Elementary Calculus
Tutorials and Examples

Most problems come from *Basic Technical Mathematics with Calculus* by Washington. The solutions to the selected problem sets are by Samuel A. Hambleton

`https://mathofcourse.com/`

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1 Tutorial 1

The following exercises are found in Washington [2, S. 3.1, 3.2, 5.1].

(1) Three vertices of a rectangle are $(5,2),(-1,2),(-1,4)$. What is the fourth?

(2) Two vertices of an equilateral triangle are $(7,1)$ and $(2,1)$. What is the third vertex?

(3) Where are all the points whose x-coordinate is equal to their y-coordinates?

(4) What is the x-coordinate of all the points on the y-axis?

(5) What is the y-coordinate of all the points on the x-axis?

(6) If the point (a,b) is in the second quadrant, in which quadrant is the point $(a,-b)$?

(7) Join the points $(-1,-2)$, $(-4,-2)$, $(7,2)$, $(2,2)$, $(-1,-2)$ in order with straight line segments. Find the distance between successive points and then identify the geometric figure formed.

(8) For the function $f(x)=2x+1$, find $f(1)$ and $f(-1)$.

(9) For the function $f(x)=5$, find $f(-2)$ and $f(0.4)$.

(10) For the function $g(t)=at^2-a^2t$, find $g\left(-\frac{1}{2}\right)$ and $g(a)$.

(11) For the function $F(H)=\frac{2H^2}{H+36.85}$, find $F(-84.466)$.

(12) Find the slope through the points $(1,0)$ and $(3,8)$.

(13) Find the slope through the points $(-1,2)$ and $(2,10)$.

(14) Find the slope through the points $(5,-3)$ and $(-2,5)$.

(15) Find the slope through the points $(3.2,-4.1)$ and $(-1.5,-10.2)$.

(16) For the line $x+2y=4$, find the x-intercept, the y-intercept, and sketch.

(17) For the line $4x-3y=12$, find the x-intercept, the y-intercept, and sketch.

(18) Calculate a few points for the function $y=x^2$ to sketch the function on the interval $[-1,10]$. Take care with the scale of the y-axis.

(19) Calculate the average slope of $y=x^2$ over the interval $[1,10]$, $[1,5]$, $[1,3]$, and $[1,1.1]$. Draw each of the line segments on the graph.

(20) It is not possible to calculate the average slope over the interval $[1,1]$. Why not?

1.1 Tutorial 1 Solutions

(1) Three vertices of a rectangle are $(5,2),(-1,2),(-1,4)$. What is the fourth?

Sol. : The line segment from $(5,2)$ to $(-1,2)$ is perpendicular to the line segment from $(-1,2)$ to $(-1,4)$. The fourth point can only be the point $(5,4)$.

(2) Two vertices of an equilateral triangle are $(7,1)$ and $(2,1)$. What is the third vertex?
Sol. : The distance between $(7,1)$ and $(2,1)$ is

$$\sqrt{(7-2)^2+(1-1)^2} = 5.$$

Letting (x,y) be the coordinates of the third point, it must be equidistant from each of $(7,1)$ and $(2,1)$. That is, the square of that distance is

$$\begin{aligned} 5^2 &= (x-7)^2+(1-y)^2, \\ 5^2 &= (x-2)^2+(1-y)^2. \end{aligned}$$

Taking the difference,

$$\begin{aligned} 0 &= (x-7)^2-(x-2)^2, \\ &= 45-10x \end{aligned}$$

so that $x = 4.5$ and y satisfies

$$(4.5-2)^2+(1-y)^2-25 = 0.$$

Simplifying,

$$(y-1)^2 = 18.75 = \frac{3\times 25}{4}$$

so that

$$y = 1 \pm \frac{5\sqrt{3}}{2}.$$

We have two possibilities for the third point: $\left(\frac{9}{2}, 1 \pm \frac{5\sqrt{3}}{2}\right)$.

(3) Where are all the points whose x-coordinate is equal to their y-coordinates?

Sol. : Let $S = \{(x,y) \ : \ y = x, \ x, y \text{ are real numbers } \}$. The set S consists of the line of points $y = x$.

(4) What is the x-coordinate of all the points on the y-axis?

Sol. : All points that are on the y-axis are the points on the line $x = 0$,

$$S = \{(0,y) \ : \ y \text{ is a real number } \}.$$

(5) What is the y-coordinate of all the points on the x-axis?

Sol. : All points that are on the x-axis are the points on the line $y = 0$, $S = \{(x,0) \ : \ x \text{ is a real number } \}$.

(6) If the point (a,b) is in the second quadrant, in which quadrant is the point $(a,-b)$?

Sol. : The point $(a,-b)$ is a reflection of the point (a,b) about the x-axis. It follows that if (a,b) is in the second quadrant, then $(a,-b)$ is in the third quadrant.

(7) Join the points $(-1,-2)$, $(-4,-2)$, $(7,2)$, $(2,2)$, $(-1,-2)$ in order with straight line segments. Find the distance between successive points and then identify the geometric figure formed.

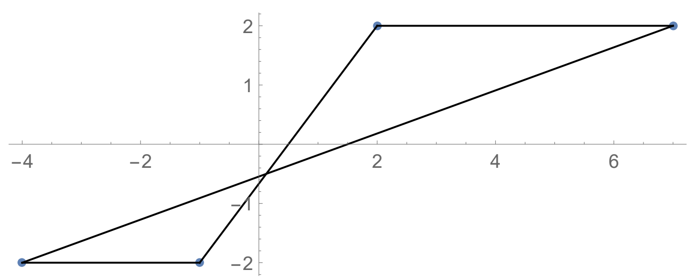
Sol. : The distances between successive points is 3,

$$\sqrt{(7-(-4))^2+(2-(-2))^2} = \sqrt{137}$$

, 5, and

$$\sqrt{(2-(-1))^2+(2-(-2))^2} = 5.$$

See the image below.



(8) For the function $f(x) = 2x+1$, find $f(1)$ and $f(-1)$.

Sol. : $f(1) = 2\times 1+1 = 3$ and $f(-1) = 2\times(-1)+1 = -1$.

(9) For the function $f(x) = 5$, find $f(-2)$ and $f(0.4)$.

Sol. : $f(-2) = 5 = f(0.4)$.

(10) For the function $g(t) = at^2 - a^2t$, find $g\left(-\frac{1}{2}\right)$ and $g(a)$.

Sol. :

$$\begin{aligned} g\left(-\frac{1}{2}\right) &= a\left(-\frac{1}{2}\right)^2 - a^2\left(-\frac{1}{2}\right), \\ &= \frac{1}{4}a + \frac{1}{2}a^2, \end{aligned}$$

$$\begin{aligned} g(a) &= a \cdot a^2 - a^2 \cdot a, \\ &= a^3 - a^3, \\ &= 0. \end{aligned}$$

(11) For the function $F(H) = \frac{2H^2}{H+36.85}$, find $F(-84.466)$.

Sol. :

$$\begin{aligned} F(-84.466) &= \frac{2(-84.466)^2}{-84.466+36.85}, \\ &= -299.668395\ldots \end{aligned}$$

(12) Find the slope through the points $(1,0)$ and $(3,8)$.

Sol. : Let $(x_1, y_1) = (1,0)$ and $(x_2, y_2) = (3,8)$. The slope of the line segment between these points is

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1}, \\ &= \frac{8-0}{3-1}, \\ &= \frac{8}{2}, \\ &= 4. \end{aligned}$$

(13) Find the slope through the points $(-1,2)$ and $(2,10)$.

Sol. : Let $(x_1, y_1) = (-1,2)$ and $(x_2, y_2) = (2,10)$. The slope of the line segment between these points is

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1}, \\ &= \frac{10 - 2}{2 - (-1)}, \\ &= \frac{8}{3}. \end{aligned}$$

(14) Find the slope through the points $(5, -3)$ and $(-2, 5)$.

Sol. : Let $(x_1, y_1) = (5, -3)$ and $(x_2, y_2) = (-2, 5)$. The slope of the line segment between these points is

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1}, \\ &= \frac{5 - (-3)}{-2 - 5}, \\ &= -\frac{8}{7}. \end{aligned}$$

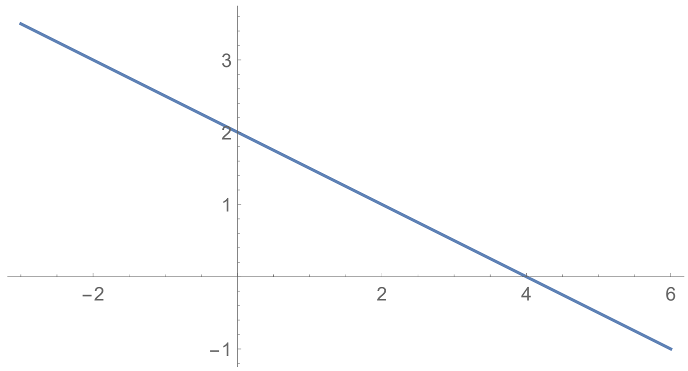
(15) Find the slope through the points $(3.2, -4.1)$ and $(-1.5, -10.2)$.

Sol. : Let $(x_1, y_1) = (3.2, -4.1)$ and $(x_2, y_2) = (-1.5, -10.2)$. The slope of the line segment between these points is

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1}, \\ &= \frac{-10.2 - (-4.1)}{-1.5 - 3.2}, \\ &= \frac{-6.1}{-4.7}, \\ &= \frac{61}{47}, \\ &= 1.29787234\ldots. \end{aligned}$$

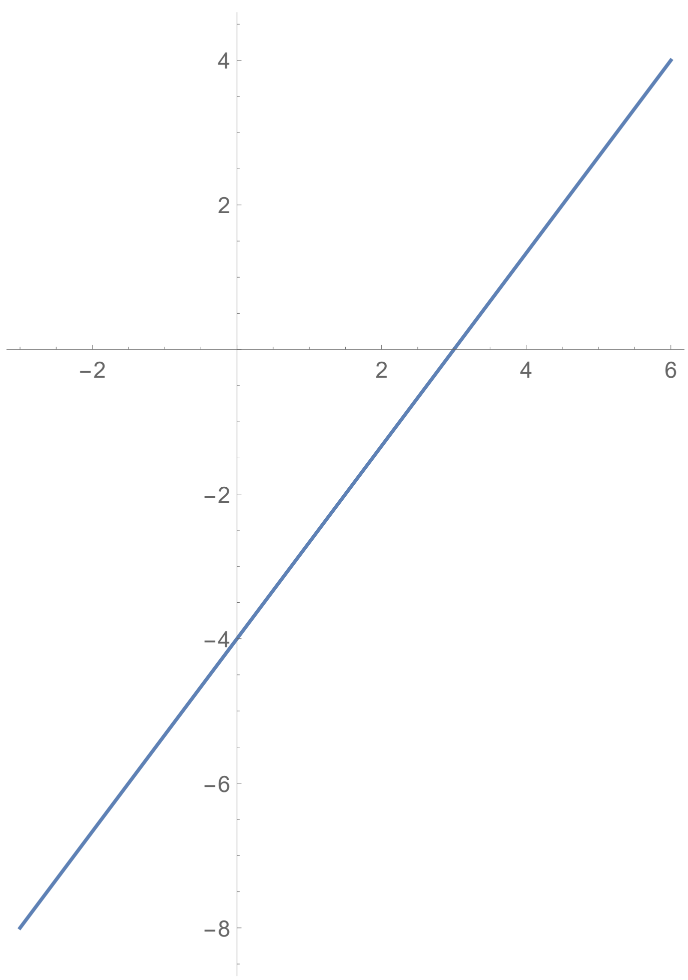
(16) For the line $x + 2y = 4$, find the x-intercept, the y-intercept, and sketch.

Sol. : When $y = 0$, $x = 4$ so the x-intercept is the point $(4,0)$. When $x = 0$, $2y = 4$ so $y = 2$ and the y-intercept is the point $(0,2)$.



(17) For the line $4x - 3y = 12$, find the x-intercept, the y-intercept, and sketch.

Sol. : When $y = 0$, $4x = 12$ so $x = 3$ so the x-intercept is the point $(3,0)$. When $x = 0$, $-3y = 12$ so $y = -4$ and the y-intercept is the point $(0,-4)$.



2 Tutorial 2

The following exercises are found in Washington [2].

(1) Simplify $2b^4b^2$.

(2) Simplify $3k^5k$.

