Linear Algebra

Workbook

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https://mathofcourse.com/

Contents

Instructions

1. **Imperative:** Print this pdf document or be prepared to annotate the pdf with a tablet. Some blank spaces for writing are a little small for large writing. If you cannot do either of these annotation options, then write notes on blank paper, noting the relevant position within the typed course notes. As you watch the lecture videos, write notes in the blank spaces. This step is very important.

2. **Optional but highly recommended:** Purchase and use *Mathematica* or obtain it through your institution. We will occasionally use this to display various graphics and verify calculations. All graphics shown in this document was produced with *Mathematica*. You will most likely find it very helpful with your studies. It is a symbolic computation tool which has full programming capabilities. E.g. Try writing

   ```
   Expand[(x+y)^3]
   ```

   then press Shift+Enter or

   ```
   s = 0;
   For[i = 0, i < 6, i++, s = s + i; Print[s]]
   ```

   You can call on *Wolfram alpha* from with in it by beginning a cell with = =.
   If your university has a license, to install this on your machine, visit:
   `wolfram.com/siteinfo/`
   Get *Mathematica* Desktop.
   Create a Wolfram ID, and download and install the software.

1 Vector Spaces

1.1 Introduction and definitions

Recall that $\mathbb{R}$ is the set of all real numbers and

$$\mathbb{C} = \left\{ a + bi \;:\; a, b \in \mathbb{R}, i = \sqrt{-1} \right\}$$

is the set of complex numbers. These sets have algebraic structures with addition and multiplication that make them *fields*, so we call $\mathbb{R}$ the field of scalars when we discuss the real numbers in the context of vector spaces. Likewise we may sometimes refer to the field $\mathbb{C}$.

In this section we will study vector spaces. Consider the following example.

Example 1.1 *Suppose we have the weights of* 8 *students in kg. The weights are listed as follows*

$$\mathbf{w} = (156, 125, 145, 134, 178, 145, 162, 193) = (w_1, w_2, \ldots, w_8).$$

The order of the entries is important. We can attribute to $\mathbf{w}$ *a magnitude and direction. Hence* $\mathbf{w}$ *is a vector in an 8-dimensional vector space* $\mathbb{R}^8$ *over the real numbers.*

Temperature and speed are scalar quantities since they have magnitude only. Force, velocity, momentum, etc are examples of vectors since they have magnitude and direction.

Vectors in 2-space $\mathbb{R}^2$, for example $\mathbf{v} = (2, -5) = 2\mathbf{i} - 5\mathbf{j}$, have two components. Vectors in 3-space $\mathbb{R}^3$, for example $\mathbf{v} = (0, 4, 2) = 4\mathbf{j} + 2\mathbf{k}$, have three components. We also consider n-space over the real numbers, $\mathbb{R}^n$.

Recall vector addition: Let

$$\begin{aligned} \mathbf{u} &= (u_1, u_2, \ldots, u_n), \\ \mathbf{v} &= (v_1, v_2, \ldots, v_n) \end{aligned}$$

be vectors in $\mathbb{R}^n$. Then the sum of $\mathbf{u}$ and $\mathbf{v}$ is defined

$$\mathbf{u}+\mathbf{v} = (u_1+v_1, u_2+v_2, \ldots, u_n+v_n).$$

To add two vectors, they must have the same number of components.

Recall that the *dot product* or *scalar product* of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + \ldots u_nv_n = \sum_{i=1}^{n} u_iv_i = \mathbf{u}\mathbf{v}^T,$$

where $\mathbf{u}\mathbf{v}^T$ denotes matrix multiplication of the matrix $\mathbf{u}$ with the transpose of matrix $\mathbf{v}$.

Let $k \in \mathbb{R}$ and let

$$\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n.$$

The scalar multiple $k\mathbf{u}$ is given by

$$k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n).$$

Let $A = [a_{ij}]$, $B = [b_{ij}]$ be $m \times n$ matrices and let $p, q \in \mathbb{R}$. A linear combination of A and B is of the form

$$\begin{aligned} pA + qB &= p\,[a_{ij}] + q\,[b_{ij}], \\ &= [pa_{ij}] + [qb_{ij}], \\ &= [pa_{ij} + qb_{ij}]. \end{aligned}$$

Note that the matrices $A, B \in \mathbb{R}^{mn}$ are mn-dimensional vectors.

Recall the following vector definitions:

- Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with $\mathbf{u} = [u_i]_{1\times n}$, $\mathbf{v} = [v_i]_{1\times n}$. Then:

  1. $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if $\mathbf{u}\cdot\mathbf{v} = 0$.
  2. The *norm* of $\mathbf{u}$ is given by $\|\mathbf{u}\| = \sqrt{\sum_{i=1}^{n} u_i^2}$.
  3. $\|\mathbf{u}\| \geq 0$ and $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

4. If $\mathbf{u} \neq \mathbf{0}$, then the *unit vector* $\widehat{\mathbf{u}} = \frac{1}{\|\mathbf{u}\|}\mathbf{u}$ is a vector of norm 1 is the direction of the vector $\mathbf{u}$.
5. If $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
6. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $(n = 3)$, the *cross product* of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}, \\
&= \det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} \mathbf{i} - \det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} \mathbf{j} + \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \mathbf{k}, \\
&= (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).
\end{aligned}$$

$\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin(\theta)$, where θ is the angle bewteen $\mathbf{u}$ and $\mathbf{v}$. Hence if $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

We have the following important inequalities for vector norms:

Theorem 1 (Cauchy-Schwartz) *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|.$$

Theorem 2 (Minkowski or triangle inequality) *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Recall that the distance between two vectors is given by $D = \|\mathbf{u} - \mathbf{v}\|$.

For proofs, see [6], pp. 17 or [5], pp. 220, 259-260.

Example 1.2 *Find the equation of the plane that passes through the points* $(1,1,2)$, $(2,3,3)$, *and* $(3,-3,3)$.

Example 1.3 *Find the distance from the point* $(2,0,0)$ *to the plane*

$$x+2y+2z=0.$$

Example 1.4 *Let* $\mathbf{u}$ *and* $\mathbf{v} \in \mathbb{R}^3$ *be parallel to the sides of a parallelogram. Show that the area of the parallelogram is* $\|\mathbf{u} \times \mathbf{v}\|$.

1.2 Vector space definition

A *vector space* V (a non-empty set of vectors) over a field $\mathbb{K}$ ($\mathbb{K}$ is usually $\mathbb{R}$) of scalars (with $0, 1$) is a set of vectors that satisfies the following axioms:

1. Addition is associative: $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$.
2. $\mathbf{0} \in V$ is neutral.
3. For all $\mathbf{u} \in V$, $-\mathbf{u} \in V$.
4. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$.
   1-4: In other words $\langle V, +\rangle$ is an abelian group.
5. For $k \in \mathbb{K}$, $\mathbf{u}, \mathbf{v} \in V$, $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}$.
6. For $a, b \in \mathbb{K}$, $\mathbf{u} \in V$, $(a+b)\mathbf{u} = a\mathbf{u}+b\mathbf{u}$.
7. For $a, b \in \mathbb{K}$, $\mathbf{u} \in V$, $(ab)\mathbf{u} = a(b\mathbf{u})$.
8. $1\mathbf{u} = \mathbf{u}$.

Theorem 3 *Let V be a vector space over the field $\mathbb{K}$.*

1. *For any $k \in \mathbb{K}$ and $\mathbf{0} \in V$, then $k\mathbf{0} = \mathbf{0}$.*
2. *For $0 \in \mathbb{K}$ and $\mathbf{u} \in V$, $0\mathbf{u} = \mathbf{0}$.*
3. *If for some $k \in \mathbb{K}$, $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$.*
4. *For any scalar $k \in \mathbb{K}$ and $\mathbf{u} \in V$, then $(-k)\mathbf{u} = k(-\mathbf{u})$.*

Examples of Vector Spaces

1. $\mathbb{K}^n$ over the field $\mathbb{K}$, e.g. $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$.
2. $\mathbb{P}(t)$, the set of all univariate polynomials

$$p(t) = a_0 + a_1 t + a_2 t^2 + \dots a_n t^n$$

   with coefficients in $\mathbb{K}$. To be a polynomial, $p(t)$ must have finite degree n. There is a correspondence: $p(t) \longleftrightarrow (a_0, a_1, \dots, a_n)$.
3. The space of all $m \times n$ matrices with entries in $\mathbb{K}$.
4. Let $F(x) = \{f(x)\}$ be the set of all functions $f : X \longrightarrow \mathbb{K}$.

1.3 Subspaces of a vector space

Let V be a vector space over the field $\mathbb{K}$ and let W be a subset of V. Then W is a *subspace* of V is W is a vector space over $\mathbb{K}$.

Theorem 4 *Let W be a subset of V. Then W is a subspace of V if the following conditions hold:*

1. $\mathbf{0} \in W$.

2. W is closed under addition and scalar multiplication.

Every vector space contains at least two subspaces, $\{\mathbf{0}\}$, and V itself.

Example 1.5 *Let V be the vector space $\mathbb{R}^3$. Let $U = \{(a,b,c) : a = b = c\}$. Show that U is a subspace of V.*

Example 1.6 *Let V be the vector space consisting of all 2×2 matrices with entries in $\mathbb{R}$. Let*

$$U = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \; : \; a, b, c \in \mathbb{R} \right\}.$$

Show that U is a subspace of V.

1.4 Inner product spaces

An *inner product* on a vector space V over the field $\mathbb{K}$ is a binary operation on V such that for each $\mathbf{u}, \mathbf{v} \in V$, the inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ is a scalar in $\mathbb{K}$ satisfying:

1. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ with $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
2. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$.
3. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and all $k_1, k_2 \in \mathbb{K}$,

$$\langle k_1\mathbf{u} + k_2\mathbf{v}, \mathbf{w} \rangle = k_1\langle \mathbf{u}, \mathbf{w} \rangle + k_2\langle \mathbf{v}, \mathbf{w} \rangle.$$

If a vector space has an inner product, then it is an *inner product space*.

Example 1.7 *Let $V = \mathbb{R}^n$. Show that $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$ is an inner product on V.*

Example 1.8 *Let* $F[a,b] = \{f(x)\}$ *be the set of all functions*

$$f : [a,b] \longrightarrow \mathbb{R},$$

where $[a,b] = \{x \in \mathbb{R} \ : \ a \leq x \leq b\}$. *Show that*

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$$

is an inner product on $F[a,b]$.

1.5 Appendix on fields

Let F be a set and let $+$ and $\times$ be binary operations on F. $\langle F, +, \times \rangle$ is a *field* if it satisfies the following conditions:

1. $+$ is associative: For all $a, b, c \in F$, $(a+b)+c = a+(b+c)$.
2. 0 is neutral: $0+a = a = a+0$ for all $a \in F$.
3. For all $a \in F$, $-a \in F$.
4. For all $a, b \in F$, $a+b = b+a$.
5. $\times$ is associative: For all $a, b, c \in F$, $(a \times b) \times c = a \times (b \times c)$.
6. 1 is neutral: $1 \times a = a = a \times 1$ for all $a \in F^*$, where $F^* = F - \{0\}$.
7. For all $a \in F^*$, $a^{-1} \in F$.
8. For all $a, b \in F$, $a \times b = b \times a$.
9. $\times$ distributes over $+$: For all $a, b, c \in F$, $a \times (b+c) = a \times b + a \times c$.

1.6 Practice exercises

1. Show that the set of rationals with addition and multiplication $\langle \mathbb{Q}, +, \times \rangle$ is a field.

2. Let $\mathbb{Q}(\sqrt{2}) = \left\{ a + b\sqrt{2} \,:\, a, b \in \mathbb{Q} \right\}$.

   (a) Show that $\mathbb{Q}(\sqrt{2})$ is a vector space over the field $\mathbb{Q}$.

