Years 10, 11 and 12 Mathematical Methods

Student Workbook and Teaching Template

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https://mathofcourse.com/

Contents

1. **Imperative:** Print this pdf document or be prepared to annotate the pdf with a tablet. Some blank spaces for writing are a little small for large writing. If you cannot do either of these annotation options, then write notes on blank paper, noting the relevant position within the typed course notes. As you watch the intructional videos, write notes in the blank spaces. This step is very important.

2. The instructor should write exercises from an appropriate textbook where the text says **Exercises/Homework**.

3. **Optional but highly recommended:** Purchase and use *Mathematica* or obtain it through your institution. We will occasionally use this to display various graphics and verify calculations. All graphics shown in this document were produced with *Mathematica*. You will most likely find it very helpful with your studies. It is a symbolic computation tool which has full programming capabilities. E.g. Try writing

   ```
   Expand[(x+y)^3]
   ```

   then press Shift+Enter or

   ```
   s = 0;
   For[i = 0, i < 6, i++, s = s + i; Print[s]]
   ```

   You can call on *Wolfram alpha* from with in it by beginning a cell with = =.
   If your school has a license, to install this on your machine, visit:
   `wolfram.com/siteinfo/`
   Get *Mathematica* Desktop.
   Create a Wolfram ID, and download and install the software.

1 Year 10 Mathematical Methods

1.1 Term 1

1.1.1 Surds and Index Laws

The decimal expansion of $\sqrt{2}$ does not terminate, nor repeat.

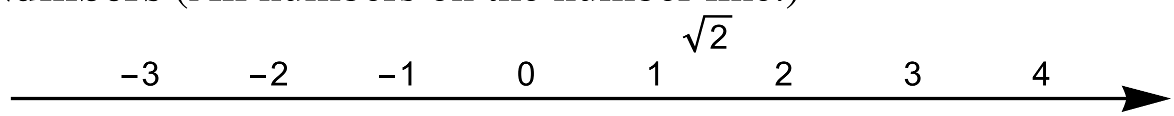
$\sqrt{2} = 1.41421356237309504880168872420969807856967187537695\ldots$

There are no whole numbers a, b with $b \neq 0$ and $\sqrt{2} = \frac{a}{b}$. We say $\sqrt{2}$ is **irrational**, meaning not rational.

Next we identify several important sets of numbers and notation for them.

$\mathbb{Z}$ **Integers** This is the set of all whole numbers $\cdots - 3, -3, -1, 0, 1, 2, \ldots$.

$\mathbb{R}$ **Real Numbers** (All numbers on the number line.)



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$\mathbb{Q}$ **Rational Numbers** $\left\{\frac{a}{b} \ : \ a, b \in \mathbb{Z}, b \neq 0\right\}$, meaning the set of all fractions a/b, where a and b are elements of ($\in$) the set of integers ($\mathbb{Z}$) and b is non-zero. Note: Integers are also rational numbers ($\mathbb{Z} \subset \mathbb{Q}$).

$\mathbb{R} - \mathbb{Q}$ or $\mathbb{R} \setminus \mathbb{Q}$ **Real Irrational Numbers** (Real numbers that are not rational. e.g. π and $\sqrt{2}$ are real numbers but not rational numbers.)

Note: A real number is rational if and only if it has a repeating decimal expansion or a terminating decimal expansion.

A **surd** is a sum of expressions of the form $a\sqrt[n]{b}$.

Example 1.1 *Which of the following real numbers are surds?*

$$\sqrt{36} \qquad \sqrt{19} \qquad \sqrt{\frac{1}{25}} \qquad \sqrt[3]{21} \qquad 4\pi \qquad \sqrt[3]{1728}$$

Example 1.2 *Simplify the following using the multiplicative property of square roots* $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

$$\sqrt{12} \qquad 3\sqrt{30} \qquad \sqrt{\frac{1}{36}} \qquad 2\sqrt{75} \qquad \frac{3\sqrt{125}}{4} \qquad \sqrt{\frac{15}{81}}$$

Exercises/Homework:

We begin this section with rationalizing the denominator of surds. We use the properties

$$\frac{x}{\sqrt{y}} = \frac{x\sqrt{y}}{\sqrt{y}\sqrt{y}} = \frac{x\sqrt{y}}{y},$$

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right) = a-b.$$

Example 1.3 *Rationalize the denominator for the following expressions.*

(a) $\frac{5}{\sqrt{3}}$

(b) $\frac{6\sqrt{5}}{\sqrt{8}}$

(c) $\frac{3\sqrt{6}}{2\sqrt{10}}$

(d) $\frac{4-\sqrt{3}}{\sqrt{15}}$

(e) $\frac{1}{5-\sqrt{3}}$

(f) $\frac{6-\sqrt{5}}{2+\sqrt{8}}$

Exercises/Homework:

We have the following index laws for real numbers a, b, c :

$$a^b a^c = a^{b+c}, \qquad a^b / a^c = a^{b-c}, \text{ for } a \neq 0$$
$$\left(a^b\right)^c = a^{bc},$$
$$(ab)^c = a^c b^c, \qquad (a/b)^c = a^c / b^c \ = \ a^c b^{-c}, \text{ for } b \neq 0$$
$$a^{-1} = \frac{1}{a}, \text{ for } a \neq 0$$
$$a^0 = 1, \qquad 0^0 = 1 \text{ (defined to be 1, but contraversial)}$$
$$\frac{1}{a^{-b}} = a^b, \text{ for } a \neq 0, \qquad a^{-b} = \frac{1}{a^b} \text{ for } a \neq 0.$$

Example 1.4 *Express the following with positive indices*

(a) a^{-3}

(b) $2x^{-3}y^4$

(c) $\frac{4}{y^{-2}}$

(d) $\frac{\left(a^{-3}b\right)^2}{3a^{-1}b^2} \times \frac{b^{-1}}{a}$

(e) $\frac{\left(5a^2b^{-1}\right)^3}{2a^4b^{-2}} \div \frac{b^{-5}}{2a^{-2}}$

Exercises/Homework:

We can write $\sqrt{3} = 3^{1/2}$ and

$$\left(\sqrt{3}\right)^2 = 3^{1/2} \times 3^{1/2} = 3^{\frac{1}{2}+\frac{1}{2}} = 3^1 = 3.$$

This allows us to use index laws to simplify surds. We have the following index laws for real number a and integers m, n:

$$\begin{aligned} a^{m/n} &= \sqrt[n]{m}, \\ a^{1/2} &= \sqrt{a}, \\ a^{1/3} &= \sqrt[3]{a}, \\ a^{1/n} &= \sqrt[n]{a}. \end{aligned}$$

Example 1.5 *Express the following in index form:*

(a) $\sqrt{11}$

(b) $\sqrt{3x^7}$

(c) $2\sqrt[4]{x^9}$

(d) $11\sqrt{7}$

Example 1.6 *Write the following in simplest surd form:*

(a) $12^{1/2}$

(b) $6^{3/2}$

Example 1.7 *Simplify:*

(a) $a^{1/5}a^{3/5}$

(b) $\left(b^2b^3\right)^{\frac{1}{4}}$

(c) $\left(\frac{x^{1/3}}{y^{1/6}}\right)^{1/4}$

Exercises/Homework:

In this section we will learn how to solve equations with one variable.

Example 1.8 *Solve the following equations for x:*

(a) $2x+9=12$

(b) $3(2x+4)=3x$

(c) $\frac{x-1}{3}=2$

(d) $3-\frac{x}{3}=8$

(e) $\frac{3-x}{4}=x-4$

Exercises/Homework:

In this section we learn how to rearrange formulas and substitute values into equations.

Example 1.9 *The volume of a sphere of radius r is given by* $V = \frac{4}{3}\pi r^3$.

(a) *Solve the equation* $V = \frac{4}{3}\pi r^3$ *for r, where r is a real number.*

(b) *If the volume of the sphere is* $42.8\ m^3$*, find the radius of the sphere.*

Example 1.10 *The area of a rectangular region adjoining a two semi-circle regions on each end of the rectangle is given by* $A = xy + \pi\left(\frac{x}{2}\right)^2$.

(a) *Solve the equation* $A = xy + \pi\left(\frac{x}{2}\right)^2$ *for y in terms of x and A.*

(b) *If* $x = 36\ m$ *and* $y = 24\ m$*, calculate A.*

Exercises/Homework:

Symbols:

$x > 3$ means x is greater than 3.
$x < 3$ means x is less than 3.
$x \geq 3$ means x is greater than or equal to 3.
$x \leq 3$ means x is less than or equal to 3.

Example 1.11 *Sketch the region on the number line corresponding to:*

(a) $\{x \in \mathbb{R} \, : \, x > 3\} = (3, \infty)$.

(b) $\{x \in \mathbb{R} \, : \, x < 3\} = (-\infty, 3)$.

(c) $\{x \in \mathbb{R} \, : \, x \geq 3\} = [3, \infty)$.

(d) $\{x \in \mathbb{R} \, : \, x \leq 3\} = (-\infty, 3]$.

Rules:

- When multiplying an inequality by a negative number, turn the symbol around. ($>$ becomes $<$, $<$ becomes $>$, $\leq$ becomes $\geq$, $\geq$ becomes $\leq$.)
- When inverting both sides of an inequality, turn the symbol around.
- Otherwise, treat solving an inequality like solving an equation.

Example 1.12 $4 > 3$ *but* $-4 \quad -3$ *and* $\frac{1}{4} \quad \frac{1}{3}$.

Example 1.13 *Solve the inequality* $3x - 6 \geq 8$ *for x.*

Example 1.14 *Solve the inequality* $-(4 - 6x) < 2(5 - x)$ *for x.*

Example 1.15 *Solve the inequality* $\frac{x}{4} - \frac{2x}{3} > -7$ *for x.*

Example 1.16 *Solve the inequality* $\frac{5}{3x} > 2$ *for x.*

Exercises/Homework:

The **greatest common divisor** of two integers a and b is written $\gcd(a,b)$. This is the greatest positive integer c such that c divides a and c divides b.

The **least common multiple** of two integers a and b is written $\text{lcm}(a,b)$. This is the least positive integer c such that a divides c and b divides c.

Theorem 1 *For any two positive integers a and b,*

$$ab = \gcd(a,b)lcm(a,b).$$

The greatest common divisor and least common multiple can be calculated efficiently using the Euclidean algorithm.

Example 1.17 *Calculate:*

(a) 4×6

(b) $\gcd(4,6)$

(c) $\frac{4\times 6}{\gcd(4,6)}$

(c) $lcm(4,6)$

(d) $\frac{1}{4}+\frac{1}{6}$

Example 1.18 *Calculate lcm*$(12,18)$ *and use it to simplify* $\frac{x+4}{12}+\frac{x-6}{18}$.

Example 1.19 *Calculate lcm*$(24,6)$ *and use it to simplify* $\frac{x-3}{6}+\frac{5x-6}{24}$.

Exercises/Homework:

Two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are **parallel** if $m_1 = m_2$.

Two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are **perpendicular** if $m_1m_2 = -1$ (or equivalently, $m_2 = -\frac{1}{m_1}$).

