrium

Let $P(t)$ be a concentration at time t of a poisonous toxin in a river. Let's assume that sunlight breaks down according to

$$\frac{dP}{dt} = -cP, \qquad c > 0,$$

but a nearby activity adds toxin to the river at a constant rate a (> 0) so that we have

$$\frac{dP}{dt} = -cP + a.$$

We can consider the equation qualitatively by consideration of the *equilibrium* or *steady state* solution(s). In other words, what solution is obtained when $\frac{dP}{dt} = 0$ and not dependent on t i.e. not locally 0. For the equation $\frac{dP}{dt} = -cP + a$, is there an equilibrium solution?

Putting $\frac{dP}{dt} = 0$, we have

$$-cP_{eq.} + a = 0$$

so that the equilibrium solution is given by

$$P_{eq.} = \frac{a}{c}.$$

If, for example, $c = 1$ and $a = 2$, then the ODE $P'(t) = -P + 2$ has the unique equilibrium solution $P = 2$, which is a solution to the ODE and tells us the behaviour of various other trajectories in relation to this solution, see Figure 2. Without solving the equation, we can see that all solutions tend towards the trajectory $P(t) = 2$. This is an example of a stable equilibrium.

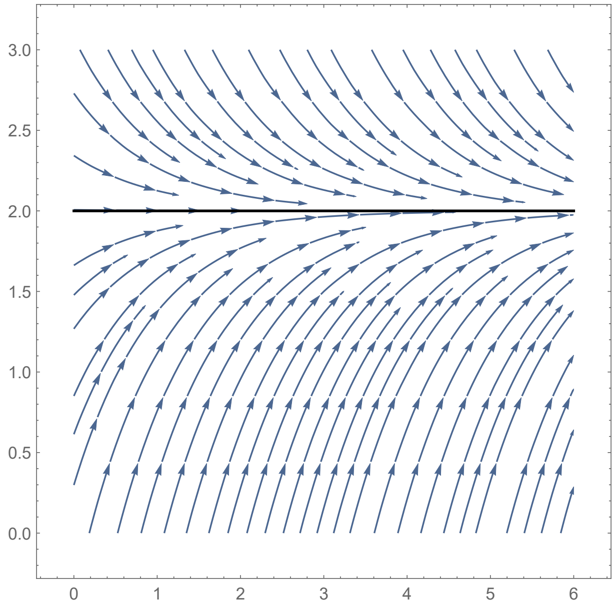


Figure 2: A plot of various trajectories satisfying $P'(t) = -P + 2$. This shows an example of a stable equilibrium at $P = 2$.

A vertically balanced pencil on the other hand is an example of an unstable equilibrium.

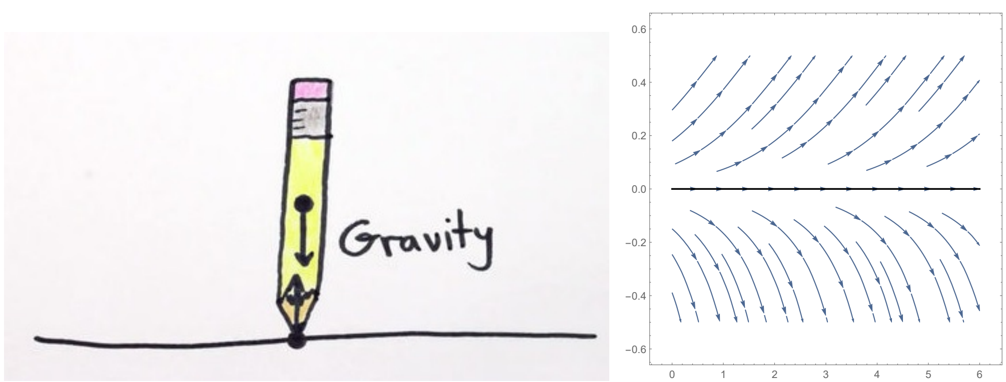


Figure 3: A vertically balanced pencil will fall due to air molecules and the impossibility of placing it exactly vertical. Right: An example of trajectories satisfying an ODE with an unstable equilibrium at $y = 0$. Solutions will lead away from the equilibrium solution as t increases.

1.4 Stability of equilibrium

What happens if the toxin is above or below equilibrium level?

For any differential equation which can be written in the form

$$\frac{dP}{dt} = F(P),$$

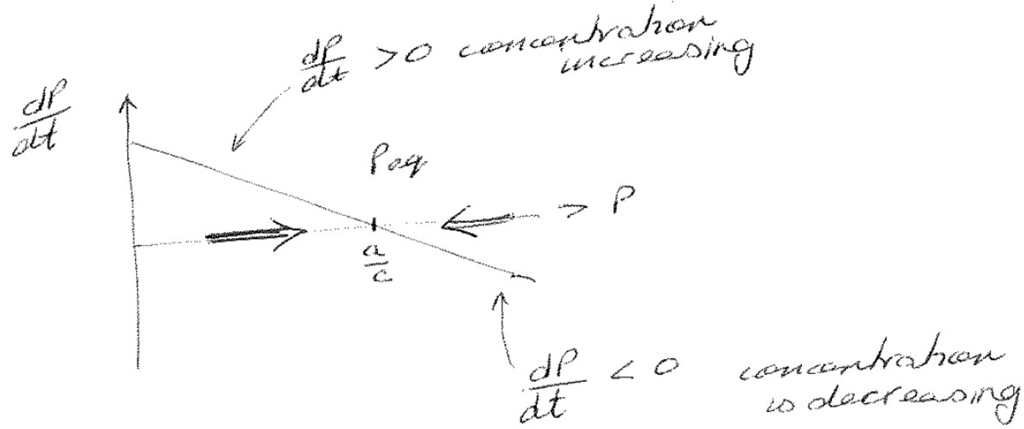


Figure 4: Concentration of toxin is always changing in a direction towards equilibrium. Therefore stable.

where $F(P)$ is a function of P, the stability of the solution can be determined from a sketch or by checking the sign of the slope $F'(P_{eq})$. If $F'(P_{eq}) > 0$, then the equilibrium is unstable. If $F'(P_{eq}) < 0$, then the equilibrium is stable. Alternatively, we can check the sign of two trajectories with an initial condition above the equilibrium solution and below the equilibrium solution.

Example 1.1 *Consider the equilibrium solutions to the logistic equation for population growth*

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{a}\right),$$

where k, a are constants. Setting $\frac{dP}{dt} = 0$, we have

Alternatively, if we let

$$F(P) = kP\left(1 - \frac{P}{a}\right),$$

and differentiate, the product rule of differentiation gives

Again, $P = 0$ is an unstable equilibrium and $P = a$ is a stable equilibrium.

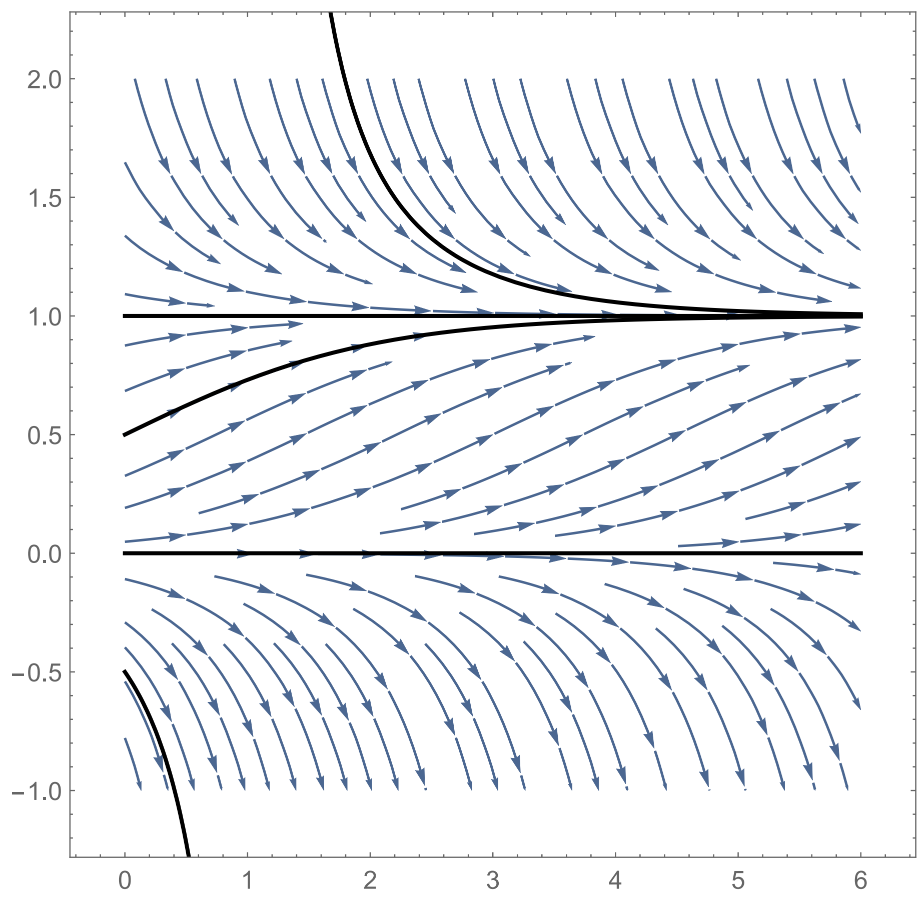


Figure 5: A plot of various trajectories $P'(t) = kP\left(1 - \frac{P}{a}\right)$, where $a = k = 1$. In black we see equilibrium solutions (horizontal lines) and some particular trajectories with initial values with $P < 0$, $0 < P < a$, and $P > a$ respectively.

Incidentally, the general solution to this logistic equation is

$$P(t) = \frac{aAe^{kt}}{Ae^{kt} - 1},$$

where A *is a constant that depends on the initial condition* $P(0)$*, see Example 2.3. Recall that* e *is Euler's number or the base of the Napier logarithm,*

$$e = 2.718281828459045\ldots$$

Not all solutions to $y' = 0$ are equilibrium solutions. We must instead have $y' = 0$ for all t.

Example 1.2 *Consider the ODE*

$$y' = y(y+2)\left(y - t^2\right).$$

Solving $y' = 0$*, we have* $y = 0$*,* $y = -2$*, and* $y = t^2$*.* $y = 0$ *and* $y = -2$ *are equilibrium solutions however* $y = t^2$ *is not since it gives* $y' = 0$ *only temporarily in trajectories. See Figure 8.*

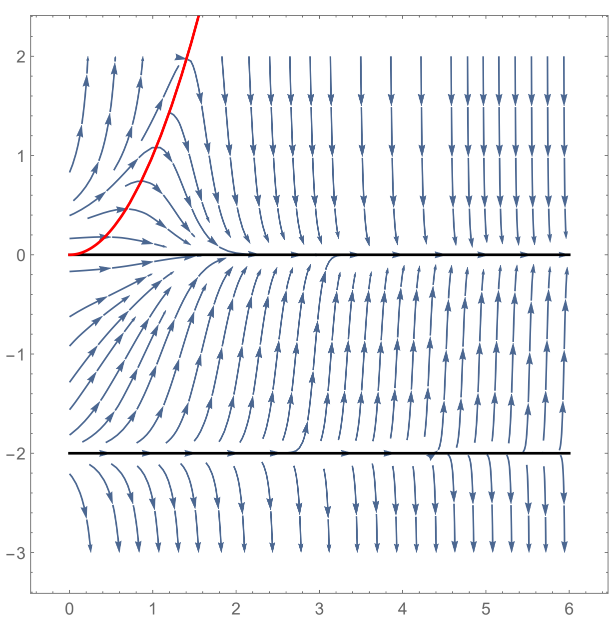


Figure 6: A plot of various trajectories $y' = y(y+2)\left(y - t^2\right)$. $y = 0$ and $y = -2$ are equilibrium solutions but $y = t^2$ is not. The equation $y = t^2$ is shown in red (not a trajectory), along which we see trajectories with turning points where $y' = 0$.

1.5 Existence and uniqueness

- (Existence and Uniqueness) The initial value problem $\frac{dy}{dt} = f(t,y)$, $y(t_0) = y_0$ with point (t_0, y_0), where f is a function, has a unique solution near the point (t_0, y_0) if f, and the partial derivatives $f_t(t,y)$ and $f_y(t,y)$ are continuous and f is differentiable.

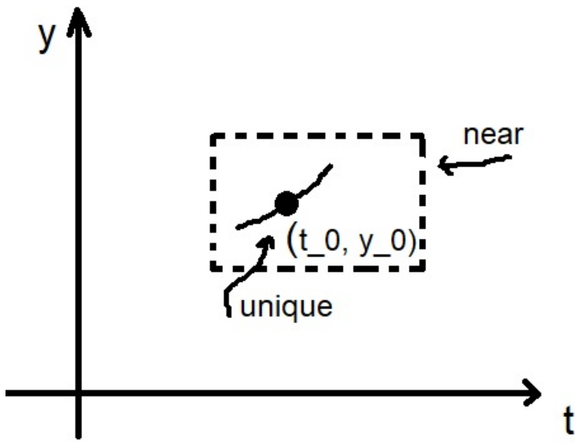


Figure 7: The Existence and Uniqueness Theorem.

- Solution curves do not cross (themselves or each other), including that they do not cross equilibrium solutions. This provides a way to give a qualitative assessment of the behaviour of solutions to ODEs without obtaining their actual solutions.

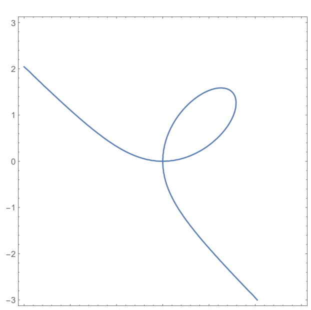


Figure 8: The folium of Descartes, $x^3 + y^3 - 3xy = 0$ cannot be a solution to an initial value problem $y' = f(x,y)$, $y(x_0) = y_0$ since if it were, a trajectory crosses itself, a contradiction. Remember that the derivatives need to exist, and they are ambiguous at the point $(0,0)$.

Example 1.3 *As an example of how qualitative analysis can be useful, consider the initial value problem*

$$y'(t) = 5(y-2), \qquad\qquad y(1) = 1.$$

Do we have $y(t) < 2$ *for all* $t > 1$ *?*

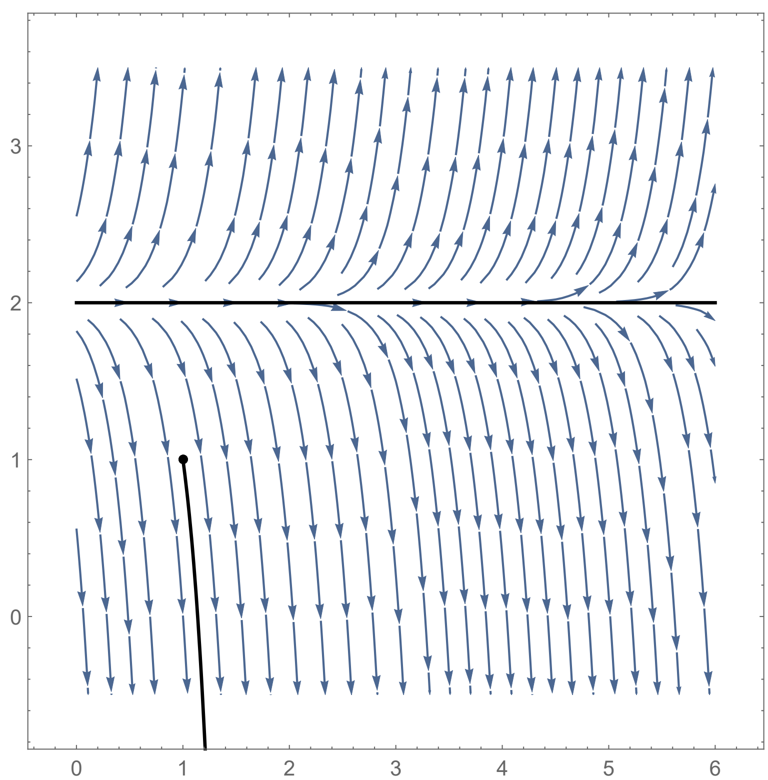


Figure 9: The solution to the IVP $y'(t) = 5(y-2)$, $y(0) = 1$ does not cross the line $y = 2$ since it is a trajectory.

So far we have seen that equilibrium solutions to ODEs can be stable or unstable. However, it is also possible for an equilibrium solution to be neither stable nor unstable. If on one side of an equilibrium trajectories tend towards it but on the other side they tend away from it, then the equilibrium is not stable nor unstable. We call such an example *semi-stable*.

Example 1.4 *Consider the ODE* $y' = 2y^2$. *We have an equilibrium solution* $y = 0$. *Setting* $F(y) = 2y^2$,

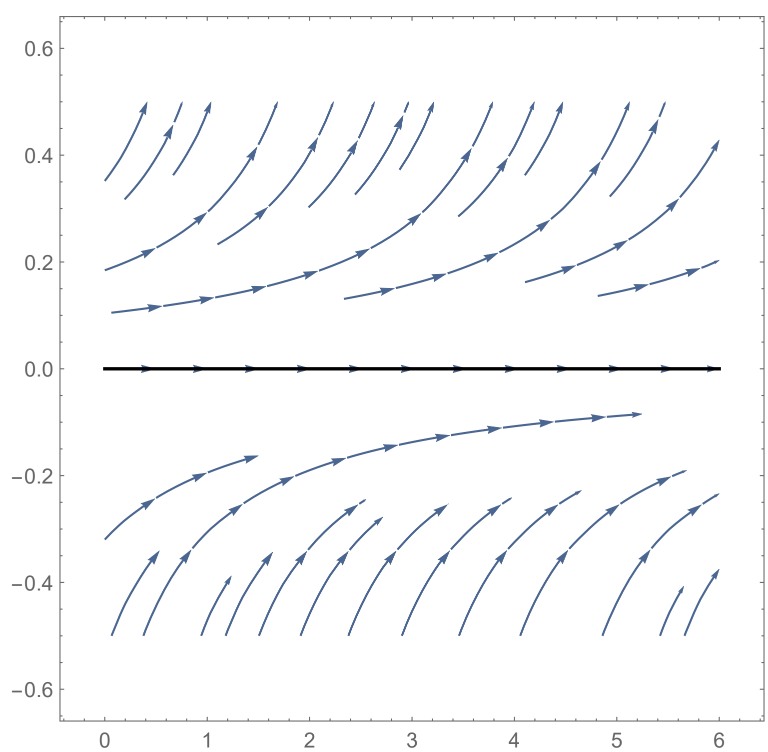


Figure 10: The the ODE $y' = 2y^2$ has a semi-stable equilibrium solution $y = 0$.

2 Separable Equations

2.1 Separable equations

If an ODE is of the form

$$\frac{dx}{dt} = F(x)G(t),$$

then we are able to separate the variables on the left and right hands sides of the equation (again in aide to memory) so that

$$\frac{dx}{F(x)} = G(t)\,dt,$$

and

$$\int \frac{dx}{F(x)} = \int G(t)\,dt.$$

Example 2.1 *Consider the equation*

$$\frac{dP}{dt} = kP + a.$$

Recall that we have the equilibrium solution $P_{eq.} = \frac{-a}{k}$ *for* $k \neq 0$*. This equilibrium is stable if* $k < 0$ *and unstable if* $k > 0$*. To find the general solution,* *This is equivalent to*

$$kP + a = \pm Ae^{kt}$$

Example 2.2 *Solve the equation $y' = 2xy + x$. See Example 4.1 for an alternative approach. We are able to separate the equation as*

2.2 Logistic equations

An example of a simple epidemic model in the logistic equation. Several systems modeled by such an equation include:

- Spread of infection in a population.
- Spread of cancer in an organ.
- Adoption of new technologies.
- Spread of a rumour.

Consider the spread of a rumour in more detail. Let ρ be the probability that a random person has heard the rumour and $1-\rho$ be the probability that they have not heard it. The rumour spreads at a rate proportional to $\rho(1-\rho)$, so we have the ODE

$$\frac{d\rho}{dt} = k\rho(1-\rho).$$

On the right of Figure 11 we see two equilibrium solutions at $\rho = 0$ and $\rho = 1$, respectively unstable and stable by consideration of the sign of the slopes of $\frac{d\rho}{dt}$ at 0 and 1. Since $0 \leq \rho \leq 1$, the solution curve look like the right hand side of Figure 11.

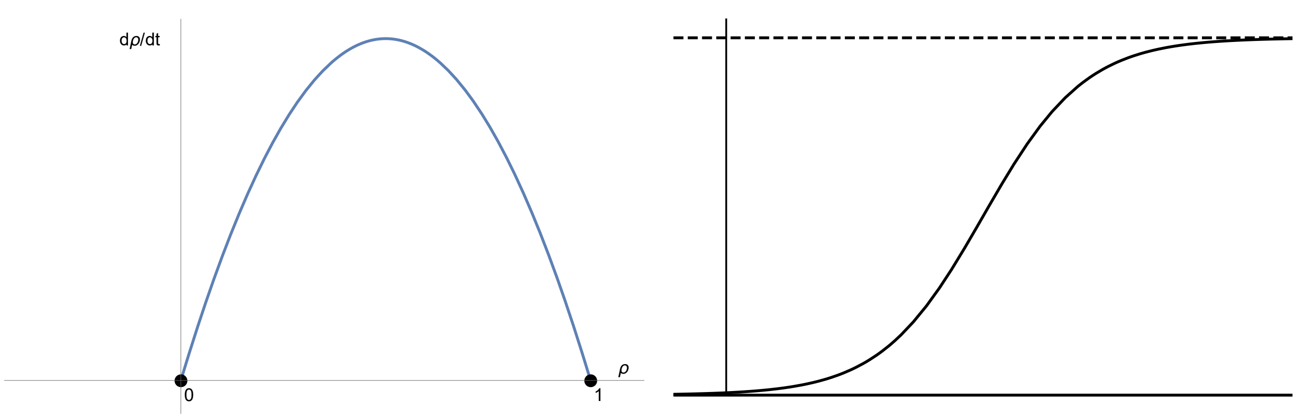


Figure 11: Left: A plot of $y = \rho(1-\rho)$, where $y = \frac{d\rho}{dt}$. Right: An example solution curve for $0 \leq \rho \leq 1$.

Example 2.3 *Next we solve the logistic equation for population growth*

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{a}\right), \tag{1}$$

where k, a *are positive constants. This equation is separable since we can separate it in the following way:*

We can use these to integrate

$$\int \frac{dP}{P(a-P)} \ = \ . \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad .$$

Finally we are able to solve for P to give the general solution to the ODE.

$$P(t) = \frac{aAe^{kt}}{Ae^{kt} - 1}.$$

2.3 Newton's law of cooling

Newton's law of cooling states that the rate of change of the temperature of a body is directly proportional to the difference in the temperatures between the body and its environment. We can model this mathematically using the ODE

$$\frac{dT}{dt} = -k\left(T - T_{amb.}\right), \qquad k > 0,$$

where $T_{amb.}$ is the ambient or surrounding temperature of the body's environment, T is the temperature of the body in degrees C, and t is time (we can select hours, minutes, seconds for the unit), and k is a constant that depends on the material of the body.

Example 2.4 *Suppose you spoon some instant coffee into a mug, tip cool tap water in the mug, and then place it in the microwave. You press a preset button for 3 minutes, walk away and then get distracted by the television. You come back after some time to find your coffee is boiling (100° C). Since it is undrinkable, you must wait until it is 70° C. The temperature in your kitchen is 25° C. After 10 seconds, the coffee has cooled to 95° C. How long must you wait to drink the coffee?*

We must solve the IVP

$$\frac{dT}{dt} = -k\left(T - 25\right), \qquad T(0) = 100, \qquad T(10) = 95.$$

and simplifying yields the general solution

$$T = 25 + Ae^{-kt}.$$

It follows that you must wait approximately 74 seconds to drink the coffee.

2.4 Existence uniqueness example for a separable IVP

Example 2.5 *Show that the initial value problem* $y' = ty^{1/3}$, $y(0) = 0$ *does not have a unique solution.*

so the solution is not unique.

3 Homogeneous Equations and Integration Techniques

3.1 Homogeneous 1st order ODEs

A first order ODE $y' = f(t,y)$ is said to be *homogeneous* if for all real non-zero s,

$$f(st,sy) = f(t,y).$$

ODEs of this form can be solving using techniques of separable equations using the substitution

$$y = tv.$$

Differentiating using the product rule, we have

$$\begin{aligned} y'(t) &= \frac{d}{dt}(tv), \\ &= v\frac{dt}{dt} + t\frac{dv}{dt}, \\ &= v + t\frac{dv}{dt}. \end{aligned}$$

We claim that the resulting differential equation

$$v + t\frac{dv}{dt} = f(t,tv)$$

is separable.

To see this, notice that since the ODE is homogeneous, we can put $s = \frac{1}{t}$ and let $y = tv$. Observe that

$$f(1,v) = . \qquad .$$

It follows that the ODE can be separated as follows:

$$\int \frac{dv}{f(1,v) - v} = \int t^{-1}\, dt. \tag{2}$$

Example 3.1 *Consider the ODE* $\frac{dy}{dt} = \frac{2y^4+t^4}{ty^3}$*. Letting* $f(t,y) = \frac{2y^4+t^4}{ty^3}$*, we see that*

$$f(st, sy) = .$$

Since $\frac{d}{dv}\left(v^4+1\right) = 4v^3$*, we are able to substitute* $dv \longrightarrow \frac{1}{4v^3} d\left(v^4+1\right)$.

Integrating the left hand side,

If we have an initial value such as $y(1) = 1$*, then we would obtain the particular solution curves* $y^4 = 2t^8 - t^4$*. By the existence and uniqueness theorem, the solutions to this IVP are not unique. See Figure 12.*

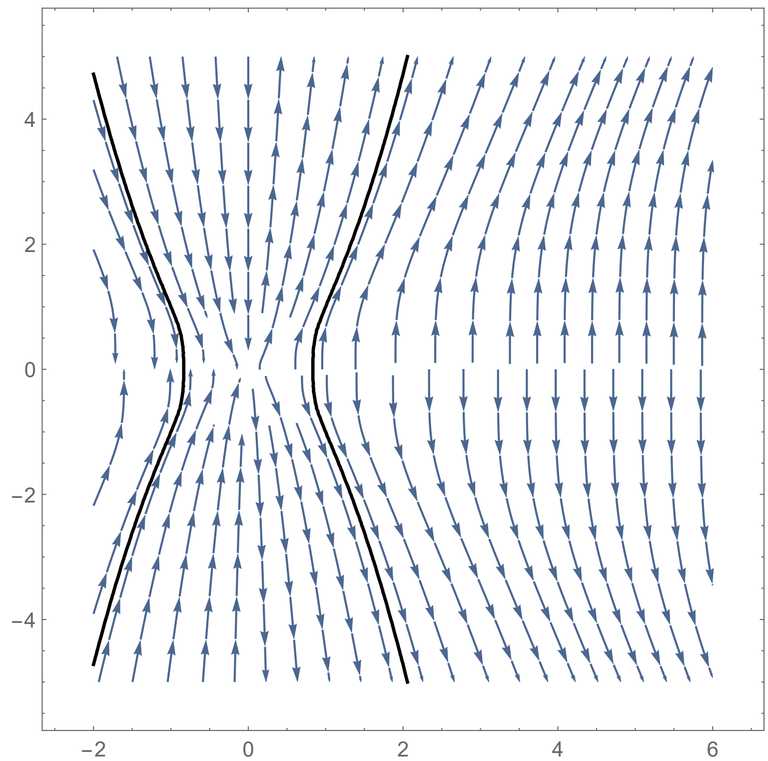


Figure 12: A plot of various trajectories satisfying $y'(t) = \frac{2y^4+t^4}{ty^3}$. In black we have a solution $y^4 = 2t^8 - t^4$ to the IVP $y'(t) = \frac{2y^4+t^4}{ty^3}$, $y(1) = 1$.