   (b) Show that $\mathbb{Q}(\sqrt{2})$ is a field.

3. Let $A = (2,9,8)$, $B = (6,4,-2)$, $C = (7,15,7)$. Show that the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular and find the point D such that $ABCD$ forms a rectangle. See pp. 160, [7].

4. Show that the planes $x + y - 2z = 1$ and $x + 3y - z = 4$ intersect in a line and find the distance from the point $C = (1,0,1)$ to this line. See pp. 173, [7].

5. Find an equation for the plane through $P_0 = (1,0,1)$ and passing through the line of intersection of the planes $x + y - 2z = 1$ and $x + 3y - z = 4$. See pp. 175, [7].

6. Show that the triangle with vertices $(-3,0,2)$, $(6,1,4)$, $(-5,1,0)$ has area $\frac{1}{2}\sqrt{333}$. See pp. 179, [7].

7. Let V be the vector space consisting of $n \times n$ matrices with real entries and let W be the subset of V consisting of diagonal matrices (if $i \neq j$, then $a_{ij} = 0$). Show that W is a subspace of V.

8. Let V be the vector space consisting of $n \times n$ matrices with real entries and let W be the subset of V consisting of symmetric ($M^T = M$) matrices. Show that W is a subspace of V.

9. Let V be the vector space of polynomials with real coefficients. Let W be the subset of V consisting of polynomials of degree less than or equal to n. Show that W is a subspace of V.

10. Show that the intersection $W_1 \cap W_2$ of subspaces W_1 and W_2 of a vector space V is a subspace of V. See pp. 17, [3].

11. Let V be the vector space of all 2×2 matrices over the real field $\mathbb{R}$. Show that W is not a subspace of V where:

    (a) W consists of all matrices with determinant 0;

    (b) W consists of all matrices A such that $A^2 = A$.

    See [6].

12. Let V be the vector space of all functions from the real field $\mathbb{R}$ in $\mathbb{R}$. Show that W is a subspace of V where:

    (a) $W = \{f \,:\, f(3) = 0\}$,

    (b) $W = \{f \,:\, f(7) = f(1)\}$,

    (c) $W = \{f \,:\, f(-x) = -f(x) \text{ (odd functions)}\}$.

    See [6].

13. Let V denote the vector space of $m \times n$ matrices over $\mathbb{R}$. Show that $\langle A, B \rangle = \operatorname{tr}\left(B^T A\right)$ is an inner product in V, where $\operatorname{tr}(A)$ denotes the trace of the matrix A, the sum of the diagonal elements of A.

14. Let $F[-\pi, \pi]$ be the vector space of functions $f : [-\pi, \pi] \longrightarrow \mathbb{R}$. We have an inner product $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx$. Calculate the inner products:

    (a) $\langle \cos(x), \sin(x) \rangle$,

    (b) $\langle \cos(x), \cos(x) \rangle$,

    (c) $\langle \sin(x), \sin(x) \rangle$.

15. Let V be the vector space of complex continuous functions on the real interval $a \leq t \leq b$, and let $\overline{p + qi} = p - qi$. Show that $\langle f, g \rangle = \int_a^b f(t) \overline{g(t)}\, dt$ is an inner product on V.

2 Linear Combinations, Spanning Sets, Orthogonality

In this section we will consider linear combinations, spanning sets, and orthogonality.

2.1 Linear combinations

A *linear combination* of n vectors in a vector space V is a sum of scalar multiples of those n vectors. Let $U = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a collection of vectors in the vector space V of a field $\mathbb{K}$ and let $k_1, k_2, \ldots, k_n \in \mathbb{K}$. Then

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_n\mathbf{u}_n \in V$$

is a linear combination of the vectors in U.

Example 2.1 *Consider the vector* $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. *Show that the vector* $\mathbf{v} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$ *is a linear combination of* $\mathbf{u}$.

Example 2.2 *Let* $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ *and* $\mathbf{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ *be vectors. Show that the vector* $\mathbf{w} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ *is a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$.

Example 2.3 *Let* $\mathbf{u} = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ -6 \\ 9 \end{pmatrix} \in \mathbb{R}^3$. *Then* $\mathbf{u}+\mathbf{v} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$ *is a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$, $7\mathbf{u}+0\mathbf{v} = \begin{pmatrix} 14 \\ 28 \\ -35 \end{pmatrix}$ *is a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$, $0\mathbf{u}-1\mathbf{v} = \begin{pmatrix} -1 \\ 6 \\ -9 \end{pmatrix}$ *is a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$, $3\mathbf{u}-5\mathbf{v} = \begin{pmatrix} 1 \\ 42 \\ -60 \end{pmatrix}$ *is a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$.

Example 2.4 *Let*

$$\mathbf{v} = \begin{pmatrix} 5 \\ -10 \\ 25 \end{pmatrix}, \quad \mathbf{u}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}.$$

Write $\mathbf{v}$ *as a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Continuing,

Example 2.5 *Determine whether the vector* $\mathbf{w} = (8,3,1)$ *is a linear combination of the vectors* $\mathbf{u} = (2,3,-1)$ *and* $\mathbf{v} = (3,0,4)$.

Theorem 5 *Let* $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ *be column vectors in* $\mathbb{R}^n$. *Then the column vector* $\mathbf{v} \in \mathbb{R}^n$ *is a linear combination of the vectors* $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ *if and only if there is a solution* X *in* $\mathbb{R}^n$ *to the matrix equation* $AX = \mathbf{v}$, *where* $A = (\mathbf{u}_1 \mid \mathbf{u}_2 \mid \ldots \mid \mathbf{u}_n)$, *if and only if*

$$rref\,(\mathbf{u}_1 \mid \mathbf{u}_2 \mid \ldots \mid \mathbf{u}_n \mid \mathbf{v})$$

does not have a row with $(0,0,\ldots,0,1)$, *where* $rref(B)$ *means the row reduced echelon form of* B.

Example 2.6 *Let* V *be the vector space consisting of polynomials with real coefficients. Determine whether* $f(t) = 3t^2 + 5t - 5$ *is a linear combination of the polynomials*

$$p_1 = t^2 + 2t + 1, \qquad p_2 = 2t^2 + 5t + 4, \qquad p_3 = t^2 + 3t + 6.$$

Continuing,

2.2 Spanning sets

We are now able to define the subspace of V consisting of all linear combinations of the n vectors, the span.

Let $U = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a collection of vectors in the vector space V of a field $\mathbb{K}$. The *span* of U or span of the vectors in U, denoted

$$\text{span}(U) = \text{span}\left(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\right),$$

is the set of all linear combinations of the vectors in U:

$$\text{span}(U) = \left\{ \mathbf{v} \in V \;:\; \mathbf{v} = \sum_{j=1}^{n} a_j \mathbf{u}_j, \text{ where } a_j \in \mathbb{K} \right\}.$$

Theorem 6 *Let U be a subset of the vector space V. Then:*

1. *$U \subseteq span(U) \subseteq V$ and $span(U)$ is a subspace of V.*
2. *If W is a subspace of V and $U \subseteq W$, then $span(U) \subseteq W$.*

Remark 2.1 *Suppose $span\left(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\right) = V$. Then for any $\mathbf{w} \in V$,*

$$span\left(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n, \mathbf{w}\right) = V.$$

Remark 2.2 *Let $U = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}, \mathbf{u}_n\}$, $\mathbf{u}_n \in U$. Suppose $span\,(U) = V$ and $\mathbf{u}_n$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}$ or $\mathbf{u}_n = \mathbf{0}$. Then*

$$span\left(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}\right) = V.$$

Note that $\mathbf{0}$ is a linear combination of the other vectors in U.

Example 2.7 *Let* $\mathbf{e}_1 = (1,0,0) = \mathbf{i}$, $\mathbf{e}_2 = (0,1,0) = \mathbf{j}$, $\mathbf{e}_3 = (0,0,1) = \mathbf{k}$ *be vectors in* $\mathbb{R}^3$. *Show that span* $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbb{R}^3$.

Example 2.8 *Consider the vectors* $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 4 \\ -3 \\ -4 \end{pmatrix}$ *in the vector space* $\mathbb{R}^3$ *over* $\mathbb{R}$. *By taking the cross product* $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} -13 \\ -4 \\ -10 \end{pmatrix}$, *we see that this vector is normal to the plane* $-13x - 4y - 10z = 0$ *so this plane is parallel to both* $\mathbf{u}$ *and* $\mathbf{v}$. *See Figure 2. In fact it is easy to show that any linear combination* $\mathbf{w}$ *of* $\mathbf{u}$ *and* $\mathbf{v}$, *so* $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$, *with* $a, b \in \mathbb{R}$, *must also be parallel to the plane* $-13x - 4y - 10z = 0$ *and all linear combinations of* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel to this plane. It is easy to see then that we cannot obtain any vector in* $\mathbb{R}^3$ *by a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$ *since not all vectors in* $\mathbb{R}^3$ *are parallel to the plane* $-13x - 4y - 10z = 0$. *For example, the vector* $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ *is not a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$. *We may think of span*$(\mathbf{u}, \mathbf{v})$ *geometrically as the set of all vectors that are parallel to the plane* $-13x - 4y - 10z = 0$. *It is clear that this is a proper subspace of* $\mathbb{R}^3$ *and not equal to* $\mathbb{R}^4$.

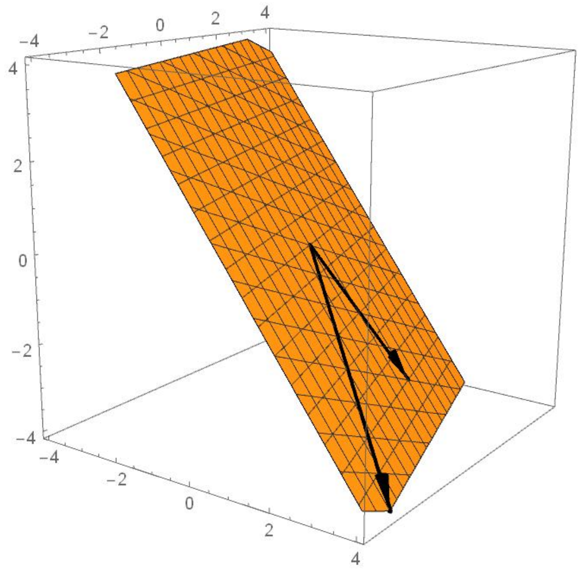

Figure 1: Example

Remark 2.3 *A spanning set is not unique. For example*

$$span(\mathbf{i},\mathbf{j},\mathbf{k}) = span\,((1,1,1),(1,1,0),(1,0,0)) = \mathbb{R}^3.$$

Remark 2.4 *The span of two non-zero vectors in* $\mathbb{R}^3$ *that are not parallel is a plane passing through the origin.*

Example 2.9 *Show that* $(2,7,8) \notin span\,((1,2,3),(1,3,5),(1,5,9))$ *by computing*

$$rref\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 9 & 8 \end{pmatrix}.$$

Hence $span\,((1,2,3),(1,3,5),(1,5,9)) \neq \mathbb{R}^3$.

Example 2.10 *Let $M_{2,2}$ be the vector space of all 2×2 matrices with real entries and let*

$$U = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then $span(U) = M_{2,2}$.