Theorem 2 *In Euclidean space:*

- *Two lines intersect in one point if and only if they are not parallel.*
- *Lines have either one intersection or infinitely many intersections (they are the same line).*
- *There is a unique line passing through two points.*

Example 1.20 *Decide whether the two lines $y = -9x - 3$ and $y = \frac{1}{9}x + 2$ are parallel, perpendicular, or neither.*

Example 1.21 *Decide whether the two lines* $y = -\frac{1}{3}x + 1$ *and* $3y + x = 2$ *are parallel, perpendicular, or neither.*

Example 1.22 *Decide whether the two lines* $y = \frac{1}{6}x + 4$ *and* $6y + x = 3$ *are parallel, perpendicular, or neither.*

Example 1.23 *Find the equation of the line that is parallel to* $y = -5x + 8$ *and passes through the point* $(2, -3)$.

Example 1.24 *Find the equation of the line that is perpendicular to* $y = -5x + 8$ *and passes through the point* $(-4, -2)$.

To decide whether two lines are parallel, perpendicular, or neither:

Step 1 Put both lines in standard form $y = mx + c$ and hence identify slopes m_1 and m_2.

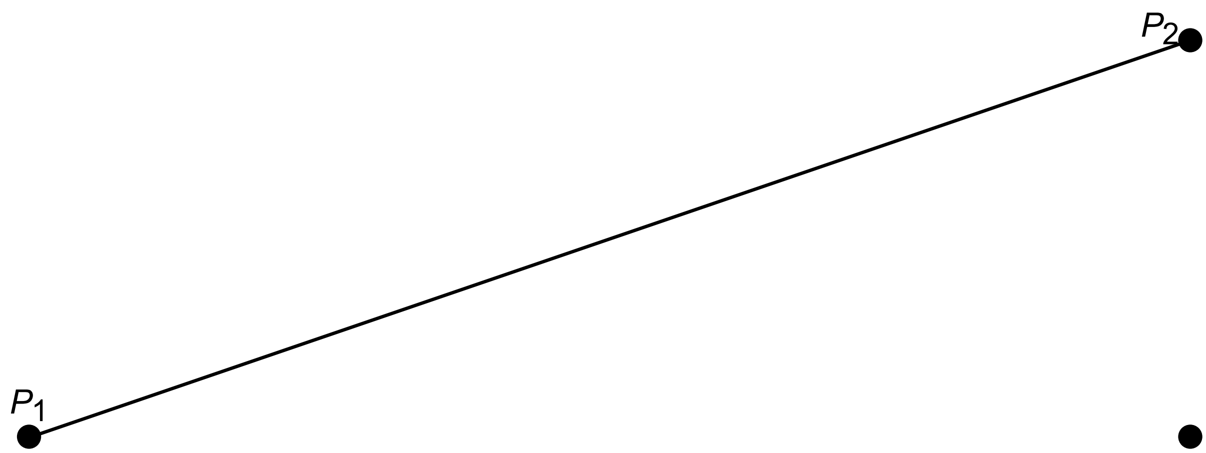
Step 2 If $m_1 = m_2$, then the lines are parallel;

Step 3 Otherwise: if $m_1 m_2 = -1$, then the lines are perpendicular;

Step 4 Otherwise: the lines are neither parallel nor perpendicular.

Exercises/Homework:

Consider the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. The distance between P_1 and P_2 is obtained by Pythagoras' theorem $a^2 + b^2 = c^2$, where $a = |x_1 - x_2|$, $b = |y_1 - y_2|$, and c is the distance between P_1 and P_2.



The formula for the distance between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$c = D(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Example 1.25 *Find the distance between the points* $(0,4)$ *and* $(-2,6)$.

The **midpoint** of the line segment connecting the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is given by

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Example 1.26 *Find the midpoint of the line segment connecting the points* $(0,4)$ *and* $(-2,6)$.

Example 1.27 *Find real numbers* a *and* b *such that the midpoint of* $(2a,a)$ *and* $(3,b)$ *is* $(4,-4)$.

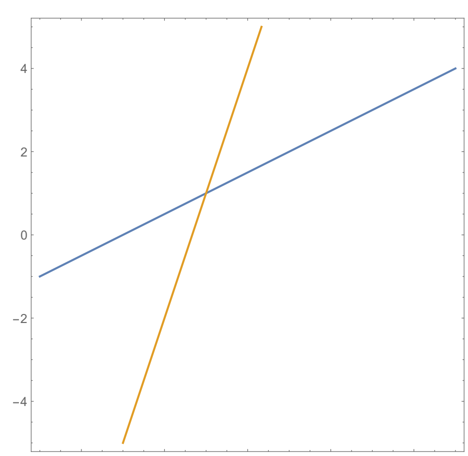
Example 1.28 *The distance between the points* $(-4,1)$ *and* $(6,a)$ *is* $4\sqrt{21}$. *Find a.*

Exercises/Homework:

Given two simultaneous linear equations that do not represent parallel lines, we learn to find the point of their intersection by substitution. That is, we solve one equation for a variable, say y, and then substitute that into the other equation and solve for the other letter, say x.

Example 1.29 *Solve the simultaneous system of linear equations*

$$\begin{aligned} 2x - 4y &= -6, \\ y &= 3x + 4. \end{aligned}$$



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Example 1.30 *Solve the simultaneous system of linear equations*

$$\begin{aligned} y &= 8x-1, \\ y &= 8x+2 \end{aligned}$$

if possible.

Example 1.31 *For which real value of k does the simultaneous system of linear equations*

$$\begin{aligned} y &= -3x-2, \\ y &= kx+6 \end{aligned}$$

(a) *have no solution?*

(b) *have one solution?*

(c) *have infinitely many solutions?*

Exercises/Homework:

Given a system of simultaneous linear equations, solving the system by elimination applies the following procedure. We multiply each equation by a number such that the coefficients of one of the variables (the coefficient of the same letter) becomes the same or of opposite sign. We then add or subtract equations so that that variable vanishes. Finally, we solve for the other variable.

Example 1.32 *Solve the system of equations*

$$\begin{aligned} 2x - 6y &= 8, \\ 3x + 4y &= 10 \end{aligned}$$

by elimination.

Example 1.33 *Solve the system of equations*

$$\begin{aligned} x + 2y &= 4, \\ 2x + 9y &= 12 \end{aligned}$$

by elimination.

Exercises/Homework:

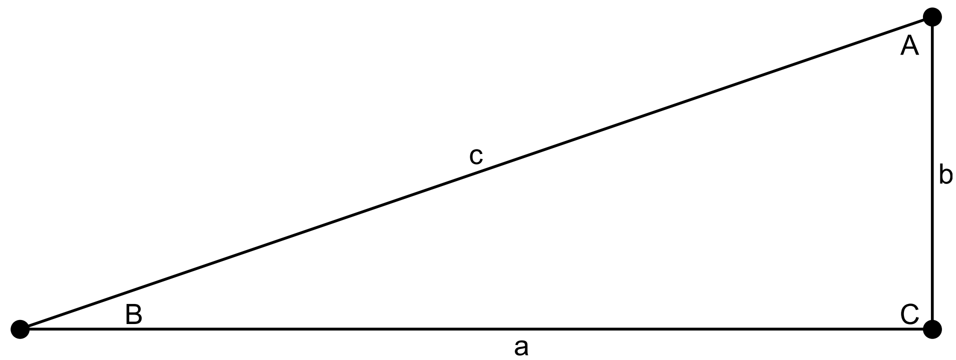
Next we consider applications of simultaneous equations.

Example 1.34 *The sum of the ages of two children Kara and Ben is* 17 *and the difference in their ages is* 5*. If Kara is older than Ben, determine their ages.*

Exercises/Homework:

1.2.1 Introduction to Trigonometry

We learn about the relationship between the angles in a right triangle and the trigonometric ratios sine, cosine and tangent (sin, cos, tan).



The trigonometric ratios sine, cosine and tangent are defined

$$\sin(B) = \frac{\text{opp}}{\text{hyp}},$$
$$\cos(B) = \frac{\text{adj}}{\text{hyp}},$$
$$\tan(B) = \frac{\text{opp}}{\text{adj}}.$$

We have the useful acronym **S**OH**C**AH**T**OA for remembering these trig. ratios.

Recall Pythagoras' theorem:

Theorem 3 *(Pythagoras) Let a,b,c be the lengths of the sides of a right triangle, where $c > a,b$ (c is the hypotenuse). Then $a^2 + b^2 = c^2$.*

Example 1.35 *Show that:*

(a) $\frac{\sin(B)}{\cos(B)} = \tan(B)$,

(b) $\sin(B) = \cos(A)$,

(c) $\cos(B) = \sin(A)$,

(d) $\tan(B) = \frac{1}{\tan(A)}$,

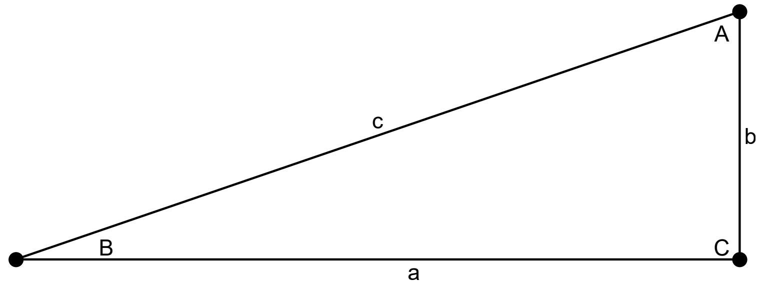
(e) $\cos^2(B) + \sin^2(B) = 1$ by Pythagoras' theorem.

Example 1.36 *Find the side length x opposite an angle of* 30° *in a right triangle with hypotenuse* 8*.*

Example 1.37 *Find the hypotenuse x in a right triangle if the triangle has side length* 4 *adjacent to an angle of* 44°.

Exercises/Homework:

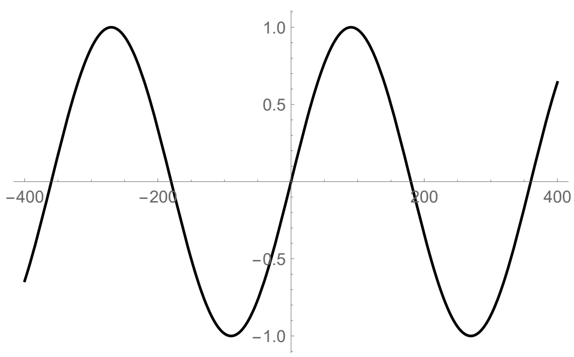
To solve a right triangle for an interior angle we use the inverse functions of sine, cosine and tangent, $(\sin^{-1}, \cos^{-1}, \tan^{-1})$.

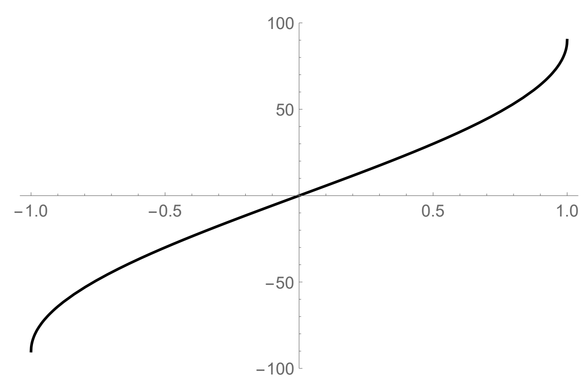


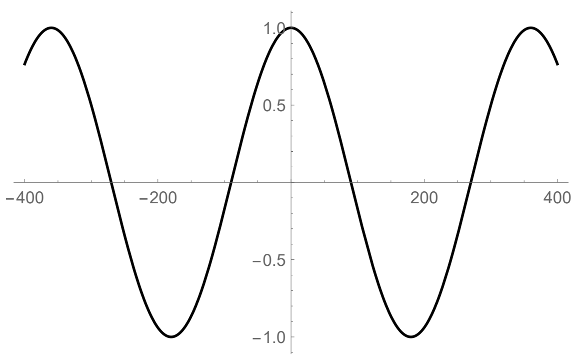
Since $\sin(B) = \frac{b}{c}$, we have $B = \sin^{-1}\left(\frac{b}{c}\right)$. This is also called arcsin.