Example 3.2 *Consider the ODE $ty'(t) = t + y$. To see that this equation is homogeneous, let $f(t,y) = \frac{t+y}{t}$.*

$$f(st, sy) \;=\; . \qquad\qquad\qquad\qquad\qquad .$$

Since the ODE is homogeneous, we let $y = tv$ and invoke (2). We first compute

$$f(1, v) - v \;=\; . \qquad\qquad\qquad\qquad\qquad .$$

We have

$$\int \frac{dv}{f(1,v)-v} = .$$

.

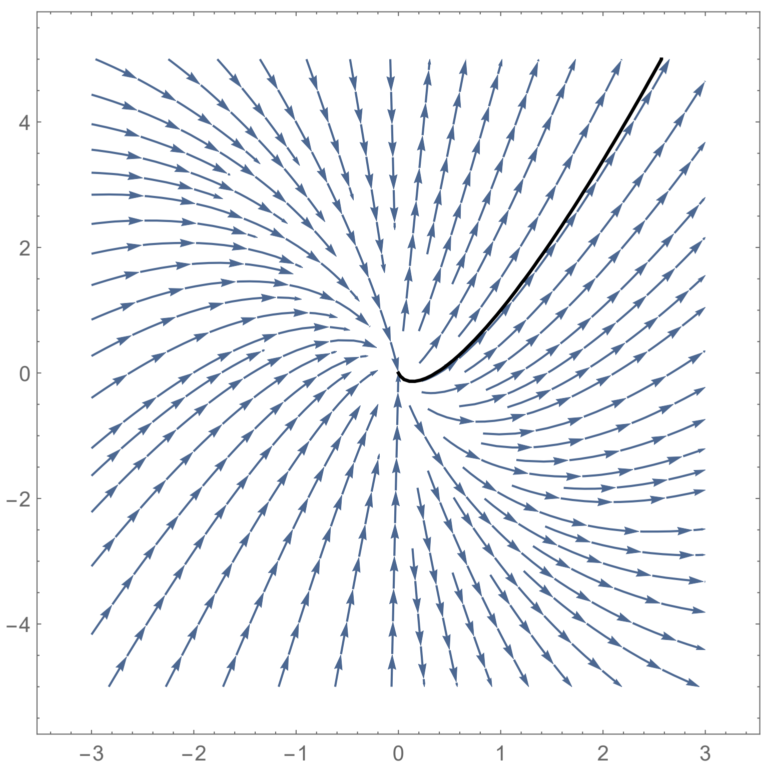


Figure 13: A plot of various trajectories satisfying $ty'(t) = t + y$. In black we have a solution $y(t) = t\log(t) + t$ to the IVP $ty'(t) = t + y$, $y(1) = 1$.

3.2 Integration by partial fractions

In this section we will digress somewhat to further reinforce our integration skills. Solving ODEs can involve functions that require skill to integrate. We will begin with integration by partial fractions since we have already used this technique to solve the logistic equation.

Suppose we must integrate $\int \frac{dx}{(x-a_1)(x-a_2)\ldots(x-a_n)}$, where $a_1, a_2, \ldots a_n$ are known constants. In this case we would integrate via partial fractions. To do so, we set

$$\frac{1}{(x-a_1)(x-a_2)\ldots(x-a_n)} = \frac{b_1}{x-a_1} + \frac{b_2}{x-a_2} + \ldots \frac{b_n}{x-a_n}$$

and we seek constants $b_1, b_2, \ldots, b_n$ such that this holds. The reason we do this is because it is easy to integrate each

$$\int \frac{b_j}{x-a_j} dx = b_j \log(x-a_j) + C,$$

for $j = 1, 2, \ldots, n$.

Example 3.3 *To compute the integral* $\int \frac{dx}{(x+2)(x-2)(x+1)}$, *we set out*

$$\frac{1}{(x+2)(x-2)(x+1)} = \frac{b_1}{x+2} + \frac{b_2}{x-2} + \frac{b_3}{x+1}.$$

We must first recombine the right hand side by multiplying each term to obtain a common denominator $(x+2)(x-2)(x+1)$. *We have*

$$\begin{aligned}
\frac{b_1}{x+2} + \frac{b_2}{x-2} + \frac{b_3}{x+1} &= \frac{b_1}{x+2} + \frac{b_2(x+1) + b_3(x-2)}{(x-2)(x+1)}, \\
&= \frac{b_1(x-2)(x+1) + (x+2)\left(b_2(x+1) + b_3(x-2)\right)}{(x+2)(x-2)(x+1)}, \\
&= \frac{b_1(x-2)(x+1) + b_2(x+1)(x+2) + b_3(x-2)(x+2)}{(x+2)(x-2)(x+1)}, \\
&= \frac{(b_1+b_2+b_3)x^2 + (3b_2 - b_1)x - 2b_1 + 2b_2 - 4b_3}{(x+2)(x-2)(x+1)}.
\end{aligned}$$

To find b_1, b_2, b_3, *we can choose a method. Either we observe that we must* *have*

$$
\begin{aligned}
b_1 + b_2 + b_3 &= 0, \\
3b_2 - b_1 &= 0, \\
-2b_1 + 2b_2 - 4b_3 &= 1
\end{aligned}
$$

in order for the numerator to be equal to 1*. We then solve the system to obtain* $b_1 = \frac{1}{4}$, $b_2 = \frac{1}{12}$, $b_3 = -\frac{1}{3}$*. This is often more work than necessary.*

Alternatively, we have

$$b_1(x-2)(x+1) + b_2(x+1)(x+2) + b_3(x-2)(x+2) = 1.$$

We choose to look at this equation when $x = 2$, -2, *and* -1*. When* $x = 2$, *we have* $12b_2 = 1$*. When* $x = -2$, *we have* $4b_1 = 1$*. When* $x = -1$, *we have* $-3b_3 = 1$*. Again* $b_1 = \frac{1}{4}$, $b_2 = \frac{1}{12}$, $b_3 = -\frac{1}{3}$*. It follows that*

$$\int \frac{dx}{(x+2)(x-2)(x+1)} = \quad . \qquad\qquad .$$

Other forms require different partial fraction expansions. For example, we set out

$$\frac{1}{(x-a_1)^2(x-a_2)} = \frac{b_1}{x-a_1} + \frac{b_2}{(x-a_1)^2} + \frac{b_3}{x-a_2},$$

and with $a_1 > 0$,

$$\frac{1}{(x^2+a_1)(x-a_2)} = \frac{b_1 x + b_2}{x^2+a_1} + \frac{b_3}{x-a_2}.$$

3.3 Integration by parts

This method of integration is useful when integrating a product of two functions in which the derivative of one of the multiplicands is related to itself. In aide to memory of the rule, we can begin with the product rule of differentiation:

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

Rearranging,

$$u\frac{dv}{dx} = \frac{d(uv)}{dx} - v\frac{du}{dx},$$

and we have the substitution

$$\begin{aligned}\int u\,dv &= \int d(uv) - \int v\,du,\\ &= uv - \int v\,du.\end{aligned}$$

Example 3.4 *To solve the ODE $y' = x^2 e^{-x}$, we integrate to obtain the following integral to which we apply integration by parts two times:*

$$y = \int x^2 e^{-x}\,dx, \qquad .$$

Hence we have the general solution

$$y = -e^{-x}\left(x^2 + 2x + 2\right) + C.$$

3.4 Integration by substitution

Example 3.5 *To solve the ODE* $\sqrt{9-x^2}\, y' = 1$*, we see that it is separable and we integrate using a substitution.*

$$y = \int \frac{dx}{\sqrt{9-x^2}}.$$

We let $x = 3\sin(\theta)$ *and differentiate to obtain* $\frac{dx}{d\theta} = 3\cos(\theta)$*. This suggests the substitution* $dx \longrightarrow 3\cos(\theta)\,d\theta$*.*

$$y = \int \frac{dx}{\sqrt{9-x^2}}, \qquad .$$

Example 3.6 *Simplify the integral* $\int \frac{x\,dx}{\sqrt{1+x^4}}$*. First we let* $u = x^2$ *and differentiate to obtain* $\frac{du}{dx} = 2x$ *so that we should perform the substitution* $dx \longrightarrow \frac{du}{2x}$*. This gives the simpler integral*

$$\int \frac{x\,dx}{\sqrt{1+x^4}} = \frac{1}{2}\int \frac{1\,du}{\sqrt{1+u^2}}$$

Next we recall the definitions of the hyperbolic functions

$$\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right), \tag{3}$$

$$\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right), \tag{4}$$

and set out the substitution $u = \sinh(v)$. *Differentiating,*

$$1 + \sinh^2(v) = . \qquad .$$

Now we are ready to simplify the integral:

$$\int \frac{x\,dx}{\sqrt{1+x^4}} = . \qquad .$$

The following example shows that these techniques can be useful in solving some second order ODES.

Example 3.7 *Solve the non-linear second order ODE*

$$\frac{d^2y}{dx^2} = 1 - \left(\frac{dy}{dx}\right)^2,$$

with $|y'| < 1$.

We discard the solutions $u = \pm 1$ *since* $|y'| < 1$.

4 Linear First Order ODEs and Bernoulli Equations

4.1 Integrating factors

A first order linear differential equation is an ODE of the form

$$y' + p(x)y = q(x), \tag{5}$$

where $y' = \frac{dy}{dx}$, $p(x)$ and $q(x)$ are functions of x. We solve linear first order ODEs by multiplying by an *integrating factor* $I(x)$ so that the resulting equation can be integrated directly using the product rule of differentiation. Let

$$I(x) = e^{\int p(x)\,dx}. \tag{6}$$

Multiplying the equation (5) by $I(x)$ gives

It follows that the resulting ODE is now separable. We have

$$yI = \int I(x)q(x)\,dx$$

so that

$$y(x) = I^{-1} \int I(x)q(x)\,dx. \tag{7}$$

It is apparent that some of the difficulty of solving a linear ODE is in the computation of the two integrals $\int p(x)\,dx$ and $\int I(x)q(x)\,dx$, which often require the techniques of integration discussed in Chapter 3.

Example 4.1 *Solve the first order linear equation* $y' = 2xy + x$ *using an integrating factor. Notice that this equation is both linear and separable. See Example 2.2 for the separable approach. First we verify that the ODE is given in the same form as* (5)*, and if not, we must rearrange so that it is. We put*

$$y' - 2xy = x$$

and identify that $p(x) = -2x$ *and* $q(x) = x$*. To form the integrating factor* I*, we must first integrate* $p(x)$ *to obtain* $\int p(x)\,dx = -x^2$*, noting that we do not require the constant of integration here at this point in the procedure. Next we multiply the ODE by* $I(x) = e^{-x^2}$ *or simply use* (7) *directly. We have*

Example 4.2 *Solve the initial value problem* $y' = x^2 + y$, $y(0) = 0$*. We can see that this equation is linear since we can put* $y' - y = x^2$*, where* $p(x) = -1$ *and* $q(x) = x^2$*. Calculation of the integrating factor is easy since we have* $I(x) = e^{\int -1\,dx} = e^{-x}$*. Using* (7) *and integration by parts as performed in Example 3.4,*

4.2 Applications

We will first consider an example from Engineering and Physics involving an electric circuit containing a resistor, inductor, battery and switch. See Figure 14.

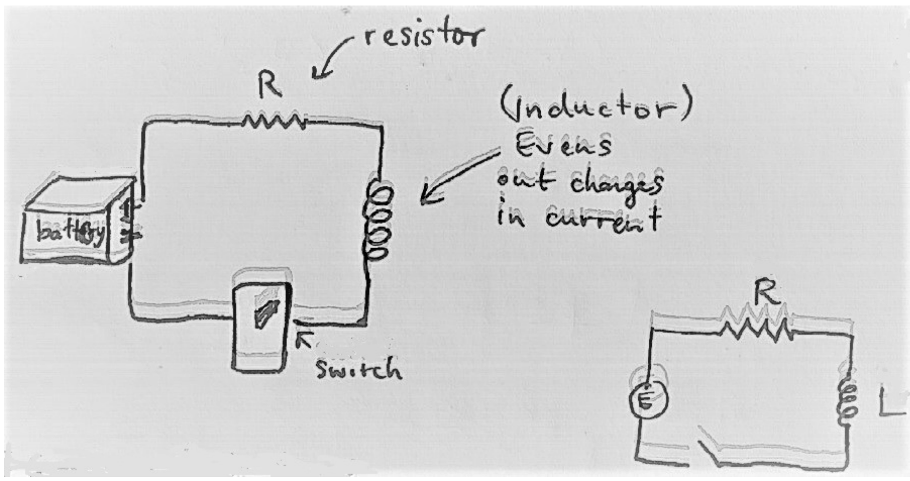


Figure 14: An electrical circuit with a resistor, inductor, battery and switch.

Example 4.3 *Consider the diagram in Figure 14. Model the current over time as the switch is turned on. Let $J(t)$ denote the current in Amps, R the resistance in Ohms, V_1 the voltage drop across the resistor, V_2 the voltage drop due to the inductor, let L denote the inductance in Henries, and let $E(t)$ denote the supplied voltage due to the battery. We have*

$$V_1 = JR, \qquad V_2 = LJ'(t).$$

Kirchoff's law tells us that

$$E(t) = V_1 + V_2,$$

and it follows that we can substitute to obtain

$$E(t) = JR + LJ'(t),$$

and dividing by L we obtain the linear first order equation

$$J'(t) + \frac{R}{L}J = \frac{1}{L}E(t). \tag{8}$$

Now suppose that initially the current is 0 *Amps, the battery supplies a constant voltage of* 60 *Volts, the resistance is* 12 *Ohms, and the inductance is* 4 *Henries. We now have an initial value problem*

$$J'(t)+3J=15, \qquad\qquad J(0)=0.$$

We have a unique stable equilibrium solution at $J=5$ Amps so we see already that the current will approach 5 *Amps as $t\longrightarrow\infty$. This equation is linear with $p(t)=3$ and $q(t)=15$. We have $I=e^{3t}$ so that*

Our next example comes from Physics.

Example 4.4 *Model the velocity of a skydiver falling from a plane taking air resistance into consideration and give an expression for terminal velocity. See [11, pp. 12 & pp. 44]. Newton's second law of motion states that the resulting force is given by $F=ma$, where m is the mass of the skydiver, and $a=v'(t)$ is the acceleration of the skydiver, or rate at which the velocity is changing with respect to time. The force acting on the skydiver due to gravity is mg, where $g=9.8$ metres per second squared, and the force due to air resistance is proportional to the velocity of the skydiver to the power of a constant c, kv^c, where k is the coefficient of friction due to air resistance. Since this force acts in the opposite direction to gravity, we have*

$$mv'(t)=mg-kv^c.$$

If $c = 1$*, then we have a linear ODE*

$$v'(t) + \frac{k}{m}v = g.$$

If $c \neq 1$*, then the equation is non-linear, however it will be separable in any case. We can see that there is a stable equilibrium solution when* $v^c = \frac{mg}{k}$*, which we can interpret as the terminal velocity of the skydiver,*

$$v_{term.} = \left(\frac{mg}{k}\right)^{1/c}. \tag{9}$$

If $c = 1$*, then clearly the integrating factor is* $I(t) = e^{\frac{k}{m}t}$ *and we have the solution*

$$v(t) = . \qquad .$$

If $c \neq 1$*, suppose that* $c = 2$*. Then we take the separable approach since the equation is non-linear. We have*

$$\frac{dv}{dt} = g - \frac{k}{m}v^2.$$

Separating the equation gives

$$\int \frac{dv}{\alpha - v^2} = \int \frac{k}{m}dt = \frac{k}{m}t + A,$$

where $\alpha = \frac{mg}{k}$*. To complete the integral on the left hand side, we observe that*

$$\int \frac{dv}{\alpha - v^2} = \frac{1}{\sqrt{\alpha}} \int \frac{d\left(\frac{v}{\sqrt{\alpha}}\right)}{1 - \left(\frac{v}{\sqrt{\alpha}}\right)^2} = \frac{1}{\sqrt{\alpha}} \int \frac{du}{1 - u^2},$$

where $u = \frac{v}{\sqrt{\alpha}}$*. Next we let* $u = tanh(\theta) = \frac{sinh(\theta)}{\cosh(\theta)}$ *since we have the identity*

$$\cosh^2(\theta) - \sinh^2(\theta) = 1.$$

Recall the definitions in (3) *and* (4)*. Differentiation gives*

It follows that

$$\int \frac{du}{1-u^2} = \quad . \qquad\qquad\qquad\qquad .$$

We obtain

$$arctanh\left(\frac{v}{\sqrt{\alpha}}\right) = \frac{k}{m}t + A,$$

and rearranging, we have

Next let us consider an example on fish keeping that is much like many chemistry problems.

Example 4.5 *An aquarium initially contains* 150 *L of water with 20 grams of salt dissolved in it. In order to accommodate a new fish, the concentration of salt must be increased from* $\frac{20\ g}{150\ L}$ *to* $\frac{1\ g}{L}$. *This new concentration is to be achieved by releasing water into the tank with a salt concentration of* $\frac{3\ g}{L}$ *at a rate of* 2 *litres per minute. At the same time, mixed water runs out of the tank at a rate of* 2 *litres per minute. Model the amount of salt* $x(t)$ *at time* t *in minutes.*

To solve for when $x(t) = 1$ gram per litre, we have

4.3 Bernoulli equations

An ODE of the form

$$y'(x) + p(x)y(x) = q(x)y^n, \tag{10}$$

where n is a real number, is called a *Bernoulli equation*.

When $n = 0$, (10) is linear. When $n = 1$, we have

$$y'(x) + (p(x) - q(x))\, y(x) = 0,$$

which is also linear.

When $n \neq 1$, the substitution $z = y^{1-n}$ makes the resulting equation linear. We have

$$z'(x) = \frac{dz}{dy}\frac{dy}{dx},$$

By substitution,

Multiplication by $(1-n)y^{-n}$ gives

$$z'(x)+(1-n)p(x)z(x)=(1-n)q(x). \tag{11}$$

which is now linear.

Example 4.6 *Find the general solution to the ODE $y'+xy=xy^2$. This is a Bernoulli equation with $p(x)=x=q(x)$ and $n=2$. We let $z=y^{-1}$ and* (11) *becomes*

Example 4.7 *Recall the logistic equation* (1) *for population growth given in Example 2.3.*

$$\frac{dP}{dt}=kP\left(1-\frac{P}{a}\right).$$

We used the fact that this equation is separable together with integration by partial fractions to find the general solution. We show how this can also be done via linear methods of Bernoulli equations.

$$\frac{dP}{dt}-kP=-\frac{k}{a}P^2.$$

We have a Bernoulli equation with

5 Exact Equations

5.1 Testing whether an equation is exact

Suppose

$$z = f(x, y)$$

is differentiable, where $x = g(t)$ and $y = h(t)$ are differentiable functions of t. Recall the chain rule:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{12}$$

Now if $f(x, y) = C$, a constant and $t = x$, then (12) becomes

$$\frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0,$$

which we can write as

$$f_x + f_y y'(x) = 0. \tag{13}$$

Conversely, if we have a differential equation of the form (13), then

$$f(x, y) = C$$

is the general solution. We say that an ODE

$$P(x, y) + Q(x, y)y'(x) = 0. \tag{14}$$

is *exact* if there exists a function $f(x, y)$ satisfying

$$P(x, y) = f_x, \qquad Q(x, y) = f_y$$

so that there exists f satisfying (13). We can test whether an ODE is exact as follows:

Proposition 5.1 *Suppose $P(x,y)$, $Q(x,y)$, $P_y(x,y)$, $Q_x(x,y)$ are continuous over a region D. The ODE*

$$P(x,y)+Q(x,y)y'(x)=0.$$

is exact if and only if for all x,y in D,

$$P_y(x,y)=Q_x(x,y).$$

The proof of this result uses Clairaut's theorem, see [2], [11, pp. 51], [14, pp. 22].

5.2 Solving exact equations

If we have an exact equation, our next step is to solve for the function $f(x,y)$. This is done by solving

$$\begin{aligned} P &= f_x, \\ Q &= f_y. \end{aligned}$$

We integrate setting

$$f=\int f_x dx,$$

where the constant of integration $G(y)$ is a function of y rather than C. We then compute the partial derivative f_y and compare with Q to determine $G(y)$. See [2] for similar problems.

Example 5.1 *Show that the following ODE is exact and then solve the IVP.*

$$y'(x) = -\frac{3x^2 + y^3}{3xy^2}, \qquad\qquad y(1) = 1.$$

Differentiating partially with respect to y,

We do not always have $G'(y) = 0$ as the following example shows.

Example 5.2 *Solve the IVP*

$$y' = \frac{-2xy}{1+x^2+3y^2}, \qquad\qquad y(0) = 1.$$

5.3 Integrating factors to make an equation exact

Occasionally an ODE can be made into an exact ODE by multiplying by some function $I(x,y)$. In general it is not always easy to find $I(x,y)$.

Example 5.3 *Solve* $y^2 - y + xy'(x) = 0$ *using the method for exact equations. We have* $P = y^2 - y$, $Q = x$. *Now* $P_y = 2y - 1$ *and* $Q_x = 1$ *so the ODE is not exact. Without knowing what* $I(x,y)$ *is, we multiply by it:*

$$I\left(y^2 - y\right) + Ixy'(x) = 0$$

We want this equation to be exact, so we seek $I(x,y)$ *such that*

$$\frac{\partial}{\partial y}\left(I\left(y^2 - y\right)\right) = \frac{\partial}{\partial x}(Ix).$$

Using the product rule,

We seek $f(x,y)$ such that

$$\begin{aligned} f_x &= p = 1 - y^{-1}, \\ f_y &= q = xy^{-2}. \end{aligned}$$

Integrating the first equation,

Note that since the original ODE is also separable, we can use integration by partial fractions to obtain the same result.

6 Second and Higher Order ODEs

6.1 Second order linear equations

A second order ODE is said to be *linear* if we can write it as

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = r(t). \tag{15}$$

If we have $r(t) = 0$, then the equation is *homogeneous*:

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0. \tag{16}$$

To make sense of the application of the term linear here, let $z(t) = y'(t)$. Then (15) becomes

$$\begin{aligned} y'(t) &= . \qquad\qquad\qquad\qquad\qquad\qquad , \\ z'(t) &= . \qquad\qquad\qquad\qquad\qquad\qquad , \end{aligned}$$

which can be written as

$$Y'(t) = \qquad\qquad\qquad\qquad\qquad\qquad , \tag{17}$$

where $Y(t) = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}$. If we let

$$A = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix}, \qquad\qquad b = \begin{pmatrix} 0 \\ r(t) \end{pmatrix},$$

then (17) becomes

$$Y'(t) = AY(t) + b,$$

which is linear. If we now suppose that p, q are constants, then the eigenvalues of the matrix A are given by the equation

$$\det(A - \lambda I) = 0. \tag{18}$$

Simplifying,

$$\det(A - \lambda I) \;=\; . \qquad\qquad\qquad\qquad\qquad\qquad .$$

Equation (18) is therefore equivalent to

$$\lambda^2 + p\lambda + q = 0, \tag{19}$$

which we call the *characteristic equation* of (15). To solve a second order linear ODE with constant coefficients, like (16) with p, q constants, we would first solve the characteristic equation (19).

Proposition 6.1 *Suppose that $y_1(t)$ and $y_2(t)$ are two linearly independent solutions to the homogeneous ODE*

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0,$$

meaning $y_1 \neq c y_2$. Then the general solution is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t),$$

where c_1, c_2 are constants.

To see this, assume that we have

$$\begin{aligned} y_1''(t) + p(t)y_1'(t) + q(t)y_1(t) &= 0, \\ y_2''(t) + p(t)y_2'(t) + q(t)y_2(t) &= 0. \end{aligned}$$

Then c_1 times the first equation plus c_2 times the second equation gives

which can be written

Letting $y_3(t) = c_1 y_1(t) + c_2 y_2(t)$, we see that $y_3(t)$ satisfies

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0.$$

6.2 Second order linear homogeneous equations with constant coefficients

A second order homogeneous ODE with constant coefficients $a\ (\neq 0), b, c$ is an equation of the form

$$ay''(t) + by'(t) + cy(t) = 0. \tag{20}$$

To see the form of the general solution, we first try $y = e^{\lambda t}$, where λ satisfies the characteristic equation (19) with $p = \frac{b}{a}$, $q = \frac{c}{a}$.

$$0 = . \qquad .$$

Since $ae^{\lambda t} \neq 0$, we must have $\lambda^2 + p\lambda + q = 0$. It follows that $y = e^{\lambda t}$ is a solution to the ODE, where λ is an eigenvalue of the matrix A or a root of the characteristic equation. For the general solution, we use Proposition 6.1. When the characteristic equation has two distinct roots, the general solution to (20) is given by

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \tag{21}$$

where $\lambda_1 \neq \lambda_2$ are the roots of

$$a\lambda^2 + b\lambda + c = 0.$$

When $\lambda_1 = \lambda_2$, the general solution is given by

$$y(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_2 t}. \tag{22}$$

We know however, that the roots of the characteristic equation are not always real numbers even when a, b, c are all real numbers. In the case that the eigenvalues are complex numbers

$$\lambda = \alpha + \beta i,$$

where α, β are real numbers, we use Euler's identity

$$e^{\theta i} = \cos(\theta) + i\sin(\theta). \tag{23}$$

Let

$$\lambda_1 = \alpha + \beta i, \qquad \lambda_2 = \alpha - \beta i.$$

Note that the complex roots of the characteristic equation are in this form. Then

$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \;=\; . \qquad\qquad .$$

We now define new constants

$$A = c_1 + c_2, \qquad B = (c_1 - c_2)\, i,$$

and obtain

$$y(t) = e^{\alpha t}\left(A\cos(\beta t) + B\sin(\beta t)\right),$$

where α is the real part of λ and β is the imaginary part of λ.

We have now demonstrated much of the following result:

Proposition 6.2 *The ODE $ay'' + by' + cy = 0$, where $a\ (\neq 0)$, b, c are constants and $\Delta = b^2 - 4ac$ has general solution given by:*

1. *if $\Delta > 0$, then $y = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$,*
2. *if $\Delta = 0$, then $y = (A + Bt)e^{\lambda t}$,*
3. *if $\Delta < 0$, then $y = e^{\alpha t}\left(A\cos(\beta t) + B\sin(\beta t)\right)$, $\alpha = \Re(\lambda)$, $\beta = \Im(\lambda)$,*

where the λ are the roots of $a\lambda^2 + b\lambda + c = 0$.

In the case that $\Delta = 0$, the quadratic formula gives repeated roots of the characteristic equation $\lambda = -\frac{b}{2a}$. We know that $y_1 = e^{-\frac{b}{2a}t}$ is one solution to the ODE. To find the other solution, we let $y_2 = uy_1$. Substitution gives

$$\begin{aligned} 0 &= a\,(uy_1)'' + b\,(uy_1)' + c\,(uy_1)\,, \\ &= a\left(u''y_1 + 2\lambda u'y_1 + \lambda^2 uy_1\right) + b\,(u'y_1 + \lambda uy_1) + cuy_1, \\ &= \left(au'' + (2a\lambda + b)u' + \left(a\lambda^2 + b\lambda + c\right)u\right)e^{-\frac{b}{2a}t}. \end{aligned} \quad .$$

Since $a\lambda^2 + b\lambda + c = 0$ and $\lambda = -\frac{b}{2a}$, this is just

$$u'' = 0,$$

and integration gives

$$u = A + Bt,$$

which completes the verification of Proposition 6.2.

Example 6.1 *Solve the second order ODE* $y'' + 2y' + y = 0$.

6.3 Second order homogeneous applications

The following example from physics illustrates the importance of these methods.