Theorem 7 *Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be three column vectors in $\mathbb{R}^3$. Then*

$$span\,(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \mathbb{R}^3$$

if and only if

$$rref\,(\mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3) = I,$$

if and only if

$$\det(\mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3) \neq 0.$$

Example 2.11 *Show that $span\,((3,3,3),(2,4,6),(-2,-10,-16)) = \mathbb{R}^3$*

Let $A = [a_{ij}]_{m \times n}$ with $a_{ij} \in \mathbb{K}$, where $\mathbb{K}$ is a field. Let

$$R_i = [a_{ij}]_{1 \times n} = (a_{i1}, a_{i2}, \ldots, a_{in}).$$

The *row space* of A, denoted $\mathrm{RS}(A)$, is given by

$$\mathrm{RS}(A) = \mathrm{span}(R_1, R_2, \ldots, R_m).$$

If B is the row reduced echelon form of A, then

$$\mathrm{RS}(A) = \mathrm{RS}(B).$$

Example 2.12 *Describe the space span* $(\mathbf{u}_1, \mathbf{u}_2)$ *with as few vectors as possible using the row space of a matrix, where* $\mathbf{u}_1 = (1,1)$, $\mathbf{u}_2 = (3,3)$.

Example 2.13 *Describe the space* $\text{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ *with as few vectors as possible using the row space of the* 4×3 *matrix* $A = \begin{pmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \\ \mathbf{u}_4^T \end{pmatrix}$, *where*

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix}, \quad \mathbf{u}_4 = \begin{pmatrix} -2 \\ -4 \\ 6 \end{pmatrix}.$$

2.3 Orthogonality

We refer to [4, pp. 263] for the following definitions and example.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be non-zero vectors in an inner product space V. If $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ if and only if $i \neq j$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an *orthogonal set* of vectors.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be non-zero unit vectors in an inner product space V. If $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ if and only if $i \neq j$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an *orthonormal set* of vectors.

Example 2.14 *Let* $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (2,1,-3)$, $\mathbf{v}_3 = (4,-5,1)$. *Show that* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *is an orthogonal set of vectors.*

Example 2.15 *Let* $F[-\pi,\pi]$ *be an inner product space of functions with inner product*

$$\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

Show that the set $\{1,\cos(x),\cos(2x),\ldots,\cos(nx)\}$ *is an orthogonal set of vectors, and find an orthonormal set of vectors.*

The square matrix Q is *orthogonal* if the column vectors of Q form an orthonormal set.

Theorem 8 *An $n \times n$ matrix Q is orthogonal if and only if $Q^T = Q^{-1}$.*

Example 2.16 *For any $\theta \in \mathbb{R}$, the rotation matrix* $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ *is orthogonal.*

Theorem 9 *Let Q be an $n \times n$ orthogonal matrix. Then*

1. *The column vectors of Q form an orthonormal basis for $\mathbb{R}^n$.*
2. $Q^T Q = I$.
3. $Q^T = Q^{-1}$.
4. $\langle Q\mathbf{u}, Q\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$.
5. $||Q\mathbf{v}|| = ||\mathbf{v}||$, *where* $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

2.4 Orthogonal matrices and QR decomposition

Now we will consider the QR decomposition of a matrix.

Recall that the square matrix Q is *orthogonal* if $Q^T = Q^{-1}$.

QR decomposition involves writing the original matrix A in the form

$$A = QR,$$

where Q is an orthogonal matrix and R is an upper triangular matrix.

This is a very useful technique, which can be used to solve linear systems, invert matrices, calculate eigenvalues, and more.

QR decompositions are defined for non-square matrices too. If A is an $m \times n$ matrix with $m \geq n$, then Q is $m \times m$ and R is $m \times n$.

For our purposes, we stick to square matrices.

Furthermore, if A is invertible, then the QR decomposition is unique if we require that the diagonal elements of R are positive.

Applications of QR decomposition

The QR decomposition is useful in many elementary tasks in linear algebra.

Suppose we want to solve the linear system $AX = b$. If we calculate the QR factorisation of A, then we can write this as

$$\begin{aligned} QRX &= b, \\ RX &= Q^T b, \end{aligned}$$

and then we can easily solve for X using back substitution. This does not involve finding the inverse of a matrix or using Gaussian elimination directly.

Likewise, if we want to invert A, we calculate its QR decomposition and immediately have $A^{-1} = R^{-1}Q^T$.

Again, because R is upper triangular, it is much easier to invert than a general matrix.

We can use QR decomposition to calculate eigenvalues.

A popular way of computing the QR decomposition is to use a Gram-Schmidt process, which you should know from computing an orthonormal basis.

We will use this process to compute an orthonormal basis of the column space of A. It is well known that a concatenation of orthonormal vectors forms an orthogonal matrix (this will be Q).

By this construction, the kth column of Q depends only on the first k columns of A. This gives R the required upper triangular structure.

Gram-Schmidt algorithm

In: An $n \times n$ matrix A.

Out: $n \times n$ matrices Q, R, where $A = QR$, Q is orthogonal, and R is upper triangular.

1. For $i = 1, 2, \ldots n$:

   (a) Let a_i be the ith column of A.

   (b) Set
$$u_i = a_i - \sum_{k=1}^{i-1} (a_i \cdot V_k) V_k.$$

   (c) Set
$$V_i = \frac{U_i}{||U_i||}.$$

2. Set
$$Q = [V_1 \; V_2 \; \ldots \; V_n],$$

   and
$$R = \begin{pmatrix} a_1 \cdot V_1 & a_2 \cdot V_1 & \ldots & a_n \cdot V_1 \\ 0 & a_2 \cdot V_2 & \ldots & a_n \cdot V_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \cdot V_n \end{pmatrix}.$$

To see that this works, firstly it is easy to see that V_i all have unit norm. By construction, they are all orthogonal – suppose that $V_1, V_2, \ldots V_{i-1}$ are mutually orthogonal. Then for $j < i$,

$$\begin{aligned} U_i \cdot V_j &= \left(a_i - \sum_{k=1}^{i-1} (a_i \cdot V_k) V_k \right) \cdot V_j, \\ &= a_i \cdot V_j - (a_i \cdot V_j) V_j \cdot V_j, \\ &= 0. \end{aligned}$$

Thus Q is indeed an orthogonal matrix.

By construction, a_i can be expressed as a linear combination of the V_ks for $k = 1, 2, \ldots i:$

$$a_i = \sum_{k=1}^{i} c_k V_k.$$

Multiplying by V_j for $j \leq i$ and using the orthonormality of the Vs gives

$$a_i \cdot V_j = c_j V_j \cdot V_j = c_j,$$

which shows that $A = QR$.

An alternative way to computing the QR decomposition is to use a technique we have seen earlier: Householder transformations.

Recall that for any two vectors X and Y of equal norm, there exists a Householder matrix (which is orthogonal) H such that $HX = Y$.

Therefore we take successive matrices which transform each of the columns of A (starting from the diagonal element) into multiples of e_1.

Example Find the QR decomposition of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Continuing,

Continuing,

3 Linear Independence and Bases

3.1 Linear independence

Definition 1 *Let $U = \{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n\}$ be a collection of vectors in the vector space V over the field $\mathbb{K}$. Then the vectors $\mathbf{u}_j \in U$ are* linearly dependent *if there exist $a_1, a_2, \ldots, a_n \in \mathbb{K}$ (not all zero) such that*

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n = \mathbf{0}.$$

If U is not linearly dependent, then U is linearly independent.

Definition 2 *Let $U = \{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n\}$ be a collection of vectors in the vector space V over the field $\mathbb{K}$. Then the vectors $\mathbf{u}_j \in U$ are* linearly independent *if whenever*

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n = \mathbf{0},$$

then we must have $a_1 = a_2 = \cdots = a_n = 0$.

Remark 3.1 *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be a set of vectors in the vector space V. If $\mathbf{0} \in S$, then S is linearly dependent.*

This holds since $0\mathbf{v}_1 + 0\mathbf{v}_2 \cdots + 1\mathbf{v}_j + \ldots 0\mathbf{v}_m = \mathbf{0}$ but not all of the scalar coefficients are zero, where $\mathbf{v}_j = \mathbf{0}$ for some j: $1 \leq j \leq m$.

Remark 3.2 *Let $S = \{\mathbf{v}\}$ be a set of vectors in the vector space V consisting of one non-zero vector $\mathbf{v}$. Then S is linearly independent.*

Clearly $k\mathbf{v} = \mathbf{0}$ if and only if $k = 0$ since $\mathbf{v} \neq \mathbf{0}$.

Remark 3.3 *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be a set of vectors in the vector space V and suppose that $\mathbf{v}_i = k\mathbf{v}_j$ for some $i \neq j$ and $k \in \mathbb{K}$. Then S is linearly dependent.*

Remark 3.4 *Let* $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ *be a set of non-zero vectors in the vector space* V. *Then* S *is linearly dependent if and only if there exists a non-zero scalar* k *such that* $\mathbf{v}_1 = k\mathbf{v}_2$.

Example 3.1 *Let* $S_1 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$ *and* $S_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$. *Show that* S_1 *is linearly dependent while* S_2 *is linearly independent.*

Remark 3.5 *Let* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ *be a set of linear independent vectors in the vector space* V *and suppose that* W *is a subset of* S. *Then* W *is linearly independent.*

Example 3.2 *Let* $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}$ *be vectors in* $\mathbb{R}^3$. *Show that* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *are linearly dependent.*

Example 3.3 *Let* $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$ *be vectors in* $\mathbb{R}^3$. *Show that* $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *is linearly independent.*

Remark 3.6 *Any two vectors in* $\mathbb{R}^3$ *are linearly dependent if and only if they lie on the same line through the origin.*

Remark 3.7 *Any three vectors in* $\mathbb{R}^3$ *are linearly dependent if and only if they lie on the same plane through the origin. (Remember we can move vectors without changing their magnitude and direction.)*

3.2 Basis of a vector space

See [4, pp. 12] for the following:

Definition 3 *A matrix is in row reduced echelon form if the following four conditions hold:*

1. *All rows (if any) consisting entirely of zeros appear at the bottom of the matrix.*
2. *The first non-zero number starting from the left in any row not consisting entirely of zeros is* 1.
3. *If two successive rows do not consist entirely of zeros, then the first* 1 *in the lower row occurs farther to the right than the first* 1 *in the higher row.*
4. *Any colum containing the first* 1 *in a row has zeros everywhere else.*

Definition 4 *If only Conditions 1,2, and 3 hold, then the matrix is in row echelon form.*

For example, the matrices

$$I, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

are in r.r.e.f. The matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

is only in row echelon form because the leading 1s or *pivots* do not have 0s above them.

Theorem 10 *The non-zero rows of a matrix that is in row reduced echelon form are linearly independent.*

Definition 5 *Let* $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ *be a set of vectors in a vector space* V *over* $\mathbb{K}$*. Then* B *is a* **basis** *of* V *if both:*

1. *span*$(B) = V$,
2. *The vectors in* B *are linearly independent.*

Remark 3.8 *A basis of a vector space* V *is not unique in general (there may be exceptions for unusual fields of scalars like* $\mathbb{K} = \{0, 1 \pmod 2\}$*.)*

Theorem 11 *Suppose* $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ *is a basis for the vector space* V*. Then for all* $\mathbf{v} \in V$*, there is a unique ordered n-tuple of scalars* $(a_1, a_2, \ldots, a_n)$ *such that* $\sum_{j=1}^{n} a_j \mathbf{u}_j = \mathbf{v}$*. (Write can write* $\mathbf{v}$ *as a unique linear combination of the vectors in* B*.)*

Theorem 12 *If B_1 and B_2 are bases for a vector space V, then they have the same number of elements.*

Definition 6 *The* **dimension** *of a vector space V, denoted $\dim(V)$, is the number of elements in any basis of V if this number is finite.*

Remark 3.9 *The trivial vector space $\{\mathbf{0}\}$ is defined to have dimension 0.*

Definition 7 *If $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a basis of a vector space V, then B is an* **orthogonal basis** *of V if for all $\mathbf{u}_i, \mathbf{u}_j \in B$ with $i \neq j$, then $\mathbf{u}_i \cdot \mathbf{u}_j = 0$.*

Definition 8 *If $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of a vector space V, then B is* **orthonormal** *if all $\mathbf{u}_i \in B$ satisfy $\|\mathbf{u}_i\| = 1$.*

Example 3.4 *Let $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. $B = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an orthonormal basis for $\mathbb{R}^3$ and $\dim\left(\mathbb{R}^3\right) = 3$.*

Remark 3.10 *Orthonormal bases are not unique.*

Definition 9 *Let $\mathbf{e}_j$ be the $n \times 1$ matrix whose entries a_{i1} are 0 if $i \neq j$ and 0 if $i = j$. Then $B = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is an orthonormal basis for $\mathbb{R}^n$. B is called the* **standard basis** *for $\mathbb{R}^n$. $\dim\left(\mathbb{R}^n\right) = n$.*

Example 3.5 *Let $M_{2,3}$ be the vector space of all 2×3 matrices wth real entries. Let*

$$B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Then B is a basis for $M_{2,3}$. The dimension of $M_{2,3}$ is equal to 6.