(3) Simplify $\frac{m^5}{m^3}$.

(4) Simplify $\frac{3s}{s^4}$.

(5) Simplify $\left(x^8\right)^3$.

(6) Simplify $(ax)^5$.

(7) Simplify $\left(\frac{3}{n^3}\right)^3$.

(8) Simplify with rational denominators $\sqrt{10}\sqrt{2}$.

(9) Simplify with rational denominators $\sqrt{7}\sqrt{21}$.

(10) Simplify with rational denominators $\left(5\sqrt{2}\right)^2$.

(11) Simplify with rational denominators $\left(2-\sqrt{5}\right)\left(2+\sqrt{5}\right)$.

(12) Factor $7x-7y$.

(13) Factor $6b-24$.

(14) Factor $15x^2-3x$.

(15) Factor $90p^3-15p^2$.

(16) Factor $2x+4y-8z$.

(17) Factor x^2-9.

(18) Factor x^4-16.

(19) Factor x^2+6x+5.

(20) Factor s^2-s-56.

(21) Factor x^2+2x+1.

(22) Factor $2n^2-13n-7$.

(23) Graph $y=2x-4$.

(24) Graph $s=7-2t$.

(25) Graph $y=x^2$.

(26) Graph $V=s^3$.

(27) For which values of x is $f(x)=6x-15$ continuous?

(28) For which values of x is $f(x)=\frac{2}{x^2-7x}$ continuous?

(29) Evaluate $\lim_{x\longrightarrow 3}(9x-16)$.

(30) Evaluate $\lim_{x\longrightarrow 5}\sqrt{x^2-9}$.

(31) Evaluate $\lim_{x\longrightarrow 0}\frac{x^2+x}{x}$.

(32) Evaluate $\lim_{x\longrightarrow -1}\frac{x^2-1}{3x+3}$.

(33) Calculate y' using the limit definition for $y=5x-2$.

(34) Calculate y' using the limit definition for $y=x^2-1$.

(35) Calculate y' using the limit definition for $y=x^3+4x-3\pi$.

(36) Calculate y' for $y=3x^2-2x$ and evaluate at the point $(-1,5)$.

(37) At what point on the curve of $y=2x^2-16x$ is there a tangent line that is horizontal?

(38) Find the derivative of $y=x^8$.

(39) Find the derivative of $f(x)=-4x^{11}$.

(40) Find the derivative of $y = 5x^4 - 3\pi$.

(41) Find the derivative of $p = 5r^3 - 2r + 1$.

(42) Find the derivative of $y = 6x^2 - 8x + 1$ at the point $(2, 9)$.

(43) Find the derivative of $y = 2x^3 + 9x - 7$ at the point $(-2, -41)$.

(44) Find the slope of the tangent of $y = 35x - 2x^4$ when $x = 2$.

(45) Find the derivative of $y = 6\sqrt{x}$.

(46) Find the derivative of $v = \frac{3}{5t^3}$.

(47) Find the derivative of $y = \frac{3}{\sqrt[3]{x}} + 4x^2$.

(48) Find the derivative of $y = x\sqrt{x} - \frac{6}{x}$.

(49) Find the derivative of $f(x) = 2x^{-3} - 3x^{-2}$.

(50) For $y = x^3 + 7x^2$, find y' and y''.

(51) For $y = x^3 - 6x^4$, find y' and y''.

(52) For $y = 2x^7 - x^6 - 3x$, find y''.

(53) For $y = 2x + \sqrt{x}$, find y''.

2.1 Tutorial 2 Solutions

(1) Simplify $2b^4b^2$.

Sol. : $2b^4b^2 = 2b^{4+2} = 2b^6$.

(2) Simplify $3k^5k$.

Sol. : $3k^5k = 3k^{5+1} = 3k^6$.

(3) Simplify $\frac{m^5}{m^3}$.

Sol. : $\frac{m^5}{m^3} = m^{5-3} = m^2$.

(4) Simplify $\frac{3s}{s^4}$.

Sol. : $\frac{3s}{s^4} = 3s^{1-4} = 3s^{-3} = \frac{3}{s^3}$.

(5) Simplify $\left(x^8\right)^3$.

Sol. : $\left(x^8\right)^3 = x^{8\times 3} = x^{24}$.

(6) Simplify $(ax)^5$.

Sol. : $(ax)^5 = a^5x^5$.

(7) Simplify $\left(\frac{3}{n^3}\right)^3$.

Sol. : $\left(\frac{3}{n^3}\right)^3 = \frac{3^3}{n^{3\times 3}} = \frac{27}{n^9}$.

(8) Simplify with rational denominators $\sqrt{10}\sqrt{2}$.

Sol. : $\sqrt{10}\sqrt{2} = \sqrt{2^2 \times 5} = 2\sqrt{5}$.

(9) Simplify with rational denominators $\sqrt{7}\sqrt{21}$.

Sol. : $\sqrt{7}\sqrt{21} = \sqrt{7^2 \times 3} = 7\sqrt{3}$.

(10) Simplify with rational denominators $\left(5\sqrt{2}\right)^2$.

Sol. : $\left(5\sqrt{2}\right)^2 = 5^2\sqrt{2}^2 = 25 \times 2 = 50$.

(11) Simplify with rational denominators $\left(2-\sqrt{5}\right)\left(2+\sqrt{5}\right)$.

Sol. : $\left(2-\sqrt{5}\right)\left(2+\sqrt{5}\right) = 2^2 - \sqrt{5}^2 = 4 - 5 = -1$.

(12) Factor $7x - 7y$.

Sol. : $7x - 7y = 7(x - y)$.

(13) Factor $6b - 24$.

Sol. : $6b - 24 = 6(b - 4)$.

(14) Factor $15x^2 - 3x$.

Sol. : $15x^2 - 3x = 3x(5x - 1)$.

(15) Factor $90p^3 - 15p^2$.

Sol. : $90p^3 - 15p^2 = 15p^2(6p - 1)$.

(16) Factor $2x + 4y - 8z$.

Sol. : $2x + 4y - 8z = 2(x + 2y - 4z)$.

(17) Factor $x^2 - 9$.

Sol. : $x^2 - 9 = (x + 3)(x - 3)$.

(18) Factor $x^4 - 16$.

Sol. : $x^4 - 16 = \left(x^2 + 4\right)\left(x^2 - 4\right)$.

(19) Factor $x^2 + 6x + 5$.

Sol. : $x^2 + 6x + 5 = (x + 1)(x + 5)$.

(20) Factor $s^2 - s - 56$.

Sol. : $s^2 - s - 56 = (s+7)(s-8)$.

(21) Factor $x^2 + 2x + 1$.

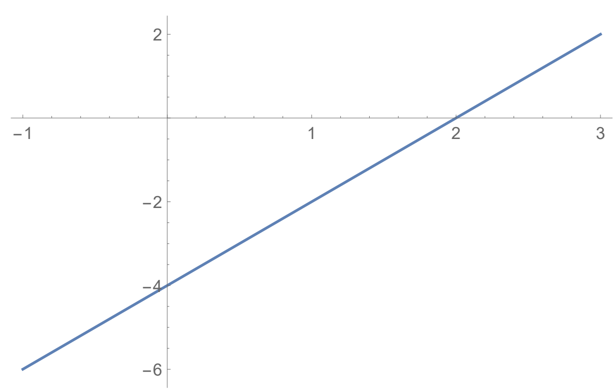
Sol. : $x^2 + 2x + 1 = (x+1)^2$.

(22) Factor $2n^2 - 13n - 7$.

Sol. : $2n^2 - 13n - 7 = (2n+1)(n-7)$.

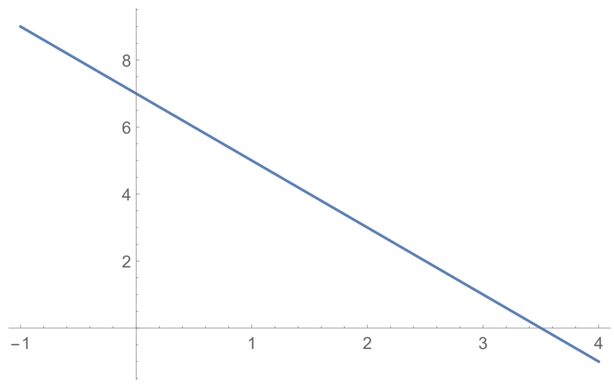
(23) Graph $y = 2x - 4$.

Sol. :



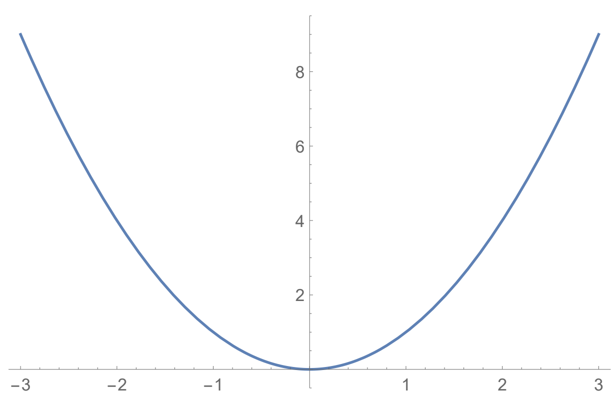
(24) Graph $s = 7 - 2t$.

Sol. :



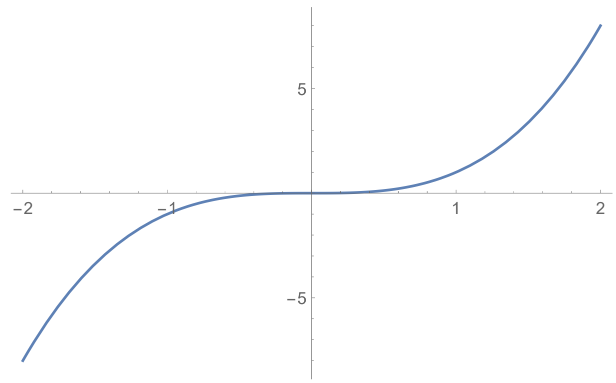
(25) Graph $y = x^2$.

Sol. :



(26) Graph $V = s^3$.

Sol. :



(27) For which values of x is $f(x) = 6x - 15$ continuous?

Sol. : $f(x)$ is continuous for all real numbers: $x \in (-\infty, \infty)$ or we can write $f(x)$ is continuous for all for all $x \in \mathbb{R}$.

(28) For which values of x is $f(x) = \frac{2}{x^2 - 7x}$ continuous?

Sol. : $f(x) = \frac{2}{x(x-7)}$ is continuous at all points x that are real numbers not equal to 0 or 7 since $f(x)$ is undefined for zero denominators. Alternatively we can write $f(x)$ is continuous for all $x \in \mathbb{R} \setminus \{0, 7\}$. The intervals over which $f(x)$ is continuous are $(-\infty, 0)$, $(0, 7)$, and $(7, \infty)$.

(29) Evaluate $\lim_{x \longrightarrow 3} (9x - 16)$.

Sol. : $\lim_{x \longrightarrow 3} (9x - 16) = 9 \times 3 - 16 = 11$.

(30) Evaluate $\lim_{x \longrightarrow 5} \sqrt{x^2-9}$.

Sol. : $\lim_{x \longrightarrow 5} \sqrt{x^2-9} = \sqrt{25-9} = \sqrt{16} = 4.$

(31) Evaluate $\lim_{x \longrightarrow 0} \frac{x^2+x}{x}$.

Sol. : $\lim_{x \longrightarrow 0} \frac{x^2+x}{x} = \lim_{x \longrightarrow 0} \frac{x(x+1)}{x} = \lim_{x \longrightarrow 0} (x+1) = 1.$

(32) Evaluate $\lim_{x \longrightarrow -1} \frac{x^2-1}{3x+3}$.

Sol. : $\lim_{x \longrightarrow -1} \frac{x^2-1}{3x+3} = \lim_{x \longrightarrow -1} \frac{(x+1)(x-1)}{3(x+1)} = \lim_{x \longrightarrow -1} \frac{(x-1)}{3} = -\frac{2}{3}.$

(33) Calculate y' using the limit definition for $y = 5x-2$.