Example 6.2 *Consider a spring-mass system as shown in Figure 15. Use the laws of Physics to model the motion of the mass at the end of the spring. We let $y(t)$ be the vertical displacement of the mass at time t from the*

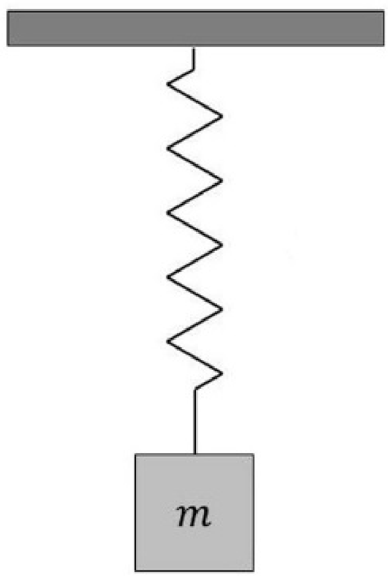


Figure 15: A spring attached to a mass.

equilibrium position hanging under gravity. The restoring force towards equilibrium is given by Hooke's law

$$F_r = -kd,$$

where d is the displacement from equilibrium and k is the spring constant particular to the spring. This displacement d is given by $d = s + y(t)$, where s is the difference in equilibria of the spring horizontally and vertically under gravity. The force due to gravity is

$$F_g = mg,$$

where $g = 9.8m/s^2$.

Hence the sum of these forces on the mass is

$$F = .$$

Newton's law of motion states that force is mass times acceleration, $F = ma$, so we have

Now initially we have $F = 0$ when $y = 0$ so $-k(s+0) + mg = 0$, which gives $ks = mg$. Therefore we obtain the equation

and since $m \neq 0$, we have

where $\omega = \sqrt{\frac{k}{m}}$. This is a second order linear homogeneous ODE with constant coefficients so we solve using the characteristic equation

$$\lambda^2 + \omega^2 = 0.$$

This equation has the roots $\lambda_1 = \omega i$ and $\lambda_2 = -\omega i$. Using Proposition 6.2, we obtain the general solution

$$y(t) = A\cos(\omega t) + B\sin(\omega t).$$

We can use the angle addition formula

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

to write this in a different way. We have

$$C\cos(\omega t - \phi) = . \qquad .$$

Now we can let

$$A = C\cos(-\phi), \qquad B = C\sin(-\phi),$$

so that

$$y(t) = C\cos(\omega t - \phi), \qquad \omega = \sqrt{\frac{k}{m}}. \tag{24}$$

Now assume that the mass is $m = 2$ *kg and the spring constant is* $k = 6$ *Newtons per metre. If the spring is pulled* 1 *metre from equilibrium, and the initial speed is* -1 *metre per second, then we have*

$$\omega = \sqrt{3}, \qquad y(0) = 1, \qquad y'(0) = -1.$$

We have

$$y(t) = C\cos(\sqrt{3}t - \phi).$$

When $t = 0$, $y = 1$, *so*

$$C\cos(\phi) = 1 \tag{25}$$

Differentiating, we have

$$y'(t) = -\sqrt{3}C\sin(\sqrt{3}t - \phi),$$

and since $y'(0) = -1$, *we get*

$$\sqrt{3}C\sin(\phi) = -1. \tag{26}$$

Dividing (26) *by* (25) *eliminates C. We have*

$$\sqrt{3}\tan(\phi) = -1.$$

We have two solutions ϕ *satisfying* $0 < \phi < 2\pi$. $\phi = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ *or* $\phi = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$. *This gives* $(C, \phi) = \left(-\frac{2}{\sqrt{3}}, \frac{5\pi}{6}\right)$ *or* $(C, \phi) = \left(\frac{2}{\sqrt{3}}, \frac{11\pi}{6}\right)$. *However, these solutions coincide.*

$$y(t) = \frac{2}{\sqrt{3}}\cos\left(\sqrt{3}t - \frac{11\pi}{6}\right).$$

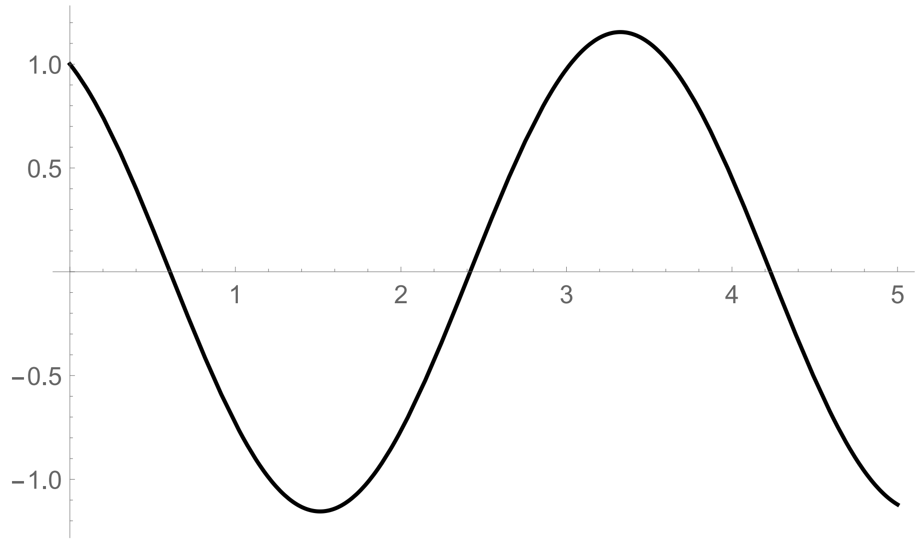


Figure 16: The motion of a spring attached to a mass under gravity in Example 6.2, where $y(t) = \frac{2}{\sqrt{3}} \cos\left(\sqrt{3}t - \frac{11\pi}{6}\right)$ is the vertical axis and t is the horizontal axis.

The previous example is not entirely realistic since we have no considered air resistance. Realistically, we have a damping force that is proportional to the speed of the mass to include in our model:

$$F_{damp.} = -\beta y'(t).$$

Letting $\frac{\beta}{m} = 2\rho$, the resulting ODE is then

$$y''(t) + 2\rho y'(t) + \omega^2 y(t) = 0.$$

The general solution is given by

$$y(t) = Ae^{-\rho t} \cos\left(\sqrt{\omega^2 - \rho^2}t - \phi\right).$$

The component of exponential decay makes the mass tend back to equilibrium over time.

6.4 Second order non-homogeneous equations

Next we consider second order linear ODEs of the form

$$y''(t) + py'(t) + qy(t) = r(t),$$

where p and q are constants. This equation is not homogeneous, however we use the techniques of homogeneous equations to find a part of the general solution. The procedure is as follows:

1. Solve the homogeneous equation $y''(t) + py'(t) + qy(t) = 0$, denoting the result by y_H.
2. Guess a particular solution y_P to $y''(t) + py'(t) + qy(t) = r(t)$.
3. Write the general solution as

$$y(t) = y_H + y_P.$$

We will exhibit this technique on an example involving a forced pendulum.

Example 6.3 *Consider a pendulum oscillating under gravity. In the absence of air resistance, we have* $F = ma = m\frac{d^2(r\theta)}{dt^2} = -mg\sin(\theta)$. *This can be written as*

$$\theta''(t) + \frac{g}{r}\sin(\theta) = 0,$$

which is a non-linear ODE. However, for small θ, *the Taylor series gives the convenient approximation* $\sin(\theta) \approx \theta$ *and so we consider the ODE*

$$\theta''(t) + \frac{g}{r}\theta = 0.$$

We now wish to include air resistance and a forcing function $r(t) = 2t + 1$ *and hence we instead have*

$$\theta''(t) + p\theta'(t) + \frac{g}{r}\theta(t) = 2t + 1.$$

Assume that $g = 9.8$, $r = 4.9$ *metres, and* $p = 1$. *Initially* $\theta(0) = \frac{\pi}{4}$, $\theta'(0) = -1$. *Thus we have the initial value problem*

$$\theta''(t) + \theta'(t) + 2\theta(t) = 2t + 1, \qquad \theta(0) = \frac{\pi}{4}, \qquad \theta'(0) = -1.$$

We begin by solving the homogeneous equation

$$\theta''(t) + \theta'(t) + 2\theta(t) = 0.$$

The characteristic equation is

For the particular solution, we guess $\theta_P = at^2 + bt + c$ *and differentiate twice to obtain*

6.5 n-th order homogeneous equations with constant coefficients

To solve an n-th order homogeneous equations with constant coefficients, we follow the same procedure as for second order homogeneous equations with constant coefficients. Consider the following example.

Example 6.4 *Solve the* 4*-th order ODE* $y^{(4)}(t) - y^{(2)}(t) - 6y(t) = 0$.

7 Systems of ODEs

7.1 How systems of ODEs arise

Consider the **Predator-Prey Population Model**.

$$\frac{dr(t)}{dt} = ar(t) - br(t)f(t),$$
$$\frac{df(t)}{dt} = cr(t)f(t) - df(t),$$

where $r(t)$ is the number of rabbits and $f(t)$ is the number of foxes and a, b, c and d are constants.

We say that these equations are coupled, meaning that we cannot solve either one independently.

A chemical reaction where reactant P is converted to a product C through two intermediates.

$$P \quad \rightarrow \quad A \quad \rightarrow \quad B \quad \rightarrow \quad C$$

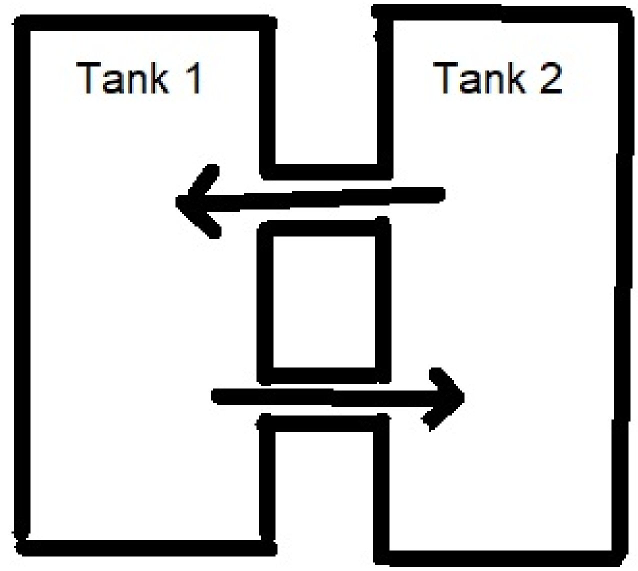
If p is the concentration of P and
a the concentration of A
b the concentration of B
c the concentration of C,
then a simple kinetic model would be

$$\begin{aligned} \frac{dp}{dt} &= -k_0 p, \\ \frac{da}{dt} &= k_0 p - k_1 a, \\ \frac{db}{dt} &= k_1 a - k_2 b, \end{aligned}$$

where k_i are reaction rates, which are assumed to be constant.

A Mixing Problem

Two tanks, one with water, the other syrup are connected by two flow pipes. In one pipe fluid flows from Tank 1 to Tank 2 with flow rate r_1.
In the other fluid flows in the opposite direction with rate r_2.
If y_1 is the amount of syrup in Tank 1
y_2 is the amount of syrup in Tank 2
and V_1 is the volume in Tank 1
and V_2 is the volume in Tank 2.



We can write this as a system of ODEs:

$$\begin{aligned} \frac{dy_1(t)}{dt} &= r_2\frac{y_2}{V_2} - r_1\frac{y_1}{V_1}, \\ \frac{dy_2(t)}{dt} &= r_1\frac{y_1}{V_1} - r_2\frac{y_2}{V_2}, \\ \frac{dy_1}{dt} &= -\frac{dy_2}{dt}. \end{aligned}$$

We have seen earlier that we can write this system as a matrix equation. Let

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}, \qquad A = \begin{pmatrix} -\frac{r_1}{V_1} & \frac{r_2}{V_2} \\ \frac{r_1}{V_1} & -\frac{r_2}{V_2} \end{pmatrix}.$$

Then $Y' = AY$, which we may solve by taking eigenvalues and eigenvectors of the matrix A.

7.2 An example of a second order linear ODE via a coupled system

Recall Example 6.2 on the spring-mass system. We found that

$$y''(t) + \omega^2 y(t) = 0.$$

If we let $z(t) = y'(t)$, then $z'(t) = y''(t)$ so that

To calculate the eigenvectors of A, we seek eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ corresponding to the eigenvalues λ_1 and λ_2 such that $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$ and $A\mathbf{x}_2 = \lambda_2\mathbf{x}_2$.

We can write these equations as

$$(A - \lambda_1 I)\mathbf{x}_1 = \mathbf{0}, \qquad (A - \lambda_2 I)\mathbf{x}_2 = \mathbf{0}.$$

We solve for $\mathbf{x}_1$ and $\mathbf{x}_2$ independently. In the case that $\lambda_1 = \omega i$, we seek $\mathbf{x}_1$ such that

The general solution to the ODE can be expressed using these eigenvalues and eigenvectors as

$$Y(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2.$$

Notice that the first row is

$$y(t) = A\cos(\omega t) + B\sin(\omega t),$$

and the second row is

$$y'(t) = B\omega\cos(\omega t) - A\omega\sin(\omega t),$$

where

$$A = c_1 + c_2, \qquad B = (c_1 - c_2)I.$$

Obtaining an expression for $y'(t)$ also in the general solution can be useful in solving initial value problems because we are given $y'(0)$. Recall that in Example 6.2, we had

$$\omega = \sqrt{3}, \qquad y(0) = 1, \qquad y'(0) = -1.$$

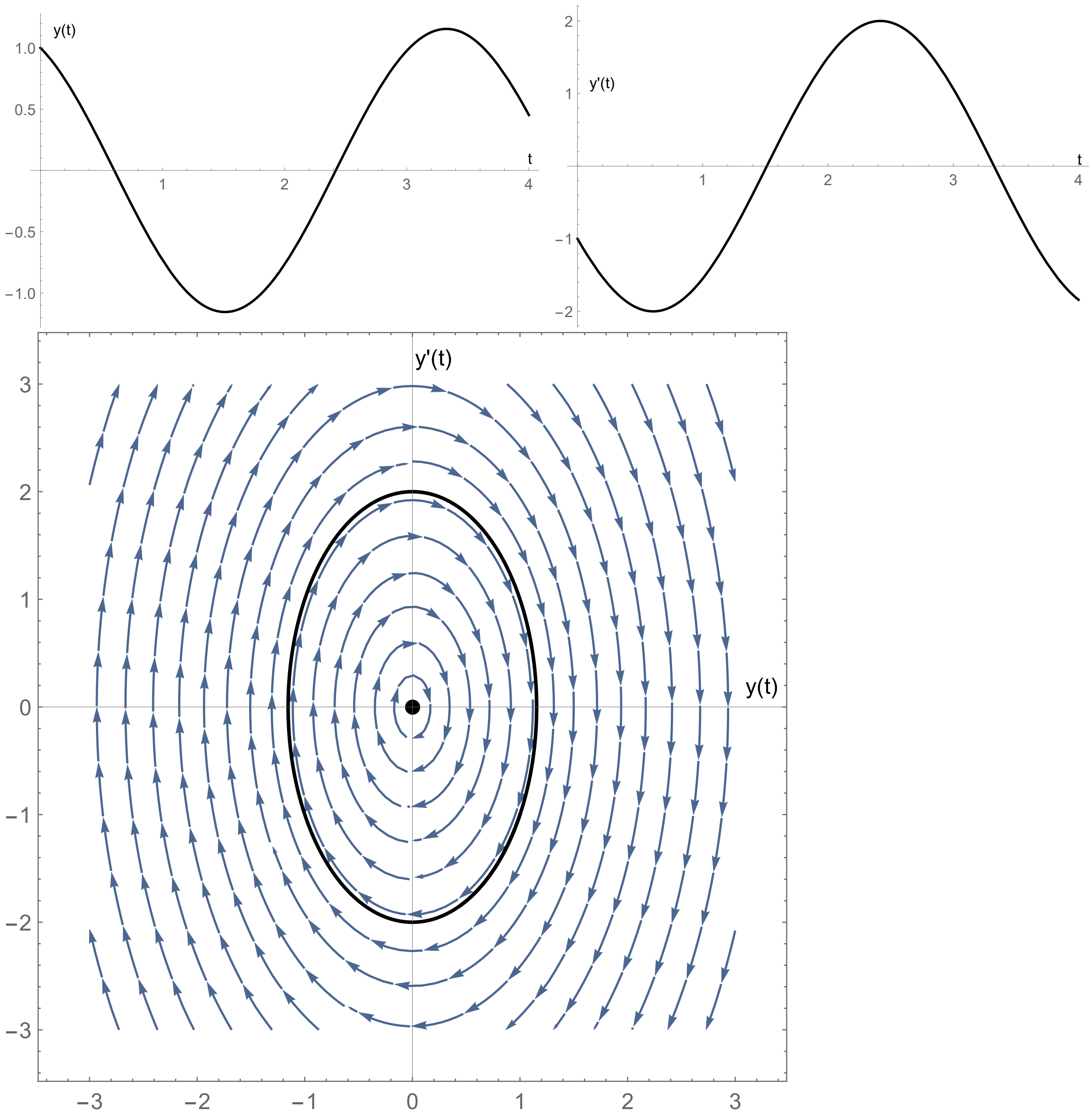


Figure 17: Left: A plot of the solution $y(t) = \cos\left(\sqrt{3}t\right) - \frac{1}{\sqrt{3}}\sin\left(\sqrt{3}t\right)$ to the IVP. Right: A plot of the derivative $y'(t) = -\sqrt{3}\sin\left(\sqrt{3}t\right) - \cos\left(\sqrt{3}t\right)$ of the solution to the IVP. Bottom: A plot of the phase portrait for $y''(t) + \omega^2 y(t) = 0$ showing the critical point $(0,0)$ and the solution to the IVP in black.

7.3 Phase portraits

The blue arrows shown in Figure 17 are called *trajectories* that correspond to a particular initial condition. A collection of all of the trajectories corresponding to each initial condition is called the *phase portrait*. Remember, **Trajectories do not cross one another** as it would violate existence and uniqueness. A system of the form

$$Y' = AY, \qquad\qquad Y = \begin{pmatrix} y \\ z \end{pmatrix},$$

where A is a 2×2 matrix with scalar entries has a single point trajectory $(y,z) = (0,0)$ since $y = 0$, $z = 0$ is a solution to the system. We call such a point trajectory a *critical point*. We aim to classify critical points into six types.

When the eigenvalues λ_1, λ_2 of the matrix A are real and distinct, we have general solution

$$Y = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t}$$

and the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ of A have real entries that can be interpreted as straight line trajectories.

Example 7.1 *Consider the initial value problem*

$$Y' = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} Y, \qquad Y(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{27}$$

The vectors $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *can be interpreted as direction vectors of lines passing through the critical point* $(0,0)$. *These are given by the parametric equations*

See Figure 18.

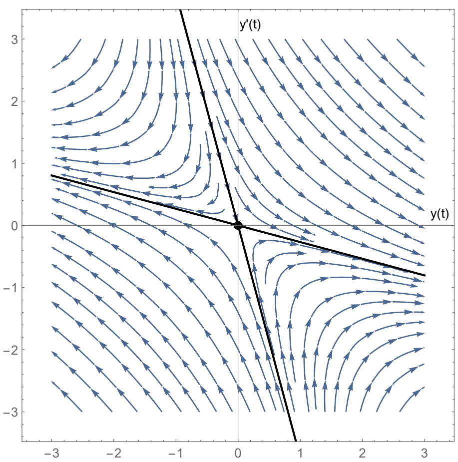


Figure 18: A plot of the phase portrait for the system given in (27) showing the two straight line trajectories obtained from the two eigenvectors of A.

The general solution to the ODE is given by

$$Y = c_1 e^{\sqrt{3}t} \begin{pmatrix} 1 \\ -2+\sqrt{3} \end{pmatrix} + c_2 e^{-\sqrt{3}t} \begin{pmatrix} 1 \\ -2-\sqrt{3} \end{pmatrix}.$$

If $c_1 = 0$*, then the trajectory* $Y = c_2 e^{-\sqrt{3}t} \begin{pmatrix} 1 \\ -2-\sqrt{3} \end{pmatrix}$ *points towards the critical point* $(0,0)$ *indicating a stable equilibrium. If* $c_2 = 0$*, then the trajectory* $Y = c_1 e^{\sqrt{3}t} \begin{pmatrix} 1 \\ -2+\sqrt{3} \end{pmatrix}$ *points away from the critical point* $(0,0)$ *indicating an unstable equilibrium. In this case, the critical point* $(0,0)$ *is said to be a* saddle.

If the system $Y' = AY$ has two equal eigenvalues and linearly independent eigenvectors, then we have a *proper node*. If $\lambda > 0$, then it is unstable. If $\lambda < 0$, then it is stable.

Example 7.2 *Consider the system*

$$Y' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} Y. \tag{28}$$

We have $\lambda_1 = \lambda_2 = 2$ *and despite equal eigenvalues we still have two linearly independent eigenvectors* $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. *See the phase portrait in Figure 19.*

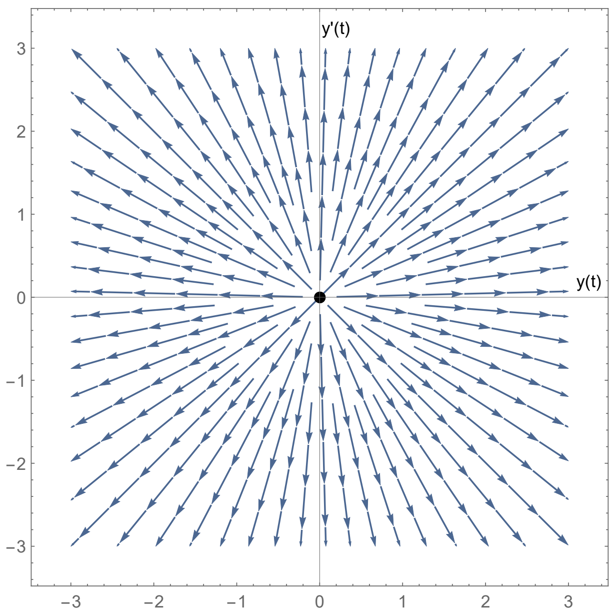
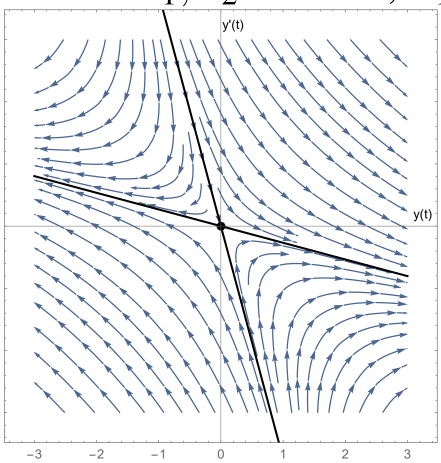


Figure 19: A plot of the phase portrait for the system given in (28) showing the two straight line trajectories obtained from the two eigenvectors of A.

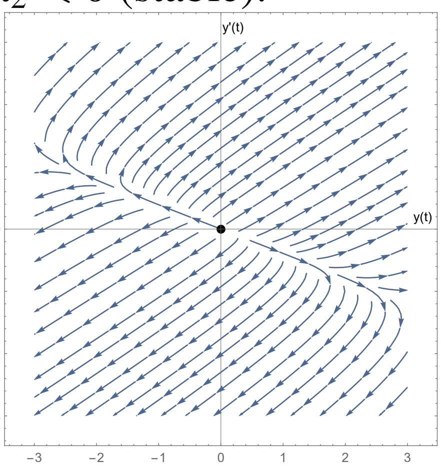
7.4 The six classifications of critical points of linear systems

We classify the six types of phase portraits for the system $Y' = AY$ below.

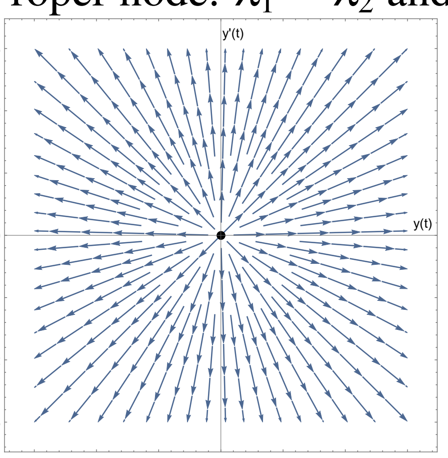
1. Saddle. λ_1, λ_2 are real, $\lambda_1 > 0$, $\lambda_2 < 0$ or $\lambda_1 < 0$, $\lambda_2 > 0$.



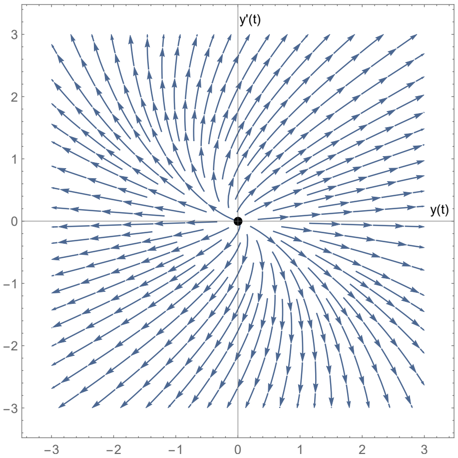
2. Improper node. λ_1, λ_2 are real, $\lambda_1 > 0$, $\lambda_2 > 0$ (unstable) or $\lambda_1 < 0$, $\lambda_2 < 0$ (stable).



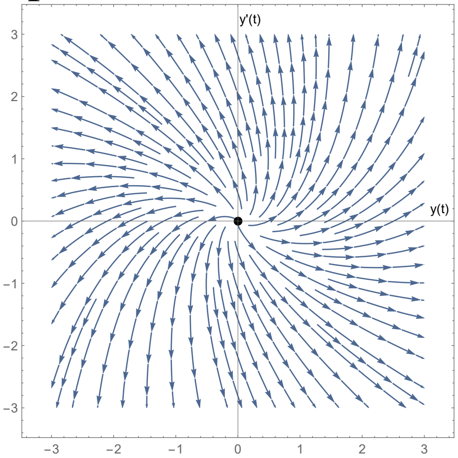
3. Proper node. $\lambda_1 = \lambda_2$ and linearly independent eigenvectors.



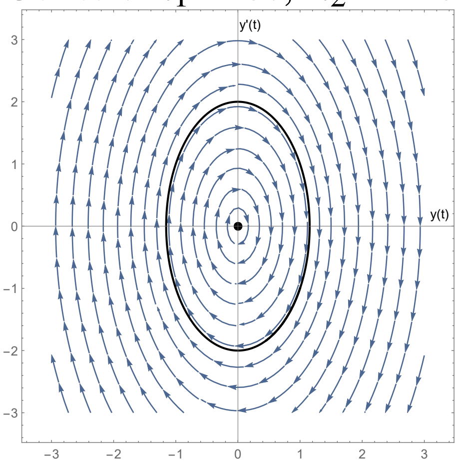
4. Inflected node. $\lambda_1 = \lambda_2$ and only one linearly independent eigenvector.



5. Spiral. $\lambda_1 = a + bi$, $\lambda_2 = a - bi$, $a \neq 0$, $b \neq 0$.



6. Center. $\lambda_1 = bi$, $\lambda_2 = -bi$.



Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The trace of A and the determinant of A are given by

$$\operatorname{tr}(A) = a + d, \qquad\qquad \det(A) = ad - bc.$$

Recall that the characteristic equation can be expressed in terms of the trace and determinant of the matrix A.

$$\det(A - \lambda I) \; = \; . \qquad\qquad\qquad\qquad .$$

Using the quadratic formula we have

$$\lambda = \frac{1}{2}\left(\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}\right).$$

We can use this formula to classify the type and stability of the six kinds of critical points of systems. See Figure 22.