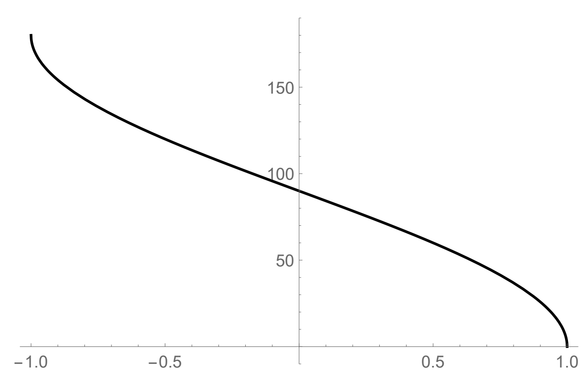
Similarly, $\cos(B) = \frac{a}{c}$ so $B = \cos^{-1}\left(\frac{a}{c}\right)$. This is also called arccos.

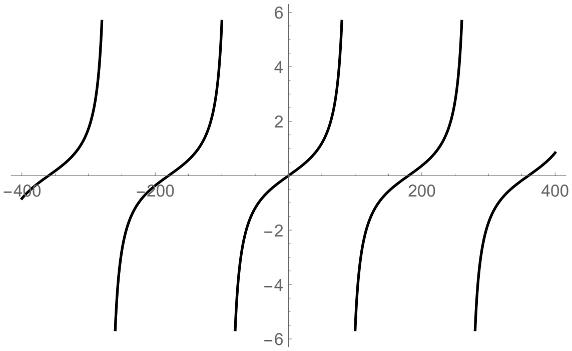
$\tan(B) = \frac{b}{a}$ so $B = \tan^{-1}\left(\frac{b}{a}\right)$. This is also called arctan.

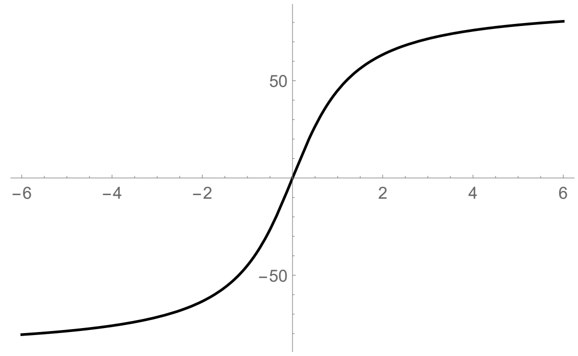












Example 1.38 *A right triangle has hypotenuse of length* 2 *and sides of length* 1 *and x. Solve for the angle adjacent to the side of length x, and then solve for x.*

Exercises/Homework:

We consider some applications of trigonometry.

Example 1.39 *A tower stands x metres high in elevation above the ground. A man standing on the top of a 250 metre tall building looks up to the tower with an elevation angle of* 30° *to the horizontal. The horizontal distance between the man and the tower is* 420 *metres. Calculate the elevation x of the tower.*

Exercises/Homework:

True Bearings $(^\circ T)$ are measured **clockwise from North**.

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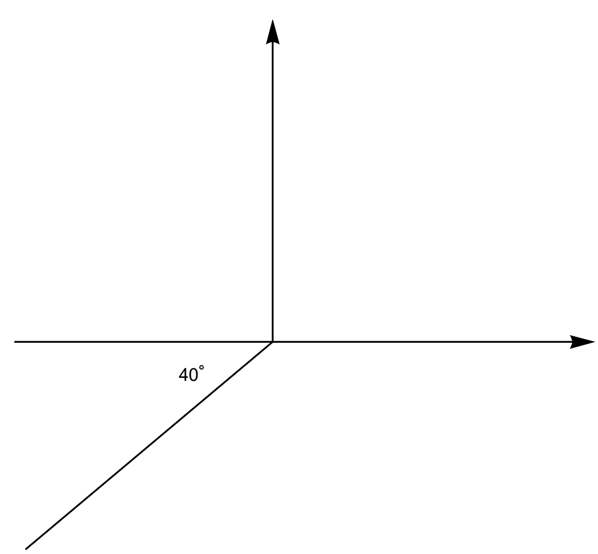
Recall that the mathematical convention is to measure angles from the positive end of the *x*-axis counter-clockwise.

Example 1.40 *Consider the points* O, A, B *show below. If the line segment OA makes an angle of* 40° *South of West, what is the true bearing of the point A from O?*



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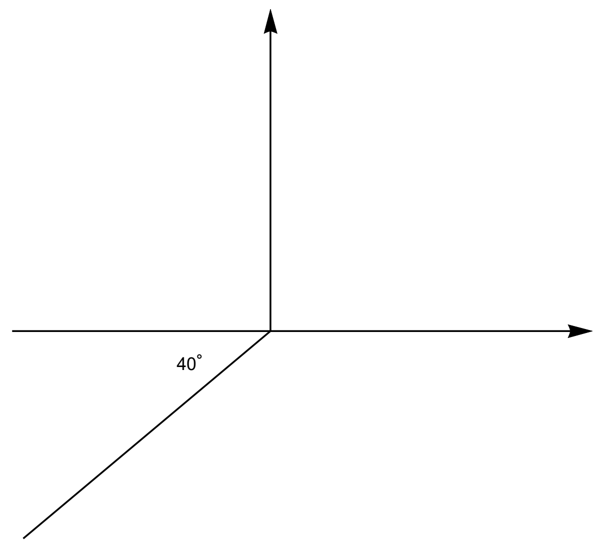
Example 1.41 *Consider the points O, A, B show below. If the line segment OA makes an angle of* $40°$ *South of West, what is the true bearing of the point O from A?*



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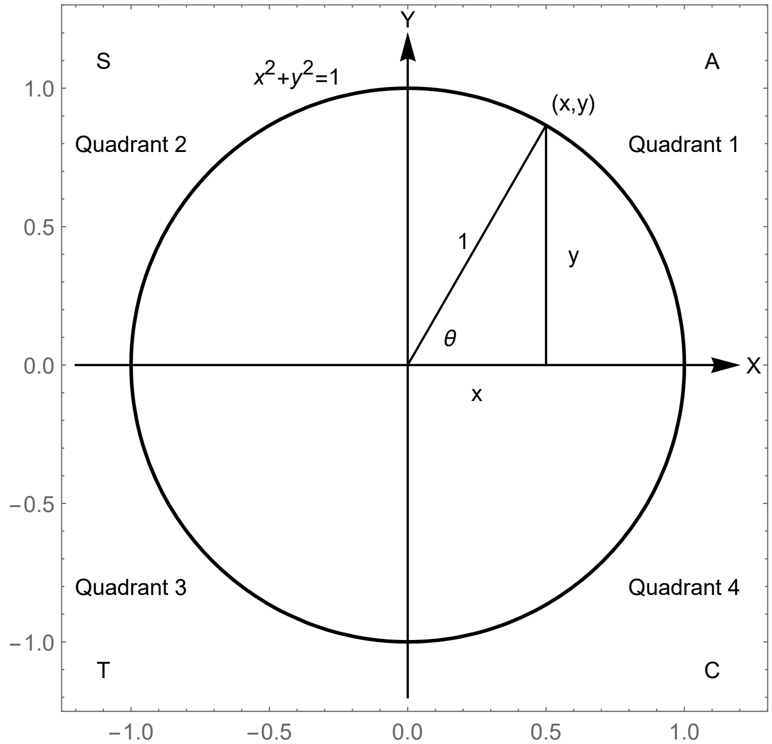
Example 1.42 *A boat travels North-East for* 5 *km followed by a true bearing of* $20°$ *for* 10 *km. Find the true bearing of the boat from the original position.*

Exercises/Homework:



Recall

$$\begin{aligned} \sin(\theta) &= \frac{\text{opp}}{\text{hyp}} = \frac{y}{1}, \\ \cos(\theta) &= \frac{\text{adj}}{\text{hyp}} = \frac{x}{1}, \\ \tan(\theta) &= \frac{\text{opp}}{\text{adj}} = \frac{y}{x}. \end{aligned}$$

For any point (x,y) on the unit circle $x^2+y^2=1$, there is an angle θ such that $(x,y)=(\cos(\theta),\sin(\theta))$. Since $x^2+y^2=1$ we again have $\cos^2(\theta)+\sin^2(\theta)=1$.

The acronym ASTC refers to the following:

For an angle θ in Quadrant 1, **All** $\sin(\theta),\cos(\theta),\tan(\theta)>0$.

For an angle θ in Quadrant 2, **Only Sine,** $\sin(\theta)>0$.

For an angle θ in Quadrant 3, **Only Tan,** $\tan(\theta)>0$.

For an angle θ in Quadrant 4, **Only Cos,** $\cos(\theta)>0$.

A **reference angle** is an angle α with $0 \leq \alpha < 90^\circ$ such that $\theta = 180^\circ \pm \alpha$, $\theta = 360^\circ - \alpha$, or $\theta = \alpha$. For example, if $\theta = 290^\circ$, then the reference angle $\alpha = 70^\circ$ so that $\theta = 360^\circ - \alpha$.

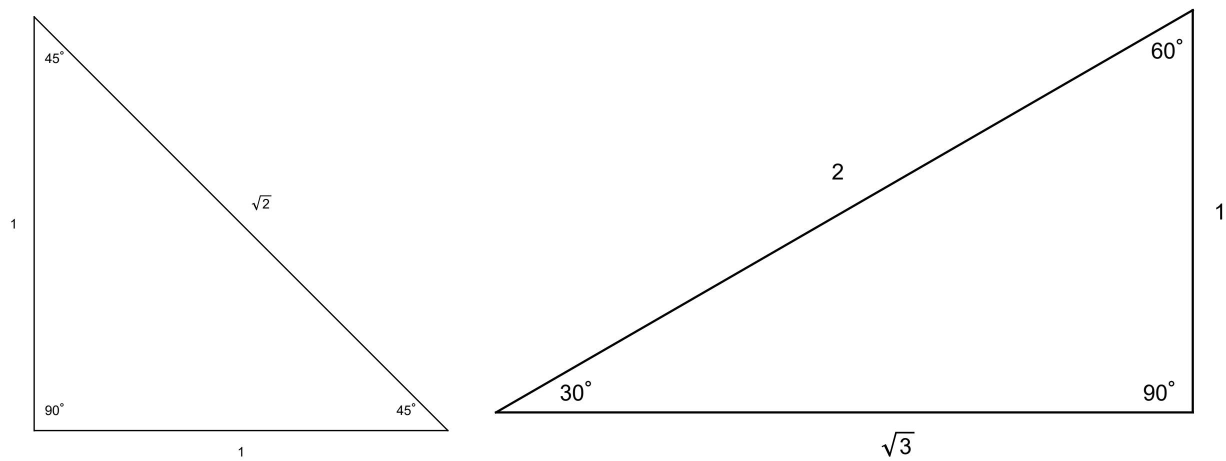
Let α be the reference angle.