Example 3.6 *Let $P_n(t)$ be the vector space consisting of all polynomials of degree $\leq n$. Let $S = \left\{1, t, t^2, \ldots, t^n\right\}$. Then S is a basis for $P_n(t)$ and $\dim\left(P_n(t)\right) = n$.*

Example 3.7 *Let* $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$ *be vectors in* $\mathbb{R}^3$. *Show that* $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *is a basis for* $\mathbb{R}^3$.

Theorem 13 *Let* V *be a vector space of dimension* n *and let* S *be a subset of* V *of* m *vectors, where* $m > n$. *Then* S *is linearly dependent.*

Theorem 14 *Let* V *be a vector space of dimension* n *and let* S *be a subset of* V *of* n *linearly independent vectors. Then* S *is a basis of* V.

Theorem 15 *Let* V *be a vector space of dimension* n. *If*

$$span\,(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = V,$$

then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is a basis of* V.

Theorem 16 *Let* V *be a vector space of finite dimension* n *and* $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ *be a set of linearly independent vectors in* V. *Then* S *can be extendend to a basis* B *of* V *so that* $S \subseteq B$.

Theorem 17 *Let W be a subspace of an n-dimensional vector space V. Then $dim(W) \leq n$.*

Example 3.8 *Let W be a subspace of $\mathbb{R}^3$. If $dim(W) = 0$, then $W = \{\mathbf{0}\}$. If $dim(W) = 1$, then W is a line through the origin. If $dim(W) = 2$, then W is a plane through the origin. If $dim(W) = 3$, then $W = \mathbb{R}^3$.*

3.3 The rank and nullspace of a matrix

Definition 10 *The* **rank** *of a matrix A, denoted $rank(A)$, is the dimension of the vector space spanned by the columns of A, the number of non-zero rows in the row-reduced echelon form of A.*

Definition 11 *The* **null-space** *of an $m \times n$ matrix A with entries in $\mathbb{K}$, denoted $NS(A)$ is the vector space consisting of the vectors*

$$NS(A) = \{X \in V \ : \ AX = \mathbf{0}\}.$$

Theorem 18 *$NS(A)$ is a sub-space of V.*

3.4 The Wronskian matrix

For what follows in this section, see pp. 155, [5].

Definition 12 *Let $F[a,b]$ be the vector space of function defined on the interval $[a,b]$. Define the* **Wronskian** *function*

$$W(f_1, f_2, \dots, f_n)(x) = \det \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix},$$

where $f^{(m)}(x)$ denotes the m-th derivative of the function $f(x)$ with respect to x.

Theorem 19 *Let $f_1, f_2, \dots, f_n \in F[a,b]$. If there exists a point $x_0 \in [a,b]$ such that $W(f_1, f_2, \dots, f_n)(x_0) \neq 0$, then $f_1, f_2, \dots, f_n$ are linearly independent.*

Example 3.9 *Show that e^x and e^{-x} are linearly independent in $F[-\infty, \infty]$.*

4 Linear Transformations

4.1 Definitions

Definition 13 *A* **map** *or* **mapping** *is an unambiguous rule f for sending the elements of the non-empty set X to the elements of the non-empty set Y, denoted $f : X \longrightarrow Y$.*

Also see Definition **??**.

Example 4.1 *Let $f : \mathbb{Z} \longrightarrow \mathbb{R}$ given by $f(x) = x^2$ is a map or* function *that sends an integer to it's square.*

Definition 14 *The* **domain** *of the map $f : X \longrightarrow Y$ is X and the* **codomain** *of f is Y or the* **target set***.*

In Example 4.1 above, the domain of f is $\mathbb{Z}$, the codomain is $\mathbb{R}$ and the **image** of the map f, denoted $f[X]$, here it is $f[\mathbb{Z}]$, is the set of all elements of $\mathbb{R}$ that are of the form $f(x)$, here $f[X] = \mathbb{Z}^2$, where $\mathbb{Z}^2$ is the set of integer squares. In another context, $\mathbb{Z}^2$ might be used to refer to the Cartesian product $\mathbb{Z} \times \mathbb{Z}$ but we don't mean that here.

Definition 15 *Let $f : X \longrightarrow Y$ be a map. The* **preimage** *of a subset W of Y, $f^{-1}(W)$ is the set of all elements x of the domain X such that $f(x) \in W$.*

Example 4.2 *Define a map $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = x^2$. Find the image of f, $f[\mathbb{R}]$. Find the preimage of $\{1, 2, 3\}$.*

Example 4.3 *Let* $V = \mathbb{R}[x]$ *be the vector space of all polynomials with real coefficients. Define a map* $D : V \longrightarrow V$ *by* $D(p(t)) = p'(t)$*, the first derivative of* $p(t)$ *w.r.t.* t*. This is an example of a map. Consider the preimage of some polynomial* $q(t)$*.*

Example 4.4 *Let* $V = \mathbb{R}[x]$ *be the vector space of polynomials with real coefficients. Define a map* $F : V \longrightarrow \mathbb{R}$ *by* $F(p(t)) = \int_0^1 p(t)\,dt$*.*

Definition 16 *A map* $f : X \longrightarrow Y$ *is said to be a* **linear map** *if:*

1. *X and Y are vector spaces over a field $\mathbb{K}$.*
2. *For all $\mathbf{u}, \mathbf{v} \in X$, $f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ (we can say f is a vector space* homomorphism*).*
3. *For all $k \in \mathbb{K}$ and $\mathbf{u} \in X$, $f(k\mathbf{u}) = kf(\mathbf{u})$.*

This is also called a **linear transformation**.

Clearly if f is a linear map, then $f(0\mathbf{v}) = \mathbf{0}$.

Example 4.5 *Let* $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *by* $F(x,y,z) = (x,y,0)$*, where we understand* $\mathbb{R}^3$ *to be the vector space* $\mathbb{R}^3$*. This map* F *projects the vector* (x,y,z) *onto the* x,y *plane.* F *is a linear map.*

Example 4.6 *Let* $G : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *by* $G(x,y) = (x+1, y+2)$*, where we understand* $\mathbb{R}^2$ *to be the vector space* $\mathbb{R}^2$*. Determine whether G is a linear map.*

Example 4.7 *Let* $V = \mathbb{R}[t]$ *be the vector space of polynomials with real coefficients. The derivative* $D : V \longrightarrow \mathbb{R}$ *is a linear map.*

The map f of Example 4.4 is also a linear map.

Theorem 20 *Let* A *be an* $m \times n$ *matrix with entries in* $\mathbb{K}$*. Let* X *and* Y *be vector spaces of dimension n and m respectively. Define a map* $f_A : X \longrightarrow Y$ *by* $f_A(\mathbf{x}) = A\mathbf{x}$*. Then matrix multiplication,* f_A*, is a linear map:*

1. $f_A(\mathbf{x}_1 + \mathbf{x}_1) = A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = f_A(\mathbf{x}_1) + f_A(\mathbf{x}_2)$.
2. *For all* $k \in \mathbb{K}$ *and* $\mathbf{x} \in X$, $f_A(k\mathbf{x}) = A(k\mathbf{x}) = kA\mathbf{x} = kf_A(\mathbf{x})$.

Change of basis theorem:

Theorem 21 *Let X and Y be vector spaces over $\mathbb{K}$ of dimension n. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ be respective bases of X and Y. Then:*

1. *There exists a unique linear mapping $f_A : X \longrightarrow Y$ satisfying*

$$f_A(\mathbf{x}_1) = \mathbf{y}_1, \qquad f_A(\mathbf{x}_2) = \mathbf{y}_2, \qquad \ldots, \qquad f_A(\mathbf{x}_n) = \mathbf{y}_n.$$

2. *There exists an $n \times n$ matrix A such that $f_A(\mathbf{x}) = A\mathbf{x}$ and*

$$A(\mathbf{x}_1 \mid \mathbf{x}_2 \mid \ldots \mid \mathbf{x}_n) = (\mathbf{y}_1 \mid \mathbf{y}_2 \mid \ldots \mid \mathbf{y}_n).$$

proof:

Definition 17 *Let X and Y be vector spaces over $\mathbb{K}$. The X and Y are* isomorphic *if there is a linear map $f : X \longrightarrow Y$ such that f is a bijection, meaning injective (if $f(\mathbf{x}) = f(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$) and surjective (For all $\mathbf{y} \in Y$, there exists $\mathbf{x} \in X$ such that $\mathbf{y} = f(\mathbf{x})$).*

Theorem 22 *Let X be the set of all polynomials of the form $ax + b$, where $a, b \in \mathbb{R}$. Then there is a vector space isomorphism $X \cong \mathbb{R}^2$ given by*

$$f(ax + b) = (a, b).$$

Theorem 23 *Let $\mathbb{C} = \mathbb{R}(i)$ be the vector space of complex numbers $ai + b$ over the field $\mathbb{R}$. There is a vector space isomorphism $\mathbb{C} \cong \mathbb{R}^2$.*

4.2 Fundamental subspaces, rank and nullity

Theorem 24 *Let X and Y be vector spaces over $\mathbb{K}$ and let $F : X \longrightarrow Y$ be a linear map. If span* $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = X$*, then*

$$span\,(F(\mathbf{x}_1), F(\mathbf{x}_2), \ldots, F(\mathbf{x}_n)) = F[X].$$

Kernel and Image:

Definition 18 *Let $f : X \longrightarrow Y$ be a linear map. Then the* **kernel** *of the map f is given by*

$$ker(f) = \{\mathbf{x} \in X \ : \ f(\mathbf{x}) = \mathbf{0}\}.$$

Theorem 25 *If $f(\mathbf{x}) = A\mathbf{x}$ for some matrix A, then $ker(f) = NS(A)$, the nullspace of A. Hence $ker(f)$ is a subspace of X.*

Definition 19

$$im(f) = \{\mathbf{x} \in X \ : \ \ exists\ \mathbf{y} \in Y\ satisfying\ f(\mathbf{x}) = \mathbf{y}\}.$$

Theorem 26 *If $f(\mathbf{x}) = A\mathbf{x}$ for some matrix A, then $im(f) = CS(A)$. Hence the image of f is a subspace of Y.*

Definition 20 *The* **rank** *of a linear map* $f : X \longrightarrow Y$ *is the dimension of the image of* f, $dim(im(f))$. *The* **nullity** *of* f *is the dimension of the kernel of* f, $nullity = dim(ker(f))$ *If* $f(\mathbf{x}) = A\mathbf{x}$ *for some matrix A, then* $nullity = dim(NS(A))$ *and* $rank = dim(CS(A))$.

Theorem 27 *Let* X *and* Y *be finite dimensional vector spaces over* $\mathbb{K}$ *with* $dim(X) = n$ *and let* $f : X \longrightarrow Y$ *be a linear map. Then*

$$rank(f) + nullity(f) = n.$$

In other words, the sum of the dimensions of the kernel of f *and the image of* f *is equal to the dimension of* X.