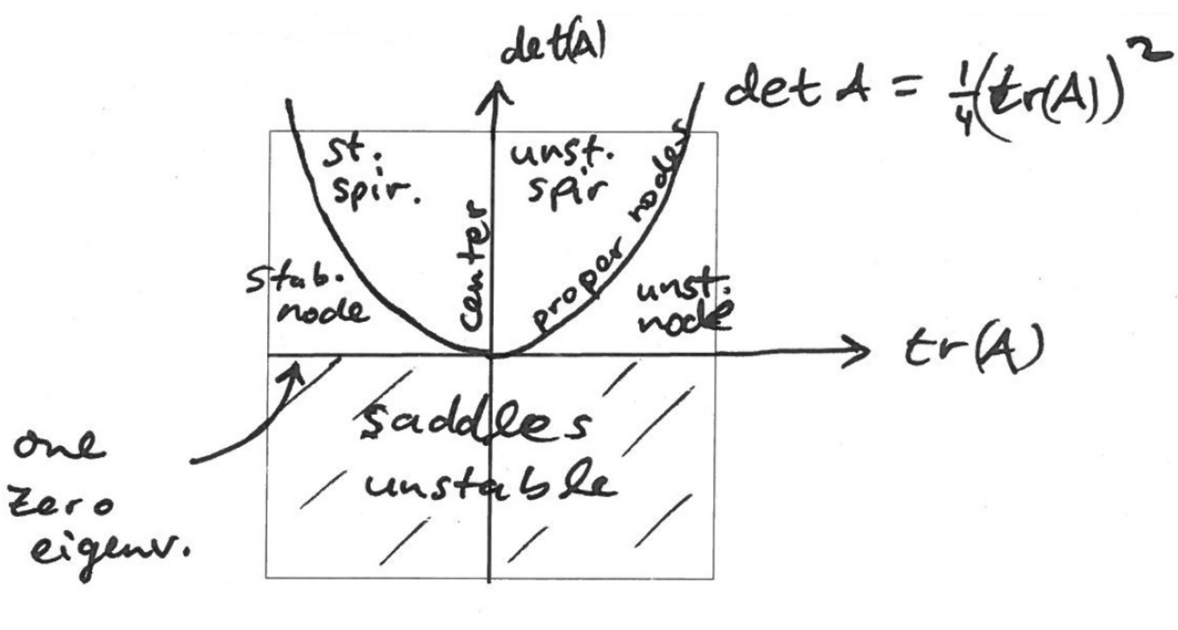


Figure 20: A stability chart.

7.5 An applied example

The following example comes from [15, pp. 323].

Example 7.3 *Let $x(t)$ and $y(t)$ denote the populations of two interacting species of fish living in a small lake. Suppose we have determined that a good model for how these populations evolve is given by the linear system*

$$\begin{aligned} x'(t) &= 2x - y, \\ y'(t) &= 6x - 3y. \end{aligned}$$

A negative population is not meaningful - if it gets to zero, we say it is extinct. If $x(0) = 200$ and $y(0) = 300$, what happens to the populations as t increases?

.

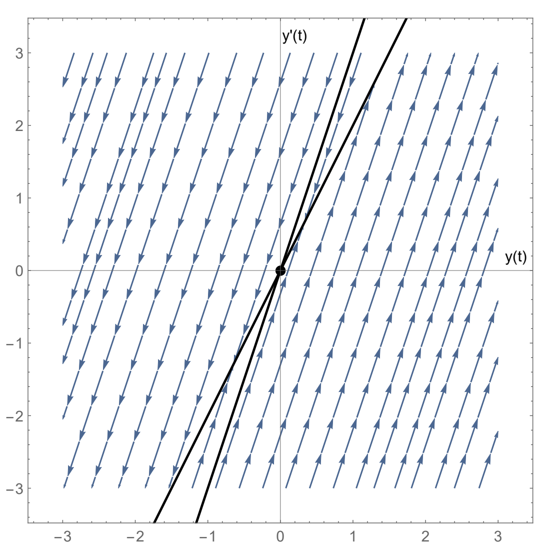


Figure 21: The stability of the critical point in Example 7.3.

8 Qualitative understanding of non-linear systems

8.1 Non-homogeneous linear systems

We can also classify the type and stability of critical points of non-homogeneous linear systems

Example 8.1 *Consider the linear system*

$$\begin{aligned} x'(t) &= 2x(t) + 4y(t) - 3, \\ y'(t) &= -x(t) - 3y(t) + 4. \end{aligned}$$

This is an example of a non-homogeneous linear system. We can write this as

Critical points of the system occur when $Y' = 0$. Hence we must solve for Y^* the linear system

This is the unique critical point of the system.

We classify this point in the same way as we did for homogeneous linear systems. We have

$$tr(A) = \qquad\qquad , \qquad\qquad \det(A) = \qquad\qquad .$$

Using the stability chart, we have a saddle at Y^* hence an unstable equilibrium.

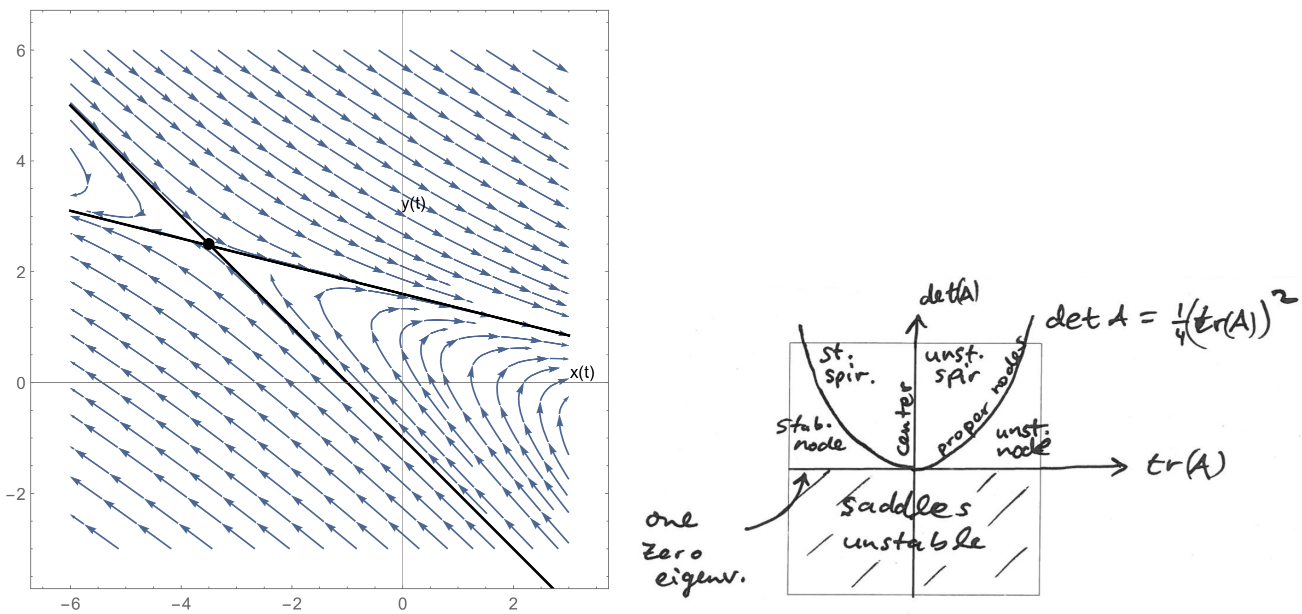


Figure 22: A saddle at $(-7/2, 5/2)$.

8.2 Solving systems via differential operators

Let D^n denote the n-th differential operator $\frac{d^n}{dt^n}$. In this section we illustrate a method of solving linear systems by transforming them into n-th order ODES. This provides a second way of solving systems of linear differential equations.

Example 8.2 *Consider the system*

$$\begin{aligned} x' + y' + 2y &= 0, \\ x' - 3x - 2y &= 0. \end{aligned}$$

Using the differential operator D, we have

Alternatively, the system can be rearranged as

$$x' = 3x + 2y,$$
$$y' = -3x - 4y,$$

.

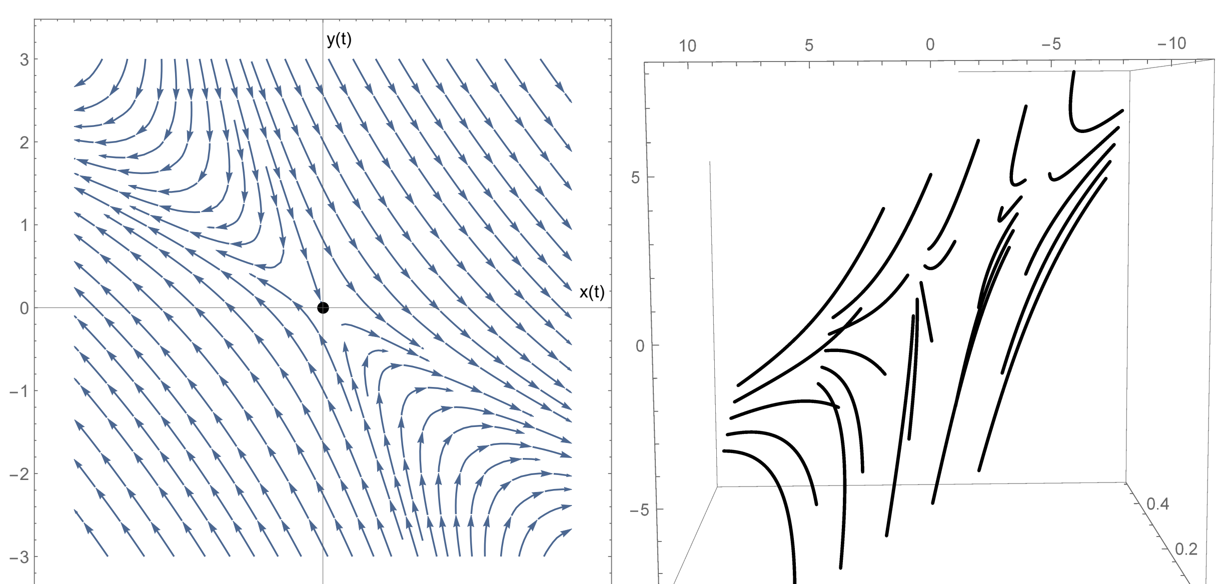


Figure 23: Left: The phase portrait for the system. Right: Trajectories shown in three variables.

8.3 Linearization of non-linear systems and applications

To understand the behaviour of a non-linear system

$$\begin{aligned} x' &= f(x,y), \\ y' &= g(x,y) \end{aligned}$$

near a critical point (x^*, y^*), we linearize the system by taking a *Jacobian matrix*. Letting $F(x,y) = \begin{pmatrix} f \\ g \end{pmatrix}$, the Jacobian is given by

$$J_F(x^*, y^*) = \begin{pmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{pmatrix}. \tag{29}$$

We then analyze the type and stability of the linearized systems for each critical point of the non-linear system

$$Y' = J_F(x^*, y^*)\, Y + b, \tag{30}$$

where b depends on the critical point (x^*, y^*).

In what follows we will revisit some of the examples considered in the introduction of Chapter 7.

Example 8.3 *Consider the predator-prey model*

$$\begin{aligned} x'(t) &= ax - bxy, \\ y'(t) &= cxy - dy, \end{aligned}$$

where $x(t)$ is the number of rabbits and $y(t)$ is the number of wolves. Suppose we have $a = 6,\ b = 1,\ c = 2,\ d = 4.$ Then we have the non-linear system

Next we linearize the system taking the Jacobian matrix

$$\begin{aligned} J_F(x^*, y^*) &= \begin{pmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{pmatrix}, \\ &= \end{aligned}$$

Substituting values of the critical points, *Near the critical points the system behaves like the linear systems*

.

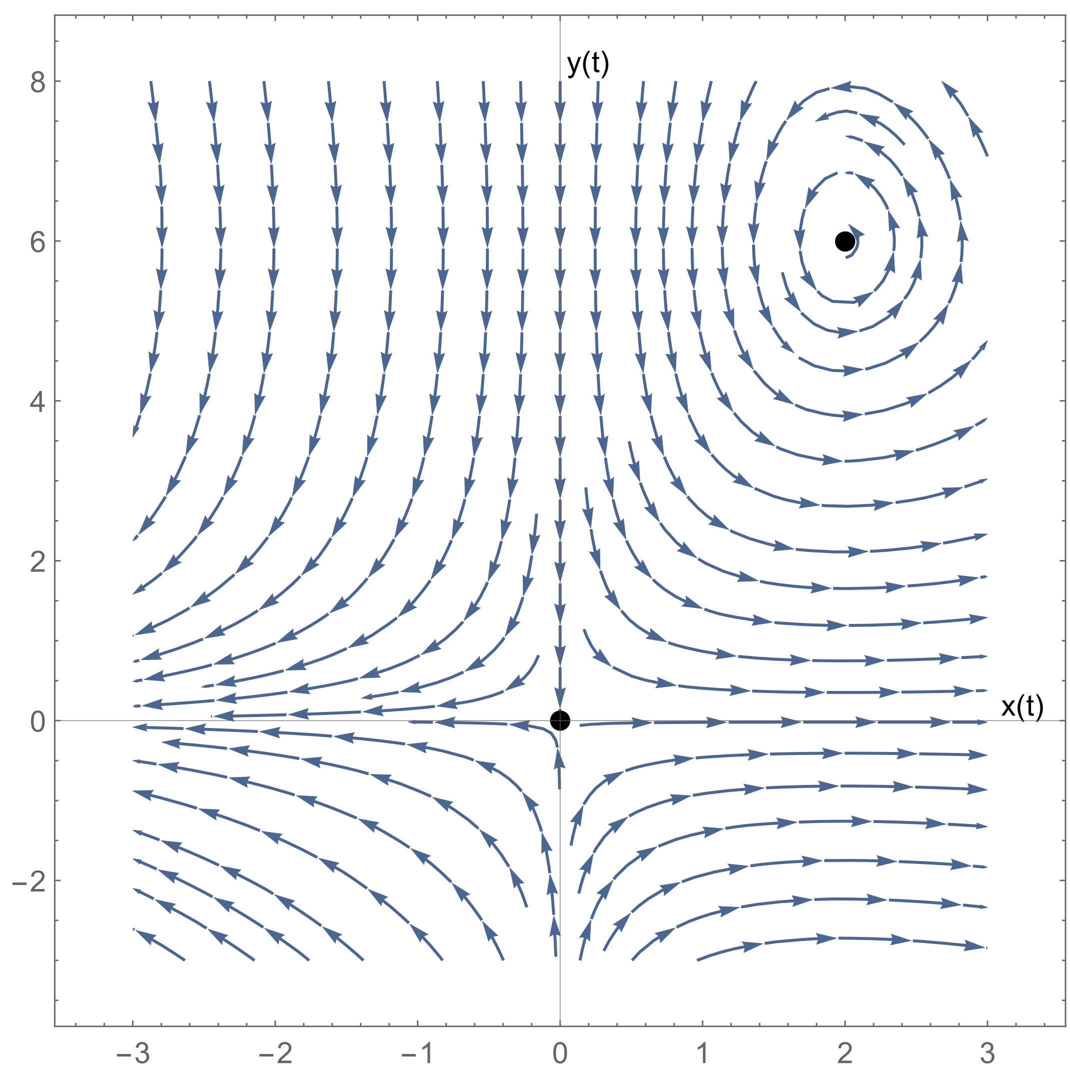


Figure 24: The phase portrait for the system. Note that $x(t) > 0$ and $y(t) > 0$.

Example 8.4 *See [15, pp. 360, Q. 5, 6]. Consider the system*

$$\begin{aligned} x'(t) &= -x, \\ y'(t) &= -4x^3 + y. \end{aligned}$$

1. *Show that* $(0,0)$ *is the only equilibrium point (critical point).*
2. *Find the linearized system near* $(0,0)$.

3. *Classify the linearized system and sketch it.*

4. *Find the general solution to* $x'(t) = -x$.

5. *Use the general solution to* $x'(t) = -x$ *to find the general solution to the system.*

Example 8.5 *See [15, pp. 362, Q. 22]. When a non-linear system depends on a parameter, then as the parameter changes, the equilibrium points can change. That is, as the parameter changes, a bifurcation can occur. Consider the one-parameter system*

$$\begin{aligned} \frac{dx}{dt} &= x^2 - a, \\ \frac{dy}{dt} &= -y\left(x^2+1\right), \end{aligned}$$

where a is a parameter.

1. *Show that for $a < 0$ the system has no equilibrium points.*

2. *Show that for $a > 0$ the system has two equilibrium points.*

3. *Show that for* $a = 0$ *the system has exactly one equilibrium point.*
4. *Find a linearisation of the equilibrium point when* $a = 0$ *and compute the eigenvalues for this point.*

9 Solving ODEs with Laplace transforms

9.1 A summary of the main tools of Laplace transforms

- Laplace transform definition:

$$\mathscr{L}(f) = \int_{t=0}^{\infty} e^{-st} f(t)\, dt,$$

- Heaviside step function:

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

- Gamma function:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1}\, dt \qquad\qquad x! = \Gamma(x+1), \text{ for } x \in \mathbb{Z}.$$

- $\mathscr{L}(k) = \frac{k}{s}$.
- $\mathscr{L}(u(t-a)) = \frac{e^{-sa}}{s}$.
- $\mathscr{L}(t^n) = \frac{n!}{s^{n+1}}$.
- $\mathscr{L}(e^{at}) = \begin{cases} \frac{1}{s-a} & \text{if } s > a \\ \text{diverges} & \text{if } s \leq a \end{cases}$
- If $Y(s) = \mathscr{L}(y)$, then

$$\mathscr{L}(y') = sY(s) - y(0), \qquad \mathscr{L}(y'') = s^2 Y(s) - sy(0) - y'(0).$$

Example 9.1 *Solve the following IVP using the Laplace transform:*

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

For comparison, first let's solve the IVP using the characteristic equation:

$$\lambda^2 - \lambda - 2 = 0.$$

Factorizing, we have $(\lambda+1)(\lambda-2)=0$ *and we then write the general solution to the ODE as*

$$y(t)=Ae^{-t}+Be^{2t}.$$

Next we use the initial values $y(0)=1$ *and* $y'(0)=0$ *to obtain the constants* A *and* B*. Beginning with* $y(0)=1$*, we see that* $y(0)=1=Ae^0+Be^0$ *so* $A+B=1$*. Using* $y'(0)=0$ *requires differentiating:*

$$y'(t)=-Ae^{-t}+2Be^{2t}.$$

Finally, $y'(0)=0=-Ae^0+2Be^0$ *so* $-A+2B=0$*. Hence we must solve the simultaneous system*

$$\begin{aligned} A+B &= 1, \\ -A+2B &= 0. \end{aligned}$$

Adding gives $3B=1$ *so that* $B=\frac{1}{3}$ *and* $A=1-B=\frac{2}{3}$*. Hence we have the solution*

$$y(t)=\frac{2}{3}e^{-t}+\frac{1}{3}e^{2t}. \tag{31}$$

to the IVP.

Now we solve the same IVP using the Laplace transform. Apply $\mathscr{L}$*, we have*

$$\mathscr{L}(y''-y'-2y) = \mathscr{L}(0)=0,$$

$$\mathscr{L}(y'')-\mathscr{L}(y')-2\mathscr{L}(y) = 0.$$

Letting $Y(s)=\mathscr{L}(y)$ *and using the rules for* $\mathscr{L}(y'')$ *and* $\mathscr{L}(y')$*,*

$$\left(s^2Y(s)-sy(0)-y'(0)\right)-(sY(s)-y(0))-2Y(s) = 0.$$

Collecting like terms and replacing $y(0) = 1$ and $y'(0) = 0$,

$$\left(s^2 - s - 2\right) Y(s) - s + 1 \;=\; 0.$$

Now it is easy to solve for $Y(s)$.

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s+1)(s-2)}.$$

Reflecting on those Laplace transforms we already know, namely

$$\mathscr{L}\left(e^{at}\right) = \begin{cases} \frac{1}{s-a} & \text{if } s > a, \\ \text{diverges} & \text{if } s \leq a, \end{cases}$$

it is clear that we must split the right hand side

$$\frac{s-1}{(s+1)(s-2)} = \frac{u}{s+1} + \frac{v}{s-2}$$

using partial fractions. Recombining by cross-multiplying,

$$\frac{s-1}{(s+1)(s-2)} = \frac{u(s-2)+v(s+1)}{(s+1)(s-2)} = \frac{(u+v)s - 2u + v}{(s+1)(s-2)}.$$

Matching coefficients gives the simultaneous system

$$\begin{aligned} u + v &= 1, \\ -2u + v &= -1. \end{aligned}$$

Subtracting equations gives $3u = 2$ so $u = \frac{2}{3}$ and $v = \frac{1}{3}$. It follows that

$$Y(s) = \frac{2}{3}\frac{1}{s-(-1)} + \frac{1}{3}\frac{1}{s-2}.$$

Hence applying the inverse Laplace transform,

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left(Y(s)\right) = \frac{2}{3}\mathscr{L}^{-1}\left(\frac{1}{s-(-1)}\right) + \frac{1}{3}\mathscr{L}^{-1}\left(\frac{1}{s-2}\right), \\ &= \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}, \end{aligned}$$

which agrees with equation (31).

Example 9.2 *Derive the Laplace transform of* $f(t) = k$.

$$
\begin{aligned}
\mathscr{L}(k) &= \int_{t=0}^{\infty} e^{-st} k \, dt, \\
&= k \int_{t=0}^{\infty} e^{-st} \, dt, \\
&= k \left[\frac{1}{-s} e^{-st} \right]_0^{\infty}, \\
&= \frac{k}{-s} \left(\lim_{t \longrightarrow \infty} \left(e^{-st} \right) - e^0 \right), \\
&= \frac{k}{-s} (0 - 1), \\
&= \frac{k}{s}.
\end{aligned}
$$

Example 9.3 *Derive the Laplace transform of* $f(t) = t^n$.

Let $F_n(s) = \mathscr{L}(t^n)$. *Using integration by parts,*

$$
\begin{aligned}
F_n(s) &= \int_{t=0}^{\infty} e^{-st} t^n \, dt, \\
&= \int_{t=0}^{\infty} t^n \frac{d}{dt} \frac{1}{-s} e^{-st} \, dt, \\
&= \left[\frac{t^n}{-s} e^{-st} \right]_0^{\infty} - \int_{t=0}^{\infty} \frac{1}{-s} e^{-st} \frac{d}{dt} t^n \, dt, \\
&= \lim_{t \longrightarrow \infty} \left(\frac{t^n}{-s e^{st}} \right) - 0 - \int_{t=0}^{\infty} \frac{n}{-s} e^{-st} t^{n-1} \, dt, \\
&= \frac{n}{s} \int_{t=0}^{\infty} e^{-st} t^{n-1} \, dt, \\
&= \frac{n}{s} F_{n-1}(s).
\end{aligned}
$$

We have used L'Hopital's rule to evaluate the limit as 0*. It is easy to show that* $F_1(s) = \frac{1}{s^2}$*. A simple induction argument shows that*

$$F_n(s) = \frac{n!}{s^{n+1}}.$$

Example 9.4 *Derive the Laplace transform of* $f(t) = e^{at}$*.*

$$\begin{aligned}
\mathscr{L}\left(e^{at}\right) &= \int_{t=0}^{\infty} e^{-st} e^{at}\, dt, \\
&= \int_{t=0}^{\infty} e^{(a-s)t}\, dt, \\
&= \left[\frac{1}{a-s} e^{(a-s)t}\right]_0^{\infty}, \\
&= \lim_{t \longrightarrow \infty} \left(\frac{1}{a-s} e^{(a-s)t}\right) - \frac{1}{a-s},
\end{aligned}$$

If $\Re(a-s) < 0$*, then* $\lim_{t\longrightarrow\infty}\left(\frac{1}{a-s}e^{(a-s)t}\right) = 0$*. Otherwise, the limit diverges. Hence if* $a < s$*, then* $\mathscr{L}\left(e^{at}\right) = \frac{1}{s-a}$*.*

Example 9.5 *Derive the Laplace transform of* $\sin(t)$ *and* $\cos(t)$ *using that of* e^{at}*.*

Let $f(t) = e^{i\alpha t}$*. We are motivated by Euler's formula*

$$e^{i\alpha t} = \cos(\alpha t) + i \sin(\alpha t).$$

Taking the Laplace transform,

$$\begin{aligned}
\mathscr{L}\left(e^{i\alpha t}\right) &= \frac{1}{s - i\alpha}, \\
&= \frac{s + i\alpha}{(s-i\alpha)(s+i\alpha)}, \\
&= \frac{s+i\alpha}{s^2+\alpha^2}, \\
&= \frac{s}{s^2+\alpha^2} + i\frac{\alpha}{s^2+\alpha^2}.
\end{aligned}$$

Using Euler's result,

$$\begin{aligned}
\mathscr{L}\left(e^{i\alpha t}\right) &= \mathscr{L}\left(\cos(\alpha t) + i\sin(\alpha t)\right), \\
&= \mathscr{L}\left(\cos(\alpha t)\right) + i\mathscr{L}\left(\sin(\alpha t)\right), \\
&= \frac{s}{s^2+\alpha^2} + i\frac{\alpha}{s^2+\alpha^2}.
\end{aligned}$$

Matching real and imaginary parts, we obtain

$$\mathscr{L}\left(\cos(\alpha t)\right) = \frac{s}{s^2+\alpha^2}, \qquad \mathscr{L}\left(\sin(\alpha t)\right) = \frac{\alpha}{s^2+\alpha^2}.$$

Example 9.6 *Solve the following IVP using the Laplace transform:*

$$x''(t) - 5x'(t) + 6x(t) = 2, \qquad x(0) = 0, \qquad x'(0) = 0.$$

Taking the Laplace transform of both sides of the ODE,

$$\begin{aligned}
\mathscr{L}\left(x''(t)\right) - 5\mathscr{L}\left(x'(t)\right) + 6\mathscr{L}\left(x(t)\right) &= \mathscr{L}\left(2\right), \\
s^2X(s) - sx(0) - x'(0) - 5\left(sX(s) - x(0)\right) + 6X(s) &= \frac{2}{s}, \\
\left(s^2 - 5s + 6\right)X(s) &= \frac{2}{s},
\end{aligned}$$

It follows that

$$\begin{aligned}
X(s) &= \frac{2}{s(s-2)(s-3)}, \\
&= \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-3}, \\
&= \frac{A(s-2)(s-3) + Bs(s-3) + Cs(s-2)}{s(s-2)(s-3)}, \\
&= \frac{s^2(A+B+C) + s(-5A-3B-2C) + 6A}{s(s-2)(s-3)}.
\end{aligned}$$

So we have the system

$$\begin{aligned} A+B+C &= 0, \\ -5A-3B-2C &= 0, \\ 6A &= 2. \end{aligned}$$

This has solution $A=\frac{1}{3}$, $B=-1$, $C=\frac{2}{3}$. *Alternatively, we can evaluate at* $s=0$, $s=2$, $s=3$ *to obtain the same result. Hence we have*

$$X(s) = \frac{1}{3}\frac{1}{s} - \frac{1}{s-2} + \frac{2}{3}\frac{1}{s-3}.$$

Applying the inverse Laplace transform now gives

$$x(t) = \frac{1}{3} - e^{2t} + \frac{2}{3}e^{3t}.$$

Example 9.7 *Solve the following IVP using the Laplace transform:*

$$y''(t) + y'(t) + y(t) = \cos(t), \qquad y(0) = 1, \qquad y'(0) = -1.$$

Taking the Laplace transform gives

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) + Y(s) &= \frac{s}{s^2+1}, \\ \left(s^2+s+1\right)Y(s) - s &= \frac{s}{s^2+1}, \end{aligned}$$

Hence

$$\begin{aligned} Y(s) &= \frac{s\left(s^2+2\right)}{\left(s^2+1\right)\left(s^2+s+1\right)}, \\ &= \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+s+1}. \end{aligned}$$

We find that $A = 0$, $B = C = 1$, $D = -1$. So

$$
\begin{aligned}
Y(s) &= \frac{1}{s^2+1} + \frac{s-1}{s^2+s+1}, \\
&= \frac{1}{s^2+1} + \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \sqrt{3}\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2},
\end{aligned}
$$

Using a table of Laplace transforms (see the next page), taking the inverse Laplace transform gives

$$
y(t) = \sin(t) + e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) - \sqrt{3}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right).
$$

9.2 Laplace transform theorems

The following results are useful when the table of Laplace transforms is insufficient to construct a Laplace transform or the inverse.