Sol. :

$$
\begin{aligned}
y' &= \lim_{h \longrightarrow 0} \frac{1}{h} \left(f(x+h) - f(x) \right), \\
&= \lim_{h \longrightarrow 0} \frac{1}{h} \left(5(x+h) - 2 - (5x-2) \right), \\
&= \lim_{h \longrightarrow 0} \frac{1}{h} \left(5x + 5h - 2 - 5x + 2 \right), \\
&= \lim_{h \longrightarrow 0} \frac{1}{h} \left(5h \right), \\
&= \lim_{h \longrightarrow 0} 5, \\
&= 5.
\end{aligned}
$$

(34) Calculate y' using the limit definition for $y = x^2 - 1$.

Sol. :

$$\begin{aligned}
y' &= \lim_{h \longrightarrow 0} \frac{1}{h}\left(f(x+h) - f(x)\right), \\
&= \lim_{h \longrightarrow 0} \frac{1}{h}\left((x+h)^2 - 1 - \left(x^2 - 1\right)\right), \\
&= \lim_{h \longrightarrow 0} \frac{1}{h}\left(x^2 + 2xh + h^2 - 1 - x^2 + 1\right), \\
&= \lim_{h \longrightarrow 0} \frac{1}{h}\left(2xh + h^2\right), \\
&= \lim_{h \longrightarrow 0} \frac{1}{h} h\left(2x + h\right), \\
&= \lim_{h \longrightarrow 0} \left(2x + h\right), \\
&= 2x.
\end{aligned}$$

(35) Calculate y' using the limit definition for $y = x^3 + 4x - 3\pi$.

Sol. : Sol. :

$$\begin{aligned}
y' &= \lim_{h \longrightarrow 0} \frac{1}{h}\left(f(x+h) - f(x)\right), \\
&= \lim_{h \longrightarrow 0} \frac{1}{h}\left((x+h)^3 + 4(x+h) - 3\pi - \left(x^3 + 4x - 3\pi\right)\right), \\
&= \lim_{h \longrightarrow 0} \frac{1}{h}\left(x^3 + 3x^2h + 3xh^2 + h^3 + 4x + 4h - 3\pi - x^3 - 4x + 3\pi\right), \\
&= \lim_{h \longrightarrow 0} \frac{1}{h}\left(3x^2h + 3xh^2 + h^3 + 4h\right), \\
&= \lim_{h \longrightarrow 0} \frac{1}{h} h\left(3x^2 + 3xh + h^2 + 4\right), \\
&= \lim_{h \longrightarrow 0} \left(3x^2 + 3xh + h^2 + 4\right), \\
&= 3x^2 + 4.
\end{aligned}$$

(36) Calculate y' for $y = 3x^2 - 2x$ and evaluate at the point $(-1, 5)$.

Sol. : $y'(x) = \frac{d}{dx}\left(3x^2 - 2x\right) = 6x - 2$. It follows that $y'(-1) = 6(-1) + 5 = -1$. Hence the slope of the curve $y = 3x^2 - 2x$ is -1 at the point $(-1, 5)$.

(37) At what point on the curve of $y = 2x^2 - 16x$ is there a tangent line that is horizontal?

Sol. : $y'(x) = 4x - 16$. When $y'(x) = 0$ we have $4(x - 4) = 0$ so that $x = 4$. The tangent line that is horizontal at the point $(4, -32)$.

(38) Find the derivative of $y = x^8$.

Sol. : $y'(x) = \frac{d}{dx}x^8 = 8x^{8-1} = 8x^7$.

(39) Find the derivative of $f(x) = -4x^{11}$.

Sol. : $f'(x) = -4 \times 11x^{11-1} = -44x^{10}$.

(40) Find the derivative of $y = 5x^4 - 3\pi$.

Sol. : $y'(x) = 5 \times 4x^{4-1} - 0 = 20x^3$.

(41) Find the derivative of $p = 5r^3 - 2r + 1$.

Sol. : $p'(r) = 5 \times 3r^{3-1} - 2 = 15r^2 - 2$.

(42) Find the derivative of $y = 6x^2 - 8x + 1$ at the point $(2, 9)$.

Sol. : $y'(x) = 12x - 8$ so $y'(2) = 12 \times 2 - 8 = 24 - 8 = 16$.

(43) Find the derivative of $y = 2x^3 + 9x - 7$ at the point $(-2, -41)$.

Sol. : $y'(x) = 6x^2 + 9$ so $y'(-2) = 6 \times (-2)^2 + 9 = 24 + 9 = 33$.

(44) Find the slope of the tangent of $y = 35x - 2x^4$ when $x = 2$.

Sol. : $y'(x) = 35 - 8x^3$ so $y'(2) = 35 - 8 \times 2^3 = 35 - 64 = -29$.

(45) Find the derivative of $y = 6\sqrt{x}$.

Sol. : We first write $y = 6x^{1/2}$. Then

$$y'(x) = 6 \times \frac{1}{2} x^{\frac{1}{2}-1} = 3x^{-\frac{1}{2}} = \frac{3}{\sqrt{x}}.$$

(46) Find the derivative of $v = \frac{3}{5t^3}$.

Sol. : We have $v = \frac{3}{5} t^{-3}$ so that $v'(t) = \frac{-9}{5} t^{-4} = -\frac{9}{5t^4}$.

(47) Find the derivative of $y = \frac{3}{\sqrt[3]{x}} + 4x^2$.

Sol. : $y'(x) = \frac{d}{dx}\left(3x^{-\frac{1}{3}} + 4x^2\right) = 3 \times \frac{-1}{3} x^{-\frac{4}{3}} + 4 \times 2x^1 = -x^{-\frac{4}{3}} + 8x = -\frac{1}{x\sqrt[3]{x}} + 8x$.

(48) Find the derivative of $y = x\sqrt{x} - \frac{6}{x}$.

Sol. : $y'(x) = \frac{d}{dx}\left(x^{\frac{3}{2}} - 6x^{-1}\right) = \frac{3}{2}x^{\frac{1}{2}} + 6x^{-2} = \frac{3}{2}\sqrt{x} + \frac{6}{x^2}$.

(49) Find the derivative of $f(x) = 2x^{-3} - 3x^{-2}$.

Sol. : $f'(x) = 2(-3)x^{-3-1} - 3(-2)x^{-2-1} = -6x^{-4} + 6x^{-3}$.

(50) For $y = x^3 + 7x^2$, find y' and y''.

Sol. :

$$\begin{aligned} y'(x) &= 3x^2 + 7(2)x^1 = 3x^2 + 14x. \\ y''(x) &= 3(2)x + 14 = 6x + 14. \end{aligned}$$

(51) For $y = x^3 - 6x^4$, find y' and y''.

Sol. :

$$\begin{aligned} y'(x) &= 3x^2 - 6(4)x^3 = 3x^2 - 24x^3. \\ y''(x) &= 3(2)x^1 - 24(3)x^2 = 6x - 72x^2. \end{aligned}$$

(52) For $y = 2x^7 - x^6 - 3x$, find y''.

Sol. :

$$\begin{aligned} y'(x) &= 2(7)x^6 - 6x^5 - 3 = 14x^6 - 6x^5 - 3. \\ y''(x) &= 14(6)x^5 - 6(5)x^4 = 84x^5 - 30x^4. \end{aligned}$$

(53) For $y = 2x + \sqrt{x}$, find y''.

Sol. :

$$\begin{aligned} y'(x) &= 2 + \frac{1}{2}x^{-\frac{1}{2}}. \\ y''(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} = -\frac{1}{4x\sqrt{x}}. \end{aligned}$$

3 Tutorial 3

The following exercises are found in Washington [2].

(1) Find the remainder by long division $\left(x^3+2x+3\right) \div (x+1)$.

(2) Find the remainder by long division $\left(2x^4-10x^2+30x-60\right) \div (x+4)$.

(3) Find the remainder by long division $\left(x^3+2x-x-2\right) \div (x-1)$.

(4) Find the remainder by long division $\left(x^5+4x^4-8\right) \div (x+1)$.

(5) Find the derivative of $f(x)=3x-x^4$.

(6) Find the derivative of $s=8t^5+5t^4$.

(7) Find the second derivative of $r=3\theta^2-\frac{20}{\sqrt{\theta}}$.

(8) Use the product rule to calculate y' for $y=2x^3\left(3x^4+x\right)$.

(9) Use the product rule to calculate $f'(x)$ for $f(x)=(3x-2)\left(4x^2+3\right)$.

(10) Use the product rule to calculate $f'(s)$ for $f(s)=\left(11s^2+3\right)\left(2s^2-1\right)$. Then expand to compute $f'(s)$.

(11) Use the product rule to calculate y' for $y=\left(3x^2-4x+1\right)\left(5-6x^2\right)$. Then expand to compute y'.

(12) Using the product rule, find the point(s) on the curve of

$$y=\left(2x^2-1\right)(1-4x)$$

for which the tangent is $y=4x-1$.

(13) Use the quotient rule to calculate y', where $y=\frac{x}{2x+3}$.

(14) Use the quotient rule to calculate y', where $y=\frac{e^2}{3x^2-5x}$.

(15) Use the quotient rule to calculate y', where $y=\frac{33x}{4x^5-3x-4}$.

(16) Use the quotient rule to calculate y', where $y = \frac{3x^3 - 8x}{2x^2 - 5x + 4}$.

(17) Use the chain rule to calculate y', where $y = (1 - 5x)^{12}$.

(18) Use the chain rule to calculate y', where $y = 8(1 - 6x)^{1.5}$.

(19) Use the chain rule to calculate y', where $y = \sqrt[4]{1 - 8x^2}$.

(20) Calculate $f'(x)$, where $f(x) = (g(x))^n$, and $g(x)$ is a function of x.

(21) Calculate $y'(x)$, where $y = \frac{\pi^3}{\sqrt{1 - 3x}}$.

(22) Calculate $y'(x)$, where $y = 9\sqrt[3]{4x^6 + 2}$.

(23) Calculate $y'(x)$, where $y = x^7(1 - 3x)^{15}$.

(24) Calculate $R'(T)$, where $R = \frac{2T^2}{\sqrt[3]{1 + 4T}}$.

3.1 Tutorial 3 Solutions

(1) Find the remainder by long division

$$\left(x^3+2x+3\right) \div (x+1).$$

Sol. :

$$
\begin{array}{r l}
 & \quad\quad\quad x^2 \quad -x+3 \\ \cline{2-2}
x+1) & \quad x^3+0x^2+2x+3 \\
 & -x^3 \quad -x^2 \\ \cline{2-2}
 & \quad\quad\quad -x^2+2x \\
 & \quad\quad\quad\quad x^2 \quad +x \\ \cline{2-2}
 & \quad\quad\quad\quad\quad\quad 3x+3 \\
 & \quad\quad\quad\quad\quad -3x-3 \\ \cline{2-2}
 & \quad\quad\quad\quad\quad\quad\quad\quad 0
\end{array}
$$

It follows that the remainder is 0. We have

$$(x+1)\left(x^2-x+3\right) = x^3+2x+3.$$

(2) Find the remainder by long division

$$\left(2x^4 - 10x^2 + 30x - 60\right) \div (x+4).$$

Sol. :

$$
\begin{array}{rl}
 & \quad 2x^3 \quad - 8x^2 + 22x \quad - 58 \\
\cline{2-2}
x+4) & \quad 2x^4 + 0x^3 - 10x^2 + 30x \quad - 60 \\
 & - 2x^4 - 8x^3 \\
\cline{2-2}
 & \qquad - 8x^3 - 10x^2 \\
 & \qquad\quad 8x^3 + 32x^2 \\
\cline{2-2}
 & \qquad\qquad 22x^2 + 30x \\
 & \qquad\quad - 22x^2 - 88x \\
\cline{2-2}
 & \qquad\qquad\qquad - 58x \quad - 60 \\
 & \qquad\qquad\qquad\quad 58x + 232 \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad\quad 172
\end{array}
$$

It follows that the remainder is 172. We have

$$
\begin{aligned}
(x+4)\left(2x^3 - 8x^2 + 22x - 58\right) + 172 &= 2x^4 - 10x^2 + 30x - 232 + 172, \\
&= \left(2x^4 - 10x^2 + 30x - 60\right).
\end{aligned}
$$

(3) Find the remainder by long division $\left(x^3+2x-x-2\right)\div(x-1)$.

Sol. :

$$
\begin{array}{r|l}
 & x^2 \quad +x+2 \\
\hline
x-1 & x^3+0x^2 \quad +x-2 \\
 & -x^3 \quad +x^2 \\
\hline
 & x^2 \quad +x \\
 & -x^2 \quad +x \\
\hline
 & 2x-2 \\
 & -2x+2 \\
\hline
 & 0
\end{array}
$$

It follows that the remainder is 0. We have

$$(x-1)\left(x^2+x+2\right)=\left(x^3+2x-x-2\right).$$

(4) Find the remainder by long division $\left(x^5+4x^4-8\right) \div (x+1)$.