Example 4.8 *Let* $A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \\ -6 & 3 & -9 \end{pmatrix}$. *The map* $f_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *given by* $f_A(\mathbf{x}) = A\mathbf{x}$ *is a linear map. Find a basis for each of* $NS(A)$ *and CS(A). Calculate the rank and nullity of the matrix A and verify that they add to* 3.

Example 4.9 *Let $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be the linear map given by*

$$F(x,y,z,t) = (x - y + z + t, 2x - 2y + 3z + 4t, 3x - 3y + 4z + 5t).$$

1. *Find a basis and dimension of $im(F)$.*
2. *Find a basis and dimension of $ker(F)$.*

Continuing,

Theorem 28 *Let* $f_A : X \longrightarrow Y$ *be a linear map with* $f_A(\mathbf{x}) = A\mathbf{x}$ *for some square matrix* A. *Then* f_A *is injective (or* one to one*) if and only if* $\det(A) \neq 0$*, if and only if* $NS(A) = \{\mathbf{0}\}$ *(or equivalently* $ker(f_A)$ *is trivial).*

5 Eigenvalues and Eigenvectors

5.1 Definitions

Eigenvalues and eigenvectors are very useful in Mathematics and Engineering - ODEs, PDEs, matrix algebra, data science, and much more.

Let A be a square $n \times n$ matrix with real entries. Then the *eigenvalues* λ and *eigenvectors* $\mathbf{x}(\neq \mathbf{0})$ are the scalars and vectors satisfying the eigenvalue-eigenvector equation

$$A\mathbf{x} = \lambda \mathbf{x}. \tag{1}$$

Any vector $\mathbf{x}$ satisfying (1) is called an eigenvector of A corresponding to the eigenvalue λ. It is easy to show that the eigenvalues $\lambda_1, \ldots \lambda_n$ of A are complex numbers satisfying $\det(A - \lambda I) = 0$. Since this is a polynomial of degree n, we will have n solutions $\lambda_i \in \mathbb{C}$ up to multiplicity (possible repeated roots) according to the fundamental theorem of algebra. It is clear then that an $n \times n$ matrix A has at most n distinct eigenvalues.

The eigenvalue-eigenvector equation can be written as $(A - \lambda I)X = \mathbf{0}$, where I is the $n \times n$ identity matrix and we assume the $a_{ij} \in \mathbb{R}$. We want non-trivial solutions $X \neq \mathbf{0}$. For this to occur, $(A - \lambda I)$ must be singular (not-invertible). This means that

$$\det(A - \lambda I) = 0,$$

and we obtain a polynomial

$$b_n \lambda^n + b_{n-1} \lambda^{n-1} + \cdots + b_0 = 0.$$

The solution gives real values λ_i or complex values that occur in complex conjugate pairs.

Example 5.1 *Find the eigenvalues of the matrix*

$$A = \begin{pmatrix} 3 & 5 & -2 & 5 & 8 \\ 2 & 1 & 1 & 0 & 12 \\ 6 & 0 & 4 & 3 & -7 \\ -3 & -4 & 1 & 0 & 8 \\ 0 & 9 & 1 & 0 & 6 \end{pmatrix}.$$

We have

$$\det(A - \lambda I) = -\lambda^5 + 14\lambda^4 + 20\lambda^3 - 304\lambda^2 + 4817\lambda - 17530.$$

The roots of this equation are

$$\lambda_1 = -7.87362, \qquad \lambda_2 = 3.87177,$$
$$\lambda_3 = 15.0574, \qquad \lambda_4, \lambda_5 = 1.47222 \pm 6.00186i.$$

Since the eigenvalues of a matrix may not be real numbers, the same is true for the eigenvectors of A. Clearly, the set of all linear combinations of the eigenvectors of real-valued A forms a subspace of $\mathbb{C}^n$.

Example 5.2 *Let* $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$, *let* $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, *and* $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. *Since*

$$A\mathbf{x}_1 = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

it is easy to see that $\mathbf{x}_1$ *is an eigenvector of* A *with corresponding eigenvalue* $\lambda_1 = 1$. *Since*

$$A\mathbf{x}_2 = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$\mathbf{x}_2$ *is an eigenvector of* A *with corresponding eigenvalue* $\lambda_2 = 4$.

To find these given the matrix A, we seek the roots of the **characteristic equation**

$$\det(A-\lambda) = \det\begin{pmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{pmatrix} = \lambda^2 - 5\lambda + 4 = 0.$$

Since we have the following factorisation of the left hand side of the characteristic equation

$$(\lambda-1)(\lambda-4) = 0,$$

the eigenvalues of A are $\lambda_1 = 1$ *and* $\lambda_2 = 4$. *To find the eigenvector* $\mathbf{x}_1$ *corresponding to* $\lambda_1 = 1$, *we have* $(A-1I)\mathbf{x}_1 = \mathbf{0}$, *so* $bvx_1 \in NS(A-I)$.

$$\begin{aligned} A - 1I &= \begin{pmatrix} 3-1 & 1 \\ 2 & 2-1 \end{pmatrix}, \\ &= \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \\ &\sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

It follows that $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. *To find the eigenvector* $\mathbf{x}_2$ *corresponding to* $\lambda_2 = 4$, *we have* $(A-4I)\mathbf{x}_2 = \mathbf{0}$, *so* $bvx_2 \in NS(A-4I)$.

$$\begin{aligned} A - 4I &= \begin{pmatrix} 3-4 & 1 \\ 2 & 2-4 \end{pmatrix}, \\ &= \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}, \\ &\sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

It follows that $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Theorem 29 *If* $\mathbf{x}$ *is an eigenvector of* A *and* k *is a scalar, then* $k\mathbf{x}$ *is an eigenvector of* A.

Theorem 30 *If* A *is a* 2×2 *matrix, then the charcterisic equation is*

$$\lambda^2 - tr(A)\lambda + \det(A) = 0.$$

Theorem 31 *Let A be a square matrix. The following are logically equivalent:*

1. λ *is an eigenvalue of* A.
2. $NS(A - \lambda I)$ *is non-trivial.*
3. *The row-reduced echelon form of* $A - \lambda I$ *has a row of zeros.*
4. $\det(A - \lambda I) = 0$.

5.2 Examples

Example 5.3 *Find all of the eigenvalues and eigenvectors of the matrix* $A = \begin{pmatrix} 5 & 3 \\ 2 & 10 \end{pmatrix}$.

Example 5.4 *Find all of the eigenvalues and eigenvectors of the matrix* $A = \begin{pmatrix} 3 & -4 \\ 2 & -6 \end{pmatrix}$.

Example 5.5 *Find all of the eigenvalues and eigenvectors of the matrix* $A = \begin{pmatrix} 1 & 1 \\ -3 & -1 \end{pmatrix}$.

Example 5.6 *Find all of the eigenvalues and eigenvectors of the matrix* $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Example 5.7 *Couple the ODE* $y''(x) + y'(x) + y(x) = 0$ *and write the corresponding characteristic equation using eigenvalues.*

5.3 Diagonalization of matrices

Definition 21 *A square matrix A is* **diagonalizable** *if there exists an invertible matrix P such that $D = P^{-1}AP$ is a diagonal matrix, equivalently $A = PDP^{-1}$.*

Theorem 32 *An $n \times n$ matrix A is diagonalizable if and only if there are n linearly independent eigenvectors of A. If A is diagonalizable, let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be the linearly independent eigenvectors of A and let*

$$P = (\mathbf{x}_1 \mid \mathbf{x}_2 \mid \ldots \mid \mathbf{x}_n).$$

Then

$$D = P^{-1}AP$$

with the diagonal entries of D being $a_{ii} = \lambda_i$.

Example 5.8 *Diagonalize the matrix $A = \begin{pmatrix} 5 & 3 \\ 2 & 10 \end{pmatrix}$.*

5.4 Singular value decomposition

Singular value decomposition is a decomposition of a (not necessarily square) matrix A, of dimension $m \times n$, into the matrices

$$A = USV^T,$$

where U is an $m \times m$ orthogonal matrix, V is an $n \times n$ orthogonal matrix, and S is an $m \times n$ diagonal matrix.

Note that since S is not (necessarily) square, it's not a 'proper' diagonal matrix; all of its non-zero values must lie on the main diagonal. These values are called the singular values of A.

.

If there exists a number σ and vectors u, and v such that

$$\begin{aligned} Av &= \sigma u, \\ A^T u &= \sigma v, \end{aligned}$$

then σ is a singular value of A, with corresponding left and right singular vectors u and v.

We can also calculate the SVD using diagonalisation. Write

$$\begin{aligned} A^T A &= V S^T U^T U S V^T, \\ &= V S^T S V^T, \end{aligned}$$

$$\begin{aligned} A A^T &= U S V^T V S^T U^T, \\ &= U S S^T U^T. \end{aligned}$$

Since S is diagonal, so is $S^T S$ and SS^T.

We now see that these are diagonalisations of $A^T A$ and AA^T.

So to find the singular value decomposition of A, we diagonalise these two (symmetric) matrices. The right singular vectors of A are the eigenvectors of $A^T A$ and the left singular vectors are the eigenvectors of AA^T.

Furthermore, the singular values of A are the square roots of the eigenvalues of $A^T A$ and AA^T (which are identical).

6 Linear Algebra Exercises

6.1 Practice Problems

1. Determine whether the vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ span the vector space $\mathbb{R}^2$.

2. Let P_2 be the vector space consisting of all polynomials of degree less than or equal to 2.

   (a) Show that if $f(x), g(x) \in P_2$, then $\text{span}(\{f(x), g(x)\}) \neq P_2$.

   (b) Show that $\text{span}(\{f(x), g(x)\})$ is a proper subspace of P_2.

3. Let

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} -2 \\ 1 \\ 3 \\ 1 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

be vectors in $\mathbb{R}^4$.

(a) Show that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent.

(b) What is the dimension of $\mathrm{span}(\{\mathbf{u}, \mathbf{v}, \mathbf{w}\})$?

(c) Let $\mathbf{p} = (0, 5, -1, 13)^T, \mathbf{q} = (5, 0, 2, -6)^T \in \mathbb{R}^4$. Show that $\mathrm{span}(\mathbf{p}, \mathbf{q})$ is a 2-dimensional subspace of $\mathrm{span}(\{\mathbf{u}, \mathbf{v}, \mathbf{w}\})$.

4. Define $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $f(x,y,z) = (x+3y-2, y+z-2)$. Show that f is a function but f is not a linear map.

5. Consider the vectors $\mathbf{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

(a) Show that $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(b) Write $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(c) Find real numbers α, β, γ such that $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = \mathbf{0}$.

6. Let $A = \begin{pmatrix} 2 & -6 \\ -1 & 3 \end{pmatrix}$.

(a) Calculate the eigenvalues and eigenvectors of the matrix A.

(b) Diagonalize A.

(c) Show that for any positive integer n, $A^n = 5^n A$.

7. Define a map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ by

$$f(x,y,z) = (3x+7y+z, 4x+8y+2z, -3x-9y, x+y+z).$$

(a) Show that f is a linear map.

(b) Find a basis for the image of f.

(c) Find a basis for the kernel of f.

8. The trace $\mathrm{tr}(z)$ of a complex number $z = a + bi$ is the sum of the conjugates. The norm $\mathrm{nm}(z))$ of a complex number $z = a + bi$ is the product of the conjugates.

   (a) Define $f : \mathbb{C} \longrightarrow \mathbb{R}^2$ by $f(z) = (\mathrm{tr}(z), \mathrm{nm}(z))$. Show that f is not a linear map.

   (b) Define $g : \mathbb{C} \longrightarrow \mathbb{R}$ by $g(z) = \Re(z) + \Im(z)$. Show that g is a linear map.

   (c) Give a basis for the image and the kernel of g.