- First Shifting: $\mathscr{L}\left(e^{at}f(t)\right) = F(s-a)$, where $F(s) = \mathscr{L}\left(f(t)\right)$.
- Times $-t$: $\mathscr{L}\left(-tf(t)\right) = \frac{d}{ds}\mathscr{L}\left(f(t)\right)$.
- Convolution: $\mathscr{L}^{-1}\left(F(s)G(s)\right) = \int_{\tau=0}^{\tau=t} f(\tau)g(t-\tau)\,d\tau$.
- Second Shifting: $\mathscr{L}\left(f(t-k)u(t-k)\right) = e^{-sk}\mathscr{L}(f(t))$,
  $\mathscr{L}^{-1}\left(e^{-ks}F(s)\right) = f(t-k)u(t-k)$, where $\mathscr{L}^{-1}\left(F(s)\right) = f(t)$.
- If $f(t)$ is periodic with period T, then $\mathscr{L}(f(t)) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$.

9.3 Table of Laplace transforms

$$\mathscr{L}(f) = \int_{t=0}^{\infty} e^{-st} f(t)\,dt$$

$$\mathscr{L}\left(y'\right) = s\mathscr{L}(y) - y(0)$$

$$\mathscr{L}\left(y''\right) = s^2\mathscr{L}(y) - sy(0) - y'(0)$$

$$\mathscr{L}\left(y^{(n)}\right) = s^n\mathscr{L}(y) - s^{n-1}y(0) - \cdots - y^{n-1}(0)$$

$$\mathscr{L}(k) = \frac{k}{s} \qquad \mathscr{L}^{-1}\left(\frac{1}{s}\right) = 1$$

$$\mathscr{L}\left(t^n\right) = \frac{n!}{s^{n+1}} \qquad \mathscr{L}^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$$

$$\mathscr{L}\left(e^{at}\right) = \frac{1}{s-a} \qquad \mathscr{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

$$\mathscr{L}\left(\cos(\alpha t)\right) = \frac{s}{s^2+\alpha^2} \qquad \mathscr{L}^{-1}\left(\frac{s}{s^2+\alpha^2}\right) = \cos(\alpha t)$$

$$\mathscr{L}\left(\sin(\alpha t)\right) = \frac{\alpha}{s^2+\alpha^2} \qquad \mathscr{L}^{-1}\left(\frac{1}{s^2+\alpha^2}\right) = \frac{1}{\alpha}\sin(\alpha t)$$

$$\mathscr{L}\left(u(t-k)\right) = \frac{e^{-ks}}{s} \qquad \mathscr{L}^{-1}\left(e^{-ks}\right) = u(t-k)$$

$$\mathscr{L}\left(t^n e^{at}\right) = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}^{-1}\left(\frac{1}{(s-a)^n}\right) = \frac{t^{n-1}}{(n-1)!}e^{at}$$

$$\mathscr{L}\left(e^{at}\cos(\alpha t)\right) = \frac{s-a}{(s-a)^2+\alpha^2} \qquad \mathscr{L}^{-1}\left(\frac{s-a}{(s-a)^2+\alpha^2}\right) = e^{at}\cos(\alpha t)$$

$$\mathscr{L}\left(e^{at}\sin(\alpha t)\right) = \frac{\alpha}{(s-a)^2+\alpha^2} \qquad \mathscr{L}^{-1}\left(\frac{\alpha}{(s-a)^2+\alpha^2}\right) = e^{at}\sin(\alpha t)$$

$$\mathscr{L}\left(e^{at}u(t-k)\right) = \frac{e^{-k(s-a)}}{s-a} \qquad \mathscr{L}^{-1}\left(\frac{e^{-k(s-a)}}{s-a}\right) = e^{at}u(t-k)$$

$$\mathscr{L}\left(t\cos(\alpha t)\right) = \frac{1}{s^2+\alpha^2} - \frac{2}{\left(s^2+\alpha^2\right)^2} \qquad \mathscr{L}^{-1}\left(\frac{s^2+\alpha^2-2}{\left(s^2+\alpha^2\right)^2}\right) = t\cos(\alpha t)$$

$$\mathscr{L}\left(t\sin(\alpha t)\right) = \frac{2\alpha s}{\left(s^2+\alpha^2\right)^2} \qquad \mathscr{L}^{-1}\left(\frac{s}{\left(s^2+\alpha^2\right)^2}\right) = \frac{1}{2\alpha}t\sin(\alpha t)$$

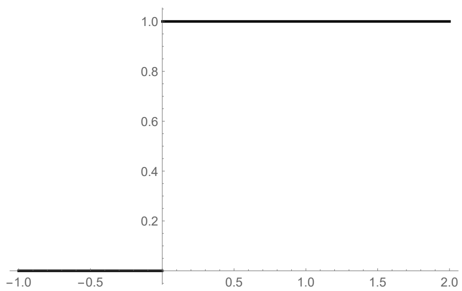
$$\mathscr{L}^{-1}\left(\frac{s^2}{\left(s^2+\alpha^2\right)^2}\right) = \frac{1}{2\alpha}t\sin(\alpha t) + \frac{1}{2}t\cos(\alpha t)$$

$$\mathscr{L}^{-1}\left(\frac{1}{\left(s^2+\alpha^2\right)^2}\right) = \frac{1}{2\alpha^3}t\sin(\alpha t) - \frac{1}{2\alpha^2}t\cos(\alpha t)$$

9.4 Solving equations with step functions

Recall the Heaviside step function:

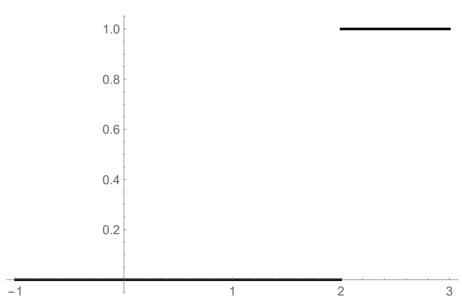
$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



We can translate the position of the step

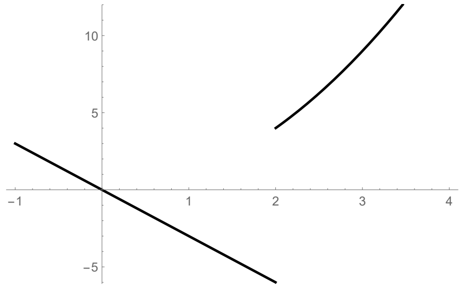
$$u(t-c) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

In this plot we have $u(t-2)$.



Here is the step down function $1 - u(t-c)$:

Using the step down function $1 - u(t-c)$, we have



which shows

$$t^2u(t-2) - 3t(1-u(t-2)) = \left(t^2+3t\right)u(t-2) - 3t.$$

Taking the Laplace transform of the step function $u(t-c)$ gives

$$\mathscr{L}\left(u(t-c)\right) = \int_0^\infty e^{-st}u(t-c)\,dt,$$

$$=$$

Example 9.8 *Calculate* $\mathscr{L}\left(1-u(t-c)\right)$.

Theorem 1 *Second Shifting theorem.*
$\mathscr{L}\left(f(t-c)u(t-c)\right) = e^{-sc}\mathscr{L}(f(t))$,
$\mathscr{L}^{-1}\left(e^{-cs}F(s)\right) = f(t-c)u(t-c)$, *where* $\mathscr{L}^{-1}\left(F(s)\right) = f(t)$.

Example 9.9 *Calculate* $\mathscr{L}\left(f(t)u(t)\right)$, *where* $f(t) = t^2$.

Example 9.10 *Let*

$$\begin{aligned} g(t) &= t^2 u(t-2) - 3t(1 - u(t-2)), \\ &= \left(t^2 + 3t\right) u(t-2) - 3t. \end{aligned}$$

Show that $f(t) = t^2 + 7t + 10$ *satisfies* $f(t-2) = t^2 + 3t$ *and then use the second shifting theorem to calculate* $\mathscr{L}(g(t))$.

Example 9.11 *Consider the RC circuit modeled by the equation*

$$2\frac{dv}{dt} + v = 3u(t-2),$$

with $v(0) = 4$. *See [4]. Use the second shifting theorem to solve this equation.*

9.5 The convolution theorem

In this section we will consider the interesting and usefull result know as the convolution theorem.

Theorem 2 *Convolution theorem.*
$\mathscr{L}^{-1}(F(s)G(s)) = \int_{\tau=0}^{\tau=t} f(\tau)g(t-\tau)\,d\tau$, *where* $f(t) = \mathscr{L}^{-1}(F(s))$ *and* $g(t) = \mathscr{L}^{-1}(G(s))$.

This allows use to avoid partial fractions for example when we calculate the inverse Laplace transform of a product of two rational functions of s.

Example 9.12 *Calculate* $\mathscr{L}^{-1}(F(s)G(s))$, *where* $F(s) = \frac{1}{s-1}$ *and* $G(s) = \frac{1}{s+2}$ *using the convolution theorem:*

$$\mathscr{L}^{-1}(F(s)G(s)) = \int_{\tau=0}^{\tau=t} e^{\tau} e^{-2(t-\tau)}\,d\tau.$$

Example 9.13 *Use the convolution theorem to calculate* $\mathscr{L}^{-1}\left(\frac{3}{(s-3)^2}\right)$.

Example 9.14 *Verify that* $\mathscr{L}^{-1}\left(\frac{s}{(s-3)^2}\right) = e^{3t}(3t+1)$ *by writing* $\frac{s}{(s-3)^2} = \frac{(s-3)+3}{(s-3)^2}$ *and using the table of Laplace transforms or Example 9.13.*

Example 9.15 *Use the convolution theorem to calculate* $\mathscr{L}^{-1}\left(\frac{1}{s^2(s+2)}\right)$.

Example 9.16 *Use the convolution theorem to calculate* $\mathscr{L}^{-1}\left(\frac{3}{s^2(s-2)}\right)$, *letting* $F(s) = \frac{3}{s^2}$ *and* $G(s) = \frac{1}{s-2}$.

10 Laplace transforms for solving systems

10.1 Revisiting Example 6.3

In this section we will solve a linear system of ODEs using the Laplace transform. This chapter does not contain very much new information. We use use the time this week to reinforce your understanding of Laplace transforms and to strengthen your skills in partial fractions.

Example 10.1 *Consider the IVP linear system*

$$X' = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} X + \begin{pmatrix} 0 \\ 2t+1 \end{pmatrix}, \qquad X(0) = \begin{pmatrix} \pi/4 \\ -1 \end{pmatrix}.$$

This is a linearized version of the second order ODE on the forced and damped pendulum of Example 6.3, where $X = \begin{pmatrix} \theta \\ \theta' \end{pmatrix} = \begin{pmatrix} \theta \\ x \end{pmatrix}$. *Recall that we obtained the solution*

$$\theta(t) = e^{\frac{-1}{2}t}\left(\frac{\pi}{4}\cos\left(\frac{1}{2}\sqrt{7}t\right) + \frac{\pi-16}{4\sqrt{7}}\sin\left(\frac{1}{2}\sqrt{7}t\right)\right) + t. \tag{32}$$

We will use this solution to compare with the solution obtained by Laplace transforms. Our first step is to set out the components

$$\theta' = x, \tag{33}$$

$$x' = -2\theta - x + 2t + 1. \tag{34}$$

We let

$$Y_1(s) = \mathscr{L}(\theta), \qquad Y_2(s) = \mathscr{L}(x).$$

We have $\mathscr{L}(2t+1) = \frac{2}{s^2} + \frac{1}{s}$. *Applying the Laplace transform to equations* (33) *and* (34) *gives*

$$\begin{aligned} \mathscr{L}(\theta') &= \mathscr{L}(x), \\ \mathscr{L}(x') &= -2\mathscr{L}(\theta) - \mathscr{L}(x) + 2\mathscr{L}(t) + \mathscr{L}(1). \end{aligned}$$

Simplifying,

$$\begin{aligned} sY_1(s) - \frac{\pi}{4} &= Y_2(s), \\ sY_2(s) + 1 &= -2Y_1(s) - Y_2(s) + \frac{2}{s^2} + \frac{1}{s}. \end{aligned}$$

Now we have an algebraic system to solve for $Y_1(s)$ and $Y_2(s)$. Rearranging gives

$$\begin{aligned} sY_1(s) - Y_2(s) &= \frac{\pi}{4}, \\ 2Y_1(s) + (s+1)Y_2(s) &= \frac{2}{s^2} + \frac{1}{s} - 1. \end{aligned}$$

Solving for $Y_1(s)$ and $Y_2(s)$, we have

$$\begin{aligned} Y_1(s) &= \frac{\pi s^3 + (\pi - 4)s^2 + 4s + 8}{4s^2\left(s^2 + s + 2\right)}, \\ Y_2(s) &= \frac{-2s^2 + (2 - \pi)s + 4}{2s\left(s^2 + s + 2\right)}. \end{aligned}$$

Now all that remains is to take the inverse Laplace transform. However, this is easier said than done! We first express the right hand sides using partial fractions. Beginning with the first equation, we seek constants a_1, a_2, b_1, b_2 satisfying

$$\frac{\pi s^3 + (\pi - 4)s^2 + 4s + 8}{4s^2\left(s^2 + s + 2\right)} = \frac{a_1 s + a_2}{4s^2} + \frac{b_1 s + b_2}{s^2 + s + 2}.$$

Recombining by cross multiplying, the numerator of the right hand side is

$$s^3\left(a_1 + 4b_1\right) + s^2\left(a_1 + a_2 + 4b_2\right) + \left(2a_1 + a_2\right)s + 2a_2 = \pi s^3 + (\pi - 4)s^2 + 4s + 8.$$

Solving gives

$$a_1 = 0, \qquad a_2 = 4, \qquad b_1 = \frac{\pi}{4}, \qquad b_2 = \frac{1}{4}(\pi - 8),$$

so that

$$Y_1(s) \;=\; \frac{1}{s^2}+\frac{\frac{\pi}{4}s+\frac{1}{4}(\pi-8)}{s^2+s+2},$$

It is clear from the table of Laplace transforms in Section 9.3 how to compute $\mathscr{L}^{-1}\left(\frac{1}{s^2}\right)=t$. *However, to use the table for the inverse Laplace transform of* $\frac{\frac{\pi}{4}s+\frac{1}{4}(\pi-8)}{s^2+s+2}$, *we must express the denominator* s^2+s+2 *as* $(s-a)^2+\alpha^2$. *Expanding this and solving for* a *and* α *shows that*

$$s^2+s+2=\left(s-\left(-\frac{1}{2}\right)\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2.$$

We seek p *and* q *such that*

$$\frac{\frac{\pi}{4}s+\frac{1}{4}(\pi-8)}{s^2+s+2}=p\frac{s-\left(-\frac{1}{2}\right)}{\left(s-\left(-\frac{1}{2}\right)\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}+q\frac{\frac{\sqrt{7}}{2}}{\left(s-\left(-\frac{1}{2}\right)\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}$$

Clearly $p=\frac{\pi}{4}$ *and* $\frac{\pi}{8}+q\frac{\sqrt{7}}{2}=\frac{1}{4}(\pi-8)$. *Solving for* q,

$$q=\frac{\pi-16}{4\sqrt{7}},$$

and we have

$$Y_1(s) \;=\; \frac{1}{s^2}+\frac{\pi}{4}\frac{s-\left(-\frac{1}{2}\right)}{\left(s-\left(-\frac{1}{2}\right)\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}+\frac{\pi-16}{4\sqrt{7}}\frac{\frac{\sqrt{7}}{2}}{\left(s-\left(-\frac{1}{2}\right)\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2},$$

Finally we are able to use the table of Laplace transforms to calculate the *inverse Laplace transform of* $Y_1(s)$.

$$\begin{aligned}\theta(t) &= \mathscr{L}^{-1}\left(\frac{1}{s^2}\right)+\frac{\pi}{4}\mathscr{L}^{-1}\left(\frac{s-\left(-\frac{1}{2}\right)}{\left(s-\left(-\frac{1}{2}\right)\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right)\\ &\quad +\frac{\pi-16}{4\sqrt{7}}\mathscr{L}^{-1}\left(\frac{\frac{\sqrt{7}}{2}}{\left(s-\left(-\frac{1}{2}\right)\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right),\\ &= t+\frac{\pi}{4}e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right)+\frac{\pi-16}{4\sqrt{7}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right).\end{aligned}$$

Notice that this agrees with (32)*. The computation of* $\mathscr{L}^{-1}\left(Y_s(s)\right)$ *is done in the just same way. We obtain*

$$x(t)=1-e^{-\frac{1}{2}t}\left(2\cos\left(\frac{\sqrt{7}}{2}t\right)+\frac{(\pi-2)}{\sqrt{7}}\sin\left(\frac{\sqrt{7}}{2}t\right)\right),$$

which is $\theta'(t)$ *since this system represents a second order equation.*

11 Partial Differential Equations (PDEs) by analytic methods

11.1 Introduction to PDEs

A *partial differential equation* (PDE) is an equation involving a function and its partial derivatives. The function depends on two or more independent variables. The *order* of the PDE is the order of the highest derivative.

Consider the heat equation,

$$\frac{\partial T}{\partial t} = k\nabla^2 T,$$

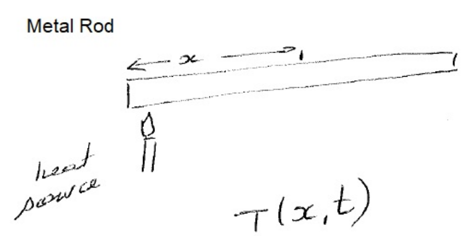
where

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

in 3 spacial dimensions (fewer letters for fewer dimensions). We can also write this as $T_t(t,x,y,z) = k\left(T_{xx}(t,x,y,z) + T_{yy}(t,x,y,z) + T_{zz}(t,x,y,z)\right)$.

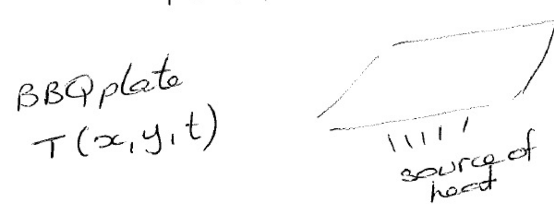
The heat equation

Heat travels through a metal rod with a heat source at one end. $\dfrac{\partial T}{\partial t} = k\dfrac{\partial^2 T}{\partial x^2}$.



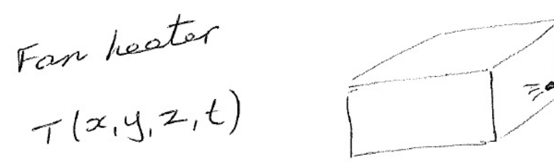
Heat passing through a 2-D plate.

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right).$$



A 3-dimensional metal

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right).$$



Notation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = u_{xx},$$
$$\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = (u_t)_x = u_{tx}.$$

Recall Clairaut's Theorem from MTH201:

Theorem 3 (Clairaut) *If f and it's 1st and 2nd partial derivatives are defined and continuous, then*

$$f_{xy} = f_{yx}.$$

The *wave equation* is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u,$$

where in 1 dimension this is simply

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

which we can write as

$$u_{tt} = c^2 u_{xx}.$$

Laplace's equation,

$$\nabla^2 u = 0 \tag{35}$$

is a time independent PDE. It arises in problems of steady-state heat transfer in a metal plate, e.g. a barbecue plate that is insulated on all sides so that no heat leaves or enters the system. We have the heat equation with $\frac{\partial T}{\partial t} = 0$, hence we obtain (35).

The *Schrödinger equation*

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0,$$

is used to find the allowed energy levels of a quantum mechanical system. ψ is the Schrödinger wave function, E is energy, V is potential energy, x is position.

Most PDEs do not have simple algebraic solutions but rather numerical approximations to the exact solution.

Example 11.1 *Let* $u(x,t) = \sin(x)\cos(ct)$. *To show that this u satisfies the wave equation,*

$$u_t(x,t) = \qquad\qquad , \qquad u_{tt}(x,t) = \qquad\qquad ,$$

$$u_x(x,t) = \qquad\qquad , \qquad u_{xx}(x,t) = \qquad\qquad .$$

Example 11.2 *Let* $u(x,y) = e^{-x}\sin(y)$. *Show that this u satisfies Laplace's equation.*

11.2 Solving PDEs by integration

The techniques of this section are much like that of Chapter 5 so you may wish to review that chapter at this point. We solve simple PDE *boundary value problems* (BVP) by integrating directly. The integrals are taken partially, where we treat all other variables as constants. Consider the following examples:

Example 11.3 *Solve the BVP*

$$u_x = x, \qquad\qquad u(0,y) = 3.$$

Here we are able to immediately integrate with respect to x.

Example 11.4 *Solve the BVP*

$$u_{xy} = y^2, \qquad\qquad u_x(x,0) = 7x, \qquad\qquad u(0,y) = 4.$$

11.3 D'Alembert's solution to the wave equation

Next we will consider a more involved problem, that of solving the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{36}$$

The solution is due to D'Alembert. We introduce

$$\begin{aligned} y &= x + ct, \\ z &= x - ct. \end{aligned}$$

By the chain rule we have

$$\frac{\partial u}{\partial t} = . \qquad .$$

and

$$\frac{\partial u}{\partial x} = . \qquad .$$

It follows that since $\frac{\partial}{\partial t} = c\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right)$ and $\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)$, we have

$$\frac{\partial^2 u}{\partial t^2} = . \qquad .$$

$$\frac{\partial^2 u}{\partial x^2} = . \qquad .$$

Expanding and simplifying,

$$\frac{\partial^2 u}{\partial y \partial z} = 0. \tag{37}$$

Integrating (37),

$$u(x,t) = f(z) + g(y) = f(x - ct) + g(x + ct). \tag{38}$$

We interpret that there are two waves moving at speed c in opposite directions.

Next, we let $u(x,0) = h(x)$ and $u_t(x,0) = k(x)$ so that

$$h(x) = . \qquad ,$$

$$k(x) = . \qquad ,$$

by the chain rule. Integrating and using $f(x) + g(x) = h(x)$,

$$cg(x) - cf(x) = . \qquad ,$$

$$cg(x) + cf(x) = . \qquad ,$$

Adding and subtracting respectively give

Dividing by $2c$,

$f(x) = . \qquad , \quad g(x) = .$

Finally we have the solution

Example 11.5 *A wave of speed 3 metres per second travels through a spring with initial conditions*

$$\begin{aligned} u(x,0) &= x^3 + x^2 - 5x + 4 + \cos(x) + 5\sin(x), \\ u_t(x,0) &= -9x^3 + 6x - 27 - 15\cos(x) - 3\sin(x). \end{aligned}$$

Calculate the wave function $u(x,t)$.

11.4 Separation of variables

Assume the solution is a product of two functions of x and t respectively.

$$u(x,t) = X(x)T(t).$$

Then taking derivatives,

$$\begin{aligned} u_x &= X'(x)T(t), \\ u_{xx} &= X''(x)T(t), \\ u_t &= X(x)T'(t). \end{aligned}$$

We use familiar methods for separable ODEs. Consider the following example:

Example 11.6 *A metal rod of length L has fixed temperature* 0 *at each end. Initially the temperature is dependent on the position*

$$u(x,0) = 6\sin\left(\frac{\pi}{L}x\right).$$

We have the following BVP

$$u_t = ku_{xx}, \qquad u(0,t) = 0, \qquad u(L,t) = 0, \qquad u(x,0) = 6\sin\left(\frac{\pi}{L}x\right),$$

where $k > 0$. Assuming that $u(x,t) = X(x)T(t)$, $u_{xx} = X''(x)T(t)$, $u_t = X(x)T'(t)$, then the heat equation gives

If $c < 0$*, then the general solution to* $X'' + \frac{c}{k}X$ *is*

(continued)

12 PDEs with finite differences

12.1 Some physics

We will begin with some definitions from multivariate calculus.

Definition 1 *Let f be a function of x, y, z. Then ∇f (grad f) is the vector*

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

and

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

is the del operator.

Definition 2 *Heat flux is the quantity of energy that flows through a surface per unit area per unit time.*

Theorem 4 (Fourier's Law of Conduction) *Let ϕ be the heat flux passing through an area A. Then*

$$\phi = -k\nabla T$$

where k is the conductivity constant of the material and the area A is perpendicular to ∇T. If Q is the energy passing through the area A in time, then

$$\frac{dQ}{dt} = \phi A.$$

Definition 3 *The Laplacian operator is the dot product*

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

so that

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

If we have only 2 position variables x, y then we may drop z:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Definition 4 *Let $\mathscr{T}$ be the temperature of a point (x,y,z) on an object in time so that $\mathscr{T}(x,y,z,t)$ is a function of x,y,z,t. The heat equation is*

$$\frac{\partial \mathscr{T}}{\partial t} = k\left(\nabla^2 \mathscr{T}\right),$$

where k is a constant.

Definition 5 *Let $\mathscr{T}$ be the temperature of a point (x,y,z) on an object in time so that $\mathscr{T}(x,y,z,t)$ is a function of x,y,z,t. An object is in thermal equilibrium if for all points (x,y,z) inside the object, the temperature at the point is not changing with respect to time. That is*

$$\frac{\partial \mathscr{T}}{\partial t} = 0.$$

It follows from the definitions that if a body is in thermal equilibrium then the temperature $\mathscr{T}(x,y,z)$ satisfies Laplace's equation

$$\nabla^2 \mathscr{T} = 0.$$

If the object is a flat 2 dimensional surface then we have

$$\frac{\partial^2 \mathscr{T}}{\partial x^2} + \frac{\partial^2 \mathscr{T}}{\partial y^2} = 0.$$

Definition 6 *A function f is a harmonic function if f is twice continuously differentiable on an open subset of $\mathbb{R}^n$ and f satisfies Laplace's equation.*

Theorem 5 *A linear combination of harmonic functions is a harmonic function.*

Let $\{f_i\}$ be a set of harmonic functions. Then for each f_i, $\bigtriangledown^2 f_i = 0$. That is

$$\frac{\partial^2 f_i}{\partial x^2} + \frac{\partial^2 f_i}{\partial y^2} + \frac{\partial^2 f_i}{\partial z^2} + \cdots = 0.$$

Let $g = \sum_{i=0}^{n} m_i f_i$.

Example 12.1 *Let $f(x,y) = x^2 - y^2$ and $g(x,y) = \ln(x^2 + y^2)$. Show that f and g are harmonic functions.*

Example 12.2 *Imagine a thin flat 1 m. $\times$ 1 m. plate. The faces are insulated so that no heat flow occurs across the faces, only across the edges. The plate is at thermal equilibrium. If we know the temperature along the edges, can we work out the temperature on the inside?*

The function

$$\mathscr{T}(x,y) = 100x(1-y^2)$$

satisfies the boundary conditions but not Laplace's equation since

.

Alternatively, the function

$$\mathscr{T}(x,y) = x^2 - y^2$$

satisfies Laplace's equation but not the boundary conditions since

Only numerical methods can compute a function to satisfy both Laplace's equation and the boundary conditions. This makes sense with example 12.1 in mind.