- If θ is in Quadrant 1, then $\theta = \alpha$.
- If θ is in Quadrant 2, then $\theta = 180^\circ - \alpha$.
- If θ is in Quadrant 3, then $\theta = 180^\circ + \alpha$.
- If θ is in Quadrant 4, then $\theta = 360^\circ - \alpha$.

Example 1.43 *Calculate* $\cos(320^\circ)$ *and* $\sin(320^\circ)$ *by considering the reference angle.*

Exercises/Homework:

Memorize the following useful triangles:



These two triangles give exact surd values for the trigonometric ratios of angles 45°, 30°, and 60°.

We have

$$\cos(45^\circ) = \qquad , \quad \cos(30^\circ) = \qquad , \quad \cos(60^\circ) = \qquad ,$$

$$\sin(45^\circ) = \qquad , \quad \sin(30^\circ) = \qquad , \quad \sin(60^\circ) = \qquad ,$$

$$\tan(45^\circ) = \qquad , \quad \tan(30^\circ) = \qquad , \quad \tan(60^\circ) = \qquad .$$

The angle addition formulas:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta),$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta),$$

give additional exact values.

Example 1.44 *Calculate* $\cos(15^\circ)$ *and* $\sin(15^\circ)$ *using the above triangles and the angle addition formulas.*

Example 1.45 *Calculate the exact surd value of* $\cos(150^\circ)$.

Example 1.46 *Find all angles* θ *with* $0 \le \theta < 360^\circ$ *such that* $\cos(\theta) = -\frac{\sqrt{3}}{2}$.

Exercises/Homework:

Like terms are terms of a polynomial with the same letters to the same powers.
Example: $4xy^2$ and $-3xy^2$ **ARE** like terms.
Example: $4x^2y$ and $-3xy^2$ are **NOT** like terms.
Example: x^2 and x are **NOT** like terms.
Example: x and 12 are **NOT** like terms.

We use the **distributive law** to expand brackets. This means multiplication distributes over addition:

$$x(y+z) = xy + xz, \qquad (x+y)z = xz + yz.$$

Notice that $2(3+5) = 2 \times 8 = 16$.
Also $2(3+5) = 2 \times 3 + 2 \times 5 = 6 + 10 = 16$.

The following are all consequences of the distributive law:

$$\begin{aligned} a(b+c) &= \\ a(b-c) &= \\ (a+b)(c+d) &= \\ &= \\ &= \end{aligned}$$

$$
\begin{aligned}
(a+b+c)(d+e+f) &= \\
(a+b)^2 &= \\
&= \\
&= \\
&= \\
(a+b)(c+d)(e+f) &= \\
&= \\
&=
\end{aligned}
$$

Example 1.47 *Expand* $(x-4)(x+8)$

Example 1.48 *Expand* $(2x-6)(3x+7)$

Exercises/Homework:

Factorizing a polynomial is the process of expressing the polynomial as a product of polynomials.

For example, $x^2 - 25 = (x+5)(x-5)$ since $a^2 - b^2 = (a+b)(a-b)$.
Similarly, $x^2 - 12 = x^2 - \sqrt{12}^2 = \left(x+\sqrt{12}\right)\left(x-\sqrt{12}\right)$.

Example 1.49 *Factorize* $3x^2 - 18x$.

Example 1.50 *Factorize* $x^2 + 8x + 15$.

Example 1.51 *Factorize* $x(x+3) - 12(x-3)$.

Exercises/Homework:

A **monic** polynomial in one variable has leading coefficient equal to 1. That is, a polynomial in the variable x has coefficient of x^n, where n is greatest, being 1.

x^2+3x+8 is monic. $3x^2-4x+12$ is not monic.

A **quadratic** polynomial in one variable is a polynomial of the form

$$ax^2+bx+c,$$

where a,b,c are specific numbers. Quadratic refers to the greatest exponent being equal to 2.

To factorize x^2+bx+c, where b,c are specific integers, we seek to find integers p,q such that

$$(x+p)(x+q)=x^2+(p+q)x+pq=x^2+bx+c$$

so that

$$c=pq, \qquad b=p+q.$$

Step 1 If $c=0$, put $x^2+bx+c=x(x+b)$. Otherwise:

Step 2 If $b=0$, $x^2+bx+c=(x+\sqrt{c})(x-\sqrt{c})$. Otherwise:

Step 3 List all of the divisor pairs (s,t) of the absolute value of c up to their order: $(1,|c|),\ldots$.

Step 4 If $c>0$, determine which pair (s,t) satisfies $s+t=|b|$. If $b>0$, put $x^2+bx+c=(x+s)(x+t)$. If $b<0$, put $x^2+bx+c=(x-s)(x-t)$.

Step 5 If $c<0$, determine which pair (s,t) satisfies $s-t=\pm b$.
If $b>0$, put $x^2+bx+c=(x+s)(x-t)$, where $s>t$. If $b<0$, put $x^2+bx+c=(x-s)(x+t)$, where $s>t$.

Example 1.52 *Factorize the monic quadratic polynomial* $x^2 - x - 20$.

Example 1.53 *Factorize the monic quadratic polynomial* $x^2 + 9x + 18$.

Example 1.54 *Factorize the monic quadratic polynomial* $x^2 + 5x - 84$.

Exercises/Homework:

We learn how to factorize expressions of the form ax^2+bx+c, where $a \neq 0$ and $a,b,c \in \mathbb{Z}$ (are integers). We demonstrate the procedure with an example.

Example 1.55 *Factorize* $10x^2-13x-3$.
We first list the divisor pairs of $|-3|=3$. *We only have* $(1,3)$.
We list the divisor pairs of the absolute value of the leading coefficient $|10|=10$. *We only have* $(1,10)$ *and* $(2,5)$. *Next we form a multiplication table where we multiply divisors:*

×	*1 10*	*2 5*
1		
3		

Since $-3<0$, *we seek a pair of products with a difference of* -13, *examining the differences within diagonals of the table.*

We use these to form the required factors
$10x^2-13x-3=(\qquad\qquad)(\qquad\qquad).$

We can expand to check our work.

Example 1.56 *Factorize* $6x^2 - 13x - 28$.

.

$\times$	*1 28*	*2 14*	*4 7*
1 *6*			
2 *3*			

Exercises/Homework:

We show how to factorize a quadratic polynomial in one variable by completing the square. The factors we obtain do not always have integer coefficients.

Example 1.57 *Factorize* $3x^2+19x+20$ *by completing the square.*

In our first step, we write ax^2+bx+c *as* $a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$ *since* $a \neq 0$. *In other words, we factor out the leading coefficient of the polynomial so that inside the brackets we have a monic quadratic polynomial. We have:*

$3x^2+19x+20 = 3\left(x^2+\frac{19}{3}x+\frac{20}{3}\right)$.

Next we calculate $\frac{1}{2}$ *of the coefficient of x in the monic quadratic inside the brackets.*

$\frac{1}{2}\frac{19}{3} = \frac{19}{6}$.

We place this inside a square: $\left(x+\frac{19}{6}\right)^2$. *Since the expansion of* $\left(x+\frac{19}{6}\right)^2$ *contains* $\left(\frac{19}{6}\right)^2$ *which is not in the original quadratic* $3x^2+19x+20$, *we must subtract* $\left(\frac{19}{6}\right)^2$ *from our new expression so that we get an equal expression.*

$$
\begin{aligned}
3x^2+19x+20 &= 3\left(x^2+\frac{19}{3}x+\frac{20}{3}\right), \\
&= 3\left(\left(x+\frac{19}{6}\right)^2-\left(\frac{19}{6}\right)^2+\frac{20}{3}\right).
\end{aligned}
$$

Now we tidy the remaining terms. Since $-\left(\frac{19}{6}\right)^2+\frac{20}{3}=-\frac{121}{36}$, we have

$$
\begin{aligned}
3x^2+19x+20 &= 3\left(\left(x+\frac{19}{6}\right)^2-\frac{121}{36}\right), \\
&= 3\left(\left(x+\frac{19}{6}\right)^2-\left(\frac{11}{6}\right)^2\right).
\end{aligned}
$$

We have completed the square but it remains to use the property $a^2-b^2=(a+b)(a-b)$ to factorize the quadratic polynomial.

$$
\begin{aligned}
3x^2+19x+20 &= 3\left(x+\frac{19}{6}+\frac{11}{6}\right)\left(x+\frac{19}{6}-\frac{11}{6}\right), \\
&= 3\,(x+5)\left(x+\frac{8}{6}\right), \\
&= (x+5)\,(3x+4).
\end{aligned}
$$

.

Example 1.58 *Factorize* $6x^2 + 5x - 56$ *by completing the square.*

.

Exercises/Homework:

We aim to solve equations of the form $ax^2+bx+c=0$ for x, where a,b,c are specific numbers, by first factorizing the quadratic expression.

Assuming there are real numbers p,q,r,s such that

$$ax^2+bx+c=(px+q)(rx+s)=0,$$

then we get $px+q=0$ or $rx+s=0$ so that $x=-\frac{q}{p}$ or $x=-\frac{-s}{r}$.

Notice that since $a=pr$ and $a\neq 0$, we have $p,r\neq 0$.

We can find real numbers p,q,r,s such that $ax^2+bx+c=(px+q)(rx+s)$ when $b^2-4ac\geq 0$.

Example 1.59 *Solve the equation* $2x^2+3x-27=0$ *by factorization.*

Exercises/Homework:

To solve a quadratic equation by completing the square, we first complete the square, writing $ax^2+bx+c=P^2-Q^2$, where $P=p_1x+p_2$ is a linear polynomial in x and Q is a number.

Since $P^2-Q^2=(P+Q)(P-Q)=0$, we have $P+Q=0$ or $P-Q=0$ and hence solve these equations for x.

Example 1.60 *Solve the equation* $2x^2+3x-27=0$ *by completing the square.*

Example 1.61 *Solve the equation* $15x^2 + 28x - 32 = 0$ *by completing the square.*

Exercises/Homework:

Assume that $ax^2+bx+c=0$, where $a \neq 0$. Factoring a from the expression on the left, we have

$$a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=0.$$

Half of $\frac{b}{a}$ is $\frac{b}{2a}$ so that

Simplifying,

Simplifying again,

.

We call $\Delta = b^2 - 4ac$ the **discriminant**.

Solving the equation for x,

We have the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 1.62 *Solve the equation* $15x^2 + 28x - 32 = 0$ *by using the quadratic formula.*

Exercises/Homework:

We learn to translate a word problem into a quadratic equation, solve the equation, and write the solution.

Example 1.63 *Mike is* 5 *years younger than Nina. The product of their ages is* 266*. How old are Mike and Nina?*

Example 1.64 *Margaret wants to fence an area of her yard adjacent to the house for a dog yard with three sides of a rectangle consisting of fencing and one side of the rectangle being the side of the house. Margaret has only* 128 *metres of fencing material. What are the dimension of the rectangle that give maximum area to the dog yard?*

Exercises/Homework:

1.3.1 Time Series

A **time-series** is a sequence of points in which the consecutive differences in the independent variable is constant, and the independent variable represents time. When plotted, line segments connect consecutive points of a time-series.

Example 1.65 $S = \{t_n, s_n\} = \{(1.2, -7.8), (1.4, -6.4), (1.6, -4.1), (1.8, -5.3)\}$ *is an example of a time-series. If*

$$\frac{s_{n+1} - s_n}{t_{n+1} - t_n} = m$$

is constant for all n, then the time-series is linear. Since $t_{n+1} - t_n$ *is constant for all n in a time-series, a linear time-series has* $s_{n+1} - s_n = k$ *(a constant) for all n.*

Example 1.66 *Plot the data* $S = \{(1.2, -7.8), (1.4, -6.4), (1.6, -4.1), (1.8, -5.3)\}$, *where the dependent variable represents a percentage change in share price over that time interval.*

Exercises/Homework:

A **bivariate data set** is a set of points (x_n, y_n) relating the dependent variable Y to the independent variable X.