6.2 A Linear Algebra Exam

This is intended to be completed in 90 Minutes

This is an open-book, open-notes, and open-tutorial-solutions exam.

Full working must be shown on the pages provided.

Permitted materials: A pocket calculator or graphics calculator.

Mobile phones and laptops are not permitted. Please switch phones off.

Name:

1. Consider the vectors

$$\mathbf{u} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

in the vector space $\mathbb{R}^2$.

(a) Calculate $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v}$. **(3 Marks)**

(b) Explain whether $\mathbf{w}$ is an element of $\text{span}(\mathbf{u}, \mathbf{v})$. **(2 Marks)**

(c) Determine whether $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. **(2 Marks)**

2. Consider the vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$$

in the vector space $\mathbb{R}^2$.

(a) Write the vector $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. **(4 Marks)**

(b) Are the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ linearly independent?
Explain why or why not. **(2 Marks)**

(c) Show that $\text{span}(\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}) = \mathbb{R}^2$. **(7 Marks)**

(d) What is the dimension of the vector space $\text{span}(\{\mathbf{u}, \mathbf{v}, \mathbf{w}\})$?
. **(2 Marks)**

3. Let V be the vector space over $\mathbb{R}$ consisting of all polynomials in the variable x with real coefficients. Determine whether the polynomials

$$f(x) = x^2 + 3, \qquad g(x) = -x^2 + x + 2, \qquad h(x) = x^2 - 2x - 7$$

are linearly independent. **(10 Marks)**

4. Let

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 3 & 2 & -7 \end{pmatrix}.$$

(a) Calculate a basis for the nullspace of A and a basis for the column space of A.

. **(6 Marks)**

(b) What is the rank of A and the nullity of A? **(2 Marks)**

(c) Consider the linear map $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by

$$F\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Calculate the dimension of the image of F and the dimension of the kernel of F. **(2 Marks)**

(d) Show that the image of F is a subspace of $\mathbb{R}^3$. **(8 Marks)**

Space for working.

5. Let $A = \begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}$.

(a) Calculate the eigenvalues and eigenvectors of A. **(12 Marks)**

(b) Express the matrix A as $A = PDP^{-1}$, where P is a 2×2 invertible matrix and D is a diagonal matrix. **(4 Marks)**

(c) Use Part (b) to calculate $A^4 = AAAA$, and hence A^{100}. Explain your reasoning.

. **(4 Marks)**

7 Tutorial Problems

7.1 Problem Set 1

Summary.

- Let $A = [a_{ij}]$, $B = [b_{ij}]$ be $m \times n$ matrices and let $p, q \in \mathbb{R}$. A linear combination of A and B is of the form

$$\begin{aligned} pA + qB &= p\,[a_{ij}] + q\,[b_{ij}]\,, \\ &= [pa_{ij}] + [qb_{ij}]\,, \\ &= [pa_{ij} + qb_{ij}]\,. \end{aligned}$$

- Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with $\mathbf{u} = [u_i]_{1\times n}$, $\mathbf{v} = [v_i]_{1\times n}$. Then:

  1. The *dot product* of $\mathbf{u}$ and $\mathbf{v}$ is given by

  $$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + \ldots u_nv_n = \sum_{i=1}^{n} u_iv_i.$$

  2. $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if $\mathbf{u}\cdot\mathbf{v} = 0$.
  3. The *norm* of $\mathbf{u}$ is given by $\|\mathbf{u}\| = \sqrt{\sum_{i=1}^{n} u_i^2}$.
  4. $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
  5. If $\mathbf{u} \neq \mathbf{0}$, then the *unit vector* $\widehat{\mathbf{u}} = \frac{1}{\|\mathbf{u}\|}\mathbf{u}$ is a vector of norm 1 is the direction of the vector $\mathbf{u}$.
  6. If $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$, then $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

7. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $(n = 3)$, the *cross product* of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}, \\
&= \det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} \mathbf{i} - \det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} \mathbf{j} + \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \mathbf{k}, \\
&= \left(u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \right).
\end{aligned}$$

- A vector space V over a field $\mathbb{K}$ of scalars (with $0, 1$) is a set of vectors that satisfies the following axioms:

  1. Addition is associative: $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$.
  2. $\mathbf{0} \in V$ is neutral.
  3. For all $\mathbf{u} \in V$, $-\mathbf{u} \in V$.
  4. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$.
     In other words $\langle V, + \rangle$ is an abelian group.
  5. For $k \in \mathbb{K}$, $\mathbf{u}, \mathbf{v} \in V$, $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}$.
  6. For $a, b \in \mathbb{K}$, $\mathbf{u} \in V$, $(a+b)\mathbf{u} = a\mathbf{u}+b\mathbf{u}$.
  7. For $a, b \in \mathbb{K}$, $\mathbf{u} \in V$, $(ab)\mathbf{u} = a(b\mathbf{u})$.
  8. $1\mathbf{u} = \mathbf{u}$.

The following problems are found in Lipschutz and Lipson [6].

(1) Let

$$\mathbf{u} = \begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}.$$

Find

(a) $5\mathbf{u} - 2\mathbf{v}$,

(b) $-2\mathbf{u} + 4\mathbf{v} - 3\mathbf{w}$.

(2) Write the vector $\mathbf{v} = (1, -2, 5)$ as a linear combination of the vectors

$$\mathbf{u}_1 = (1, 1, 1), \qquad \mathbf{u}_2 = (1, 2, 3), \qquad \mathbf{u}_3 = (2, -1, 1).$$

(3) Find $\mathbf{u} \cdot \mathbf{v}$ where:

(a) $\mathbf{u} = (2, -5, 6)$, $\mathbf{v} = (8, 2, -3)$.

(b) $\mathbf{u} = (4, 2, -3, 5, -1)$, $\mathbf{v} = (2, 6, -1, -4, 8)$.

(4) Let

$$\mathbf{u} = (5,4,1), \qquad \mathbf{v} = (3,-4,1), \qquad \mathbf{w} = (1,-2,3).$$

Which pair of vectors, if any, are orthogonal?

(5) Find k such that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, where:

(a) $\mathbf{u} = (1,k,-3)$, $\mathbf{v} = (2,-5,4)$.

(b) $\mathbf{u} = (2,3k,-4,1,5)$, $\mathbf{v} = (6,-1,3,7,2k)$.

(6) Find $\|\mathbf{u}\|$, where

(a) $\mathbf{u} = (3,-12,-4)$,

(b) $\mathbf{u} = (2,-3,8,-7)$.

(7) Let

$$\mathbf{u} = (3,-4), \qquad \mathbf{v} = (4,-2,-3,8), \qquad \mathbf{w} = \left(\frac{1}{2},\frac{2}{3},-\frac{1}{4}\right).$$

Calculate $\widehat{\mathbf{u}}$, $\widehat{\mathbf{v}}$, and $\widehat{\mathbf{w}}$.

(8) Let

$$\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}, \qquad \mathbf{v} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}, \qquad \mathbf{w} = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}.$$

Calculate:

(a) $\mathbf{u} \times \mathbf{v}$,

(b) $\mathbf{u} \times \mathbf{w}$.

(9) Find $\mathbf{u} \times \mathbf{v}$, where:

(a) $\mathbf{u} = (1,2,3)$, $\mathbf{v} = (4,5,6)$.

(b) $\mathbf{u} = (-4,7,3)$, $\mathbf{v} = (6,-5,2)$.

(10) Let $\mathbf{u} = (1,2,-2)$, $\mathbf{v} = (3,-12,4)$, $k = -3$.

(a) Calculate $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\|\mathbf{u}+\mathbf{v}\|$, $\|k\mathbf{u}\|$.

(b) Verify that $\|k\mathbf{u}\| = |k|\ \|\mathbf{u}\|$ and $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

(11) Find k such that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, where:

(a) $\mathbf{u} = (3,k,-2)$, $\mathbf{v} = (6,-4,-3)$.

(b) $\mathbf{u} = (5,k,-4,2)$, $\mathbf{v} = (1,-3,2,2k)$.

(c) $\mathbf{u} = (1,7,k+2,-2)$, $\mathbf{v} = (3,k,-3,k)$.

(12) Given

$$\mathbf{u} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}, \qquad \mathbf{v} = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}, \qquad \mathbf{w} = 4\mathbf{i} + 7\mathbf{j} + 2\mathbf{k},$$

calculate:

(a) $2\mathbf{u} - 3\mathbf{v}$.

(b) $3\mathbf{u} + 4\mathbf{v} - 2\mathbf{w}$

(c) $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\cdot\mathbf{w}$, $\mathbf{v}\cdot\mathbf{w}$.

(d) $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\|\mathbf{w}\|$.

(13) Given

$$\mathbf{u} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}, \qquad \mathbf{v} = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}, \qquad \mathbf{w} = 4\mathbf{i} + 7\mathbf{j} + 2\mathbf{k},$$

calculate:

(a) $\mathbf{u} \times \mathbf{v}$.

(b) $\mathbf{u} \times \mathbf{w}$.

(c) $\mathbf{v} \times \mathbf{w}$.

(14) Find a unit vector $\widehat{\mathbf{u}}$ orthogonal to:

(a) $\mathbf{v} = (1,2,3)$, $\mathbf{w} = (1,-1,2)$.

(b) $\mathbf{v} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{w} = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

(15) Consider the following theorem: W is a subspace of a vector space V if the following two conditions hold:

– $\mathbf{0} \in W$,

– If $\mathbf{u}, \mathbf{v} \in W$ and $k \in \mathbb{K}$, then $\mathbf{u}+\mathbf{v} \in W$ and $k\mathbf{u} \in W$.

Now for the following W, show that W is not a subset of $V = \mathbb{R}^3$ and that the theorem does not hold.

(a) $W = \{(a,b,c) \,:\, a \geq 0\}$.

(b) $W = \left\{(a,b,c) \,:\, a^2+b^2+c^2 = 1\right\}$.

(16) Let $V = \mathbf{P}(x)$, the vector space consisting of polynomials $q_n(x) = \sum_{j=0}^{n} a_j x^j$ where $a_j \in \mathbb{R}$ and n is a non-negative integer. Determine whether W is a subspace of V, where:

(a) W consists of all polynomials with integer coefficients.

(b) W consists of all polynomials with degree ≥ 6 and the zero polynomial.

(c) W consists of all polynomials with only even powers of x.

(17) Let V be the vector space of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$. Show that W is a subspace of V, where:

(a) $W = \{f(x) \ : \ f(1) = 0\}$.

(b) $W = \{f(x) \ : \ f(1) = f(3)\}$

(c) $W = \{f(x) \ : \ f(-t) = -f(t)\}$, *odd functions*.

(18) Suppose $\mathbf{u}, \mathbf{v} \in V$. Simplify $4(5\mathbf{u} - 6\mathbf{v}) + 2(3\mathbf{u} + \mathbf{v})$.

(19) Show that the axiom $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ of a vector space can be derived from the other axioms.

(20) Let V be the set of ordered pairs (a, b) of real numbers. Show that V is not a vector space over $\mathbb{R}$, where addition and scalar multiplication are defined by:

(a) $(a,b) + (c,d) = (a+d, b+c)$ and $k(a,b) = (ka, kb)$.

(b) $(a,b) + (c,d) = (a+c, b+d)$ and $k(a,b) = (a,b)$.

(c) $(a,b) + (c,d) = (0,0)$ and $k(a,b) = (ka, kb)$.

(d) $(a,b) + (c,d) = (ac, bd)$ and $k(a,b) = (ka, kb)$.

7.2 Problem Set 2

Summary.