Sol. :

$$
\begin{array}{rl}
 & \; x^4+3x^3-3x^2+3x-3 \\
\cline{2-2}
x+1 & \big) \; x^5+4x^4+0x^3+0x^2+0x-8 \\
 & \underline{-x^5 \;\; -x^4} \\
 & \qquad\quad 3x^4+0x^3 \\
 & \qquad \underline{-3x^4-3x^3} \\
 & \qquad\qquad\quad -3x^3+0x^2 \\
 & \qquad\qquad\quad \underline{\;\;\, 3x^3+3x^2} \\
 & \qquad\qquad\qquad\qquad 3x^2+0x \\
 & \qquad\qquad\qquad\quad \underline{-3x^2-3x} \\
 & \qquad\qquad\qquad\qquad\qquad -3x-8 \\
 & \qquad\qquad\qquad\qquad\qquad \underline{\;\;\, 3x+3} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad -5
\end{array}
$$

It follows that the remainder is -5. We have

$$(x+1)\left(x^4+3x^3-3x^2+3x-3\right)-5=\left(x^5+4x^4-8\right).$$

(5) Find the derivative of $f(x)=3x-x^4$.

Sol. : $f'(x)=3-4x^3$.

(6) Find the derivative of $s=8t^5+5t^4$.

Sol. : $s'(t)=40t^4+20t^3$.

(7) Find the second derivative of $r=3\theta^2-\frac{20}{\sqrt{\theta}}$.

Sol. : $r'(\theta) = 6\theta + 10\theta^{-\frac{3}{2}}$ so

$$r''(\theta) = 6 - 15\theta^{-\frac{5}{2}}.$$

(8) Use the product rule to calculate y' for $y = 2x^3\left(3x^4 + x\right)$.

Sol. : We let $u = x^3$ and $v = 3x^4 + x$.

$$\begin{aligned}
y'(x) &= \frac{d}{dx}\left(2x^3\left(3x^4+x\right)\right), \\
&= 2\frac{d}{dx}\left(x^3\left(3x^4+x\right)\right), \\
&= 2\left(x^3\frac{d}{dx}\left(3x^4+x\right) + \left(3x^4+x\right)\frac{d}{dx}x^3\right), \\
&= 2\left(x^3\left(12x^3+1\right) + \left(3x^4+x\right)3x^2\right), \\
&= 2\left(12x^6+x^3+9x^6+3x^3\right), \\
&= 42x^6+8x^3.
\end{aligned}$$

(9) Use the product rule to calculate $f'(x)$ for $f(x) = (3x-2)\left(4x^2+3\right)$.

Sol. : We let $u = 3x - 2$ and $v = 4x^2 + 3$.

$$\begin{aligned}
y'(x) &= \frac{d}{dx}(3x-2)\left(4x^2+3\right), \\
&= (3x-2)\frac{d}{dx}\left(4x^2+3\right) + \left(4x^2+3\right)\frac{d}{dx}(3x-2), \\
&= (3x-2)(8x) + \left(4x^2+3\right)(3), \\
&= 36x^2 - 16x + 9.
\end{aligned}$$

(10) Use the product rule to calculate $f'(s)$ for

$$f(s) = \left(11s^2+3\right)\left(2s^2-1\right).$$

Then expand to compute $f'(s)$.

Sol. : We let $u = 11s^2+3$ and $v = 2s^2-1$.

$$\begin{aligned} f'(s) &= \frac{d}{ds}\left(11s^2+3\right)\left(2s^2-1\right), \\ &= \left(2s^2-1\right)\frac{d}{ds}\left(11s^2+3\right)+\left(11s^2+3\right)\frac{d}{ds}\left(2s^2-1\right), \\ &= \left(2s^2-1\right)(22s)+\left(11s^2+3\right)(4s), \\ &= 88s^3-10s. \end{aligned}$$

Alternatively,

$$\begin{aligned} f'(s) &= \frac{d}{ds}\left(11s^2+3\right)\left(2s^2-1\right), \\ &= \frac{d}{ds}\left(22s^4-5s^2-3\right), \\ &= 88s^3-10s. \end{aligned}$$

(11) Use the product rule to calculate y' for $y = \left(3x^2-4x+1\right)\left(5-6x^2\right)$. Then expand to compute y'.

Sol. : We let $u = 3x^2-4x+1$ and $v = 5-6x^2$.

$$\begin{aligned} y'(x) &= \frac{d}{dx}\left(3x^2-4x+1\right)\left(5-6x^2\right), \\ &= \left(3x^2-4x+1\right)\frac{d}{dx}\left(5-6x^2\right)+\left(5-6x^2\right)\frac{d}{dx}\left(3x^2-4x+1\right), \\ &= \left(3x^2-4x+1\right)(-12x)+\left(5-6x^2\right)(6x-4), \\ &= -72x^3+72x^2+18x-20. \end{aligned}$$

Alternatively,

$$\begin{aligned} y'(x) &= \frac{d}{dx}\left(3x^2-4x+1\right)\left(5-6x^2\right), \\ &= \frac{d}{dx}\left(-18x^4+24x^3+9x^2-20x+5\right), \\ &= -72x^3+72x^2+18x-20. \end{aligned}$$

(12) Using the product rule, find the point(s) on the curve of

$$y = \left(2x^2-1\right)(1-4x)$$

for which the tangent is $y = 4x-1$.

Sol. : We let $u = 2x^2-1$ and $v = 1-4x$.

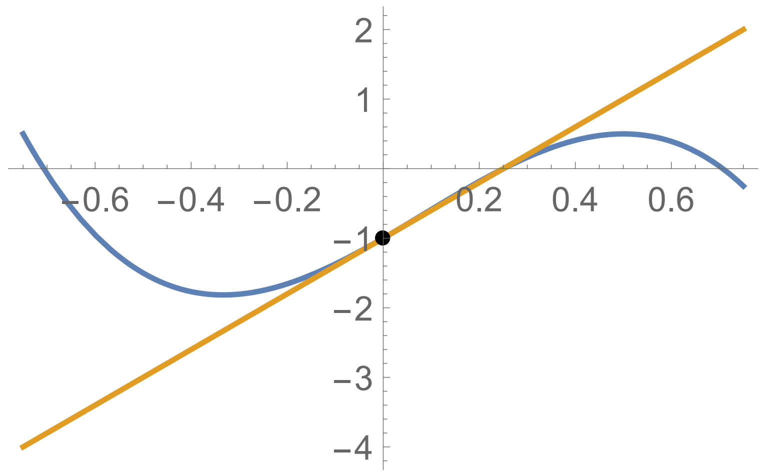
$$\begin{aligned} y'(x) &= \frac{d}{dx}\left(2x^2-1\right)(1-4x), \\ &= \left(2x^2-1\right)\frac{d}{dx}(1-4x)+(1-4x)\frac{d}{dx}\left(2x^2-1\right), \\ &= \left(2x^2-1\right)(-4)+(1-4x)(4x), \\ &= -8x^2+4+4x-16x^2, \\ &= -24x^2+4x+4, \end{aligned}$$

Since the slope of the tangent $y = 4x-1$ is equal to 4, we seek x such that $y'(x) = 4$. That is,

$$-24x^2+4x+4 = 4.$$

Simplifying, $-4x(6x-1) = 0$ and we see that we must have $x = \frac{1}{6}$ or $x = 0$. If $x = \frac{1}{6}$, then $y\left(\frac{1}{6}\right) = \left(2\left(\frac{1}{6}\right)^2-1\right)\left(1-4\left(\frac{1}{6}\right)\right) = -\frac{17}{54}$. However, the point $\left(\frac{1}{6}, -\frac{17}{54}\right)$ is not on the tangent line $y = 4x-1$ so we cannot have $x = \frac{1}{6}$. If $x = 0$, then $y(0) = (0-1)(1-0) = -1$. The point

$(0,-1)$ is on both the curve $y=\left(2x^2-1\right)(1-4x)$ and the tangent line $y=4x-1$. Hence $(0,-1)$ is the required point.



(13) Use the quotient rule to calculate y', where $y=\frac{x}{2x+3}$.

Sol. : We let $u=x$ and $v=2x+3$ so that $y=\frac{u}{v}$ and $y'=\frac{1}{v^2}\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)$.

$$
\begin{aligned}
y'(x) &= \frac{d}{dx}\frac{x}{2x+3}, \\
&= \frac{1}{(2x+3)^2}\left((2x+3)\frac{d}{dx}x-x\frac{d}{dx}(2x+3)\right), \\
&= \frac{1}{(2x+3)^2}(2x+3-2x), \\
&= \frac{3}{(2x+3)^2}.
\end{aligned}
$$

(14) Use the quotient rule to calculate y', where $y = \frac{e^2}{3x^2-5x}$.

Sol. : We let $u = e^2$ and $v = 3x^2 - 5x$ so that $y = \frac{u}{v}$ and $y' = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)$.

$$\begin{aligned}
y'(x) &= \frac{d}{dx}\frac{e^2}{3x^2-5x}, \\
&= \frac{1}{(3x^2-5x)^2}\left(\left(3x^2-5x\right)\frac{d}{dx}e^2 - e^2\frac{d}{dx}\left(3x^2-5x\right)\right), \\
&= \frac{1}{(3x^2-5x)^2}\left(\left(3x^2-5x\right)0 - e^2\left(6x-5\right)\right), \\
&= \frac{-e^2\left(6x-5\right)}{(3x^2-5x)^2}.
\end{aligned}$$

(15) Use the quotient rule to calculate y', where $y = \frac{33x}{4x^5-3x-4}$.

Sol. : We let $u = 33x$ and $v = 4x^5 - 3x - 4$ so that $y = \frac{u}{v}$ and $y' = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)$.

$$\begin{aligned}
y'(x) &= \frac{d}{dx}\frac{33x}{4x^5-3x-4}, \\
&= \frac{1}{(4x^5-3x-4)^2}\left(\left(4x^5-3x-4\right)\frac{d}{dx}(33x) - (33x)\frac{d}{dx}\left(4x^5-3x-4\right)\right), \\
&= \frac{1}{(4x^5-3x-4)^2}\left(33\left(4x^5-3x-4\right) - (33x)\left(20x^4-3\right)\right), \\
&= \frac{-528x^5-132}{(4x^5-3x-4)^2}.
\end{aligned}$$

(16) Use the quotient rule to calculate y', where $y = \frac{3x^3-8x}{2x^2-5x+4}$.

Sol. : We let $u = 3x^3 - 8x$ and $v = 2x^2 - 5x + 4$ so that $y = \frac{u}{v}$ and $y' = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)$.

$$\begin{aligned}
y'(x) &= \frac{d}{dx}\frac{3x^3-8x}{2x^2-5x+4}, \\
&= \frac{1}{(2x^2-5x+4)^2}\left((2x^2-5x+4)\frac{d}{dx}(3x^3-8x)-(3x^3-8x)\frac{d}{dx}(2x^2-5x+4)\right), \\
&= \frac{1}{(2x^2-5x+4)^2}\left((2x^2-5x+4)(9x^2-8)-(3x^3-8x)(4x-5)\right), \\
&= \frac{(6x^4-30x^3+52x^2-32)}{(2x^2-5x+4)^2}.
\end{aligned}$$

(17) Use the chain rule to calculate y', where $y=(1-5x)^{12}$.

Sol. : Let $u=1-5x$. Then

$$\begin{aligned}
y'(x) &= \frac{dy}{dx}, \\
&= \frac{dy}{du}\frac{du}{dx}, \\
&= \frac{du^{12}}{du}\frac{d}{dx}(1-5x), \\
&= 12u^{11}(-5), \\
&= 60(1-5x)^{11}.
\end{aligned}$$

(18) Use the chain rule to calculate y', where $y=8(1-6x)^{1.5}$.

Sol. : Let $u=1-6x$. Then

$$\begin{aligned}
y'(x) &= \frac{d}{dx}8(1-6x)^{1.5}, \\
&= 8\frac{du^{1.5}}{du}\frac{du}{dx}, \\
&= 8(1.5)u^{\frac{1}{2}}(-6), \\
&= -72\sqrt{1-6x}.
\end{aligned}$$

(19) Use the chain rule to calculate y', where $y = \sqrt[4]{1-8x^2}$.