A **scatter plot** is a plot of points in a bivariate data set, where the independent variable is shown on the horizontal axis and the dependent variable is shown on the vertical axis.

An **outlier** is a point of a bivariate data set which is deemed to be isolated from other points of the data set.

A bivariate data set has **correlation** if the points closely fit a line. Correlations are described as **strong correlation** or **weak correlation** depending on how well they fit a line. The data has **positive correlation** if the slope of the line of best fit is positive. The data has **negative correlation** if the slope of the line of best fit is negative.

Example 1.67 *Consider the bivariate data set* $S = \{(1,3),(4,2),(5,1),(6,0),(9.-2)\}$. *Draw a scatter plot and describe any correlation observed.*

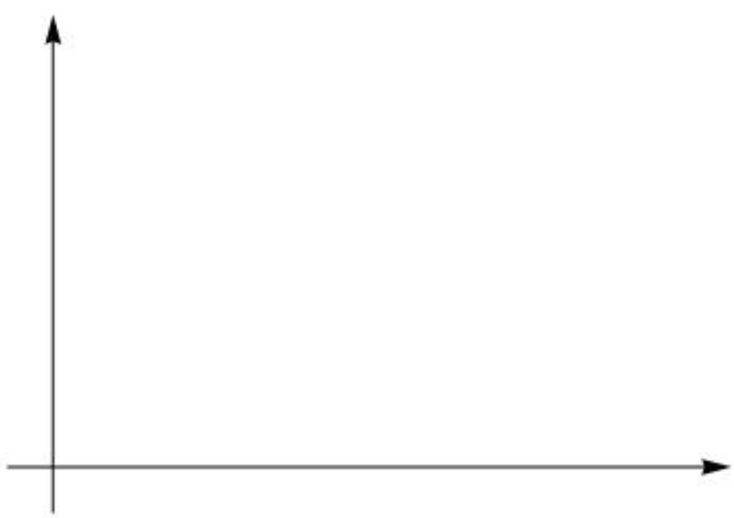
Exercises/Homework:

Given a bivariate data set S, we can guess a line of best fit by choosing two points (x_1, y_1) and (x_2, y_2) and hence a line connecting them such that half of the points in the data set S are above the line and half of the points in the data set S are below the line. The points (x_1, y_1) and (x_2, y_2) do not need to be in S.

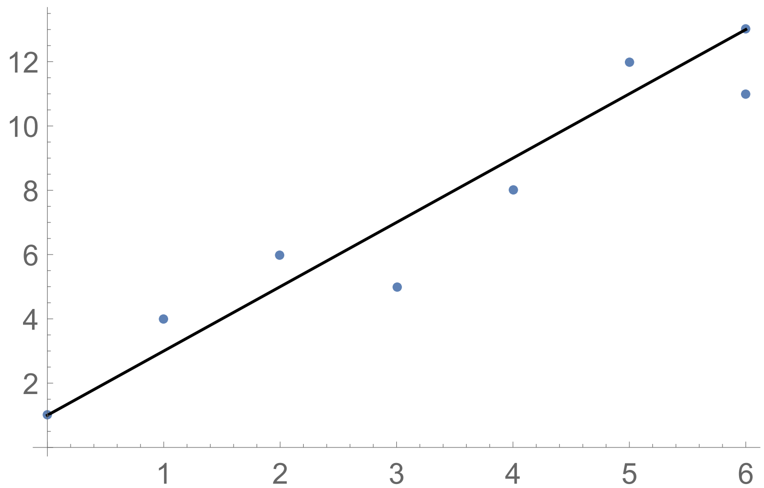
Recall that the slope of the line passing through two points (x_1, y_1) and (x_2, y_2) is given by $Y = mX + c$, where $m = \frac{y_2 - y_1}{x_2 - x_1}$ and $c = y_1 - mx_1$ or $c = y_2 - mx_2$. These c values are the same.

The construction of points in the data range with a line of best fit is called **interpolation**. The construction of points outside the data range with a line of best fit is called **extrapolation**.

Example 1.68 *Consider the bivariate data set* $S = \{(1,4), (2,6), (3,5), (4,8), (5,12), (6,11)\}$. *Draw a scatter plot, guess a line of best fit, and describe any correlation observed.*



Guessing the points $(0,1)$ and $(6,13)$ to find the equation of a line of best fit,



Exercises/Homework:

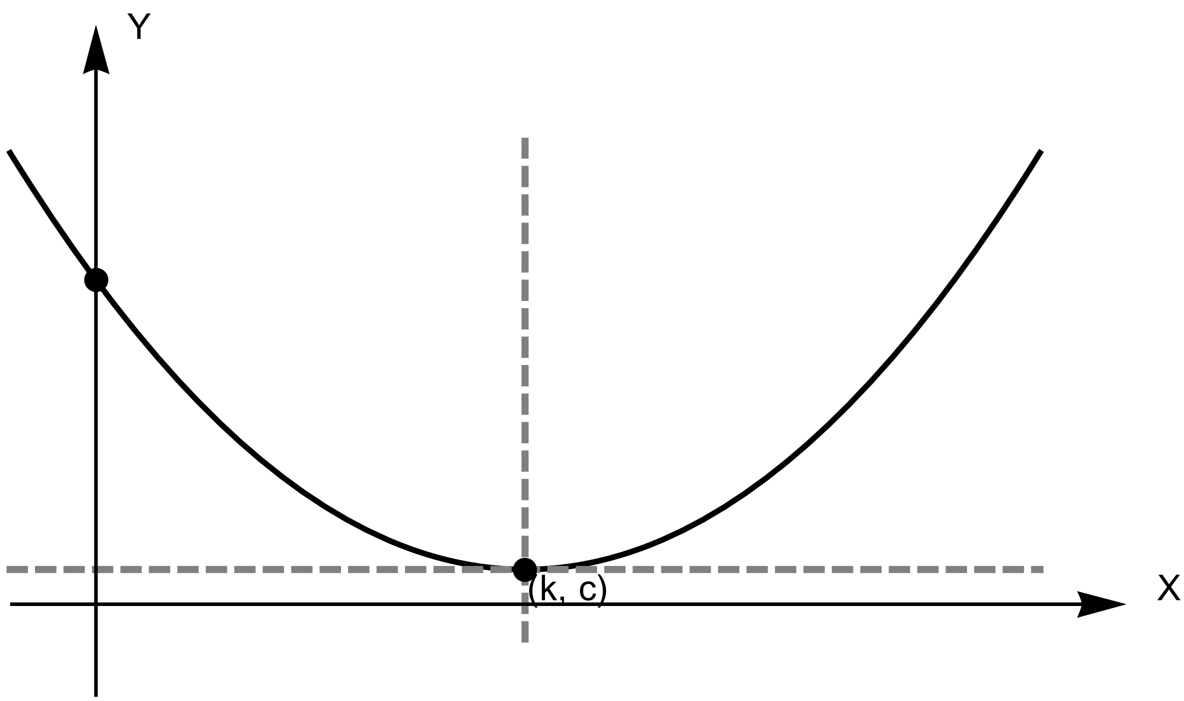
We study parabolas with an equation of the form $y = (x-k)^2 + c$ or $y = -(x-k)^2 + c$, where c and k are particular real numbers.

Note that we can use these techniques more generally since

$$y = (x-k)^2 + c = x^2 - 2kx + \left(k^2 + c\right).$$

Since $(x-k)^2 \geq 0$ and $(x-k)^2 = 0$ precisely when $x = k$, we see that $y = (x-k)^2 + c \geq c$ and $y = c$ precisely when $x = k$. This means that the point (k,c) is the **minimum turning point** of the parabola $y = (x-k)^2 + c$.

Similarly, (k,c) is the **maximum turning point** of the parabola $y = -(x-k)^2 + c$.



Features of $y = (x-k)^2 + c$:

$x - k = 0$ or $x = k$ is the **axis of symmetry**.

When $x = 0$, we have $y = k^2 + c$ so the point $\left(0, k^2 + c\right)$ is the **y-intercept**, which is the intersection with the y-axis.

When $y = 0$, $(x-k)^2 + c = 0$ so $(x-k)^2 = -c$. If $c \leq 0$, then $-c \geq 0$ so that $x - k = \pm\sqrt{-c}$ and we have $x = k \pm \sqrt{-c}$ so that the points $\left(k - \sqrt{-c}, 0\right)$ and $\left(k + \sqrt{-c}, 0\right)$ are the **x-intercepts**, the points of intersection with the x-axis.

Example 1.69 *Consider the parabola shown below. Find the minimum turning point, x-intercepts, the y-intercept, and the axis of symmetry.*

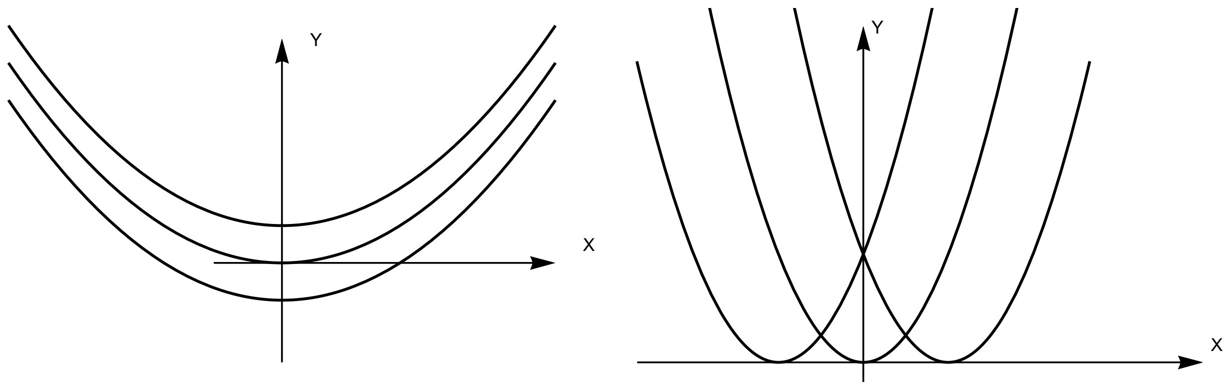
.

Example 1.70 *Sketch the parabola* $y = -(x+4)^2 - 6$*, identifying the maximum turning point, x-intercepts (if real), the y-intercept, and the axis of symmetry.*

Exercises/Homework:

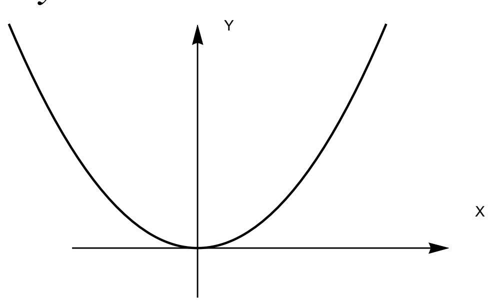
We discuss sketching parabolas of the form $y = a(x-h)^2 + k$ from the point of view of transformations of the parabola $y = x^2$.

Consider up/down and left/right translations of the parabola $y = x^2$.