12.2 Finite differences

Consider the central difference approximation of a second derivative. Let $h = x_{i+1} - x_i = x_i - x_{i-1}$. Then

$$f^{(1)}(x_i) \approx \frac{1}{2h}\Big(f(x_{i+1}) - f(x_{i-1})\Big) \tag{39}$$

$$f^{(2)}(x_i) \approx \frac{1}{h^2}\Big(f(x_{i+1}) - 2f(x_i) + f(x_{i-1})\Big) \tag{40}$$

We do something similar for partial derivatives.

$$\frac{\partial f(x_i, y_i)}{\partial x} \approx \frac{1}{2h}\Big(f(x_{i+1}, y_i) - f(x_{i-1}, y_i)\Big) \tag{41}$$

$$\frac{\partial f(x_i, y_i)}{\partial y} \approx \frac{1}{2h}\Big(f(x_i, y_{i+1}) - f(x_i, y_{i-1})\Big) \tag{42}$$

$$\frac{\partial^2 f(x_i, y_i)}{\partial x^2} \approx \frac{1}{h^2}\Big(f(x_{i+1}, y_i) - 2f(x_i, y_i) + f(x_{i-1}, y_i)\Big) \tag{43}$$

$$\frac{\partial^2 f(x_i, y_i)}{\partial y^2} \approx \frac{1}{h^2}\Big(f(x_i, y_{i+1}) - 2f(x_i, y_i) + f(x_i, y_{i-1})\Big) \tag{44}$$

Then we approximate Laplace's equation

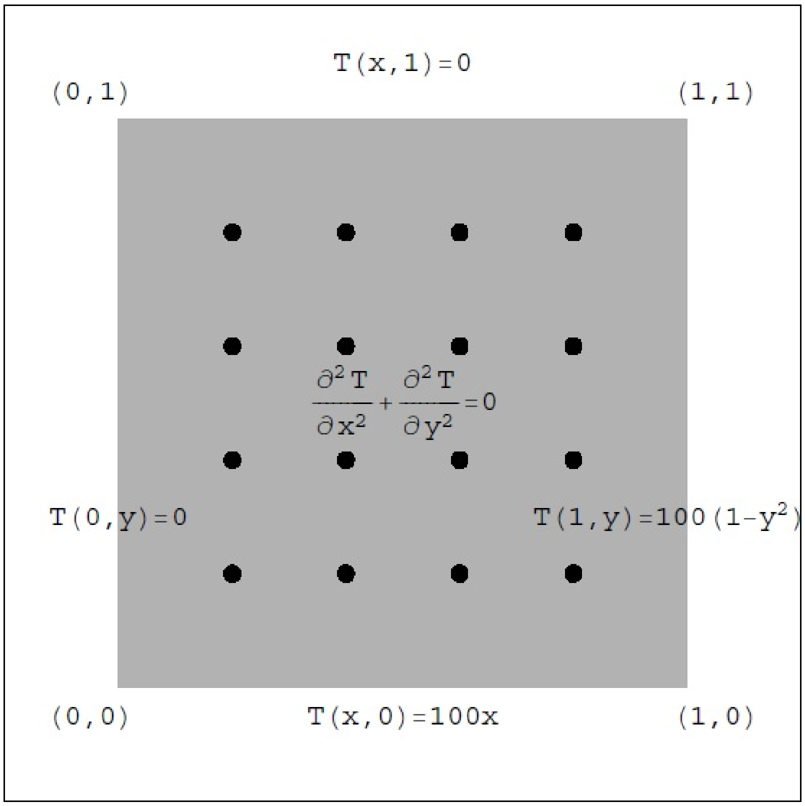
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \approx \quad . \qquad\qquad .$$

This simplifies to

We may think of this as applying the 'template'

	1	
1	-4	1
	1	

to the grid points on the metal plate



Let $T_1, T_2, \ldots T_{16}$ be the temperatures at the grid points from left to right and top to bottom respectively.

First we need the temperatures on the boundary in line with the grid points.

$\mathscr{T}(0.2,0) = \quad , \ \mathscr{T}(0.4,0) = \quad , \ \mathscr{T}(0.6,0) = \quad , \ \mathscr{T}(0.8,0) = \quad ,$

$\mathscr{T}(1,0.2) = \quad , \ \mathscr{T}(1,0.4) = \quad , \ \mathscr{T}(1,0.6) = \quad , \ \mathscr{T}(1,0.8) = \quad .$

(continued)

We now get the 16×16 matrix equation

$$
\begin{pmatrix}
4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4
\end{pmatrix}
\cdot
\begin{pmatrix}
T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \\ T_{11} \\ T_{12} \\ T_{13} \\ T_{14} \\ T_{15} \\ T_{16}
\end{pmatrix}
=
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 36 \\ 0 \\ 0 \\ 0 \\ 64 \\ 0 \\ 0 \\ 0 \\ 84 \\ 20 \\ 40 \\ 60 \\ 176
\end{pmatrix}.
$$

We may solve for the T_i using , *Python*, *Mathematica*, *Matlab*, etc. In , *Python*,

```
>>> A=array([[4,-1,0,0,-1,0,0,0,0,0,0,0,0,0,0,0],
            [-1,4,-1,0,0,-1,0,0,0,0,0,0,0,0,0,0],
            [0,-1,4,-1,0,0,-1,0,0,0,0,0,0,0,0,0],
            [0,0,-1,4,0,0,0,-1,0,0,0,0,0,0,0,0],
            [-1,0,0,0,4,-1,0,0,-1,0,0,0,0,0,0,0],
            [0,-1,0,0,-1,4,-1,0,0,-1,0,0,0,0,0,0],
            [0,0,-1,0,0,-1,4,-1,0,0,-1,0,0,0,0,0],
            [0,0,0,-1,0,0,-1,4,0,0,0,-1,0,0,0,0],
            [0,0,0,0,-1,0,0,0,4,-1,0,0,-1,0,0,0],
            [0,0,0,0,0,-1,0,0,-1,4,-1,0,0,-1,0,0],
            [0,0,0,0,0,0,-1,0,0,-1,4,-1,0,0,-1,0],
            [0,0,0,0,0,0,0,-1,0,0,-1,4,-1,0,0,-1],
            [0,0,0,0,0,0,0,0,-1,0,0,0,4,-1,0,0],
            [0,0,0,0,0,0,0,0,0,-1,0,0,-1,4,-1,0],
            [0,0,0,0,0,0,0,0,0,0,-1,0,0,-1,4,-1],
            [0,0,0,0,0,0,0,0,0,0,0,-1,0,0,-1,4]])

>>> b=array([[0],[0],[0],[36],[0],[0],[0],[64],[0],[0],[0],[84],[20],[40],
             [60],[176]])

>>> solve(A,b)
```

We obtain

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \\ T_{11} \\ T_{12} \\ T_{13} \\ T_{14} \\ T_{15} \\ T_{16} \end{pmatrix} = \begin{pmatrix} 5.1322721 \\ 10.65424518 \\ 17.05500613 \\ 25.04816126 \\ 9.87484322 \\ 20.42970247 \\ 32.5176181 \\ 47.13763889 \\ 13.93739829 \\ 28.6721034 \\ 45.44812489 \\ 66.98477621 \\ 17.20264656 \\ 34.87318795 \\ 53.61800184 \\ 74.15069451 \end{pmatrix}$$

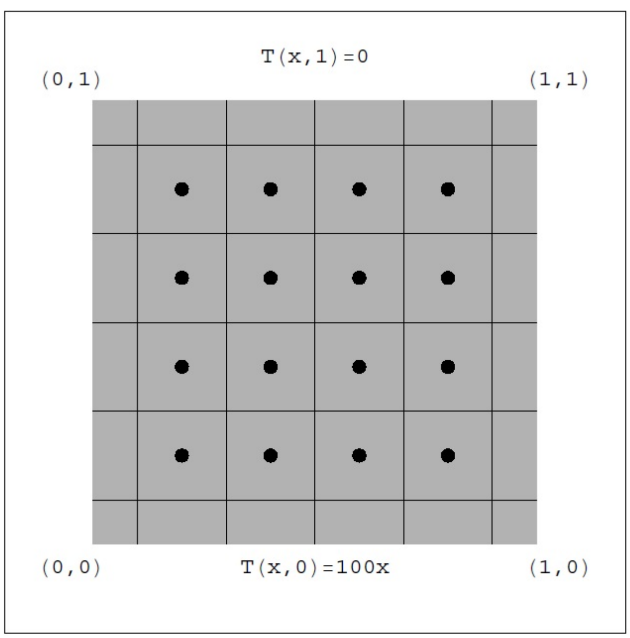
Here we have a plot of the temperature at the position (x,y) in cm.

An array of the temperatures by positions is

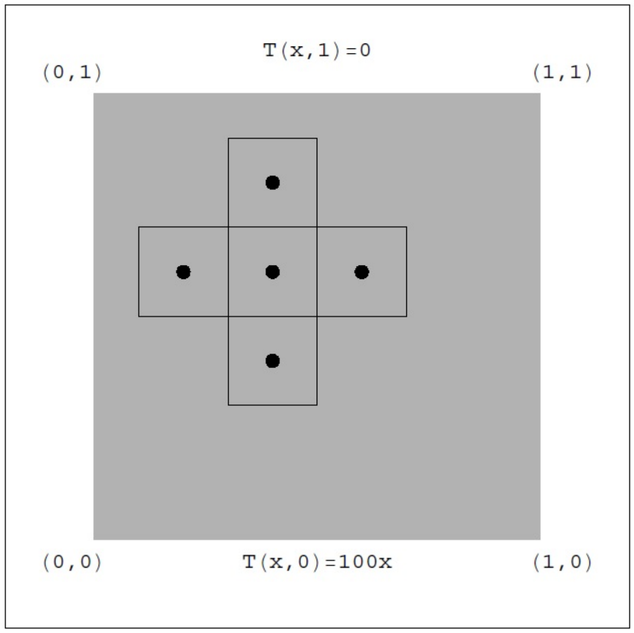
0	0	0	0	0	0
0	5.13	10.65	17.05	25.05	36
0	9.87	20.43	32.52	47.14	64
0	13.93	28.67	45.45	66.98	84
0	17.20	34.87	53.61	74.15	96
0	20	40	60	80	100

12.3 Is this physically reasonable?

Now take our metal slab and place cells of width h around each grid point



but centre our template above a particular point $T(x_i, y_j)$.



Consider the heat flow on the top face of the centre cell flowing into the cell above it. We with denote this heat flow as $H_N = \frac{dQ}{dt}$ according to Theorem 4 (The rate at which energy is traveling through the North face of the cell) We can compute this heat flow as

$$H_N = (-\text{ conductivity constant }) \times (\text{ width of cell }) \times (\text{ thickness if metal }) \times \frac{\partial T(x_i, y_j + \frac{1}{2}h)}{\partial y}$$

ignoring the vector part of this. That is

$$H_N = -khw\frac{\partial T(x_i, y_j + \frac{1}{2}h)}{\partial y}$$

We may do the same for the South, East, and West faces of the cell centered at (x_i, y_j)

$$H_N = -khw\frac{\partial T(x_i, y_j + \frac{1}{2}h)}{\partial y} \approx -khw(\frac{1}{h}(T(x_i, y_{j+1}) - T(x_i, y_j))) = -kw(T(x_i, y_{j+1}) - T(x_i, y_j))$$

$$H_S = -khw\frac{\partial T(x_i, y_j - \frac{1}{2}h)}{\partial y} \approx -khw(\frac{1}{h}(T(x_i, y_{j-1}) - T(x_i, y_j))) = -kw(T(x_i, y_{j-1}) - T(x_i, y_j))$$

$$H_E = -khw\frac{\partial T(x_i + \frac{1}{2}h, y_j)}{\partial x} \approx -khw(\frac{1}{h}(T(x_{i+1}, y_j) - T(x_i, y_j))) = -kw(T(x_{i+1}, y_j) - T(x_i, y_j))$$

$$H_W = -khw\frac{\partial T(x_i - \frac{1}{2}h, y_j)}{\partial x} \approx -khw(\frac{1}{h}(T(x_{i-1}, y_j) - T(x_i, y_j))) = -kw(T(x_{i-1}, y_j) - T(x_i, y_j))$$

Heat flows across the faces of each cell but since each cell is in thermal equilibrium, The total heat flow into the cell is equal to the total heat flow out of the cell (the rate at which energy enters and leaves the cell in time is 0). That is

$$H_N + H_S + H_E + H_W = 0$$

so that

$$-kw(T(x_i, y_{j+1}) - T(x_i, y_j)) - kw(T(x_i, y_{j-1}) - T(x_i, y_j))$$

$$-kw(T(x_{i+1}, y_j) - T(x_i, y_j)) - kw(T(x_{i-1}, y_j) - T(x_i, y_j)) = 0$$

Simplifying,

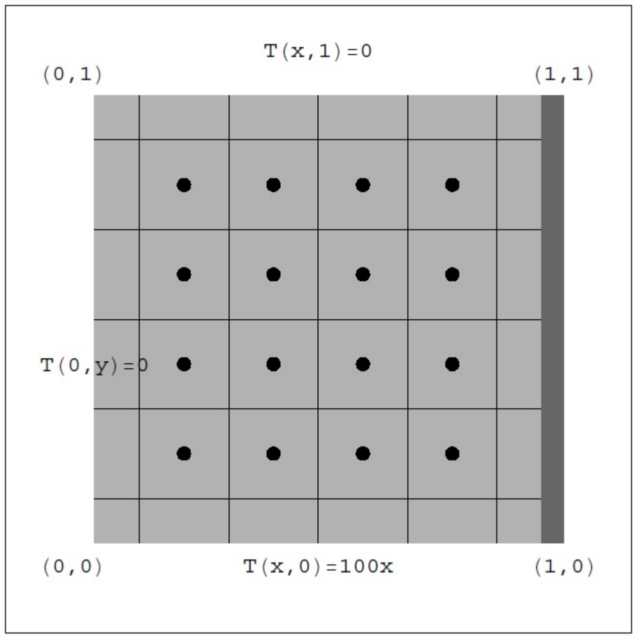
$$-T(x_i, y_{j+1}) - T(x_i, y_{j-1}) - T(x_{i+1}, y_j) - T(x_{i-1}, y_j) + 4T(x_i, y_j) = 0$$

but this is precisely what we obtain from Laplace's equation!

12.4 Neumann boundary conditions

Carl Neumann (Boundary conditions on the edge of the domain)

Now suppose the boundary $x = 1$ is insulated so that no heat may flow across this boundary. The temperature gradient across this edge is 0. That is we replace the boundary condition $\mathscr{T}(1,y) = 100(1-y^2)$ with $\frac{\partial \mathscr{T}(x,y)}{\partial x} = 0$.



Definition 7 *For any point (x,y) on any edge, the outward normal vector is denoted $\nu(x,y)$ and the derivative in the direction of $\nu(x,y)$ is called the normal derivative and denoted $\mathscr{T}_\nu(x,y)$ given by*

$$\mathscr{T}_\nu(x,y) = \nabla \mathscr{T}(x,y) \cdot \frac{\nu(x,y)}{||\nu||} = (\frac{\partial \mathscr{T}(x,y)}{\partial x}\hat{\imath} + \frac{\partial \mathscr{T}(x,y)}{\partial y}\hat{\jmath}) \cdot \frac{\nu(x,y)}{||\nu||}$$

Note: $\mathscr{T}_\nu(x,y)$ is a directional derivative in which ν is a normal vector.

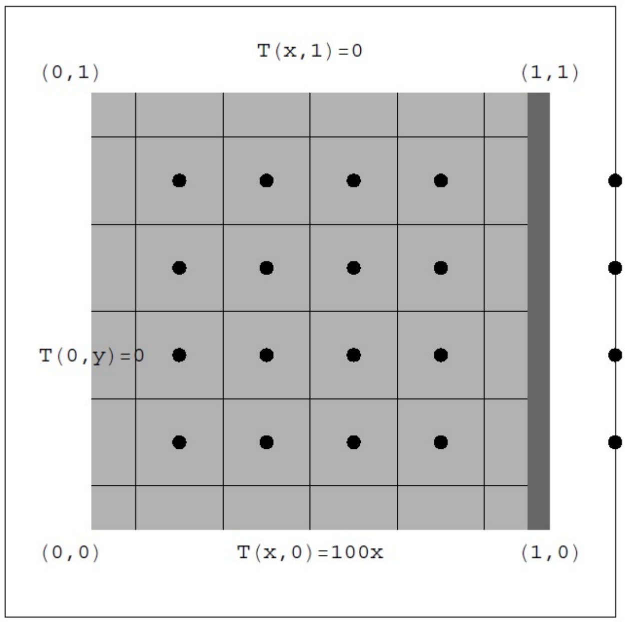
In our problem, $\hat{\imath}$ is the normal vector to the East face (the insulated face) so

$$\mathscr{T}_\nu(1,y) = \bigtriangledown \mathscr{T}(1,y) \cdot \hat{\imath} = (\frac{\partial \mathscr{T}(1,y)}{\partial x}\hat{\imath} + \frac{\partial \mathscr{T}(1,y)}{\partial y}\hat{\jmath}) \cdot \hat{\imath} = \frac{\partial \mathscr{T}(1,y)}{\partial x} = 0$$

So the new boundary condition is also written as

$$\mathscr{T}_v(1,y) = \frac{\partial \mathscr{T}(1,y)}{\partial x} = 0$$

This is a Neumann boundary condition. With Neumann boundary conditions, temperatures at the grid points along the insulated face are now unknown. Equations are derived by imagining a (potentially curved depending on the geometry) column of fictitious points outside the plate.



The boundary condition $\frac{\partial \mathscr{T}(1,y)}{\partial x} = 0$ is approximated by

$$\frac{\partial \mathscr{T}(1,y)}{\partial x} \approx \frac{\mathscr{T}(1+h,y_j) - \mathscr{T}(1-h,y_j)}{2h} = 0$$

so that

$$\mathscr{T}(1+h,y_j) \approx \mathscr{T}(1-h,y_j)$$

We then place our template over the insulated boundary approximating the temperature at the fictitious points by $\mathscr{T}(1+h,y_j) \approx \mathscr{T}(1-h,y_j)$.

Using the template along the insulated face we have

$$-2\mathscr{T}(x_{i-1},y_j) - \mathscr{T}(x_i,y_{j+1}) - \mathscr{T}(x_i,y_{j-1}) + 4\mathscr{T}(x_i,y_j) = 0$$

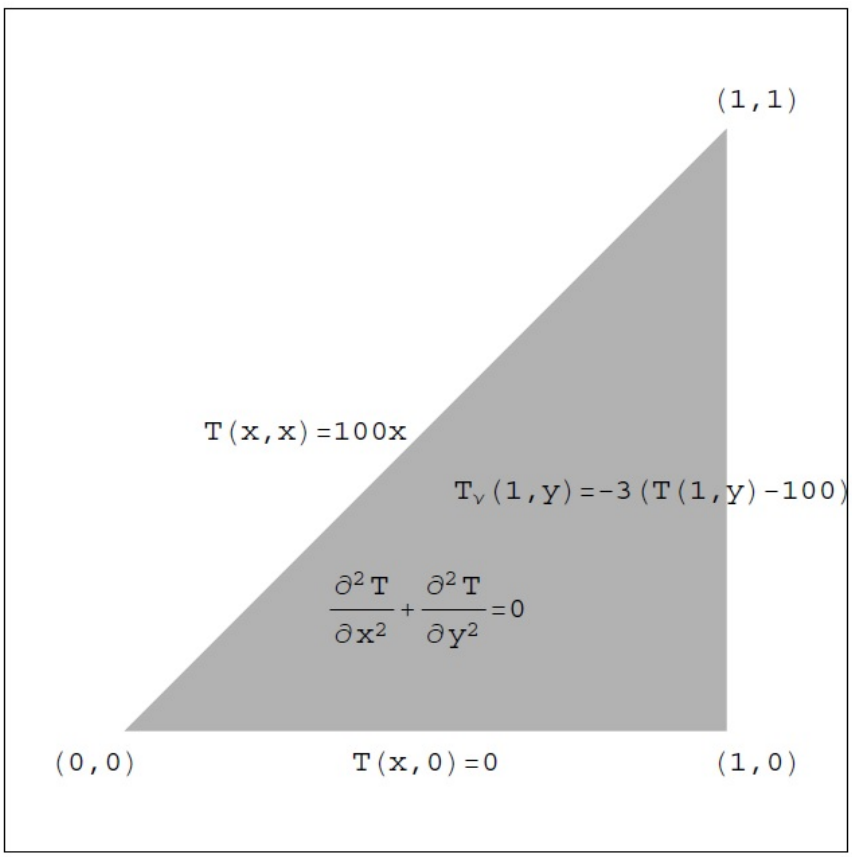
and of course this will change the shape of our matrix equation $Ax = b$ and A will now be a 20 by 20 matrix!

12.5 Another Neumann boundary condition

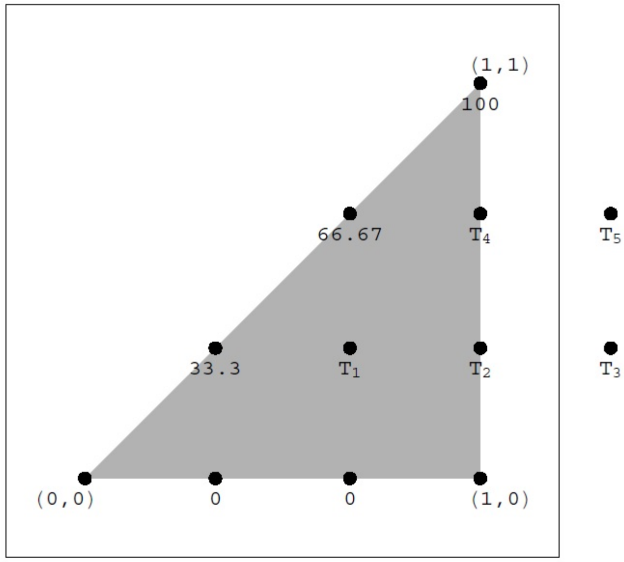
A thin triangular metal plate is insulated on both faces and heat can flow only across the edges. In appropriate 2D coordinates the plate occupies the area

$$\{(x,y) : 0 \leq x \leq 1, \text{ and } 0 \leq y \leq x\}$$

and the temperature at (x,y) is $\mathscr{T}(x,y)$. The plate is in thermal equilibrium $\frac{\partial^2 \mathscr{T}}{\partial x^2} + \frac{\partial^2 \mathscr{T}}{\partial y^2} = 0$ The boundary conditions are $\mathscr{T}(x,0) = 0$, $\mathscr{T}(x,x) = 100x$, $\mathscr{T}_v(1,y) = -3(\mathscr{T}(1,y) - 100)$, ($h = \frac{1}{3}$ m.)



Place the grid points



Using the template we have

Simplifying,

The Neumann boundary condition is $\mathscr{T}_v(1,y) = -3(\mathscr{T}(1,y) - 100)$ so

Simplifying the boundary condition,

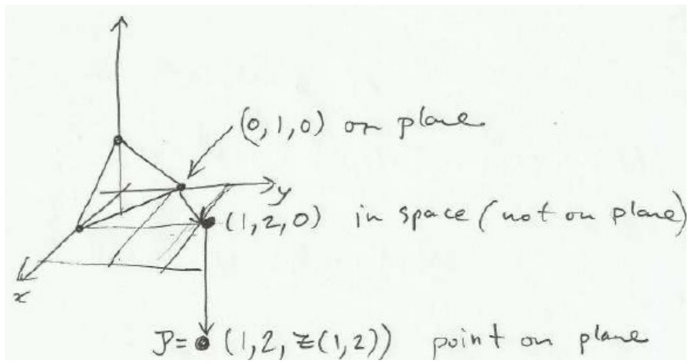
Replacing T_3 and T_5 in the three equations obtained from the template,

Solving this matrix equation,

$$\begin{pmatrix} T_1 \\ T_2 \\ T_4 \end{pmatrix} = \begin{pmatrix} 40.1041 \\ 60.4166 \\ 82.2911 \end{pmatrix}.$$

12.6 Directional Derivatives

Consider the plane $x+y+z=1$. What is the slope of the plane in the direction of the vector $(1,1)$?



Starting from the point $(0,1,0)$ on the plane, stepping off the plane in the direction of the vector $(1,1)$ brings us to the point $(0,1,0)+(1,1,0)=(1,2,0)$, which is not on the plane $x+y+z=1$. In order to return to the plane from $(1,2,0)$ along a vertical path, we compute the point $(1,2,z(1,2))=(1,2,-2)$ since $z(1,2)=1-1-2=-2$. Stepping off the plane in the direction of the vector $(1,1)$ gives us the *run* of our slope, equal to $\sqrt{2}$, the length of the vector $(1,1)$. Returning to the plane gives us the *rise* of our slope, $-2-0=-2$. The slope in the direction of the vector $(1,1)$ is given by

$$\frac{rise}{run} = \frac{-2}{\sqrt{2}} = -\sqrt{2}.$$

Now suppose we have a surface $\mathscr{S}: z=f(x,y)$ and we wish to calculate the slope of the surface at the point $(a,b,f(a,b))$ in the direction of the vector $u=(u_1,u_2)$. The tangent plane to the surface at the point $(a,b,f(a,b))$ is given by

$$\mathscr{P}: Z=f(a,b)+f_x(a,b)(X-a)+f_y(a,b)(Y-b), \tag{45}$$

where we have used uppercase X,Y,Z for the plane to distinguish from the points (x,y,z) lying on the surface $\mathscr{S}$. Like previously, we step off the surface at the point $(a,b,f(a,b))$ in the direction of the vector $u=(u_1,u_2)$, the norm of this gives us $run = ||u||$. Now we are at the position $(a+u_1,b+u_2,f(a,b))$. We return to the plane via a vertical path, bringing us to the point

$$\begin{aligned}(a+u_1,b+u_2,Z(a+u_1,b+u_2)) &= (a+u_1,b+u_2,f(a,b)+f_x(a,b)(a+u_1-a)+f_y(a,b)(b+u_2-b)),\\ &= (a+u_1,b+u_2,f(a,b)+f_x(a,b)u_1+f_y(a,b)u_2),\end{aligned}$$

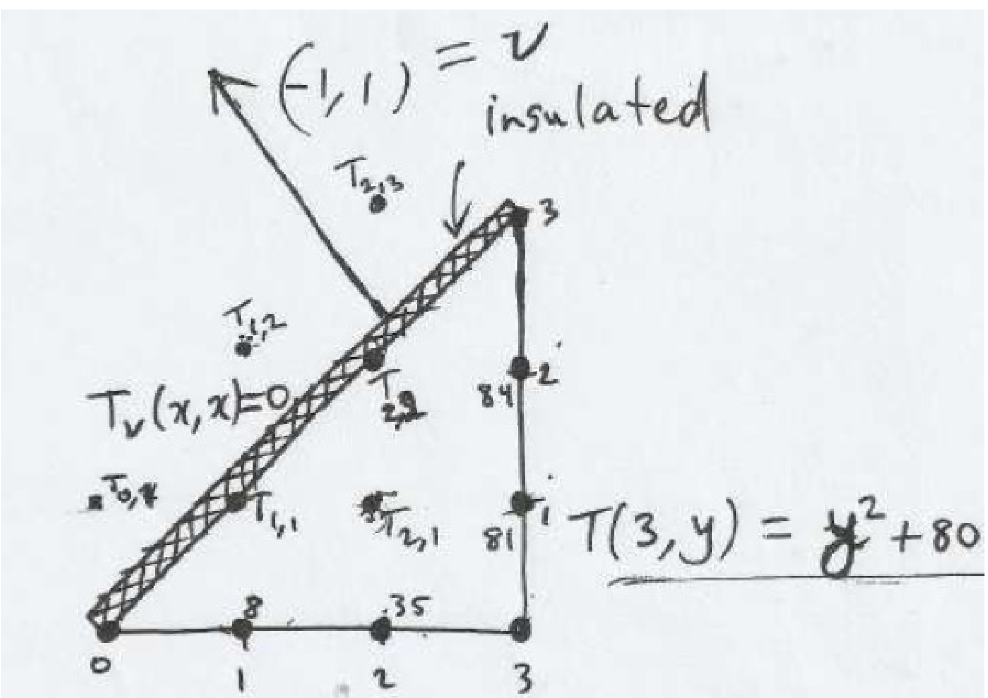
on the tangent plane $\mathscr{P}$. The difference in the z values of this point and the point $(a+u_1,b+u_2,f(a,b))$ give us the $rise = f_x(a,b)u_1+f_y(a,b)u_2$. Finally, the slope of the surface $\mathscr{S}$ at the point $(a,b,f(a,b))$ in the direction of the vector $u=(u_1,u_2)$ is given by

$$\begin{aligned}\frac{rise}{run} &= \frac{f_x(a,b)u_1+f_y(a,b)u_2}{||u||},\\ &= \frac{(f_x(a,b),f_y(a,b))\cdot u}{||u||},\\ f_u(a,b) &= \frac{\nabla f(a,b)\cdot u}{||u||}.\end{aligned}$$

Example 12.3 *Find the directional derivative of* $f(x,y) = 4 - x^2 - 4y^2$ *at* $(a,b,f(a,b))$ *in the direction of the vector* $(1,1)$.

12.7 Thermal equilibrium with various Neumann boundary conditions

Example 12.4 *Consider the steady state heat distribution in a thin triangular metal plate with the diagonal side insulated:*



Use finite differences to approximate the temperature at the points $(1,1)$, $(2,1)$, $(2,2)$.