Sol. : We can write $y = \left(1-8x^2\right)^{1/4}$ and let $u = 1-8x^2$ so that $y = u^{1/4}$. Then

$$\begin{aligned} y'(x) &= \frac{dy}{dx}, \\ &= \frac{dy}{du}\frac{du}{dx}, \\ &= \frac{1}{4}u^{-\frac{3}{4}}(-16x), \\ &= -4x\left(1-8x^2\right)^{-\frac{3}{4}}. \end{aligned}$$

(20) Calculate $f'(x)$, where $f(x) = (g(x))^n$, and $g(x)$ is a function of x.

Sol. :

$$\begin{aligned} f'(x) &= \frac{d}{dg}(g(x))^n g'(x), \\ &= n(g(x))^{n-1} g'(x). \end{aligned}$$

(21) Calculate $y'(x)$, where $y = \frac{\pi^3}{\sqrt{1-3x}}$.

Sol. :

$$
\begin{aligned}
y'(x) &= \pi^3 \frac{d}{dx}(1-3x)^{\frac{1}{2}}, \\
&= \pi^3 \frac{d}{d(1-3x)}(1-3x)^{\frac{1}{2}} \frac{d}{dx}(1-3x), \\
&= \pi^3 \frac{1}{2}(1-3x)^{-\frac{1}{2}}(-3), \\
&= \frac{-3\pi^3}{2}(1-3x)^{-\frac{1}{2}}, \\
&= \frac{-3\pi^3}{2\sqrt{1-3x}}.
\end{aligned}
$$

(22) Calculate $y'(x)$, where $y = 9\sqrt[3]{4x^6+2}$.

Sol. :

$$
\begin{aligned}
y'(x) &= 9\frac{d}{dx}\left(4x^6+2\right)^{\frac{1}{3}}, \\
&= 9\frac{1}{3}\left(4x^6+2\right)^{-\frac{2}{3}} \frac{d}{dx}\left(4x^6+2\right), \\
&= 3\left(4x^6+2\right)^{-\frac{2}{3}}\left(24x^5\right), \\
&= 72x^5\left(4x^6+2\right)^{-\frac{2}{3}}.
\end{aligned}
$$

(23) Calculate $y'(x)$, where $y = x^7(1-3x)^{15}$.

Sol. : We require both the product rule and the chain rule.

$$\begin{aligned}
y'(x) &= x^7 \frac{d}{dx}(1-3x)^{15} + (1-3x)^{15}\frac{d}{dx}x^7, \\
&= x^7 \frac{d}{d(1-3x)}(1-3x)^{15}\frac{d}{dx}(1-3x) + (1-3x)^{15}7x^6, \\
&= x^7 15(1-3x)^{14}(-3) + (1-3x)^{15}7x^6, \\
&= -45x^7(1-3x)^{14} + 7x^6(1-3x)^{15}, \\
&= x^6(1-3x)^{14}\left(-45x + 7(1-3x)\right), \\
&= x^6(7-66x)(1-3x)^{14},
\end{aligned}$$

(24) Calculate $R'(T)$, where $R = \frac{2T^2}{\sqrt{1+4T}}$.

Sol. : We require both the quotient rule and the chain rule.

$$\begin{aligned}
R'(T) &= 2\frac{d}{dT}\frac{T^2}{(1+4T)^{\frac{1}{2}}}, \\
&= 2\frac{1}{\left((1+4T)^{\frac{1}{2}}\right)^2}\left((1+4T)^{\frac{1}{2}}\frac{d}{dT}T^2 - T^2\frac{d}{dT}(1+4T)^{\frac{1}{2}}\right), \\
&= \frac{2}{1+4T}\left((1+4T)^{\frac{1}{2}}2T - T^2\frac{d}{d(1+4T)}(1+4T)^{\frac{1}{2}}\frac{d}{dT}(1+4T)\right), \\
&= \frac{2}{1+4T}\left(2T(1+4T)^{\frac{1}{2}} - \frac{1}{2}T^2(1+4T)^{-\frac{1}{2}}(4)\right), \\
&= \frac{2}{1+4T}\left(2T\sqrt{1+4T} - 2\frac{T^2}{\sqrt{1+4T}}\right), \\
&= \frac{4T}{1+4T}\left(\frac{1+4T-T}{\sqrt{1+4T}}\right), \\
&= \frac{4T(1+3T)}{(1+4T)\sqrt{1+4T}}.
\end{aligned}$$

4 Tutorial 4

The following exercises are found in Washington [2].

(1) Solve the right triangle with $A = 78.7°$, $a = 7600$, where c is the hypotenuse.

(2) Solve the right triangle with $B = 32.1°$, $c = 23.8$, where c is the hypotenuse.

(3) Solve the right triangle with $a = 9.908$, $c = 12.63$, where c is the hypotenuse.

(4) Plot points of $y = \sin(x)$ by evaluating the points $(x, \sin(x))$ for several $x\frac{\pi}{4}k$, $k \in \mathbb{Z}$.

(5) Plot points of $y = \cos(x)$ by evaluating the points $(x, \cos(x))$ for several $x\frac{\pi}{4}k$, $k \in \mathbb{Z}$.

(6) Plot points of $y = 3\cos(x)$ by evaluating the points $(x, 3\cos(x))$ for several $x\frac{\pi}{4}k$, $k \in \mathbb{Z}$.

(7) Plot points of $y = -4\sin(x)$ by evaluating the points $(x, -4\sin(x))$ for several $x\frac{\pi}{4}k$, $k \in \mathbb{Z}$.

(8) Prove that $\frac{\sin(x)}{\tan(x)} = \cos(x)$.

(9) Prove that $\sin(x)\sec(x) = \tan(x)$.

(10) Prove that $\cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$.

(11) Express the following angles in radians: $15°$, $120°$.

(12) Express the following angles in radians: $210°$, $99°$.

(13) Express the following angles in degrees: $\frac{3\pi}{5}$, $\frac{3\pi}{2}$.

(14) Express the following angles in degrees: $\frac{3\pi}{10}$, $\frac{11\pi}{6}$.

(15) Express the following angles in radians, and round to the number of significant figures given: $84.0°$.

(16) Express the following angles in radians, and round to the number of significant figures given: $478.5°$.

(17) Find the derivative of $y = 3\sin(7x)$.

(18) Find the derivative of $y = 5\sin(7-3t)$.

(19) Find the derivative of $y = \cos(1-x)$.

(20) Find the derivative of $y = 4\cos\left(6x^2+5\right)$.

(21) Find the derivative of $y = 3\sin^3\left(2x^4+1\right)$.

(22) Find the derivative of $y = \cos^2\left(\sqrt{x}\right)$.

(23) Find the derivative of $y = 6\sin(x)\cos(4x)$.

(24) Find the derivative of $y = \left(x-\cos^2(x)\right)^4$.

(24) Find the derivative of $T = \frac{1-3z}{\sin(\pi z)}$.

(25) Find the derivative of $y = \cos^3(4x)\sin^2(2x)$.

(26) Find the derivative of $y = 5x\sin(5x)+\cos(5x)$.

(27) Show that $\frac{d^4 \sin(x)}{dx^4} = \sin(x)$.

(28) Find the derivative of $y = \tan(x)$.

(29) Differentiate with respect to x : $2xy^3$.

(30) Differentiate with respect to x : $\frac{y^2}{x+1}$.

(31) Differentiate implicitly to calculate $\frac{dy}{dx}$: $14x-7y = 112$.

(32) Differentiate implicitly to calculate $\frac{dy}{dx}$: $x^2+2y^2-11 = 0$.

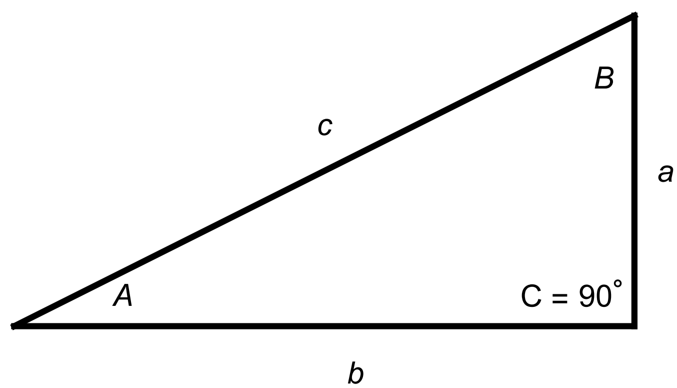
(33) Differentiate implicitly to calculate $\frac{dy}{dx}$: $x^{2/3}+y^{2/3} = 5$.

(34) Find the derivative of the implicit function
$x\cos(2y)+\sin(x)\cos(y) = 1$.

4.1 Tutorial 4 Solutions

(1) Solve the right triangle with $A = 78.7^\circ$, $a = 7600$, where c is the hypotenuse.

Sol. : We have $A + B + C = 180^\circ$ so $B = 180^\circ - 90^\circ - 78.7^\circ = 11.3^\circ$. Also, $\sin(A) = \frac{a}{c}$ so $c = \frac{a}{\sin(A)} = \frac{7600}{0.980615} = 7750.24$. Since $a^2 + b^2 = c^2$, we have $b = \sqrt{c^2 - a^2} = \sqrt{7750.24^2 - 7600^2} = 1518.62$.



5 Tutorial 5

The following exercises are found in Washington [2].

(1) Solve for $x: 2^x = 16$.

(2) Solve for $x: 3^x = \frac{1}{81}$.

(3) Solve for $x: \pi^{2x} = 20$.

(4) Solve for $x: e^{-3x} = 100$.

(5) Solve for $x: 6\log_{30}(x) = -3$.

(6) Solve for $x: x^{\log_{10}(x)} = 1000x^2$.

(7) Solve for $x: \log_2(x) + \log_2(7) = \log_2(21)$.

(8) Solve for $x: 3\log_{10}(2x-1) = 1$.

(9) Solve for $x: \log_{10}\left(12x^2\right) - \log_{10}(3x) = 3$.

(10) Solve for $x: \log_e(4x-1) - 3\log_e(9) = 2\log_e(3)$.

(11) Differentiate $y = 4^{6x}$.

(12) Differentiate $y = 10^{x^6}$.

(13) Differentiate $y = 0.6\log_e\left(e^{5x} + 3\right)$.

(14) Differentiate $y = 5x^2e^{2x}$.

(15) Differentiate $p = \left(3e^{2n} + e^2\right)^8$.

(16) Differentiate $y = \log_{10}\left(x^4\right)$.

(17) Differentiate $y = \log_2(9x)$.

(18) Differentiate $y = \log_7\left(x^2 + 1\right)$.

(19) Differentiate $y = 2\log_e\left(3x^2 - 1\right)$.

(20) Differentiate $s = \log_e \left(\sin^2(t)\right)$.

(21) Differentiate $y = \log_e (4x-3)^3$.

(22) Differentiate $y = 6x^2 \log_e(5x)$.

(23) Differentiate $y = \frac{8\log_e(x)}{x}$.

(24) Differentiate $y = \log_e \left(\frac{2x}{1+x}\right)$.

(25) Differentiate $y = \sqrt{x + \log_e(3x)}$.

5.1 Tutorial 5 Solutions

(1) .

Sol. : .

6 Tutorial 6

The following exercises are found in Washington [2].

(1) Let $y = 2 + 6x - 3x^2$. Find all x such that $y(x)$ is increasing. Find all x such that $y(x)$ is decreasing.

(2) Let $y = 2 + 6x - 3x^2$. Find any local maxima or minima.

(3) Let $y = 2 + 6x - 3x^2$. Find all values for which $y(x)$ is concave up, concave down.

(4) Sketch $y = 2 + 6x - 3x^2$.

(5) Let $y = x^4 - 6x^2$. Find all x such that $y(x)$ is increasing. Find all x such that $y(x)$ is decreasing.

(6) Let $y = x^4 - 6x^2$. Find any local maxima or minima.

(7) Let $y = x^4 - 6x^2$. Find all values for which $y(x)$ is concave up, concave down.

(8) Sketch $y = x^4 - 6x^2$.

(9) Sketch $y = 4x^2 - 16x - 20$.

(10) Sketch $y = x^3 - 9x^2 + 15x + 1$.

(11) Sketch $y = x^5 - 20x^2$.

(12) Sketch a continuous curve that satisfies $f(1) = 0$, $f'(x) > 0$ for all x, $f''(x) < 0$ for all x.