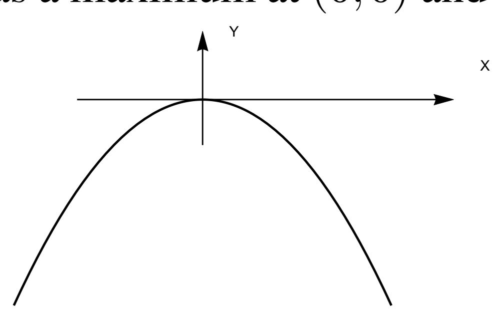
$y = ax^2$ is a **dilation** of $y = x^2$ by a.

If $a > 0$, then the parabola $y = ax^2$ has a minimum at $(0,0)$ and we say the



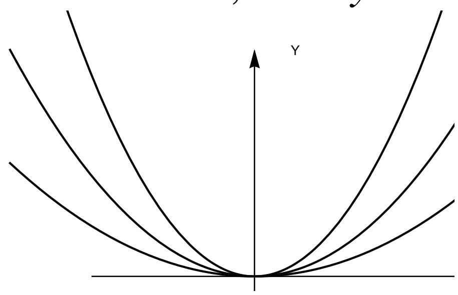
parabola is **concave up**.

If $a < 0$, then the parabola $y = ax^2$ has a maximum at $(0,0)$ and we say the



parabola is **concave down**.

If the dilation has $a > 1$, then $y = x^2$ is stretched upwards to become $y = ax^2$. If the dilation has $1 > a > 0$, then $y = x^2$ is compressed downwards to become $y = ax^2$.



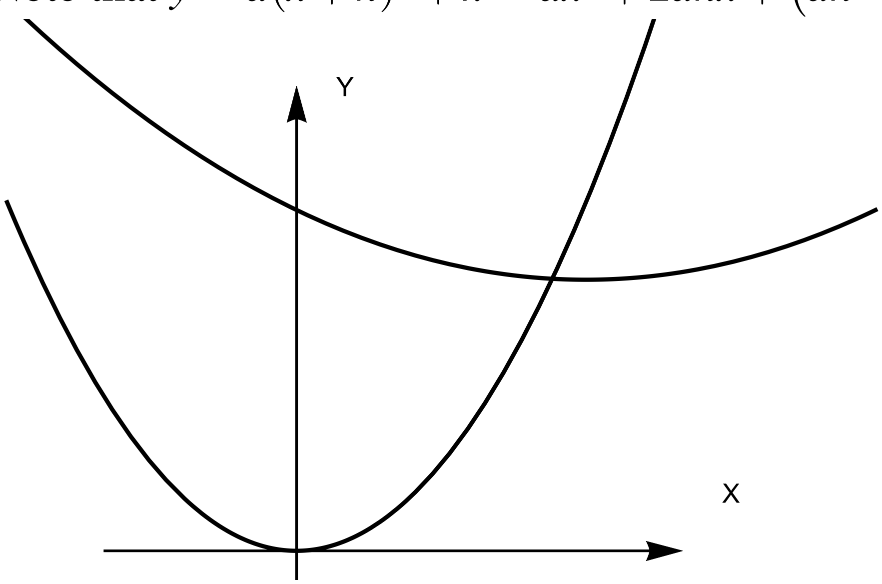
$y = (x-h)^2$ is a **right translation** of $y = x^2$, h units right.

$y = (x+h)^2$ is a **left translation** of $y = x^2$, h units left.

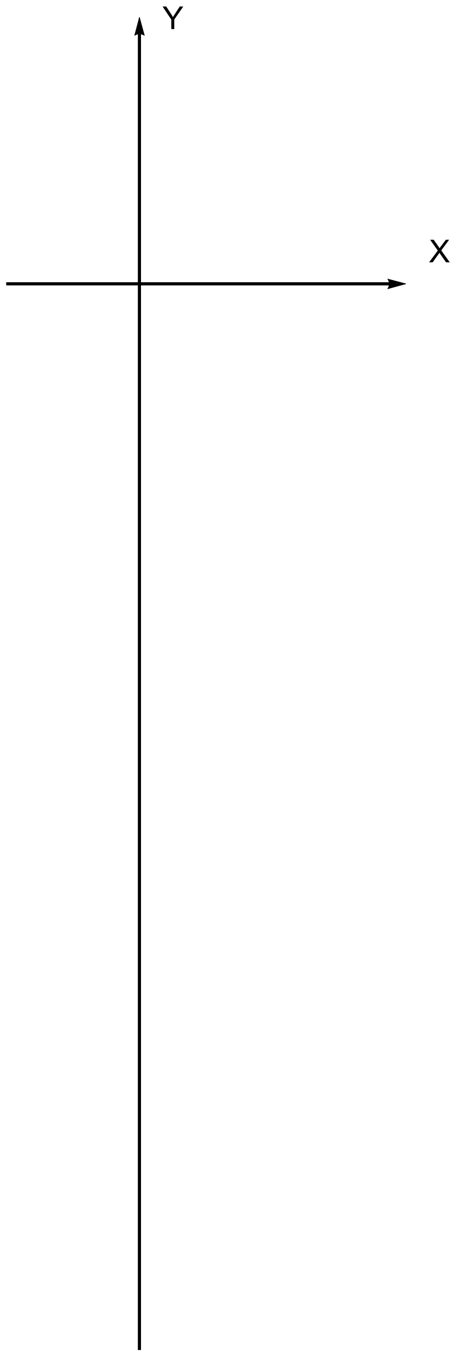
$y = x^2 + k$ is an **upwards translation** of $y = x^2$, k units upwards.

$y = x^2 - k$ is an **downwards translation** of $y = x^2$, k units downwards.

$y = a(x+h)^2 + k$ is a combination of translations and a dilation of $y = x^2$. Note that $y = a(x+h)^2 + k = ax^2 + 2ahx + \left(ah^2 + k\right)$.



Example 1.71 *Describe the transformations that transform* $y = x^2$ *into* $y = -2(x-3)^2 + 4$ *and sketch the parabola* $y = -2(x-3)^2 + 4$.



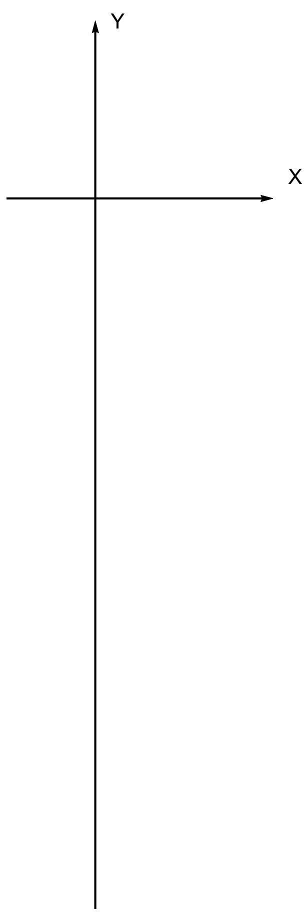
Exercises/Homework:

To sketch $y = ax^2 + bx + c$, with $a \neq 0$, suppose we are able to factorize the right hand side as

$$y = ax^2 + bx + c = (px + q)(rx + s).$$

When $y = 0$ we get the x-intercepts by solving $px + q = 0$ and $rx + s = 0$ so that $x = -\frac{q}{p}$, $x = -\frac{s}{r}$. We have the two x-intercept points $\left(-\frac{q}{p}, 0\right)$ and $\left(-\frac{s}{r}, 0\right)$. When $x = 0$ we have $y = c$ so we have the y-intercept point $(0, c)$.

Example 1.72 *Sketch the parabola* $y = x^2 + 4x - 21$ *by factorization.*



Exercises/Homework:

If we have $y = a(x-h)^2 + k$, then expanding,

$$y = a(x-h)^2 + k,$$

$$=$$

$$=$$

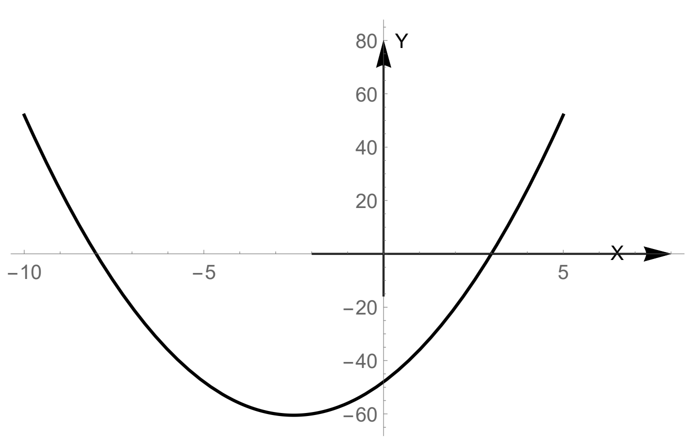
If given $y = ax^2 + bx + c$, then to complete the square is to find h and k such that

$$y = ax^2 + bx + c = a(x-h)^2 + k.$$

We have

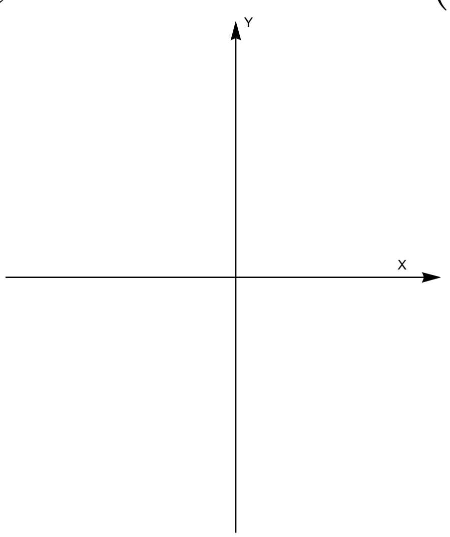
$$h = \qquad\qquad , \qquad\qquad k = \qquad\qquad .$$

Example 1.73 *Let $y = 2x^2 + 10x - 48$. Complete the square using the formulas for h and k above.*



Example 1.74 *Let* $y = 2x^2 + 10x - 48$. *Complete the square without using formulas.*

Example 1.75 *Use the completed square*
$y = 2x^2 + 10x - 48 = 2\left(x + \frac{5}{2}\right)^2 - \frac{121}{2}$ *to sketch the parabola.*



Exercises/Homework:

Let $y = ax^2 + bx + c$.