Along the boundaries $y = 0$ and $x = 3$, we have $T(x,0) = 9x^2 - 1$ and $T(3,y) = y^2 + 80$. We have

$$\begin{aligned} T_{1,0} &= \qquad , & T_{2,0} &= \qquad , \\ &= \qquad , & &= \qquad , \\ T_{3,1} &= \qquad , & T_{3,2} &= \qquad , \\ &= \qquad , & &= \qquad . \end{aligned}$$

Since the diagonal side of the plate is insulated, we have

$$T_v(x,x) = T_{(-1,1)}(x,x) = 0.$$

This directional derivative is

Therefore along the insulated face $y = x$ we have $\frac{\partial T}{\partial y} = \frac{\partial T}{\partial x}$.

Moving the template over each of the points $(1,1)$, $(2,1)$, $(2,2)$, we find the equations

The fictitious points are the points $(0,1)$, $(1,2)$, and $(2,3)$. We deduced that $\frac{\partial T}{\partial y}(x,x) = \frac{\partial T}{\partial x}(x,x)$ since the diagonal face is insulated. Approximating this,

$$\frac{\partial T}{\partial x}(x,x) \approx \frac{T(x+h,x) - T(x-h,x)}{2h} \approx \frac{T(x,x+h) - T(x,x-h)}{2h}$$

This gives us

Finally, we have the matrix equation

$$\begin{pmatrix} -4 & 2 & 0 \\ 1 & -4 & 1 \\ 0 & 2 & -4 \end{pmatrix} \begin{pmatrix} T_{1,1} \\ T_{2,1} \\ T_{2,2} \end{pmatrix} = \begin{pmatrix} -16 \\ -116 \\ -168 \end{pmatrix},$$

and solving gives $T_{1,1} \approx 31$, $T_{2,1} \approx 54$, $T_{2,2} \approx 69$.

13 Exam Practice

13.1 Exam 1

1. A population P of koalas is observed to grow at a rate proportional to its size when the population is small, but if the population is large in comparison with available resources, then the population will decrease. The rate of change of the population is consistent with the logistic population model,

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P, \tag{46}$$

where t is measured in years.

(a) Suppose that $\frac{P}{N} \approx 0$. Use techniques for solving separable equations to find the general solution to Equation 46 when $\frac{P}{N} \approx 0$.
.
(5 Marks)

(b) Suppose that $\frac{P}{N} \approx 0$. Use techniques for solving first order linear equations to find the general solution to Equation 46 when $\frac{P}{N} \approx 0$.
.
(5 Marks)

(c) Now suppose that $0 < \frac{P}{N} < 1$ and $\frac{P}{N}$ is not close to 0. Find the general solution to Equation 46 using techniques for Bernoulli equations.
.
(5 Marks)

(d) Using the general solution to Equation 46 obtained in Part (c), if the carrying capacity is $N = 1000$ koalas, and initially the population is 1245 koalas and after 1 year the population is 1100, determine the population after 5 years.
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(5 Marks)

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2. Consider the spring-mass system in which a mass of 2 kg is suspended from the ceiling by a spring so that gravity pulls the mass vertically downwards. The spring is known to have spring constant 4 N/m and the force due to air resistance is $-4\,y'\,N$. Assume that there is a forcing factor $f(t) = 6$ so that the motion of the mass is modeled by

$$ay'' + by' + cy = f(t), \tag{47}$$

where a, b, c are real constants. Initially we have $y(0) = 2$, $y'(0) = 2$.

(a) Determine the constants a, b, c. **(5 marks)**

(b) Find the solution $y(t)$ by adding a homogeneous solution y_H to a particular solution y_P. **(10 marks)**

(c) By letting $z = y'$ and $Y = \begin{pmatrix} y \\ z \end{pmatrix}$, find matrices A and b such that $Y' = AY + b$ is the coupled first order linear system corresponding to Equation (49).

. **(5 marks)**

(d) For the matrix A found in Part (c), find the eigenvalues and eigenvectors of A.

. **(5 marks)**

(e) For the matrix A found in Part (d), calculate the trace and determinant of A, and use the stability chart to classify the stability of the equilibrium point (critical point). **(5 marks)**

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3. Consider the second order linear initial value problem (IVP):

$$y'' + 5y' + 4y = 0, \qquad y(0) = 0, \qquad y'(0) = -1.$$

(a) Use the Laplace transform to solve the IVP. Incorporate the method of partial fractions in your solution. **(6 marks)**

(b) Use the convolution theorem to find the inverse Laplace transform of the function

$$H(s) = \frac{-1}{s^2 + 5s + 4}.$$

(8 marks)

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4. A tall-fenced paddock is home to two competing species goats and kangaroos. Letting $x(t)$ denote the number of goats and $y(t)$ the number of kangaroos, the competing species model

$$\begin{aligned} x'(t) &= x(165 - x - 2y), \\ y'(t) &= y(195 - 3x - y) \end{aligned}$$

governs the rate of change of the populations of each species.

(a) Find all equilibrium solutions (critical points) of the system.

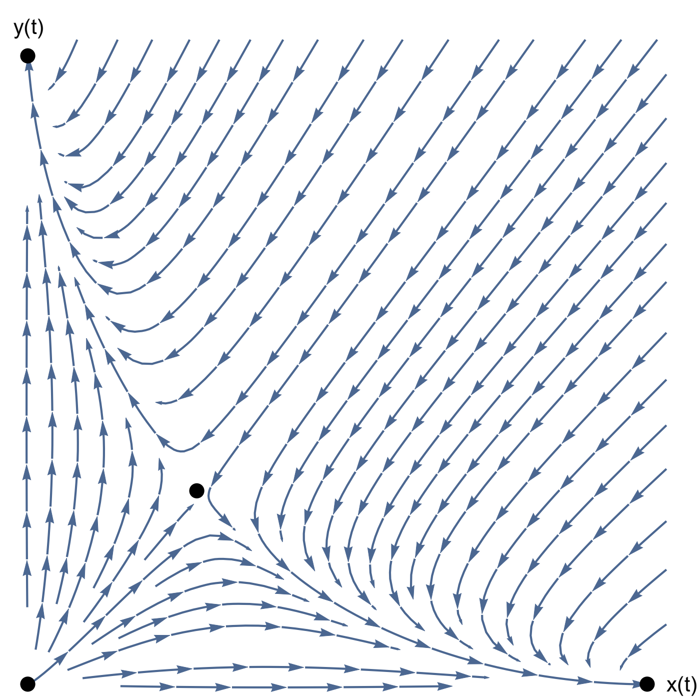
. **(4 marks)**

(b) Calculate the Jacobian matrix of the linearized system and use this to classify the stability of the critical point(s) found in Part (a).

. **(8 marks)**

(c) If $x(0) = 50$ and $y(0) = 20$, see the phase portrait below with critical points shown in black, determine which species goes extinct first after a considerable amount of time has elapsed (as $t \longrightarrow \infty$).

. **(4 marks)**



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5. Let $u(x,y)$ represent the vertical displacement of a string of length π, which is placed on the interval $[0,\pi]$, at position x and time t. Assuming proper units for length, times, and the constant k, the wave-equation models the displacement $u(x,t)$:

$$u_{tt} = c^2 u_{xx}.$$

The boundary conditions are given by

$$u(0,t) = u(\pi,t) = 2\cos(4ct)$$

for $t \geq 0$, with initial displacement

$$u(x,0) = 2\cos(4x),$$

and initial velocity

$$u_t(x,0) = 0$$

for $0 \leq x \leq \pi$. Solve the solve the equation for $u(x,t)$ using D'Alembert's method. **(10 marks)**

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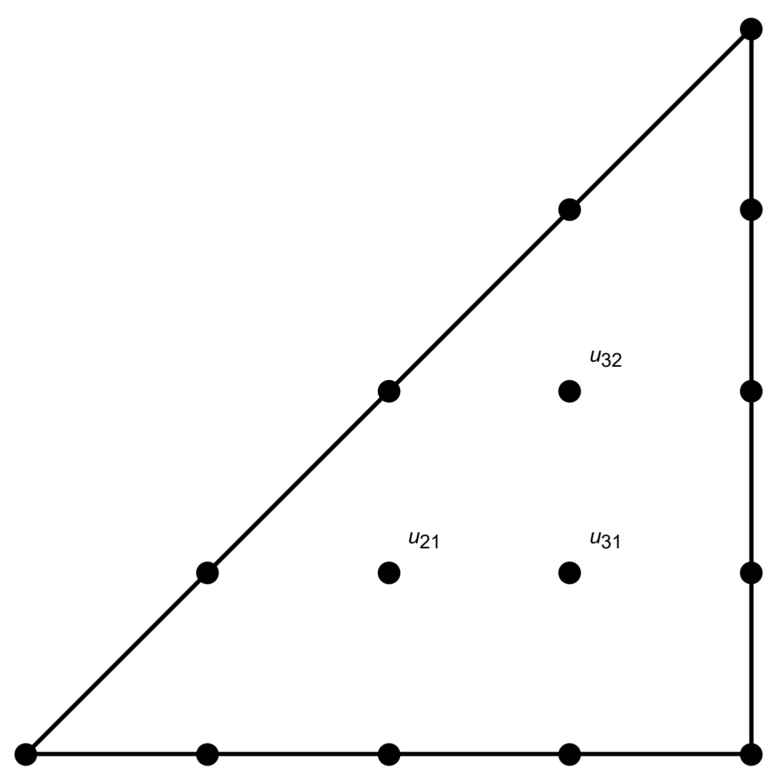
6. Consider the problem of determining the steady-state heat distribution in a thin triangular metal plate with width and height 4 metres. The temperature $u(x,y)^\circ\, C$ along the three boundaries of the plate are given by the equations

$$\begin{aligned} u(x,0) &= 2x^2 - 15x + 68, \\ u(4,y) &= y^2 - 2y + 40, \\ u(x,x) &= 68 + 3x - 2x^2. \end{aligned}$$

The plate is in thermal equilibrium so that the temperature inside the plate satisfies Laplace's equation. Use the method of finite differences to write a system of equations for which the solution approximates the temperature at the coordinates $(2,1)$, $(3,1)$, $(3,2)$. See the figure below. **(10 marks)**



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13.2 Exam 2

1. A population $P = P(t)$ of wild boar are living in the forest. The population is changing according to the harvested logistic model,

$$\frac{dP}{dt} = 0.4\left(1 - \frac{P}{100}\right)P - H, \tag{48}$$

where t is measured in years and H is a harvesting parameter which depends on how often hunters kill the boar.

(a) First suppose that $\frac{P}{100} \approx 0, H \approx 0$ ($\frac{P}{100}$ and H are is approximately zero). Use techniques for solving separable equations to find the general solution to Equation 48 in this case. **(5 Marks)**

(b) If $H = 6$, find all equillibrium solutions to Equation (48), sketch the slope field of $P(t)$ (equilibrium solutions and several trajectories) and determine the approximate population of boar after very many years have passed if $P(0) = 40$.
. **(10 Marks)**

(c) By considering equilibrium solutions to Equation (48), find all real parameters H such that the population of boar is destined to die out irrespective of the initial population $P(0)$. **(5 Marks)**

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2. Consider the second order differential equation

$$y'' + y' - y = f(y,t), \tag{49}$$

where y is a function of t and $f(y,t)$ is a function of y and t.

(a) If $f(y,t) = 2t + 1$, find the general solution to Equation (49).
. **(10 marks)**

(b) Now assume that $f(y,t) = y^3$. Write a system of coupled first order differential equations corresponding to Equation (49) by letting $z = y'$. **(5 marks)**

(c) Find all critical (equillibrium) points of the system and classify their stability, where $f(y,t) = y^3$. **(5 marks)**

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3. Consider the second order linear initial value problem (IVP):

$$y'' + 3y' + 2y = 0, \qquad y(0) = 1, \qquad y'(0) = -1.$$

(a) Solve the IVP without use the Laplace transform. **(5 marks)**

(b) Use the Laplace transform to solve the IVP. **(10 marks)**

(c) Recall that the hyperbolic sine function is given by

$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right).$$

Use the definition of the Laplace transform to show that

$$\mathscr{L}(\sinh(t)) = \frac{1}{s^2 - 1}$$

for all real $s > 1$. **(5 marks)**

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4. Two species of mammals share a small island. Letting $x(t)$ denote the population of Species X in thousands, and $y(t)$ the population of Species Y in thousands. It is observed that the populations of the two species are related by the following system:

$$\begin{aligned} x'(t) &= x(3x+2y-24), \\ y'(t) &= y(-9x+2y-8). \end{aligned}$$

governs the rate of change of the populations of each species.

(a) Find all equilibrium solutions (critical points) of the system.

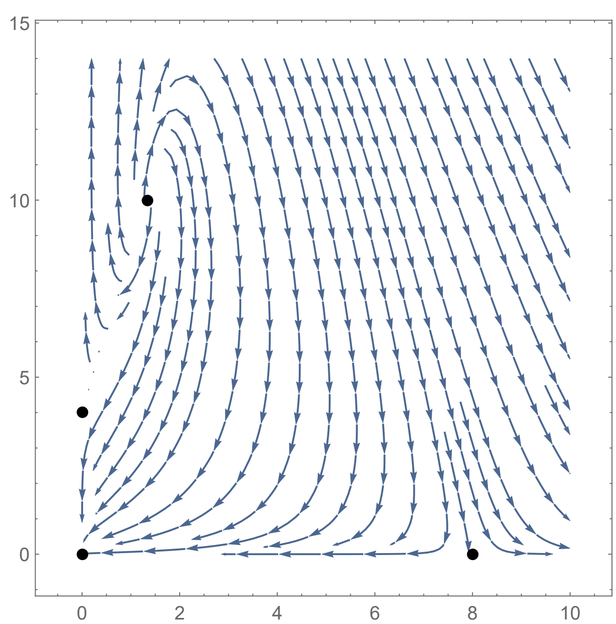
. **(4 marks)**

(b) Calculate the Jacobian matrix of the linearized system about each critical point and use this to classify the stability of the critical point(s) found in Part (a).

. **(12 marks)**

(c) If $x(0) = 9$ (meaning 9000) and $y(0) = 6$ (meaning 6000), see the phase portrait below with critical points shown in black and $x(t)$ being the horizontal axis, determine which species goes extinct first after a considerable amount of time has elapsed (as $t \longrightarrow \infty$).

. **(4 marks)**



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5. Let $u(x,y)$ represent the vertical displacement of a string of length π, which is placed on the interval $[0,\pi]$, at position x and time t. Assuming proper units for length, times, and the constant k, the wave-equation models the displacement $u(x,t)$:

$$u_{tt} = c^2 u_{xx}.$$

The boundary conditions are given by

$$u(0,t) = u(\pi,t) = 0$$

for $t \geq 0$, with initial displacement

$$u(x,0) = 2\sin(2x),$$

and initial velocity

$$u_t(x,0) = 0$$

for $0 \leq x \leq \pi$.

(a) Solve the solve the equation for $u(x,t)$ using separation of variables.

. **(15 marks)**

(b) Solve the solve the equation for $u(x,t)$ using D'Alembert's method.

. **(5 marks)**

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6. Consider the problem of determining the steady-state heat distribution in a thin rectangular metal plate with width 2 metres and height 1 metre. The temperature $u(x,y)^\circ\ C$ along the boundaries of the plate are given by the equations

$$\begin{aligned} u(x,0) &= 5x+2, \\ u(2,y) &= 12-2y+2y^3, \\ u(x,1) &= 3x^2-x+2, \\ u(0,y) &= 2. \end{aligned}$$

The plate is in thermal equilibrium so that the temperature inside the plate satisfies Laplace's equation. Use the method of finite differences to write a system of equations for which the solution approximates the temperature u_1, u_2, u_3 at the three interior coordinates $(0.5, 0.5)$, $(1.0, 0.5)$, $(1.5, 0.5)$. Express the system as $A\mathbf{x} = \mathbf{b}$, where A is a diagonally dominant symmetric matrix. **(20 marks)**

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14 Exercises

14.1 Problem Set 1

(1) Give the general solution to each of the following ODEs.

(a) $2x' + 3 = 0$.

(b) $2x' + 3x = 0$.

(c) $3P' = P$.

(2) Solve each of the following IVPs.

(a) $2x' + 3 = 0$, $x(0) = 1$.

(b) $2x' + 3x = 0$, $x(4) = 2$.

(c) $3P' = P$, $P(0) + P(6) = 2$.

(3) State the order of each of the following ODEs and whether they are linear or non-linear.

(a) $x'''(t) + 5x'(t) = 7x''(t)$.

(b) $\frac{d^2P}{dt^2} + 5\left(\frac{dP}{dt}\right)^2 = 2$.

(c) $\sin(y') = \cos(y'') + \log(y)$.

(4) Radioactive carbon-14 is present in small quantities in all living matter, and is constantly replenished from the atmosphere. When an organism dies, the carbon-14 decays to stable carbon-12. Suppose that $C(t)$ is the mass of carbon-14 (measured in grams) present in a sample of bone at time t (measured in years). The radioactive decay is described by the differential equation

$$\frac{dC}{dt} = -kC.$$

(a) If the half-life of the decay is 5730 years, calculate the relative rate of decay k to 3 significant figures.

(b) In January this year, a team of scientists found a human bone fragment which contained 89% of the amount of carbon-14 contained in living human bone. How old is the bone fragment?

(5) Two researchers are investigating the metabolism of a new drug. Let $D(t)$ represent the amount of drug in the patient at time t. The initial amount of drug is 100 g. After 1 hour the amount is 75 g.

(a) Researcher A believes the drug behaves like alcohol and decays linearly. Write down an ODE that describes this behaviour, give its general solution and use the information given to determine the rate of decay.

(b) Researcher B believes the drug behaves like caffeine and decays exponentially. Write down an ODE that describes this behaviour, give its general solution and use the information given to determine $D(t)$. Also, convert the decay rate into a half-life.

(6) Find the particular solution of $\frac{dx}{dt} = x^2 \cos(t)$, $x(0) = 1$.

(7) An organism of length L has a surface area proportional to L^2. If we assume that its growth rate is also proportional to its surface area, we can model this as $\frac{dL}{dt} = kL^2$, where k is a constant of proportionality. If $L(0) = 1$, how does the length vary with time?

(8) Find the general solution of $\frac{dy}{dx} = x^2 y^3$.

(9) Find the general solution of the following IVPs.

(a) $P' = 2P - 5$, $P(0) = 3$.

(b) $y' = 2 - 3y$, $y(1) = 0$.

(10) Find the equilibrium solution of the following ODEs if possible:

(a) $P' = 3P - 5$.

(b) $y' = 1.2y + 12$.

(c) $\frac{dA}{dt} = 2$.

(d) $\frac{dx}{dt} = 3x^2 - 2$.

(e) $\frac{dP}{dt} = \cos(P)$.

(f) $\frac{dQ}{dt} = e^{-Q} - Q$.

(11) Find the equilibrium solution of the equation $\frac{dQ}{dt} = 0.6Q - 3.0$ and determine its stability.

(12) Analyze the equilibrium solution(s) of the equations:

(a) $2\frac{dy}{dt} - (y-1)(y-5) = 0$.

(b) $\frac{dP}{dt} = \cos(P)$.

(c) $\frac{dP}{dt} = P^3 - 3P^2 + 3$.

(d) $\frac{dP}{dt} = P^2 - \cos(P)$.

14.2 Problem Set 2

(1) An epidemic spreads from 20% of the population to affecting 70% of the population in just 12 days. Use the logistic equation $\frac{dp}{dt} = rp(1-p)$.

(a) Estimate the value of the intrinsic growth rate r.

(b) After how many days was exactly half the population infected?

(c) What proportion of the population is infected after 20 days?

(2) Solve the initial value problem (IVP)

$$\frac{dx}{dt} = 0.1x(4-x),\ x(0) = 1$$

and calculate t when $x = 2$ and $x(10)$.

(3) Five mice in a stable population of 500 are intentionally infected with a contagious disease to test a theory of epidemic spread that postulates the rate of change of the infected population is proportional to the product of the number of mice who have the disease and the number who are disease free. Assuming the theory is correct, how long will it take half the population to contract the disease?

(4) A population of bacteria grows according to the differential equation

$$\frac{dP}{dt} = 0.03P\left(1 - \frac{P}{2000}\right).$$

When $t = 0$, the population is 200 g. Find the population P at time t.

(5) Solve the ODE $y' = \frac{y}{x+\sqrt{xy}}$.

(6) Solve the ODE $y' = \frac{2xy}{y^2-x^2}$.

(7) Solve the ODE $\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)^2 + 4 = 0$, where $\left|\frac{dy}{dx}\right| > 2$ for all $x \neq 0$.

(8) Recall that the imaginary number i is equal to $\sqrt{-1}$ and that the Taylor series of $e^x, \sin(x), \cos(x)$ about $x = 0$ are respectively given by

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots, \\ \sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots, \\ \cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \dots. \end{aligned}$$

(a) Verify Euler's identity $e^{ix} = \cos(x) + i\sin(x)$.

(b) Show that $\sinh(2x) = 2\sinh(x)\cosh(x)$.

(c) Use the above identity to show that $\sin(2x) = 2\sin(x)\cos(x)$.

(9) Find the general solution to the ODE $y' = xy\sin(x)$.

14.3 Problem Set 3

(1) Find the general solution to the ODE $y' = \frac{2xy}{x^2-y^2}$.

(2) Solve the IVP

$$y' = \frac{x^2+y^2}{xy}, \qquad y(1) = -2.$$

(3) Solve the IVP

$$y' = \frac{x^2y-y}{y+1}, \qquad y(3) = -1.$$

(4) Find the general solution to the ODE $y' = \frac{x^4+3x^2y^2+y^4}{x^3y}$.

(5) Show that $\int \cos^2(x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$.

(6) Calculate $\int x^5\sqrt{1+x^2}\,dx$ by substitution.

(7) Calculate $\int x^2\cos(x)\,dx$ by parts.

(8) Calculate $\int \frac{x\,dx}{x^2-5x+6}\,dx$ by partial fractions.

(9) Calculate $\int \frac{1\,dx}{(x^2+2)(x+3)}\,dx$ by partial fractions.

(10) Show that $e^x = \cosh(x)+\sinh(x)$, $e^{-x} = \cosh(x)-\sinh(x)$, and hence

$$\cosh^2(x) - \sinh^2(x) = 1.$$

(11) Calculate $\int arsinh(x)\,dx$ by integration by parts.

14.4 Problem Set 4

(1) Find the general solution to the ODE $y' + y = y^2$ using methods for linear 1st order ODEs.

(2) Find the general solution to the ODE $y' + xy = 6x\sqrt{y}$ using methods for linear 1st order ODEs.

(3) Find the general solution to the ODE $y' + y = y^{-2}$ using methods for linear 1st order ODEs.

(4) What constant interest rate is required if an initial deposit placed into an account that accrues interest compounded continuously is to double in value in six years?

(5) A yeast grows at a rate proportional to its present size. If the original amount doubles in two hours, in how many hours will it triple?

(6) A depositor places \$10,000 in a certificate of deposit which pays 6 percent interest per annum, compounded continuously. How much will be in the account at the end of seven years, assuming no additional deposits or withdrawals?

(7) Determine the interest rate required to double an investment in eight years under continuous compounding.

(8) A body of unknown temperature is placed in a refrigerator at a constant temperature of 0° F. If after 20 minutes the temperature of the body is 40° F and after 40 minutes the temperature of the body is 20° F, find the initial temperature of the body.

(9) A tank initially holds 10 gal of fresh water. At $t = 0$, a brine solution containing $\frac{1}{2}$ lb of salt per gallon is poured into the tank at a rate of 2 gal/min, while the well stirred mixture leaves the tank at the same rate. Find the amount and concentration of salt in the tank at time t.

(10) An RC circuit has an emf of 5 volts, a resistance of 10 ohms, a capacitance of 10^{-2} farads, and an initial charge of 5 coulombs on the capacitor. Find the transient current and the steady state current.

(11) Solve the IVP $y' = -2y + e^t y^3$, $y(0) = 1$.

14.5 Problem Set 5

(1) Find the general solution to the ODE

$$y' = \frac{2 + ye^{xy}}{2y - xe^{xy}}$$

using methods for exact equations.

(2) Find the general solution to the ODE $x + \sin(y) + (x\cos(y) - 2y)\,y' = 0$.

(3) Solve $y - xy' = 0$ using methods for exact equations.

(4) Convert $y' = 2xy - x$ into an exact ODE and find the general solution.

(5) Solve $t^2 - x - tx' = 0$ using methods for exact equations.

(6) Solve $y'' - y = 0$.

(7) Find the general solution to $y'' + 2y' + 2y = 0$.

(8) Solve the IVP $y'' - 3y' - 5y = 0$, $y(0) = 1$, $y'(0) = -1$.

(9) Find the general solution to $x'' - 3x' + x = 0$.

(10) Find the general solution to $x'' + 25x = 0$.

(11) Find the general solution to $x'' + x' + 2x = 0$.

(12) Solve $y'' - 36y = 0$.

14.6 Problem Set 6

(1) Find the general solution to $y'' - 5y' + 6y = 2x - 7$.

(2) Find the general solution to $y'' - 5y' + 6y = 132x^2 - 388x + 1077$.

(3) Find the general solution to $y'' - 5y' + 6y = 13e^{5x}$.

(4) Find the general solution to $y' - y = e^x$ using a sum consisting of a solution to $y' - y = 0$ and a particular solution to $y' - y = e^x$.

(5) Solve $y' - y = e^x$ using methods for exact equations.

(6) Find the general solution to $y'' - 2y' + y = x^2 - 1$.

(7) Find the general solution to $y'' - 2y' + y = 3e^{2x}$.

(8) Find the general solution to $y'' - 2y' + y = 4\cos(x)$.

(9) Find the general solution to $y'' - 2y' + y = 3e^x$.

(10) Find the general solution to $y'' - 2y' + y = xe^x$.

(11) Find the general solution to $y^{(3)} - 3y^{(2)} + 3y' - y = e^x + 1$.

(12) Solve the IVP $x'' + 3x' + 2x = 1 - 2t^2$, $x(0) = 0$, $x'(0) = -4$.

(13) Solve the IVP $x'' + 4x' + 4x = 2e^{-2t}$, $x(0) = 1$, $x'(0) = 1$.

(14) Consider a spring-mass system with spring constant $k = 6$ N/m and damping coefficient $b = 5$ kg/sec. The 1 kg mass is lifted up one metre and given a downward velocity of 8 m/sec. Without forcing, we have the homogeneous equation $x'' + 5x' + 6x = 0$. With a periodic force, we instead have $x'' + 5x' + 6x = 4\sin(t)$. Model the motion of the mass in the forced system.

(15) The vertical motion of a car along a bumpy road is modeled by the equation

$$2x'' + bx' + 3x = 4\sin\left(\frac{t}{2}\right).$$

How large would the damping coefficient b have to be so that the long term oscillatory up-and-down motion of the car would have a vertical amplitude less than 0.2 m?

14.7 Problem Set 7

From Schuam's Outlines, Chapter 14:

(1) A mass of 0.4 g is hung onto a spring and stretches it 3 cm from its natural length. Find the spring constant.

(2) A 20 g mass is suspended from the end of a vertical spring having spring constant of 2880 dynes/cm and is allowed to reach equilibrium. It is then set into motion by stretching the spring 3 cm from its equilibrium position and releasing the mass with an initial velocity of 10 cm/sec in the downward direction. Find the position of the mass at any time t if there is no external force and no air resistance.

(3) For Question (2) above, determine:

- **(a)** the circular frequency,
- **(b)** the natural frequency,
- **(c)** the period.

(4) An RCL circuit connected in series with $R = 6$ ohms, $C = 0.02$ farad, and $L = 0.1$ henry has an applied voltage $E(t) = 6$ volts. Assuming no initial current and no initial charge at $t = 0$ when the voltage is first applied, find the subsequent charge on the capacitor and the current in the circuit.