(13) Sketch a continuous curve such that $f(0) = 2$, $f'(x) > 0$, $f''(x) < 0$ for all x, and $f(x) \longrightarrow 4$ as $x \longrightarrow \infty$.

(14) A rectangular corral is to be enclosed with 2400 m of fencing. Find the maximum possible area of the corral.

(15) A small oil refinery estimates that its daily profit P (in dollars) from refining x barrels of oil is $P = 8x - 0.02x^2$. How many barrels should be refined for maximum daily profit, and what is the maximum profit?

(16) The sum of the length l and width w of a rectangular table top is to be 280 cm. Determine l and w if the area of the table top is to be a maximum.

(17) The rectangular animal display area in a zoo is enclosed by chain-link fencing and divided into two areas by internal fencing parallel to one of the sides. What dimensions will give the maximum area for the display if a total of 120 m of fencing are used?

(18) What is the minimum slope of the curve $y = x^5 - 10x^2$?

6.1 Tutorial 6 Solutions

(1) .

Sol. : .

7 Tutorial 7

The following exercises are found in Washington [2].

(1) Calculate $\int 2x\,dx$.

(2) Calculate $\int 5x^4\,dx$.

(3) Calculate $\int 0.6y^5\,dy$.

(4) Calculate $\int \frac{4}{\sqrt{x}}\,dx$.

(5) Calculate $\int (1-3x)\,dx$.

(6) Calculate $\int x(x-2)^2\,dx$.

(7) Calculate $\int \left(x^{1/3}+x^{1/5}+x^{-1/7}\right)\,dx$.

(8) Find y in terms of x, where $\frac{dy}{dx}=8x+1$, curve passes through $(-1,4)$.

(9) Is $\int 3x^2\,dx = x^3$? Explain.

(10) Find the approximate area under the curve of the equation $y=2x+1$ by dividing the indicated intervals into n subintervals and then adding up the areas of the inscribed rectangles. There are two values of n for each exercise and therefore two approximations for each area. The height of each rectangle may be found by evaluating the function for the proper value of x. Interval: between $x=0$ and $x=2$ for:

(a) $n=4$, $\Delta x=0.5$.

(b) $n=10$, $\Delta x=0.2$.

(11) Find the exact area under the curve $y=2x+1$ between $x=0$ and $x=3$.

(12) Find the approximate area under the curve of the equation $y=9-x^2$ by dividing the indicated intervals into n subintervals and then adding up the areas of the inscribed rectangles. There are two values of n

for each exercise and therefore two approximations for each area. The height of each rectangle may be found by evaluating the function for the proper value of x. Interval: between $x = 2$ and $x = 3$ for:

(a) $n = 5$, $\Delta x = 0.2$.

(b) $n = 10$, $\Delta x = 0.1$.

(13) Find the exact area under the curve $y = 9 - x^2$ between $x = 2$ and $x = 3$.

(14) Calculate the definite integral $\int_0^3 6x\,dx$.

(15) Calculate the definite integral $\int_0^2 4x^3\,dx$.

(16) Calculate the definite integral $\int_4^9 \left(p^{3/2} - 3\right) dp$.

(17) Calculate the definite integral $\int_{1.2}^{1.6} \left(5 + \frac{6}{x^4}\right) dx$.

(18) Calculate the definite integral $\int_1^2 \left(3x^5 - 2x^3\right) dx$.

(19) Approximate the value of the integral $\int_0^1 \left(1 - x^2\right) dx$ by use of the trapezoidal $\int_a^b f(x)\,dx \approx (\Delta x) \sum \frac{1}{2} \left(f(x_j) + f(x_{j+1})\right)$ rule with $n = 3$, and then check by direct integration.

(20) Approximate the value of the integral $\int_3^8 \sqrt{1+x}\,dx$ by use of the trapezoidal rule with $n = 5$, and then check by direct integration.

(21) Approximate the value of the integral $\int_0^8 x^{1/3}\,dx$ by use of Simpson's rule,

$$\int_a^b f(x)\,dx = \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right),$$

with $n = 2$, and then check by direct integration. Round to three significant digits.

7.1 Tutorial 7 Solutions

(1) Calculate $\int 2x\,dx$.

Sol. $\int 2x\,dx = x^2 + C$.

(2) Calculate $\int 5x^4\,dx$.

Sol. $\int 5x^4\,dx = x^5 + C$.

(3) Calculate $\int 0.6y^5\,dy$.

Sol. $\int 0.6y^5\,dy = 0.1y^6 + C$.

(4) Calculate $\int \frac{4}{\sqrt{x}}\,dx$.

Sol. $\int \frac{4}{\sqrt{x}}\,dx = \int 4x^{-1/2}\,dx = \frac{4}{1-1/2}x^{1-1/2} + C = 8x^{1/2} + C$.

(5) Calculate $\int (1-3x)\,dx$.

Sol. $\int (1-3x)\,dx = x - \frac{3}{2}x^2 + C$.

(6) Calculate $\int x(x-2)^2\,dx$.

Sol. $\int x(x-2)^2\,dx = \int x^3 - 4x^2 + 4x\,dx = \frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2 + C$.

(7) Calculate $\int \left(x^{1/3} + x^{1/5} + x^{-1/7}\right)\,dx$.

Sol. $\int \left(x^{1/3}+x^{1/5}+x^{-1/7}\right)dx = \frac{3}{4}x^{4/3}+\frac{5}{6}x^{6/5}+\frac{7}{6}x^{6/7}+C.$

(8) Find y in terms of x, where $\frac{dy}{dx}=8x+1$, curve passes through $(-1,4)$.

Sol. $y=\int \frac{dy}{dx}dx=\int 8x+1\,dx=4x^2+x+C$. Since the curve $y=4x^2+x+C$ passes through $(-1,4)$, we have $4=4(-1)^2+(-1)+C$ so that $C=1$ and we get $y=4x^2+x+1$.

(9) Is $\int 3x^2\,dx=x^3$? Explain.

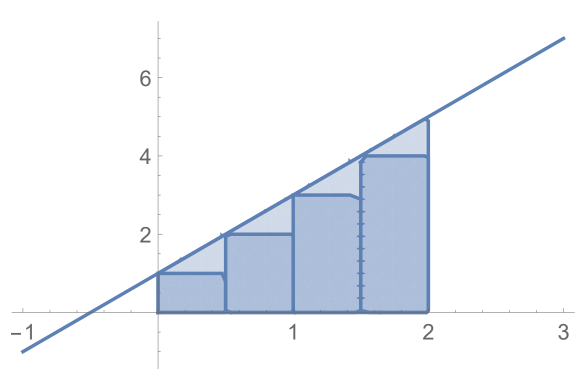
Sol. No, this is only one of infinitely many solutions. We have $\int 3x^2\,dx=x^3+C$ in general, where C is a constant.

(10) Find the approximate area under the curve of the equation $y=2x+1$ by dividing the indicated intervals into n subintervals and then adding up the areas of the inscribed rectangles. There are two values of n for each exercise and therefore two approximations for each area. The height of each rectangle may be found by evaluating the function for the proper value of x. Interval: between $x=0$ and $x=2$ for:

(a) $n=4$, $\Delta x=0.5$.

(b) $n=10$, $\Delta x=0.2$.

Sol. (a)



With $n=4$ and inscribed rectangles of width 0.5, the area A under the curve is approximated by

$$\begin{aligned} A &\approx (2\times 0+1)\times 0.5+(2\times 0.5+1)\times 0.5+(2\times 1+1)\times 0.5+(2\times 1.5+1)\times 0.5, \\ &= (1+2+3+4)\times 0.5, \\ &= 5. \end{aligned}$$

With $n=10$ and inscribed rectangles of width 0.2, the area A under the curve is approximated by

$$\begin{aligned} A \approx\ & (2\times 0+1)\times 0.2+(2\times 0.2+1)\times 0.2+(2\times 0.2\times 2+1)\times 0.2 \\ & (2\times 0.2\times 3+1)\times 0.2+(2\times 0.2\times 4+1)\times 0.2+(2\times 0.2\times 5+1)\times 0.2 \\ & +(2\times 0.2\times 6+1)\times 0.2+(2\times 0.2\times 7+1)\times 0.2+(2\times 0.2\times 8+1)\times 0.2 \\ & +(2\times 0.2\times 9+1)\times 0.2 \\ =\ & (1.0+1.4+1.8+2.2+2.6+3.0+3.4+3.8+4.2+4.6)\times 0.2, \\ =\ & 5.6. \end{aligned}$$

The actual area is given by the definite integral:

$$A=\int_0^2 2x+1\,dx=\left[x^2+x\right]_0^2=2^2+2-\left(0^2+0\right)=6.$$

(11) Find the exact area under the curve $y=2x+1$ between $x=0$ and $x=3$.

Sol.

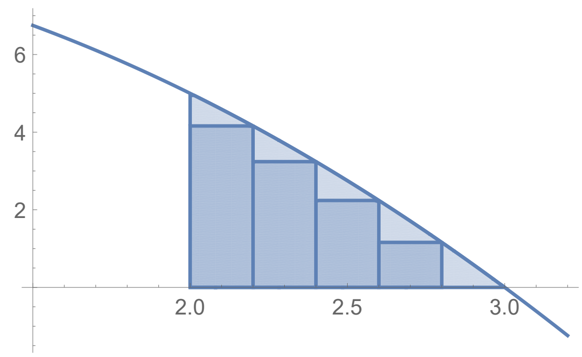
$$\int_0^3 2x+1\,dx=\left[x^2+x\right]_0^3=3^2+3-\left(0^2+0\right)=12.$$

(12) Find the approximate area under the curve of the equation $y=9-x^2$ by dividing the indicated intervals into n subintervals and then adding up the areas of the inscribed rectangles. There are two values of n for each exercise and therefore two approximations for each area. The height of each rectangle may be found by evaluating the function for the proper value of x. Interval: between $x=2$ and $x=3$ for:

(a) $n=5$, $\Delta x=0.2$.

(b) $n=10$, $\Delta x=0.1$.

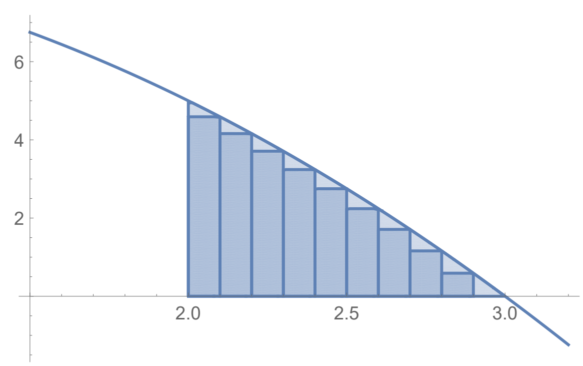
Sol. (a)



With $n = 5$, $\Delta x = 0.2$,

$$\int_2^3 9 - x^2\,dx \approx \sum_{j=1}^{5} \left(9 - (2 + 0.2 \times j)^2\right) \times 0.2,$$
$$\approx 2.16$$

(b)



With $n = 10$, $\Delta x = 0.1$,

$$\int_2^3 9 - x^2\,dx \approx \sum_{j=1}^{10} \left(9 - (2 + 0.1 \times j)^2\right) \times 0.1,$$
$$\approx 2.415$$

(13) Find the exact area under the curve $y = 9 - x^2$ between $x = 2$ and $x = 3$.

$$\begin{aligned}\int_2^3 9 - x^2\,dx &= \left[9x - \frac{1}{3}x^3\right]_2^3, \\ &= 9(3) - \frac{1}{3}(3)^3 - \left(9(2) - \frac{1}{3}(2)^3\right), \\ &= 2.6666\ldots.\end{aligned}$$

(14) Calculate the definite integral $\int_0^3 6x\,dx$.

Sol.

$$\int_0^3 6x\,dx = \left[3x^2\right]_0^3 = 3(3)^2 - 3(0)^2 = 27.$$

(15) Calculate the definite integral $\int_0^2 4x^3\,dx$.

Sol.

$$\int_0^2 4x^3\,dx = \left[x^4\right]_0^2 = (2)^4 - (0)^4 = 16.$$

(16) Calculate the definite integral $\int_4^9 \left(p^{3/2} - 3\right)\,dp$.