Completing the square,

$$y = a(\qquad\qquad\qquad\qquad),$$
$$=$$
$$=$$
$$=$$
$$=$$

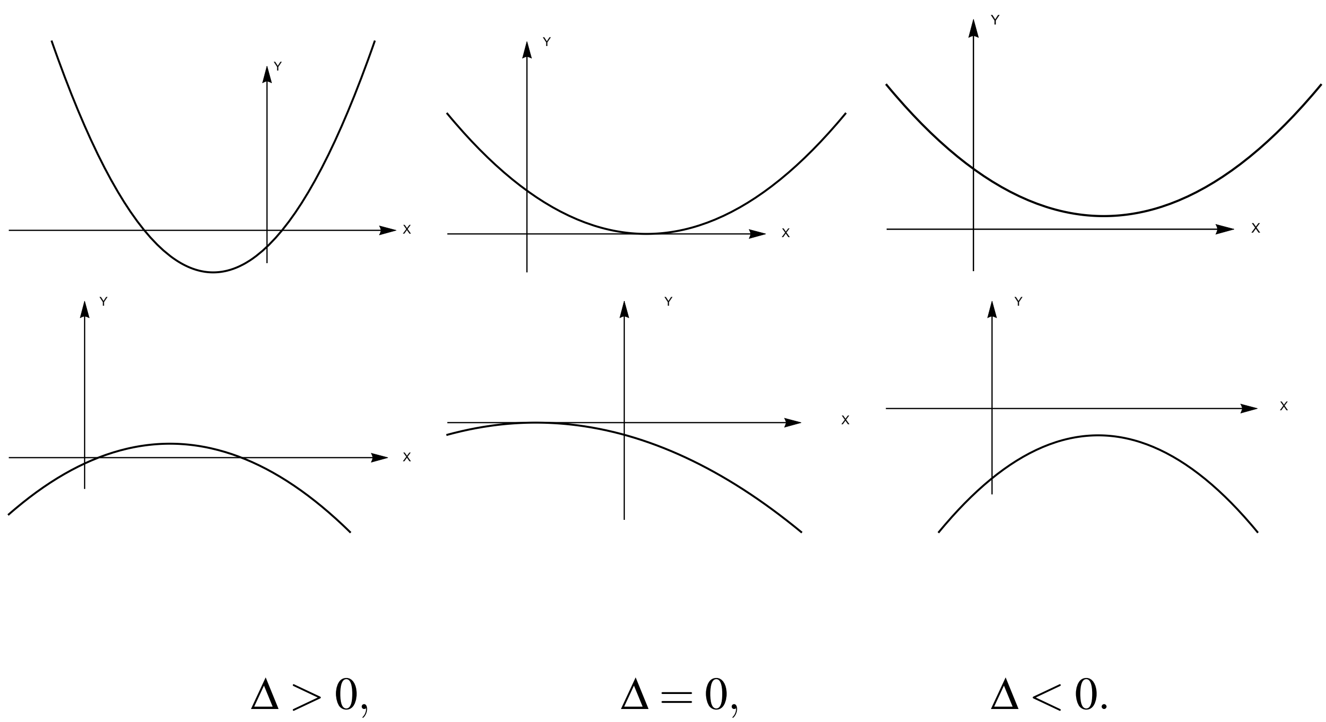
If $y = 0$, then

. This is called the **quadratic formula**.

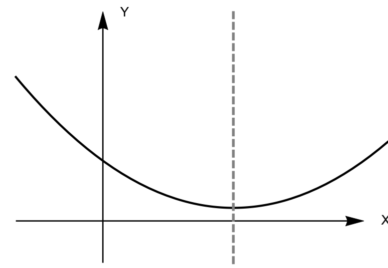
Let $\Delta = b^2 - 4ac$. This is called the **discriminant** of the quadratic polynomial $ax^2 + bx + c$.

Theorem 4 *Let Δ be the discriminant of $f(x) = ax^2 + bx + c$.*

- *If $\Delta > 0$, then $f(x)$ has two distinct real roots; the sketch of the parabola crosses the x-axis.*
- *If $\Delta = 0$, then $f(x)$ has one distinct real root; it is repeated and the quadratic is of the form $f(x) = a(x-h)^2$. The sketch of the parabola touches the x-axis at a tangent.*
- *If $\Delta < 0$, then $f(x)$ has no real roots; there are two non-real roots. The sketch of the parabola is above or below the x-axis entirely.*



$\Delta > 0,$ $\qquad\Delta = 0,$ $\qquad\Delta < 0.$



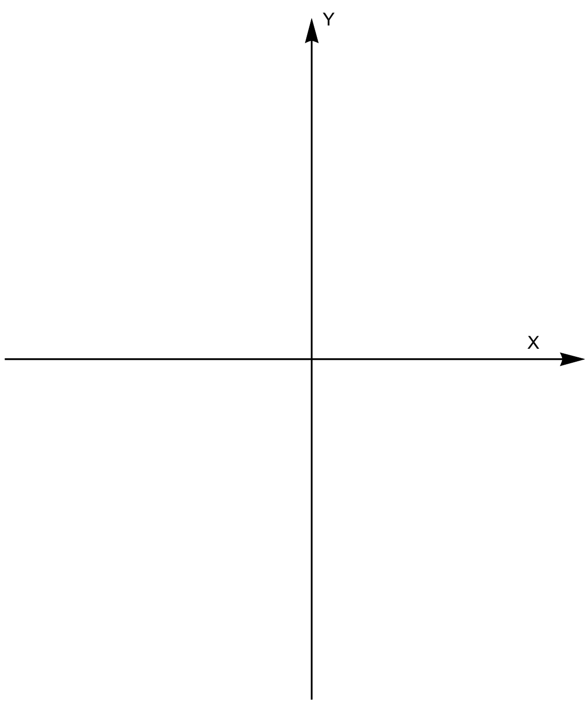
The vertical line $x = \frac{-b}{2a}$ is the **axis of symmetry**.

The point $\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right)$ is the **turning point**; if $a > 0$, then minimum and the parabola is **concave up**, if $a < 0$, then maximum and the parabola is **concave down**.

The y-intercept point is $(0, c)$.

If $\Delta \geq 0$, then the x-intercept point(s) is/are $\left(\frac{-b-\sqrt{\Delta}}{2a}, 0\right)$, $\left(\frac{-b+\sqrt{\Delta}}{2a}, 0\right)$.

Example 1.76 *Sketch* $y = 3x^3 - 18x - 48$ *using the above formulas.*

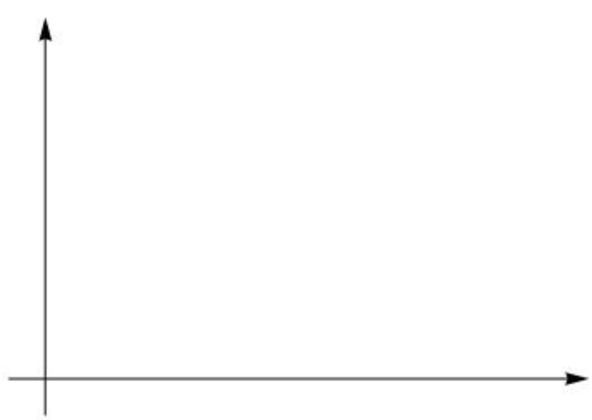


Exercises/Homework:

Example 1.77 *A stone is tossed vertically upwards at* $15\,m/\,s$ *and accelerates downwards due to gravity at* $10\,m/\,s^2$ *so that* $u = 15$ *and* $a = -10$*. Let* y *be the height of the stone after* t *seconds. If the height* y *is given by*

$$y = ut + \frac{1}{2}at^2,$$

find the maximum height of the stone and sketch the parabola relating y *to* t.



Example 1.78 *A gardener wants to erect a rectangular fenced area using* 50 *m of fencing. Determine the dimensions that give maximum area to the fenced region.*

Exercises/Homework:

A **function** f is a rule for sending (assigning) elements of a set X to elements of a set Y, and we put $y = f(x)$ or equivalently say (x, y) is in f, such that:

- For all elements x in X, there is a y in Y with $y = f(x)$, meaning f sends x to y.
- If y_1 and y_2 are in Y with $f(x) = y_1$ and $f(x) = y_2$, then we must have $y_1 = y_2$. This means that f is unambiguous and x is never sent to two different elements of Y. Rephrased, this is known as the vertical line test: If a vertical line intersects with the graph of a function in the x, y-plane, then it intersects exactly once.

If f is a function sending elements of X to elements of Y, we write $f : X \longrightarrow Y$ and state the rule for sending elements x to y; e.g. $f(x) = y^2$. The set X is called the **domain** of the function f and Y is called the **co-domain** of the function f. The **range** of a function f is the subset of Y of elements that we sent to Y, the outputs of f.

Example 1.79 *Let* $X = \{1,2,3\}$ *and* $Y = \{4,5,-2,0\}$. *We define the function* $f : X \longrightarrow Y$ *by* $f(1) = 5$, $f(2) = 4$, $f(3) = -2$. f *is an example of a function with domain* $\{1,2,3\}$, *co-domain* $\{4,5,-2,0\}$, *and range* $\{-2,4,5\}$. *Notice that the range is a subset of the co-domain.*

Example 1.80 *Let X be the set of all real numbers x satisfying* $-1 \leq x \leq 1$ *and let Y be the set of all real numbers. Does the relation* $x^2 + y^2 = 1$ *correspond to a function?*

Example 1.81 $[-1,1]$ *refers to the set all real numbers satisfying* $-1 \leq x \leq 1$. $\mathbb{R}$ *refers to the set of real numbers. Define* $f : [-1,1] \longrightarrow \mathbb{R}$ *by* $f(x) = -\sqrt{1-x^2}$. *Is* f *a function? Sketch the relation* f.

Example 1.82 *Show that* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *given by* $f(x) = x^2$ *is a function.*

Example 1.83 *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be given by* $f(x) = 3x^2 - 2x + 4$ *be a function. Calculate* $f(1)$, $f(0)$, *and* $f(5)$.

Exercises/Homework:

A **polynomial in one variable** x **with real coefficients** is an expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0,$$

where:

- n is a non-negative whole number.
- $a_0, a_1, a_2, \ldots, a_n$ are real numbers called **coefficients**.

The **coefficient of** x^n is the real number multiplying by x^n in an expression.

In this section we will refer to polynomials in one variable x with real coefficients simply as **polynomials**.

Note: A polynomial is a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0.$$

The **degree** of a polynomial $f(x)$ is the greatest exponent n among all of the terms of the polynomial. We write $\deg(f) = n$.

a_n is called the **leading coefficient** of f, where $\deg(f) = n$ and a_n is the coefficient of a_nx^n.

a_0 is called the **constant coefficient** or **constant term** of f. a_0 is the only term of f that is a real number.

Theorem 5 *If $f(x)$ and $g(x)$ are polynomials and k_1, k_2 are real numbers, then $k_1f(x)$, $k_2g(x)$, $k_1f(x)+k_2g(x)$, $k_1f(x)g(x)$, $k_1f(g(x))$, and $k_1g(f(x))$ are also polynomials.*

Exercises/Homework:

Example 1.84 *Expand the polynomial* $f(x) = (x+2)(x-3)$ *in two ways.*

Example 1.85 *Expand the polynomial*
$f(x) = \left(x^2+2x-5\right)\left(x^2+6x-2\right)$.

Example 1.86 *Expand the polynomial* $f(x) = (x+1)(x-3)(x-1)$.

Exercises/Homework:

Let $f(x)$ and $g(x)$ be polynomials, where $\deg(f) \geq \deg(g)$. We learn how to find polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x)q(x) + r(x),$$

where $\deg(q), \deg(r) \leq \deg(f)$. The polynomial $q(x)$ is called the **quotient** and $r(x)$ is called the **remainder**.

Note: If $f(x) = g(x)q(x) + r(x)$, then $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$.

Example 1.87 *Let* $f(x) = x^3 - 2x^2 - 2x - 3$ *and let* $g(x) = x^2 + x + 1$. *Use long division to calculate* $\frac{f(x)}{g(x)}$.

Example 1.88 *Let* $f(x) = 2x^3 - 9x^2 + 20x - 7$ *and let* $g(x) = x^2 - 3x + 4$. *Use long division to find polynomials* $q(x)$ *and* $r(x)$ *such that* $f(x) = g(x)q(x) + r(x)$.

Example 1.89 *Given that* $x - 1$ *is a factor of* $f(x) = x^3 - 2x^2 - 5x + 6$, *factorize* $f(x)$ *as a product of* 3 *linear factors.*

Exercises/Homework:

In this section we will introduce some important results for polynomial quotients and remainders.