(5) An RCL circuit connected in series with $R = 6$ ohms, $C = 0.02$ farad, and $L = 0.1$ henry has no applied voltage. Find the subsequent current in the circuit if the initial charge on the capacitor is $\frac{1}{10}$ coulomb and the initial current is zero.

(6) An RCL circuit connected in series with a resistance of 16 ohms, a capacitor of 0.02 farad, and an inductance of 2 henries has an applied voltage $E(t) = 100\sin(3t)$. Assuming no initial current and no initial charge on the capacitor, find an expression for the current flowing through the circuit at any time t.

(7) For Question (6) above, determine the steady-state current in the circuit and express this in the form $\pm A\cos(\omega t - \phi)$.

14.8 Problem Set 8

(1) Consider the homogeneous ODE $y'' + 3y' + 2y = 0$, which has the general solution $y(t) = Ae^{-t} + Be^{-2t}$. Let $z = y'$ and $Y = \begin{pmatrix} y \\ z \end{pmatrix}$.

(a) Write the ODE as a coupled first order system $Y' = AY$, where A is a 2×2 matrix.

(b) Find the eigenvalues λ_1 and λ_2 of the matrix A.

(c) Find the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ of the matrix A.

(d) Simplify the general solution $Y(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2$.

(2) Consider the homogeneous ODE $y'' + 2y' + 2y = 0$, which has the general solution

$$y(t) = e^{-t}\left(A\cos(t) + B\sin(t)\right), \quad y(t) = -e^{-t}\left((A-B)\cos(t) + (A+B)\sin(t)\right).$$

Let $z = y'$ and $Y = \begin{pmatrix} y \\ z \end{pmatrix}$.

(a) Write the ODE as a coupled first order system $Y' = AY$, where A is a 2×2 matrix.

(b) Find the eigenvalues λ_1 and λ_2 of the matrix A.

(c) Find the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ of the matrix A.

(d) Simplify the general solution $Y(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2$ using Euler's formula.

(3) Consider the homogeneous ODE $y'' + 2y' + y = 0$, which has the general solution

$$y(t) = (A + Bt)e^{-t}, \qquad y'(t) = -(A - B + Bt)e^{-t}.$$

Let $z = y'$ and $Y = \begin{pmatrix} y \\ z \end{pmatrix}$.

(a) Write the ODE as a coupled first order system $Y' = AY$, where A is a 2×2 matrix.

(b) Find the eigenvalues λ_1 and λ_2 of the matrix A.

(c) Find the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ of the matrix A.

(d) Write an expression for the general solution $Y(t)$.

(4) Sketch the phase portrait for $y'' + 3y' + 2y = 0$, where y corresponds to the horizontal axis and $z = y'$ corresponds to the vertical axis. Use the stability chart to classify the system $Y' = AY$, where $Y = \begin{pmatrix} y \\ z \end{pmatrix}$.

(5) Sketch the phase portrait for $y'' + 2y' + 2y = 0$, where y corresponds to the horizontal axis and $z = y'$ corresponds to the vertical axis. Use the stability chart to classify the system $Y' = AY$, where $Y = \begin{pmatrix} y \\ z \end{pmatrix}$.

(6) Sketch the phase portrait for $y'' + 2y' + y = 0$, where y corresponds to the horizontal axis and $z = y'$ corresponds to the vertical axis. Use the stability chart to classify the system $Y' = AY$, where $Y = \begin{pmatrix} y \\ z \end{pmatrix}$.

14.9 Problem Set 9

(1) Consider the non-homogeneous linear system of first order differential equations

$$\begin{aligned} x'(t) &= x(t) + 2y(t) + 3, \\ y'(t) &= 4x(t) + 5y(t) + 6. \end{aligned}$$

(a) Find the critical point of the system.

(b) Classify its stability.

(2) Consider the non-homogeneous linear system of first order differential equations

$$\begin{aligned} x'(t) &= x(t) + 2y(t) + 3, \\ y'(t) &= 4x(t) + 5y(t) + 6. \end{aligned}$$

Use differential operators to solve the system.

(3) A pond initially contains $x = 3$ carnivorous piranha fish and $y = 12$ electric yellow cichlids, known to give birth from their mouth - dozens per female every few months. The system is governed by the equations

$$\begin{aligned} x'(t) &= 3x - 4xy, \\ y'(t) &= xy - 9y. \end{aligned}$$

(a) Determine critical points of the system.

(b) Calculate the Jacobian matrix at the critical points.

(c) Classify the stability of the linearized system.

(d) Determine the long term behaviour of the system.

(4) Consider the non-linear second order differential equation

$$x'' + \mu\left(x^2 - 1\right)x' + x = 0,$$

which models the Van der Pol oscillator circuits for radios, where $\mu > 0$. See [11, pp. 200]. This system exhibits a limit cycle, a periodic solution.

(a) Determine critical points of the system.

(b) Calculate the Jacobian matrix at the critical points.

(c) Determine the type and stability of any critical point that is inside this limit cycle.

(5) In a competing species model the growth rates of both populations are negatively affected by their interaction. Suppose the two populations are cooperating, instead of competing. The following equations represent such a model:

$$\begin{aligned} x' &= 0.2x\left(1-\frac{x}{N}\right)+0.1xy, \\ y' &= 0.6y\left(1-\frac{y}{3}\right)+0.05xy, \end{aligned}$$

where $N > 0$. See [11, pp. 200].

(a) What do you expect to happen to the populations x and y over the long term?

(b) Let the carrying capacity N of the population x be equal to 4. Find all equilibrium points for the system. Is there an equilibrium where the populations coexist?

(c) Evaluate the Jacobian matrix at each equilibrium and determine its type. If the initial conditions $x(0)$ and $y(0)$ are both positive, what must happen to a solution as $t \longrightarrow \infty$?

(c) Let $N = 10$, and again find an equilibrium where the two populations coexist. What happens now to a solution starting in the positive quadrant?

14.10 Problem Set 10

Summary

- Laplace transform definition:

$$\mathscr{L}(f) = \int_{t=0}^{\infty} e^{-st} f(t)\, dt,$$

- Heaviside step function:

$$u(x) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

- Gamma function:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1}\, dt \qquad\qquad x! = \Gamma(x+1), \text{ for } x \in \mathbb{Z}.$$

- $\mathscr{L}(k) = \frac{k}{s}$.
- $\mathscr{L}(u(t-a)) = \frac{e^{-sa}}{s}$.
- $\mathscr{L}(t^n) = \frac{n!}{s^{n+1}}$.
- $\mathscr{L}(e^{at}) = \begin{cases} \frac{1}{s-a} & \text{if } s > a \\ \text{diverges} & \text{if } s \leq a \end{cases}$
- If $Y(s) = \mathscr{L}(y)$, then

$$\mathscr{L}(y') = sY(s) - y(0), \qquad\qquad \mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0).$$

(1) Solve the following IVP using the Laplace transform:

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

(2) Derive the Laplace transform of $f(t) = k$.

(3) Derive the Laplace transform of $f(t) = t^n$.

(4) Derive the Laplace transform of $f(t) = e^{at}$.

(5) Derive the Laplace transform of $\sin(t)$ and $\cos(t)$ using that of e^{at}.

(6) Solve the following IVP using the Laplace transform:

$$x''(t) - 5x'(t) + 6x(t) = 2, \qquad x(0) = 0, \qquad x'(0) = 0.$$

(7) Solve the following IVP using the Laplace transform:

$$y''(t) + 2y'(t) + y(t) = e^{-t}, \qquad y(0) = y_0, \qquad y'(0) = y_1.$$

(8) Solve the following IVP using the Laplace transform:

$$y''(t) + y'(t) + y(t) = \cos(t), \qquad y(0) = 1, \qquad y'(0) = -1.$$

14.11 Problem Set 11

The problems are found in Schuam's Outlines.

$Y(s) = \mathscr{L}(f(x)) = \int_0^\infty e^{-sx} f(x)\,dx.$

(1) Show that $\mathscr{L}\left(x^2\right) = \frac{2}{s^3}$ using integration by parts.

(2) Show that $\mathscr{L}\left(e^{ax}\right) = \frac{1}{s-a}$ for $s > a$ using the definition of the Laplace transform.

(3) Calculate $\mathscr{L}(f(x))$, using the definition of the Laplace transform, where

$$f(x) = \begin{cases} e^x & \text{if } x \leq 2, \\ 3 & \text{if } x > 2. \end{cases}$$

(4) Calculate $\mathscr{L}\left(2x^2 - 3x + 4\right)$.

(5) Calculate $\mathscr{L}\left(xe^{4x}\right)$ using the definition of the Laplace transform.

(6) Calculate $\mathscr{L}\left(e^{2x}\right)$ using the definition of the Laplace transform.

(7) Calculate $\mathscr{L}(x)$ using the definition of the Laplace transform.

(8) Calculate $\mathscr{L}\left(xe^{-8x}\right)$ using the definition of the Laplace transform.

(9) Calculate $\mathscr{L}^{-1}\left(\frac{1}{s-8}\right)$.

(10) Calculate $\mathscr{L}^{-1}\left(\frac{1}{\sqrt{s}}\right)$.

(11) Calculate $\mathscr{L}^{-1}\left(\frac{s}{(s-2)^2+9}\right)$.

(12) Calculate $\mathscr{L}^{-1}\left(\frac{-2}{s-2}\right)$.

(13) Calculate $\mathscr{L}^{-1}\left(\frac{s}{(s+1)^2+5}\right)$.

(14) Calculate $\mathscr{L}^{-1}\left(\frac{2s^2}{(s-1)(s^2+1)}\right)$.

(15) Solve the IVP $y' - 5y = 0$, $y(0) = 2$ using the Laplace transform.

(16) Solve the IVP $y' + y = \sin(x)$, $y(0) = 1$ using the Laplace transform.

(17) Solve the IVP $\frac{dN}{dt} = 0.05N$, $N(0) = 20000$ using the Laplace transform.

(18) Solve the IVP $y' + 5y = 0$, $y(1) = 0$ using the Laplace transform.

(19) Solve the IVP $y'' + y' + y = 0$, $y(0) = 4$, $y'(0) = -3$ using the Laplace transform.

(20) Solve the IVP $y' + y = 4\cos(2x)$, $y(0) = 0$ using the Laplace transform.

14.12 Problem Set 12

(1) Solve the system

$$\begin{aligned} u' - 2v &= 3, \\ v' + v - u &= -x^2,\ u(0) = 0,\ v(0) = -1. \end{aligned}$$

(2) Solve the system

$$\begin{aligned} u' + 5u - 12v &= 0, \\ v' + 2u - 5v &= 0,\ u(0) = 8,\ v(0) = 3. \end{aligned}$$

(3) Solve the system

$$\begin{aligned} y' - z &= 0, \\ y - z' &= 0,\ y(0) = 1,\ z(0) = 1. \end{aligned}$$

(4) Solve the system

$$\begin{aligned} u' + 4u - 6v &= 0, \\ v' + 3u - 5v &= 0,\ u(0) = 3,\ v(0) = 2. \end{aligned}$$

(5) Solve the system

$$\begin{aligned} y' + z &= x, \\ z' - y &= 0,\ y(0) = 1,\ z(0) = 0. \end{aligned}$$

(6) Solve the system

$$\begin{aligned} x'(t) &= x(t) + 2y(t) + 3, \\ y'(t) &= 4x(t) + 5y(t) + 6 \end{aligned}$$

that we solved in a previous tutorial but using Laplace transforms. Assume $x(0) = 1$, $y(0) = -1$.

(7) Solve the system

$$\begin{aligned} x'(t) &= 3x(t) + y(t) + 4t - 1, \\ y'(t) &= 2x(t) - y(t) + t + 2 \end{aligned}$$

using Laplace transforms. Assume $x(0)=0$, $y(0)=1$.

(8) Use Laplace transforms to solve the system

$$\begin{aligned} y'+z &= x, \\ z'+4y &= 0, \end{aligned}$$

$y(0)=1$, $z'(0)=-1$. See [3, pp. 250].

(9) Use Laplace transforms to solve the system

$$\begin{aligned} z''+y' &= \cos(x), \\ y''-z &= \sin(x), \end{aligned}$$

$z(0)=-1$, $z'(0)=-1$, $y(0)=1$, $y'(0)=0$. See [3, pp. 251].

(5) Use the convolution theorem to find the inverse Laplace transform of $H(s)=\frac{1}{s^2(s+1)}$.

(6) Use the convolution theorem to find the inverse Laplace transform of $H(s)=\frac{1}{(s^2+1)^2}$.

(10) Find the Laplace transform of $u(t-1)\left(t^2+2\right)$. See [11, pp. 239].

(8) A harmonic oscillator with natural frequency $\omega_0=2$, initially at rest, is forced by the ramp function

$$f(t)=\begin{cases} t & \text{if } 0<t<1, \\ 0 & \text{if } t>1. \end{cases}$$

Solve the IVP

$$x''+4x=f(t), \qquad x(0)=0, \qquad x'(0)=0.$$

See [11, pp. 240].

14.13 Problem Set 13

Exercises 1 to 9 are found in [3].

(1) Verify that $u(x,t) = \left(55 + 22x^6 + x^{12}\right)\sin(2t)$ satisfies the PDE

$$12x^4 u_{tt} - x^5 u_{xtt} = -4u_{xx}.$$

(2) A function is called harmonic if it satisfies Laplace's equation; that is, $u_{xx} + u_{yy} = 0$. Which of the following functions are harmonic:

(a) $3x + 4y + 1$,

(b) $e^{3x}\cos(3y)$,

(c) $e^{3x}\cos(4y)$,

(d) $\log\left(x^2 + y^2\right)$,

(e) $\sin\left(e^x\right)\cos\left(e^y\right)$.

(3) Find the general solution to $u_x = \cos(y)$ if $u(x,y)$ is a function of x and y.

(4) Find the general solution to $u_y = \cos(y)$ if $u(x,y)$ is a function of x and y.

(5) Find the general solution to $u_y = 3$ if $u(x,y)$ is a function of x and y, and $u(x,0) = 4x + 1$.

(6) Find the general solution to $u_x = 2xy + 1$ if $u(x,y)$ is a function of x and y, and $u(0,y) = \cosh(y)$.

(7) Find the general solution to $u_{xx} = 3$ if $u(x,y)$ is a function of x and y.

(8) Find the general solution to $u_{xy} = 8xy^3$ if $u(x,y)$ is a function of x and y.

(9) Find the general solution to $u_{xyx} = -2$ if $u(x,y)$ is a function of x and y.

(10) Let $u(x,y)$ represent the vertical displacement of a string of length π, which is placed on the interval $[0,\pi]$, at position x and time t. Assuming proper units for length, times, and the constant k, the wave-equation models the displacement $u(x,t)$:

$$u_{tt} = c^2 u_{xx}.$$

Using the method of separation of variables, solve the equation for $u(x,t)$ if the boundary conditions

$$u(0,t) = u(\pi,t) = 0$$

for $t \geq 0$ are imposed, with initial displacement

$$u(x,0) = 5\sin(3x),$$

and initial velocity

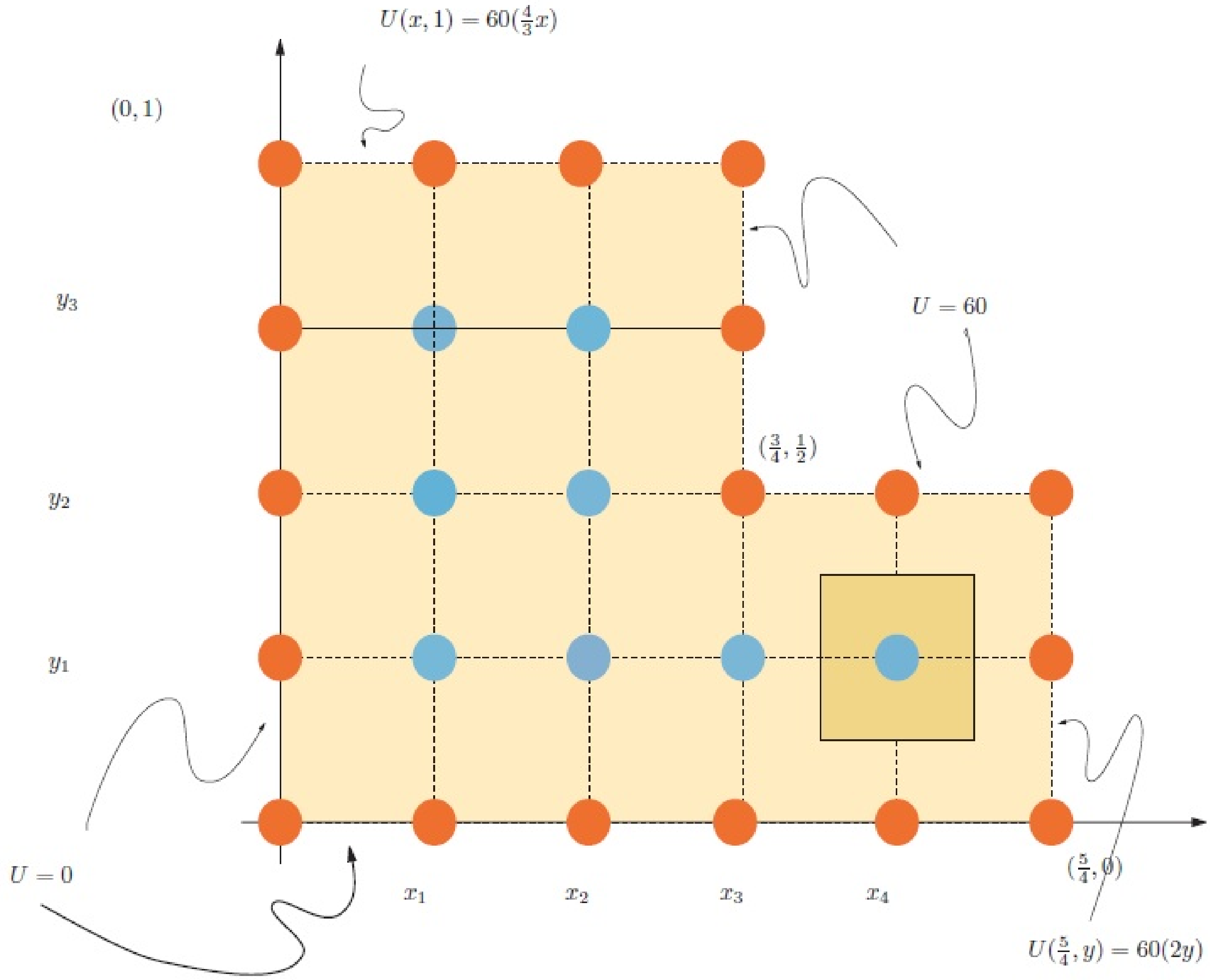
$$u_t(x,0) = 0$$

for $0 \leq x \leq \pi$.

(11) Solve the problem given in Question 10 via D'Alembert's method.

14.14 Problem Set 14

(1) Consider the steady state heat problem and the grid shown below.



(a) On the diagram, fill in the known temperatures at the grid points on the boundary.

(b) Write down the equations that must be satisfied at the points u_{11}, u_{23}, and u_{41}.

(c) The unknown temperatures are ordered by rows from left to right starting from the bottom; i.e. $u = (u_{11}, u_{21}, \ldots, u_{23})^T$. Write the system of equations in the form $Au = b$ by filling in the following diagram.

$$
\left[\begin{array}{cccc|cc|cc}
\_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \\
\_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \\
\_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \\
\_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \\
\hline
\_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \\
\_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \\
\hline
\_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \\
\_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_
\end{array}\right]
\left[\begin{array}{c}
\_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \hline \_\_ \\ \_\_ \\ \hline \_\_ \\ \_\_
\end{array}\right]
=
\left[\begin{array}{c}
\_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \hline \_\_ \\ \_\_ \\ \hline \_\_ \\ \_\_
\end{array}\right]
$$

(2) Consider the problem of determining the steady-state heat distribution in a thin rectangular metal plate with dimensions 0.3 m. wide by 0.5 m. high. The temperature $T(x,y)^\circ\ C$ along the boundaries of the plate are given by the equations

$$T(x,0) = x^2 + 15x + 39, \qquad T(0,y) = 39 - 18y,$$
$$T(x,0.5) = 30, \qquad T(0.4,y) = -40y^2 - 7.18y + 43.59.$$

The plate is in thermal equilibrium so that the temperature inside the plate satisfies

$$T_{xx}(x,y) + T_{yy}(x,y) = 0.$$

Write a matrix equation whose solution approximates the temperature at the eight interior coordinates $(0.1a, 0.1b)$, where $a \in \{1,2\}$, $b \in \{1,2,3,4\}$.

(3) Consider the problem of determining the steady-state heat distribution in a thin triangular metal plate with width and height 4 metres. The temperature $u(x,y)^\circ\ C$ along the three boundaries of the plate are given by the equations

$$
\begin{aligned}
u(x,0) &= x^2 + 40, \\
u(4,y) &= 56 - y^2 - 8y, \\
u(x,x) &= 40 - 3x^2 + 4x.
\end{aligned}
$$

The plate is in thermal equilibrium so that the temperature inside the plate satisfies Laplace's equation. Write a system of equations for which the solution approximates the temperature at the coordinates $(2,1)$, $(2,2)$, $(2,3)$.

14.15 Problem Set 15

Summary: To solve $y' = f(x_n, y_n)$, numerically, where $h = \Delta x$,

- **Euler's Method:**

$$\begin{aligned} x_{n+1} &= x_n + h, \\ y_{n+1} &= y_n + hf(x_n, y_n). \end{aligned}$$

- **Modified Euler's Method:**

$$\begin{aligned} x_{n+1} &= x_n + h, \\ y_{n+1} &= y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\right). \end{aligned}$$

- **RK2:**

$$\begin{aligned} x_{n+1} &= x_n + h, \\ k_1(h, x_y, y_n) &= f(x_n, y_n), \\ k_2(h, x_y, y_n) &= f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1(h, x_y, y_n)\right), \\ y_{n+1} &= y_n + hk_2(h, x_y, y_n). \end{aligned}$$

- **RK4:**

$$\begin{aligned} x_{n+1} &= x_n + h, \\ k_1(h, x_y, y_n) &= f(x_n, y_n), \\ k_2(h, x_y, y_n) &= f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1(h, x_y, y_n)\right), \\ k_3(h, x_y, y_n) &= f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2(h, x_y, y_n)\right), \\ k_4(h, x_y, y_n) &= f(x_n + h, y_n + hk_3(h, x_y, y_n)), \\ y_{n+1} &= y_n + \frac{h}{6}\left(k_1(h, x_y, y_n) + 2k_2(h, x_y, y_n) + 2k_3(h, x_y, y_n) + k_4(h, x_y, y_n)\right). \end{aligned}$$

(1) Solve the initial value problem $y' = y^2 + 1$, $y(0) = 0$ analytically.

(2) Use Euler's method to solve $y' = y^2 + 1$, $y(0) = 0$ with $h = 0.1$ for $x \in [0, 1]$.

(3) Use RK4 to solve $y' = y^2 + 1$, $y(0) = 0$ with $h = 0.1$ for $x \in [0, 1]$.

(4) Use RK4 to solve $y' = y^2 + 1$, $y(0) = 0$ with $h = 0.1$ for $y(1)$ without entering all previous values of $y(x)$ into memory.

(5) Solve the initial value problem $y' = -y$, $y(0) = 0$ analytically.

(6) Use Euler's method to solve $y' = -y$, $y(0) = 0$ with $h = 0.1$ for $x \in [0, 1]$.

(7) Solve the initial value problem $y' = 5x^4$, $y(0) = 0$ analytically.

(8) Use RK4 to solve $y' = 5x^4$, $y(0) = 0$ with $h = 0.1$ for $x \in [0, 1]$.

14.16 Problem Set 16

(1) An epidemic spreads from 20% of the population to affecting 70% of the population in just 12 days. Use the logistic equation $\frac{dp}{dt} = rp(1-p)$.

(a) Estimate the value of the intrinsic growth rate r.

(b) After how many days was exactly half the population infected?

(c) What proportion of the population is infected after 20 days?

(2) Solve the IVP

$$y' = \frac{x^2+y^2}{xy}, \qquad y(1) = -2.$$

(3) Find the general solution to the ODE $y' + xy = 6x\sqrt{y}$ using methods for linear 1st order ODEs.

(4) Solve $t^2 - x - tx' = 0$ using methods for exact equations.

(5) Solve the IVP $y'' - 3y' - 5y = 0$, $y(0) = 1$, $y'(0) = -1$.

(6) Find the general solution to $y'' - 2y' + y = 4\cos(x)$.

(7) Consider the homogeneous ODE $y'' + 3y' + 2y = 0$, which has the general solution $y(t) = Ae^{-t} + Be^{-2t}$. Let $z = y'$ and $Y = \begin{pmatrix} y \\ z \end{pmatrix}$.

(a) Write the ODE as a coupled first order system $Y' = AY$, where A is a 2×2 matrix.

(b) Find the eigenvalues λ_1 and λ_2 of the matrix A.

(c) Find the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ of the matrix A.

(d) Simplify the general solution $Y(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2$

(8) Consider the homogeneous ODE $y'' + 2y' + y = 0$, which has the general solution

$$y(t) = (A+Bt)e^{-t}, \qquad y'(t) = -(A-B+Bt)e^{-t}.$$

Let $z = y'$ and $Y = \begin{pmatrix} y \\ z \end{pmatrix}$.

(a) Write the ODE as a coupled first order system $Y' = AY$, where A is a 2×2 matrix.

(b) Find the eigenvalues λ_1 and λ_2 of the matrix A.

(c) Find the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ of the matrix A.

(d) Write an expression for the general solution $Y(t)$.

(9) A pond initially contains $x = 3$ carnivorous piranha fish and $y = 12$ electric yellow cichlids, known to give birth from their mouth - dozens per female every few months. The system is governed by the equations

$$\begin{aligned} x'(t) &= 3x - 4xy, \\ y'(t) &= xy - 9y. \end{aligned}$$

(a) Determine critical points of the system.

(b) Calculate the Jacobian matrix at the critical points.

(c) Classify the stability of the linearized system(s).

(d) Determine the long term behaviour of the system.

(10) Solve the following IVP using the Laplace transform:

$$x''(t) - 5x'(t) + 6x(t) = 2, \qquad x(0) = 0, \qquad x'(0) = 0.$$

(11) Use the convolution theorem to find the inverse Laplace transform of $H(s) = \frac{1}{s^2(s+1)}$.

(12) Find the Laplace transform of $u(t-1)\left(t^2+2\right)$.

(13) A metal rod of length L has fixed temperature 0 at each end. Initially the temperature is dependent on the position

$$u(x,0) = 6\sin\left(\frac{\pi}{L}x\right).$$

Solve the following BVP

$$u_t = ku_{xx}, \qquad u(0,t) = 0, \qquad u(L,t) = 0, \qquad u(x,0) = 6\sin\left(\frac{\pi}{L}x\right),$$

where $k > 0$.

(14) A wave of speed 3 metres per second travels through a spring with initial conditions

$$\begin{aligned} u(x,0) &= x^3 + x^2 - 5x + 4 + \cos(x) + 5\sin(x), \\ u_t(x,0) &= -9x^3 + 6x - 27 - 15\cos(x) - 3\sin(x). \end{aligned}$$

Calculate the wave function $u(x,t)$.

(15) Consider the problem of determining the steady-state heat distribution in a thin rectangular metal plate with dimensions 0.3 m. wide by 0.5 m. high. The temperature $T(x,y)^\circ\, C$ along the boundaries of the plate are given by the equations

$$T(x,0) = x^2 + 15x + 39, \qquad T(0,y) = 39 - 18y,$$
$$T(x,0.5) = 30, \qquad T(0.3,y) = -40y^2 - 7.18y + 43.59.$$

The plate is in thermal equilibrium so that the temperature inside the plate satisfies

$$T_{xx}(x,y) + T_{yy}(x,y) = 0.$$

Write a matrix equation whose solution approximates the temperature at the eight interior coordinates $(0.1a, 0.1b)$, where $a \in \{1,2\}$, $b \in \{1,2,3,4\}$.

15 Books & Notes

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