Sol.

$$\begin{aligned}\int_4^9 \left(p^{3/2} - 3\right)\,dp &= \left[\frac{2}{5}p^{5/2} - 3p\right]_4^9, \\ &= \frac{2}{5}(9)^{5/2} - 3(9) - \left(\frac{2}{5}(4)^{5/2} - 3(4)\right), \\ &= \frac{2}{5}3^5 - 27 - \frac{2}{5}2^5 + 12 = \frac{347}{5}.\end{aligned}$$

(17) Calculate the definite integral $\int_{1.2}^{1.6}\left(5+\frac{6}{x^4}\right)\,dx$.

Sol.

$$\begin{aligned}\int_{1.2}^{1.6}\left(5+\frac{6}{x^4}\right)\,dx &= \left[5x-2x^{-3}\right]_{1.2}^{1.6},\\ &= 5(1.6)-2(1.6)^{-3}-\left(5(1.2)-2(1.2)^{-3}\right),\\ &= 2.66913\ldots.\end{aligned}$$

(18) Calculate the definite integral $\int_{1}^{2}\left(3x^5-2x^3\right)\,dx$.

Sol.

$$\begin{aligned}\int_{1}^{2}\left(3x^5-2x^3\right)\,dx &= \left[\frac{1}{2}x^6-\frac{1}{2}x^4\right]_1^2,\\ &= \frac{1}{2}(2)^6-\frac{1}{2}(2)^4-\left(\frac{1}{2}(2)^6-\frac{1}{2}(2)^4\right),\\ &= 24.\end{aligned}$$

(19) Approximate the value of the integral $\int_0^1 \left(1-x^2\right)\,dx$ by use of the trapezoidal

$$\int_a^b f(x)\,dx \approx (\Delta x)\sum \frac{1}{2}\left(f(x_j)+f(x_{j+1})\right)$$

rule with $n=3$, and then check by direct integration.

Sol. Let $f(x)=1-x^2$. With $n=3$, we have $\Delta x = \frac{1}{3}(1-0)=\frac{1}{3}$ and $x_0=0$, $x_1=\frac{1}{3}$, $x_2=\frac{2}{3}$, $x_3=1$.

$$\begin{aligned}
\int_0^1 \left(1-x^2\right) &\approx \frac{1}{3}\sum_{j=0}^{j=2}\frac{1}{2}\left(f(x_j)+f(x_{j+1})\right),\\
&\approx \frac{1}{3}\sum_{j=0}^{j=2}\frac{1}{2}\left(2-x_j^2-x_{j+1}^2\right),\\
&\approx \frac{1}{6}\left(2-x_0^2-x_1^2+2-x_1^2-x_2^2+2-x_2^2-x_3^2\right),\\
&\approx \frac{1}{6}\left(6-x_0^2-2x_1^2-2x_2^2-x_3^2\right),\\
&\approx \frac{1}{6}\left(6-0^2-\frac{2}{9}-\frac{8}{9}-1\right),\\
&\approx \frac{35}{54},\\
&\approx 0.648148\ldots.
\end{aligned}$$

The actual definite integral is

$$\begin{aligned}
\int_0^1 \left(1-x^2\right) &= \left[x-\frac{1}{3}x^3\right]_0^1,\\
&= 1-\frac{1}{3}(1)^3-\left((0)-\frac{1}{3}(0)^3\right),\\
&= \frac{2}{3}.
\end{aligned}$$

(20) Approximate the value of the integral $\int_3^8 \sqrt{1+x}\,dx$ by use of the trapezoidal rule with $n = 5$, and then check by direct integration.

Sol. Let $f(x) = \sqrt{1+x}$. With $n = 5$, we have $\Delta x = \frac{1}{5}(8-3) = 1$ and $x_0 = 3$, $x_1 = 4$, $x_2 = 5$, $x_3 = 6$, $x_4 = 7$, $x_5 = 8$. The trapezoidal rule gives

$$\begin{aligned}
\int_3^8 \sqrt{1+x}\,dx &\approx 1\sum_{j=0}^{j=4} \frac{1}{2}\left(f(x_j) + f(x_{j+1})\right), \\
&\approx \frac{1}{2}\left(f(x_0) + f(x_5)\right) + \sum_{j=1}^{4} f(x_j), \\
&\approx \frac{1}{2}\left(f(3) + f(8)\right) + f(4) + f(5) + f(6) + f(7), \\
&\approx \frac{1}{2}\left(\sqrt{4} + \sqrt{9}\right) + \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{8}, \\
&\approx \frac{5}{2} + \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{8}, \\
&\approx 12.6597\ldots.
\end{aligned}$$

Direct integration gives

$$\begin{aligned}
\int_3^8 \sqrt{1+x}\,dx &= \left[\frac{2}{3}(x+1)^{3/2}\right]_3^8, \\
&= \frac{2}{3}(8+1)^{3/2} - \frac{2}{3}(3+1)^{3/2}, \\
&= 38/3, \\
&= 12.6666\ldots.
\end{aligned}$$

(21) Approximate the value of the integral $\int_0^8 x^{1/3}\,dx$ by use of Simpson's rule,

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right),$$

with $n=2$, and then check by direct integration. Round to three significant digits.

Sol. Let $f(x)=x^{1/3}$. With $n=2$, Simpson's rule gives:

$$\begin{aligned}
\int_0^8 x^{1/3}\,dx &\approx \frac{2}{3}(f(0)+4f(2)+f(4))+\frac{2}{3}(f(4)+4f(6)+f(8)),\\
&\approx \frac{2}{3}(f(0)+4f(2)+2f(4)+4f(6)+f(8)),\\
&\approx \frac{2}{3}\left(0+4\cdot 2^{1/3}+2\cdot 4^{1/3}+4\cdot 6^{1/3}+2\right),\\
&\approx 11.6553\ldots,\\
&\approx 11.7.
\end{aligned}$$

Direct integration gives

$$\int_0^8 x^{1/3}\,dx = \left[\frac{3}{4}x^{4/3}\right]_0^8 = \frac{3}{4}8^{4/3}-0 = 12.$$

8 Tutorial 8

The following exercises are found in Washington [2].

(1) Calculate $\int 3x^2\left(x^3-2\right)^6 dx$.

(2) Calculate $\int 8\sin^{1/3}(x)\ \cos(x)\, dx$.

(3) Calculate $\int \frac{1}{4-9x}\, dx$.

(4) Calculate $\int_{-1}^{3} \frac{8x^3}{x^4+1}\, dx$

(5) Calculate $\int \frac{3v^2-2v}{v^2}\, dv$.

(6) Calculate $\int xe^{-x^2}\, dx$.

(7) Calculate $\int_1^3 3e^{2x}\left(e^{-2x}-1\right)\, dx$.

(8) Calculate $\int 4\sin(2-x)\, dx$.

(9) Calculate $\int_{0.5}^{1} x^2\cot\left(x^3\right)\, dx$.

(10) Calculate $\int \sin(x)\cos^5(x)\, dx$.

(11) Calculate $\int x\sin(2x)\, dx$.

(12) Calculate $\int 3xe^x\, dx$.

(13) Calculate $\int \log_e(s)\, ds$.

(14) Calculate $\int x\sqrt{x+1}\, dx$.

(15) Calculate $\int \cos\left(\log_e(x)\right)\, dx$.

8.1 Tutorial 8 Solutions

(1) Calculate $\int 3x^2\left(x^3-2\right)^6\,dx$.

Sol. Let $u = x^3$. Then $\frac{du}{dx} = 3x^2$ so that $dx \longrightarrow \frac{du}{3x^2}$ and

$$\begin{aligned}\int 3x^2\left(x^3-2\right)^6 dx &= \int 3x^2 (u-2)^6 \frac{du}{3x^2},\\ &= \int (u-2)^6\,du,\\ &= \frac{1}{7}(u-2)^7 + C,\\ &= \frac{1}{7}\left(x^3-2\right)^7 + C.\end{aligned}$$

(2) Calculate $\int 8\sin^{1/3}(x)\ \cos(x)\,dx$.

Sol. Let $u = \sin(x)$. Then $\frac{du}{dx} = \cos(x)$ so that $dx \longrightarrow \frac{du}{\cos(x)}$ and

$$\begin{aligned}\int 8\sin^{1/3}(x)\ \cos(x)\,dx &= \int 8u^{1/3}\ \cos(x)\frac{du}{\cos(x)},\\ &= \int 8u^{1/3}\,du,\\ &= \frac{8}{4/3}u^{4/3} + C,\\ &= 6\sin^{4/3}(x) + C.\end{aligned}$$

(3) Calculate $\int \frac{1}{4-9x}\,dx$.

Sol. Let $u = 4-9x$. Then $\frac{du}{dx} = -9$ so that $dx \longrightarrow \frac{du}{-9}$ and

$$\begin{aligned} \int \frac{1}{4-9x}\,dx &= \int u^{-1}\,\frac{du}{-9}, \\ &= -\frac{1}{9}\log_e(u) + C, \\ &= -\frac{1}{9}\log_e(4-9x) + C. \end{aligned}$$

(4) Calculate $\int_{-1}^{3} \frac{8x^3}{x^4+1}\,dx$

Sol. Let $u = x^4+1$. Then $\frac{du}{dx} = 4x^3$ so that $dx \longrightarrow \frac{du}{4x^3}$ and

$$\begin{aligned} \int_{-1}^{3} \frac{8x^3}{x^4+1}\,dx &= \int_{2}^{81} \frac{8x^3}{u}\,\frac{du}{4x^3}, \\ &= 2\int_{2}^{81} u^{-1}\,du, \\ &= [\log_e(u)]_2^{82}, \\ &= \log_e(82/2), \\ &= \log_e(41). \end{aligned}$$

(5) Calculate $\int \frac{3v^2-2v}{v^2}\,dx$.

Sol.

$$\begin{aligned} \int \frac{3v^2-2v}{v^2}\,dv &= \int 3-2v^{-1}\,dv, \\ &= 3v - 2\log_e(v) + C. \end{aligned}$$

(6) Calculate $\int xe^{-x^2}\,dx$.

Sol. Let $u=-x^2$. Then $\frac{du}{dx}=-2x$ so that $dx\longrightarrow\frac{du}{-2x}$ and

$$\begin{aligned}\int xe^{-x^2}\,dx &= \int xe^{u}\,\frac{du}{-2x},\\ &= -\frac{1}{2}\int e^{u}\,du,\\ &= -\frac{1}{2}e^{u}+C,\\ &= -\frac{1}{2}e^{-x^2}+C.\end{aligned}$$

(7) Calculate $\int_1^3 3e^{2x}\left(e^{-2x}-1\right)\,dx$.

Sol.

$$\begin{aligned}\int_1^3 3e^{2x}\left(e^{-2x}-1\right)\,dx &= \int_1^3 3-3e^{2x}\,dx,\\ &= \left[3x-\frac{3}{2}e^{2x}\right]_1^3,\\ &= 3\times 3-\frac{3}{2}e^{2\times 3}-\left(3\times 1-\frac{3}{2}e^{2\times 1}\right),\\ &= 6-\frac{3}{2}e^{6}+\frac{3}{2}e^{2}.\end{aligned}$$

(8) Calculate $\int 4\sin(2-x)\,dx$.

Sol. Let $u = 2-x$. Then $\frac{du}{dx} = -1$ so that $dx \longrightarrow -du$ and

$$\begin{aligned}\int 4\sin(2-x)\,dx &= -4\int \sin(u)\,du,\\ &= 4\cos(u)+C,\\ &= 4\cos(2-x)+C.\end{aligned}$$

(9) Calculate $\int_{0.5}^{1} x^2\cot\left(x^3\right)\,dx$.

Sol. Let $u = x^3$. Then $\frac{du}{dx} = 3x^2$. Then $dx \longrightarrow \frac{du}{3x^2}$ and

$$\begin{aligned}\int_{0.5}^{1} x^2\cot\left(x^3\right)\,dx &= \int_{0.5}^{1} x^2\cot(u)\,\frac{du}{3x^2},\\ &= \frac{1}{3}\int_{0.5}^{1}\cot(u)\,du,\\ &= \left[\frac{1}{3}\log_e(\sin(u))\right]_{0.5}^{1},\\ &= \frac{1}{3}\log_e(\sin(1)) - \frac{1}{3}\log_e(\sin(0.5)),\\ &= \frac{1}{3}\log_e\left(\frac{\sin(1)}{\sin(0.5)}\right).\end{aligned}$$

(10) Calculate $\int \sin(x)\cos^5(x)\,dx$.