- Let $U = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a collection of vectors in the vector space V of a field $\mathbb{K}$. The *span* of U or span of the vectors in U, denoted

$$\operatorname{span}(U) = \operatorname{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n),$$

is the set of all linear combinations of the vectors in U:

$$\operatorname{span}(U) = \left\{ \mathbf{v} \in V \ : \ \mathbf{v} = \sum_{j=1}^{n} a_j \mathbf{u}_j, \text{ where } a_j \in \mathbb{K} \right\}.$$

- Let U be a subset of the vector space V. Then:

  1. $U \subseteq \operatorname{span}(U) \subseteq V$ and $\operatorname{span}(U)$ is a subspace of V.
  2. If W is a subspace of V and $U \subseteq W$, then $\operatorname{span}(U) \subseteq W$.

- Let $A = [a_{ij}]_{m\times n}$ with $a_{ij} \in \mathbb{K}$, where $\mathbb{K}$ is a field. Let

$$R_i = [a_{ij}]_{1\times n} = (a_{i1}, a_{i2}, \ldots, a_{in}).$$

The *row space* of A, denoted $\operatorname{RS}(A)$, is given by

$$\operatorname{RS}(A) = \operatorname{span}(R_1, R_2, \ldots, R_m).$$

If B is the row reduced echelon form of A, then

$$\operatorname{RS}(A) = \operatorname{RS}(B).$$

- Let $U = \{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n\}$ be a collection of vectors in the vector space V over the field $\mathbb{K}$. Then the vectors $\mathbf{u}_j \in U$ are *linearly dependent* if there exist $a_1, a_2, \ldots, a_n \in \mathbb{K}$, (not all zero), such that

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n = 0.$$

If U is not linearly dependent, then U is *linearly independent*.

- The non-zero rows of a matrix that is in row reduced echelon form are linearly independent.

Example 7.1 *Consider the vectors* $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 4 \\ -3 \\ -4 \end{pmatrix}$ *in the vector space* $\mathbb{R}^3$ *over* $\mathbb{R}$. *By taking the cross product* $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} -13 \\ -4 \\ -10 \end{pmatrix}$, *we see that this vector is normal to the plane* $-13x - 4y - 10z = 0$ *so this plane is parallel to both* $\mathbf{u}$ *and* $\mathbf{v}$. *See Figure 2. In fact it is easy to show that any linear combination* $\mathbf{w}$ *of* $\mathbf{u}$ *and* $\mathbf{v}$, *so* $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$, *with* $a, b \in \mathbb{R}$, *must also be parallel to the plane* $-13x - 4y - 10z = 0$ *and all linear combinations of* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel to this plane. It is easy to see then that we cannot obtain any vector in* $\mathbb{R}^3$ *by a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$ *since not all vectors in* $\mathbb{R}^3$ *are parallel to the plane* $-13x - 4y - 10z = 0$. *For example, the vector* $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ *is not a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$. *We may think of span*$(\mathbf{u}, \mathbf{v})$ *geometrically as the set of all vectors that are parallel to the plane* $-13x - 4y - 10z = 0$. *It is clear that this is a proper subspace of* $\mathbb{R}^3$ *and not equal to* $\mathbb{R}^4$.

See Lipschutz and Lipson [6] for some of the following problems.

(1) Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v} \in \mathbb{R}^3$. Write $\mathbf{v} = \begin{pmatrix} 9 \\ -3 \\ 16 \end{pmatrix}$ as a linear combination of

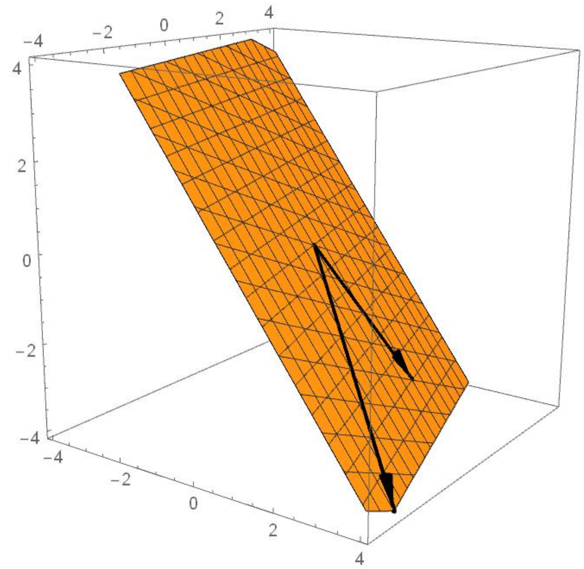

Figure 2: Example

the vectors

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad \mathbf{u}_2 = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}, \qquad \mathbf{u}_3 = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}.$$

(2) Consider the following vectors in $V = \mathbb{R}^3$:

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 5 \\ 8 \end{pmatrix}.$$

Show that $\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = V$.

(3)

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 8 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_4 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 20 \end{pmatrix}.$$

Show that $\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ is not equal to $\mathbb{R}^4$. Are the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ linearly dependent or linearly independent?

(4) Suppose that the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent. Show that the vectors $\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}, \mathbf{u}-2\mathbf{v}+\mathbf{w}$ are also linearly independent.

(5) Show that the vector space $V = \mathbb{P}(t)$ of polynomials with real coefficients cannot be spanned by a finite number of polynomials.

(6) Consider the vectors $\mathbf{u} = (1,2,3), \mathbf{v} = (2,3,1) \in \mathbb{R}^3$.

1. Write $\mathbf{w} = (1,3,8)$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
2. Write $\mathbf{w} = (2,4,5)$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
3. Find k such that $\mathbf{w} = (1,k,4)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
4. Find conditions on a,b,c such that $\mathbf{w} = (a,b,c)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(7) Determine whether the polynomials

$$u = t^3 - 4t^2 + 3t + 3, \quad v = t^3 + 2t^2 + 4t - 1, \quad w = 2t^3 - t^2 - 3t + 5$$

are linearly dependent or linearly independent.

(8) Consider the following three subspaces of $\mathbb{R}^3$:

$$\begin{aligned} U_1 &= \operatorname{span}\{(1,1,-1),(2,3,-1),(3,1,-5)\}, \\ U_2 &= \operatorname{span}\{(1,-1,-3),(3,-2,-8),(2,1,-3)\}, \\ U_3 &= \operatorname{span}\{(1,1,1),(1,-1,3),(3,-1,7)\}. \end{aligned}$$

Which of U_1, U_2, U_3 are equal?

7.3 Problem Set 3

Summary.

- Let $U = \{\mathbf{u}_1, \mathbf{u}_2, \dots \mathbf{u}_n\}$ be a collection of vectors in the vector space V over the field $\mathbb{K}$. Then the vectors $\mathbf{u}_j \in U$ are *linearly dependent* if there exist $a_1, a_2, \dots, a_n \in \mathbb{K}$ (not all zero) such that

  $$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n = 0.$$

  If U is not linearly dependent, then U is *linearly independent*.

- Let $U = \{\mathbf{u}_1, \mathbf{u}_2, \dots \mathbf{u}_n\}$ be a collection of vectors in the vector space V over the field $\mathbb{K}$. Then the vectors $\mathbf{u}_j \in U$ are *linearly independent* if whenever

  $$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n = 0,$$

  then we must have $a_1 = a_2 = \cdots = a_n = 0$.

- The non-zero rows of a matrix that is in row reduced echelon form are linearly independent.

- Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be a collection of vectors in a vector space V over $\mathbb{K}$. Then B is a *basis* of V if both:

  1. $\text{span}(B) = V$,
  2. The vectors in B are linearly independent.

- A basis of a vector space V is not unique in general (there may be exceptions for unusual fields of scalars like $\mathbb{K} = \{0, 1 \pmod 2\}$).

- If B_1 and B_2 are bases for a vector space V, then they have the same number of elements.

- The *dimension* of a vector space V, denoted $\dim(V)$, is the number of elements in any basis of V.

- If $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a basis of a vector space V, then B is an *orthogonal basis* of V if for all $\mathbf{u}_i, \mathbf{u}_j \in B$ with $i \neq j$, then $\mathbf{u}_i \cdot \mathbf{u}_j = 0$.
- If $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of a vector space V, then B is *orthonormal* if all $\mathbf{u}_i \in B$ satisfy $\|\mathbf{u}_i\| = 1$.
- Let $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. $B = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an orthonormal basis for $\mathbb{R}^3$ and $\dim\left(\mathbb{R}^3\right) = 3$.
- Orthonormal bases are not unique.
- Let $\mathbf{e}_j$ be the $n \times 1$ matrix whose entries a_{i1} are 0 if $i \neq j$ and 0 if $i = j$. Then $B = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is an orthonormal basis for $\mathbb{R}^n$.
- The *rank* of a matrix A, denoted $\operatorname{rank}(A)$, is the dimension of the vector space spanned by the columns of A, the number of non-zero rows in the row-reduced echelon form of A.
- Let $U = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be n vectors in a vector space V of dimension m. If $n > m$, then the vectors in U are linearly dependent.
- The *null-space* of an $m \times n$ matrix A with entries in $\mathbb{K}$, denoted $\mathrm{NS}(A)$ is the vector space consisting of the vectors $\mathrm{NS}(A) = \{X \in V \ : \ AX = \mathbf{0}\}$. $\mathrm{NS}(A)$ is a sub-space of V.

See Lipschutz and Lipson [6] for some of the following problems.

(1) Determine whether or not **u** and **v** are linearly dependent, where:

(a) $\mathbf{u} = (1,2)$, $\mathbf{v} = (3,-5)$.

(b) $\mathbf{u} = (1,2,-3)$, $\mathbf{v} = (4,5,-6)$.

(c) $\mathbf{u} = (1,-3)$, $\mathbf{v} = (-2,6)$.

(d) $\mathbf{u} = (2,4,-8)$, $\mathbf{v} = (3,6,-12)$.

(2) Determine whether or not **u** and **v** are linearly dependent, where:

(a) $u(t) = 2t^2 + 4t - 3$, $v(t) = 4t^2 + 8t - 6$.

(b) $u(t) = 2t^2 - 3t + 4$, $v(t) = 4t^2 - 3t + 2$.

(c) $\mathbf{u} = \begin{pmatrix} 1 & 3 & -4 \\ 5 & 0 & -1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -4 & -12 & 16 \\ -20 & 0 & 4 \end{pmatrix}$.

(d) $\mathbf{u} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$.

(3) Determine whether or not the vectors $\mathbf{u} = (1,1,2)$, $\mathbf{v} = (2,3,1)$, $\mathbf{w} = (4,5,5) \in \mathbb{R}^3$ are linearly dependent.

(4) Determine whether or not each of the following list of vectors in $\mathbb{R}^3$ is linearly dependent:

(a) $\mathbf{u}_1 = (1,2,5)$, $\mathbf{u}_2 = (1,3,1)$, $\mathbf{u}_3 = (2,5,7)$, $\mathbf{u}_4 = (3,1,4)$.

(b) $\mathbf{u} = (1,2,5)$, $\mathbf{v} = (2,5,1)$, $\mathbf{w} = (1,5,2)$.

(c) $\mathbf{u} = (1,2,3)$, $\mathbf{v} = (0,0,0)$, $\mathbf{w} = (1,5,6)$.

(5) Determine whether or not each of the following form a basis of $\mathbb{R}^3$:

(a) $(1,1,1), (1,0,1)$.

(b) $(1,1,1), (1,2,3), (2,-1,1)$.

(c) $(1,2,3), (1,3,5), (1,0,1), (2,3,0)$.

(d) $(1,1,2),(1,2,5),(5,3,4)$.

(6) Determine whether $(1,1,1,1)$, $(1,2,3,2)$, $(2,5,6,4)$, $(2,6,8,5)$ form a basis of $\mathbb{R}^4$. If not, find the dimension of the subspace they span.

(7) Extend

$$\{\mathbf{u}_1 = (1,1,1,1), \mathbf{u}_2 = (2,2,3,4)\}$$

to a basis of $\mathbb{R}^4$.

(8) Determine whether the following vectors in $\mathbb{R}^4$ are linearly dependent or independent:

(a) $(1,2,-3,1)$, $(3,7,1,-2)$, $(1,3,7,-4)$.