Let $u = \cos(x)$. Then $\frac{du}{dx} = -\sin(x)$ so that $dx \longrightarrow \frac{du}{-\sin(x)}$ and

$$\begin{aligned}
\int \sin(x)\cos^5(x)\,dx &= \int \sin(x)u^5 \frac{du}{-\sin(x)}, \\
&= -\int u^5\,du, \\
&= -\frac{1}{6}u^6 + C, \\
&= -\frac{1}{6}\cos^6(x) + C.
\end{aligned}$$

(11) Calculate $\int x\sin(2x)\,dx$.

Sol.

$$\begin{aligned}
\int x\sin(2x)\,dx &= \int \frac{-1}{2}x\frac{d}{dx}\cos(2x)\,dx, \\
&= \frac{-1}{2}x\cos(2x) - \frac{-1}{2}\int \cos(2x)\frac{d}{dx}x\,dx, \\
&= -\frac{1}{2}x\cos(2x) + \frac{1}{2}\int \cos(2x)\,dx, \\
&= -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C.
\end{aligned}$$

(12) Calculate $\int 3xe^x \, dx$.

$$\begin{aligned}
\int 3xe^x \, dx &= 3\int x\frac{d}{dx}e^x \, dx, \\
&= 3xe^x - \int e^x \frac{d}{dx} x \, dx, \\
&= 3xe^x - \int e^x \, dx, \\
&= 3xe^x - e^x + C.
\end{aligned}$$

(13) Calculate $\int \log_e(s) \, ds$.

$$\begin{aligned}
\int \log_e(s) \, ds &= \int \log_e(s)\frac{ds}{ds} \, ds, \\
&= s\log_e(s) - \int s\frac{d}{ds}\log_e(s) \, ds, \\
&= s\log_e(s) - \int s\frac{1}{s} \, ds, \\
&= s\log_e(s) - \int 1 \, ds, \\
&= s\log_e(s) - s + C.
\end{aligned}$$

(14) Calculate $\int x\sqrt{x+1}\,dx$.

$$\begin{aligned}
\int x\sqrt{x+1}\,dx &= \frac{2}{3}\int x\frac{d}{dx}(x+1)^{3/2}\,dx,\\
&= \frac{2}{3}x(x+1)^{3/2}-\frac{2}{3}\int(x+1)^{3/2}\frac{dx}{dx}\,dx,\\
&= \frac{2}{3}x(x+1)^{3/2}-\frac{2}{3}\int(x+1)^{3/2}\,dx,\\
&= \frac{2}{3}x(x+1)^{3/2}-\frac{2}{3}\frac{2}{5}(x+1)^{5/2}+C,\\
&= \frac{2}{3}(x+1)^{3/2}\left(x-\frac{2}{5}(x+1)\right)+C,\\
&= \frac{2}{15}(x+1)^{3/2}(5x-2x-2)+C,\\
&= \frac{2}{15}(x+1)^{3/2}(3x-2)+C,
\end{aligned}$$

(15) Calculate $\int \cos(\log_e(x))\,dx$.

$$\begin{aligned}
\int\cos(\log_e(x))\,dx &= \int\cos(\log_e(x))\frac{dx}{dx}\,dx,\\
&= x\cos(\log_e(x))-\int x\frac{d}{dx}\cos(\log_e(x))\,dx,\\
&= x\cos(\log_e(x))-\int x\frac{-\sin(\log_e(x))}{x}\,dx,\\
&= x\cos(\log_e(x))+\int\sin(\log_e(x))\,dx,\\
&= x\cos(\log_e(x))+\int\sin(\log_e(x))\frac{dx}{dx}\,dx,\\
&= x\cos(\log_e(x))+x\sin(\log_e(x))-\int x\frac{d}{dx}\sin(\log_e(x))\,dx,\\
&= x\cos(\log_e(x))+x\sin(\log_e(x))-\int x\frac{\cos(\log_e(x))}{x}\,dx,\\
&= x\cos(\log_e(x))+x\sin(\log_e(x))-\int\cos(\log_e(x))\,dx.
\end{aligned}$$

It follows that

$$2\int\cos(\log_e(x))\,dx = x\cos(\log_e(x))+x\sin(\log_e(x))+C.$$

Hence

$$\int\cos(\log_e(x))\,dx = \frac{1}{2}x(\cos(\log_e(x))+\sin(\log_e(x)))+C.$$

9 Tutorial 9

$$A_{\text{between two functions}} = \int_a^b f(x) - g(x)\,dx, \qquad Vol._{rot.,x} = \pi \int_a^b [f(x)]^2\,dx,$$

$$\text{Arc length} = \int_a^b \sqrt{1+[f'(x)]^2}\,dx, \qquad \text{Mean Val.} = \bar{y} = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

$$Vol._{rot.,y} = \pi \int_a^b [f(y)]^2\,dy, \qquad \text{Surf. Area}_{rot.,x} = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx.$$

(1) Mean values

Calculate the mean value of the following functions over the specified interval. Make a rough sketch of each function and indicate the mean value.

(a) $y = x$ over $\{x \in \mathbb{R} \,:\, 0 \le x \le 10\}$.

(b) $y = x^2$ over $\{x \in \mathbb{R} \,:\, 0 \le x \le 10\}$.

(c) $y = x^3$ over $\{x \in \mathbb{R} \,:\, 0 \le x \le 10\}$.

(d) $y = \sin(x)$ over $\left\{x \in \mathbb{R} \,:\, 0 \le x \le \frac{\pi}{2}\right\}$.

(e) $y = \sin(x)$ over $\{x \in \mathbb{R} \,:\, 0 \le x \le \pi\}$.

(2) A simple application of infinitesimals

Use the method of infinitesimals to show that the formula for the area of a circle is $A = \pi r^2$, by stating with the definition of π:

$$\pi = \frac{\text{circumference}, c}{\text{diameter}, d}$$

so that $c = \pi d = 2\pi r$.

Hint: Imagine adding infinite circumferences of infinitesimal thickness and infinitesimal area, starting at the centre of a circle out to the outer-edge of the circle.

(3) Area bounded by two functions

Calculate the total area bounded by the following curves (check for intersections; try to draw them first)

(a) $y = x^2 - 6$ and $y = x$.

(b) $y = \sin(x)$ and $y = \sin(x) + 3$ over the interval $\{x \in \mathbb{R} : 0 \leq x \leq 7\}$ (can you explain the "nice", "neat" result?)

(c) $y = \sin(x)$ and $y = \cos(x)$ over the interval $\{x \in \mathbb{R} : 0 \leq x \leq 1.5\}$.

(d) $y = \cos(x)$ and $y = x$ over the interval $\left\{x \in \mathbb{R} : 0 \leq x \leq \frac{\pi}{2}\right\}$ (your tutor will assist after you have tried to find the intersection).

(e) $y = \sqrt{x}$ and $x = 0$ over the interval $\{x \in \mathbb{R} : 0 \leq x \leq 3\}$ (this is asking for the area between the curve and the y-axis).

10 Tutorial 10 - Introduction to Differential Equations

(1) Determine whether $y = e^{-x^2}$ is the general solution to $\frac{dy}{dx} + 2xy = 0$ or a particular solution.

(2) Determine whether $y = c\ln(x)$ is the general solution to $y'\ln(x) - \frac{y}{x} = 0$ or a particular solution.

(3) Solve the differential equation $\frac{dp}{dx} = \sqrt{\frac{p}{x}}$.

(4) Solve the differential equation $y^2\frac{dy}{dx} + x^3 = 0$.

(5) Solve the differential equation $y + t\frac{dy}{dt} = 3ty$.

(6) Solve the differential equation $e^{2x}\frac{dy}{dx} + e^x = 4$.

(7) Solve the initial value problem $\frac{ds}{dt} = \sec(s)$, $t = 0$ when $s = 0$.

(8) Solve the initial value problem $2x\frac{dy}{dx} = y\ln(y)$, $x = 2$ when $y = e$.

(9) Solve the initial value problem $y^2e^x + e^{-x}\frac{dy}{dx} = y^2$, $x = 0$ when $y = 2$.

(10) Solve the initial value problem $(2y\cos(y) - \sin(y))\frac{dy}{dx} = y\sin(y)$, $x = 0$ when $y = \frac{\pi}{2}$.

11 Tutorial 11 - 1st Order Linear Differential Equations, 2nd Order Homogeneous Equations, and Euler's Method

(1) Solve the differential equation $\frac{dy}{dx} + y = e^{-x}$.

(2) Solve the differential equation $\frac{dy}{dx} + 3y = e^{-3x}$.

(3) Solve the differential equation $2\frac{dy}{dx} = 5 - 6y$.

(4) Solve the differential equation $\frac{dy}{dx} = 3x^2(2-y)$.

(5) Solve the differential equation $3x\frac{dy}{dx} - y = 9x$.

(6) Solve the differential equation $y' = x^2y + 3x^2$.

(7) Solve the differential equation $y' + 2y = \sin(x)$.

(8) Solve the differential equation $3y'' + 4y' + y = 0$.

(9) Solve the differential equation $y'' - y' - 6y = 0$.

(10) Solve the differential equation $y'' + y' = 0$.

(11) Solve the differential equation $y'' - 2y' - 8y = 0$.

(12) Solve the initial value problem $4y'' - y' = 0$, $y(0) = 4$, $y'(0) = 2$.

(13) Use Euler's method:

$$y_{n+1} = y_n + f(x_n, y_n)\Delta x, \qquad (x_0, y_0) = (x_0, y(x_0)),$$

with $\Delta x = 0.3$ to solve the initial value problem $\frac{dy}{dx} = \sqrt{2x+1}$, $y(0) = 2$.

(14) Use Euler's method with $\Delta x = 0.1$ to solve the initial value problem $\frac{dy}{dx} = y(0.4x+1)$, $y(-0.2) = 2$.

12 Tutorial 12 - Exercises in Complex Numbers and Second Order Differential Equations that Use Them

(1) Express $\sqrt{-121}$ in terms of $i = \sqrt{-1}$.

(2) Express $-\sqrt{-49}$ in terms of i.

(3) Express $3\sqrt{-48}$ in terms of i.

(4) Simplify $\sqrt{(-15)^2}$ by expanding the brackets first.

(5) Simplify $\sqrt{-9}\sqrt{-16}$.

(6) Simplify $-26+\sqrt{-64}$.

(7) Simplify $5-2\sqrt{25i^2}$.

(8) Solve $x^2-2x+2=0$.

(9) Solve $3x^2-6x+4=0$.

(10) Evaluate $\sum_{n=1}^{8} i^n$.

(11) Simplify $(3-7i)+(2-i)$.

(12) Simplify $(5.4-3.4i)-(2.9i+5.5)$.

(13) Simplify $(-2.2i)(1.5i-4.0)$.

(14) Simplify $\sqrt{-6}\sqrt{-12}\sqrt{30}$.

(15) Simplify $(8+3i)(8-3i)$.

(16) Simplify $\frac{12+10i}{6-8i}$.

(17) Show that $1+i\sqrt{3}$ satisfies $x^2+4=2x$.

(18) Add graphically: $2i+(-2+3i)$.

(19) Add graphically: $(-6-3i)+(2-7i)$.

(20) Show $a+bi$, $3(a+bi)$, and $-3(a+bi)$ on the same plot, where $a=-10$, $b=-30$.

(21) Show graphically and give the polar form of $-8-15i$.

(22) Show graphically and give the polar form of $-5+12i$.

(23) Show graphically and give the polar form of $\sqrt{2}-\sqrt{2}i$.

(24) Give the rectangular form of the complex number with modulus 6 and argument $180°$.

(25) Give the rectangular form of the complex number with modulus 2.5 and argument $315°$.

(26) Give the rectangular form of the complex number with modulus 15 and argument $0°$.

(27) Give the exponential form of the complex number $576\,(\cos(135°)+i\sin(135°))$.

(28) Give the exponential form of the complex number $2.1\,(\cos(588.7°)+i\sin(588.7°))$.

(29) Give the exponential form of the complex number $16.7\,(\cos(-7.14°)+i\sin(-7.14°))$.

(30) Give both the polar and the rectangular forms of $20.0e^{1.0i}$.

(31) Give both the polar and the rectangular forms of $2.5e^{3.84i}$.

(32) Give both the polar and the rectangular forms of $0.8e^{3.0i}$.

(33) Find the general solution to the differential equation $y''+y'+y=0$.

(34) Find the general solution to the differential equation $y''-y'+3y=0$.

13 Books & Notes

References

[1] James Stewart, Calculus, 8th Ed.

[2] A. J., Washington. Basic Technical Mathematics with Calculus, SI Version.