(b) $(1,3,1,-2)$, $(2,5,-1,3)$, $(1,3,7,-2)$.

(9) Determine whether the following polynomials $u,v,w \in \mathbb{P}(t)$ are linearly dependent or independent:

(a) $u(t) = t^3 - 4t^2 + 3t + 3$, $v(t) = t^3 + 2t^2 + 4t - 1$, $w(t) = 2t^3 - t^2 - 3t + 5$.

(b) $u(t) = t^3 - 5t^2 - 2t + 3$, $v(t) = t^3 - 4t^2 - 3t + 4$, $w(t) = 2t^3 - 17t^2 - 7t + 9$.

(10) True or False: If false, give a counter-example.

(a) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ span V, then $\dim(V) = 3$.

(b) If A is a 4×8 matrix, then any six columns are linearly dependent.

(c) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent, then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent.

(d) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly dependent, then $\dim(V) \geq 4$.

(e) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ span V, then $\mathbf{w}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ span V.

(f) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly independent, then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent.

Summary.

- A *map* or *mapping* is an unambiguous rule f for sending the elements of the set X to the elements of the set Y, denoted $f : X \longrightarrow Y$.

- For example $f : \mathbb{Z} \longrightarrow \mathbb{R}$ given by $f(x) = x^2$ is a map or *function* that send an integer to it's square. The *domain* of f is X and the *codomain* of f is Y. In the example above, the domain is $\mathbb{Z}$, the codomain is $\mathbb{R}$ and the *image* of the map f, denoted $f[X]$, here it is $f[\mathbb{Z}]$, is the set of all elements of $\mathbb{R}$ that are of the form $f(x)$, here $f[X] = \mathbb{Z}^2$, where $\mathbb{Z}^2$ is the set of integer squares. In another context, $\mathbb{Z}^2$ might be used to refer to the Cartesian product $\mathbb{Z} \times \mathbb{Z}$ but we don't mean that here.

- A map $f : X \longrightarrow Y$ is said to be a linear map if:

  1. X and Y are vector spaces over a field $\mathbb{K}$.
  2. For all $\mathbf{u}, \mathbf{v} \in X$, $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ (we can say f is a vector space *homomorphism*).
  3. For all $k \in \mathbb{K}$ and $\mathbf{u} \in X$, $f(k\mathbf{u}) = kf(\mathbf{u})$.

- Let A be an $m \times n$ matrix with entries in $\mathbb{K}$. Let X and Y be vector spaces of dimension n and m respectively. Define a map $f_A : X \longrightarrow Y$ by $f_A(\mathbf{x}) = A\mathbf{x}$. Then:

  1. $f_A(\mathbf{x}_1 + \mathbf{x}_1) = A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = f_A(\mathbf{x}_1) + f_A(\mathbf{x}_2)$.
  2. For all $k \in \mathbb{K}$ and $\mathbf{x} \in X$, $f_A(k\mathbf{x}) = A(k\mathbf{x}) = kA\mathbf{x} = kf_A(\mathbf{x})$.
  3. In other words, matrix multiplication is a linear map.

- Let X and Y be vector spaces over $\mathbb{K}$ of dimension n. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ be respective bases of X and Y. Then:

1. There exists a unique linear mapping $f_A : X \longrightarrow Y$ satisfying

$$f_A(\mathbf{x}_1) = \mathbf{y}_1, \qquad f_A(\mathbf{x}_2) = \mathbf{y}_2, \qquad \ldots, \qquad f_A(\mathbf{x}_n) = \mathbf{y}_n.$$

2. There exists an $n \times n$ matrix A such that $f_A(\mathbf{x}) = A\mathbf{x}$ and

$$A(\mathbf{x}_1 \mid \mathbf{x}_2 \mid \ldots \mid \mathbf{x}_n) = (\mathbf{y}_1 \mid \mathbf{y}_2 \mid \ldots \mid \mathbf{y}_n).$$

- Let X and Y be vector spaces over $\mathbb{K}$. The X and Y are *isomorphic* if there is a linear map $f : X \longrightarrow Y$ such that f is a bijection, meaning injective (if $f(\mathbf{x}) = f(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$) and surjective (For all $\mathbf{y} \in Y$, there exists $\mathbf{x} \in X$ such that $\mathbf{y} = f(\mathbf{x})$).

- Let X be the set of all polynomials of the form $ax + b$, where $a, b \in \mathbb{R}$. Then there is a vector space isomorphism $X \cong \mathbb{R}$ given by $f(ax+b) = (a, b)$.

- Kernel and Image. Let $f : X \longrightarrow Y$ be a linear map. Then

$$\ker(f) = \{\mathbf{x} \in X \ : \ f(\mathbf{x}) = \mathbf{0}\}.$$

If $f(\mathbf{x}) = A\mathbf{x}$ for some matrix A, then $\ker(f) = \mathrm{NS}(A)$, the nullspace of A. Hence $\ker(f)$ is a subspace of X.

$$\mathrm{im}(f) = \{\mathbf{x} \in X \ : \ \text{exists } \mathbf{y} \in Y \text{ satisfying } f(\mathbf{x}) = \mathbf{y}\}.$$

If $f(\mathbf{x}) = A\mathbf{x}$ for some matrix A, then $\mathrm{im}(f) = \mathrm{CS}(A)$. Hence the image of f is a subspace of Y.

- Let X and Y be vector spaces over $\mathbb{K}$ with $\dim(X) = n$ and let $f : X \longrightarrow Y$ be a linear map. Then

$$\mathrm{rank}(f) + \mathrm{nullity}(f) = n.$$

In other words, the sum of the dimensions of the kernel of f and the image of f is equal to the dimension of X.

- Let $f_A : X \longrightarrow Y$ be a linear map with $f_A(\mathbf{x}) = A\mathbf{x}$ for some square matrix A. Then f_A is injective (or *one to one*) if and only if $\det(A) \neq 0$, if and only if $\mathrm{NS}(A) = \{\mathbf{0}\}$ (or equivalently $\ker(f_A)$ is trivial).

See Lipschutz and Lipson [6] for some of the following problems.

(1) Define $f : A \longrightarrow B$, where their rules are respectively given below and $A = \{a,b,c\}$, $B = \{x,y,z\}$. State whether each rule defines a mapping.

(a) $f(a) = y$, $f(c) = x$.

(b) $f(a) = y$, $f(b) = z$, $f(c) = x$, $f(c) = z$.

(c) $f(a) = x$, $f(b) = z$, $f(c) = x$.

(2) Consider the map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $f(x,y,z) = (yz, x^2)$. Find:

(a) $f(2,3,4)$,

(b) $f(5,-2,7)$,

(c) The pre-image of $\{(0,0)\}$.

(3) Define the map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $f(x,y) = (x+y, y)$. Show that f is a linear map.

(4) Show that the following maps are not linear:

(a) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $f(x,y) = (xy, x)$.

(b) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $f(x,y) = (x+3, 2y, x+y)$.

(c) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $f(x,y,z) = (|\, x \,|, y+z)$.

(5) Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be the linear map given by

$$f(x,y,z,t) = (x - y + z + t, x + 2z - t, x + y + 3z - 3t).$$

Find a basis and the dimension of:

(a) $\mathrm{im}(f)$.

(b) $\ker(f)$.

(6) Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear map given by

$$f(x,y,z) = (x+2y-z, y+z, x+y-2z).$$

Find a basis and the dimension of:

(a) $\operatorname{im}(f)$.

(b) $\ker(f)$.

(7) Show that the following maps are linear:

(a) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $f(x,y,z) = (x+2y-3z, 4x-5y+6z)$.

(b) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $f(x,y) = (ax+by, cx+dy)$, where $a,b,c,d \in \mathbb{R}$.

(8) Show that the following maps are not linear:

(a) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $f(x,y) = \left(x^2, y^2\right)$.

(b) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $f(x,y,z) = (x+1, y+z)$.

(c) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $f(x,y) = (xy, y)$.

(9) For each linear map f find a basis and the dimension of the kernel and the image of f.

(a) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $f(x,y,z) = (x+2y-3z, 2x+5y-4z, x+4y+z)$.

(b) $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ by $f(x,y,z) = (x+2y+3z+2t, 2x+4y+7z+5t, x+2y+6z+5t)$.

(10) For each linear map g, find a basis and the dimension of the kernel and the image of g:

(a) $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $g(x,y,z) = (x+y+z, 2x+2y+2z)$.

(b) $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $g(x,y,z) = (x+y, y+z)$.

(c) $g : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ by

$$g(x,y,z,s,t) = (x+2y+2z+s+t, x+2y+3z+2s-t, 3x+6y+8z+5s-t)$$

7.5 Problem Set 5

Summary.

- Let A be a square matrix. Then the *eigenvalues* λ and *eigenvectors* $\mathbf{x}(\neq \mathbf{0})$ are the scalars and vectors satisfying the equation

$$A\mathbf{x} = \lambda \mathbf{x}.$$

- If $\mathbf{x}$ is an eigenvector of A and k is a scalar, then $k\mathbf{x}$ is an eigenvector of A.

- Let A be a square matrix. The following are logically equivalent:
  1. λ is an eigenvalue of A.
  2. $NS(A - \lambda I)$ is non-trivial.
  3. The row-reduced echelon form of $A - \lambda I$ has a row of zeros.
  4. $\det(A - \lambda I) = 0$.

- The *characteristic polynomial* of A is $\Delta(\lambda) = \det(A - \lambda I)$ and the eigenvalues of A are the roots of the characteristic polynomial.

- A square matrix A is diagonalizable if there exists an invertible matrix P such that $D = P^{-1}AP$ is a diagonal matrix, equivalently $A = PDP^{-1}$.

- An $n \times n$ matrix A is diagonalizable if and only if there are n linearly independent eigenvectors of A.

- If A is diagonalizable, let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be the linearly independent eigenvectors of A and let

$$P = \left(\mathbf{x}_1 \mid \mathbf{x}_2 \mid \ldots \mid \mathbf{x}_n\right).$$

  Then

$$D = P^{-1}AP$$

with the diagonal entries of D being $a_{ii} = \lambda_i$.

See Lipschutz and Lipson [6] for some of the following problems.

(1) Find the characteristic polynomial of each of the following matrices:

(a) $A = \begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix}$.

(b) $B = \begin{pmatrix} 7 & -3 \\ 5 & -2 \end{pmatrix}$.

(c) $C = \begin{pmatrix} 3 & -2 \\ 9 & -3 \end{pmatrix}$.

(2) Find the characteristic polynomial of each of the following matrices:

(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 4 \\ 6 & 4 & 5 \end{pmatrix}$.

(b) $B = \begin{pmatrix} 1 & 6 & -2 \\ -3 & 2 & 0 \\ 0 & 3 & -4 \end{pmatrix}$.

(3) Let $A = \begin{pmatrix} 3 & -4 \\ 2 & -6 \end{pmatrix}$.

(a) Find all eigenvalues and corresponding eigenvectors.

(b) Find matrices P and D such that P is nonsingular and $D = P^{-1}AP$ is diagonal.

(4) Let $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. Find all eigenvalues and corresponding eigenvectors.

(5) For each of the following matrices, find all eigenvectors and corresponding linearly independent eigenvectors. When possible, find the nonsingular matrix P that diagonalizes the matrix:

(a) $A = \begin{pmatrix} 2 & -3 \\ 2 & -5 \end{pmatrix}$.

(b) $B = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}$.

(c) $C = \begin{pmatrix} 1 & -4 \\ 3 & -7 \end{pmatrix}$.

(6) Let $A = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}$. Find eigenvalues and corresponding eigenvectors.

(7) Let $A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$. Find eigenvalues and corresponding eigenvectors.

8 Books & Notes

References

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