Theorem 6 *Let $f(x)$ be a polynomial. If $f(a) = 0$, then $x - a$ is a factor of $f(x)$.*

If $g(x)$ is a factor of $f(x)$, then we say $g(x)$ **divides** $f(x)$ and equivalently, the remainder $r(x) = 0$ in $f(x) = g(x)q(x) + r(x)$.

Theorem 7 *Let $f(x)$ and $g(x)$ be polynomials with $g(x) \neq 0$ and $deg(f) \geq deg(g)$. There exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = g(x)q(x) + r(x)$, where $r(x) = 0$ or $deg(r) < deg(g)$.*

Corollary 1.1 (Bézout) *Let $f(x)$ and $x - a$ be polynomials, where a is a real number. Then the quotient and remainder polynomials satisfy $r(x) = f(a)$ and*

$$f(x) = (x - a)q(x) + f(a).$$

Example 1.90 *Let $f(x) = 3x^3 + x^2 - 10x - 8$. Factorize $f(x)$ as a product of 3 linear factors by calculating $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(2)$ and using long division.*

Example 1.91 *Show that* x^2+2 *divides* $x^4+x^3+6x^2+2x+8$.

Example 1.92 *Let* $f(x) = x^4+x^3+6x^2+2x+8$, $g(x) = x+1$. *Use Corollary 1.1 to find a polynomial* $q(x)$ *such that*
$f(x) = (x+1)q(x)+f(-1)$.

Exercises/Homework:

Theorem 8 (Null Factor Law) *Let $f(x)$ be a polynomial.*
If $f(x) = p_1(x)p_2(x)\dots p_n(x) = 0$, where the $p(x)$ are polynomials, then $p_1(x) = 0$ or $p_2(x) = 0$ or $\dots$ $p_n(x) = 0$.

Theorem 9 (Fundamental Theorem of Algebra) *Let $f(x)$ be a polynomial of degree n. Then there are exactly n complex numbers $\alpha = a + b\sqrt{-1}$, where a and b are real numbers, and it is possible that $b = 0$ so the α may include real numbers, such that*

$$f(x) = k(x - \alpha_1)(x - \alpha_2)\dots(x - \alpha_n),$$

and it is possible that the α coincide. Any non-real roots of $f(x)$ occur in conjugate pairs, meaning if $f(a + b\sqrt{-1}) = 0$, then $f(a - b\sqrt{-1}) = 0$.

Example 1.93 *Find all roots α of $f(x) = 2x^3 + x^2 - 4x - 3$ and reflect on the fundamental theorem of algebra.*

Example 1.94 *Find all roots* α *of* $f(x) = x^3 + x^2 + x + 1$ *and reflect on the fundamental theorem of algebra.*

Exercises/Homework:

1.4.1 Logarithms

$$a^x = b, \qquad \text{and} \qquad x = \log_a(b)$$

are different rearrangements of the same equation.

If $x = \log_a(b)$, then $a^x = b$. Conversely, if $a^x = b$, then $x = \log_a(b)$.

Example 1.95 *Write* $x = \log_{10}(1000)$ *in exponential form and simplify x numerically.*

Example 1.96 *Evaluate* $\log_3(81)$.

Example 1.97 *Evaluate* $\log_2(8)$.

Example 1.98 *Use a calculator to evaluate* $\log_{10}(6)$.

Example 1.99 *Evaluate* $\log_5(125)$.

Example 1.100 *Evaluate* $\log_6(216)$.

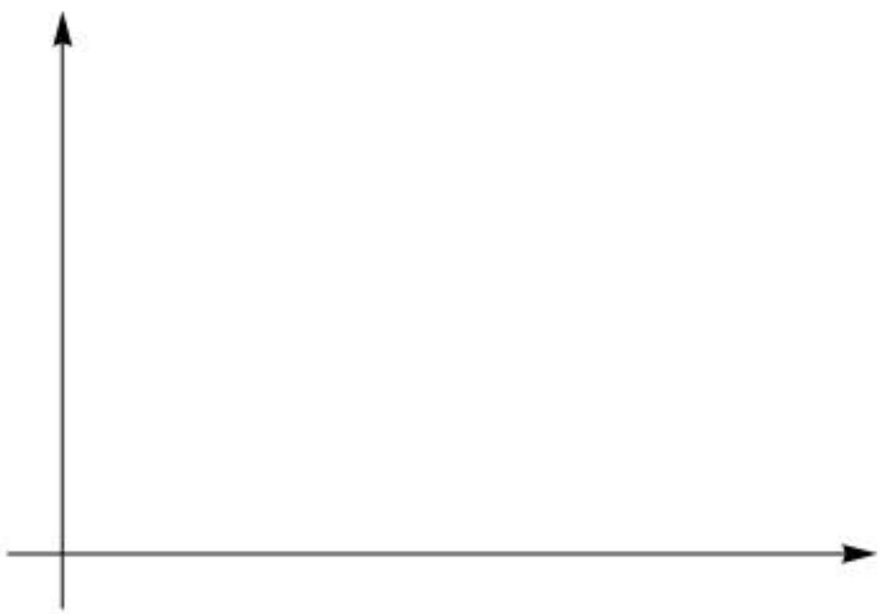
Exercises/Homework:

Data which shows exponential growth can be understood and modelled by taking the logarithm of the y-coordinate of the data points, plotting x on the horizontal axis and $\log(y)$ on the vertical axis.

Example 1.101 *Consider the data set consisting of points* (x,y)*:*

$$S = \{(1,2),(2,4),(3,8),(4,16),(5,32),(6,64)\}.$$

Let $Y = \log_2(y)$*. Then* $\{(x,Y)\} =$



In similar examples we can find a line of best fit to model such data sets.

Example 1.102 *The following table shows the population P of rabbits in a paddock over t years.*

t	0	1	2	3	4	5
P	4	32	256	2048	16384	131072

(a) *Calculate* $\log_{10}(P)$ *for the five P values in the table.*

(b) *Plot t versus* $\log_{10}(P)$.

(c) *Find the relationship between t and* $\log_{10}(P)$.

(d) *Use Part (c) to find the relationship between P and t.*

Exercises/Homework:

The exponent rule $a^b a^c = a^{b+c}$ is equivalent to the log rule $\log_a(x) + \log_a(y) = \log_a(xy)$.

Let $x = a^b$ so
Let $y = a^c$ so

$$a^b a^c =$$

Hence

$$\log_a(xy) = \log_a(x) + \log_a(y).$$

Similarly we have the following log rules:

$$\begin{aligned}
\log_a(xy) &= \log_a(x) + \log_a(y), && \text{from } a^b a^c = a^{b+c}, \\
\log_a(x/y) &= \log_a(x) - \log_a(y), && \text{from } a^b/a^c = a^{b-c}, \\
\log_a(x^n) &= n\log_a(x), && \text{from } \left(a^b\right)^c = a^{bc}, \\
\log_a(b) &= \frac{\log_c(b)}{\log_c(a)}, && \text{from } (c^y)^x = c^{xy}, \\
\log_a(1/b) &= -\log_a(b), && \text{from } \frac{1}{b} = b^{-1}, \log_a(x^n) = n\log_a(x), \\
b &= a^{\log_a(b)}, && \text{by letting } x = \log_a(b), \\
\log_a(a) &= 1, (a \neq 1) && \text{from } a^1 = a, \\
\log_a(1) &= 0, && \text{from } a^0 = 1.
\end{aligned}$$

Example 1.103 *Simplify* $\log_a(3) + \log_a(5)$.

Example 1.104 *Simplify* $3\log_a(2)$.

Example 1.105 *Calculate* $\log_2(1/8)$.

Example 1.106 *Calculate* $\log_6(36) - \log_6(4)$.

Example 1.107 *Prove that* $\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$ *using* $(c^y)^x = c^{xy}$*, where* $x = \log_a(b)$*,* $y = \log_c(a)$*,* $z = \log_c(b)$.

Exercises/Homework:

Recall that if $a^x = b$, then $x = \log_a(b)$. In this section we use this fact to solve exponential equations.

Example 1.108 *Solve* $3^x = 9$ *for x using logarithms.*

Example 1.109 *Solve* $5^x = 35$ *for x using logarithms.*

Example 1.110 *Solve* $12 \times 2^x = 4$ *for x using logarithms.*

Example 1.111 *Solve* $3^{2x+5} = 18$ *for x using logarithms.*

Exercises/Homework:

A **trial** is a probability experiment such as tossing a coin or rolling a die or a pair of dice.

The set of all possible outcomes of a probability experiment is called the **sample space**.

An **outcome** of a probability experiment is a possible result such as obtaining a 5 on rolling a die.

A collection of specific outcomes of a probability experiment is called an **event**.

The **probability of an event** x is a real number P (or $P(x)$) with $0 \leq P \leq 1$ that represents and measures the chance or likelihood of the event occurring.

If $P = 0$, then the event cannot occur.

If $P = 1$, then the event must occur.

For example, the probability of obtaining heads when tossing a fair coin is $\frac{1}{2} = 0.5$.

If two events have the same probability (chance of occurring), then we say that the events are **equally likely**.

The probability of the event A occurring is

$$P(A) = \frac{|A|}{n},$$

where n is the number of elements in the sample space and $|A|$ is the number of elements in the event space A.

Example 1.112 *A card is randomly selected from a 52-card pack of playing cards. What is the probability that the card is a spade?*

Example 1.113 *A 6-sided die is rolled. What is the probability that the outcome is greater than 1 and odd?*

Example 1.114 *A letter is randomly chosen from the word INDOOROOPILLY. What is the probability that O is chosen?*

Exercises/Homework:

A **set** is a collection of objects (usually numbers) such that there are no repeated elements and there is no order of elements.

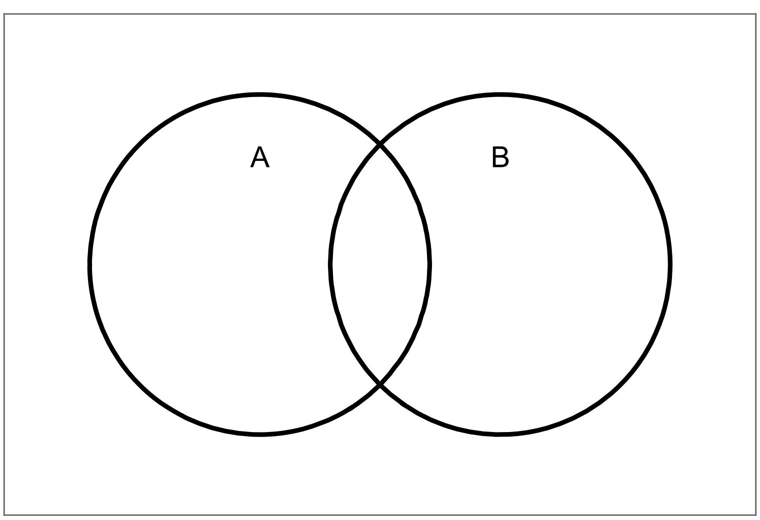
The **universal set** $\mathscr{U}$ is the set of all elements we could possibly be referring to. In probability this is the set of all possible outcomes, the sample space.

A **Venn diagram** is an illustration with intersecting circles used for displaying sets and their elements, common elements, and the universal set. We often list the elements of sets within the drawn circles. However, sometimes the number of elements are instead written in the circles.

Example 1.115 *Let*

$$\begin{aligned} A &= \{1,2,-5,8,6,0\}, \\ B &= \{9,3,7,1,5,-4,2,8\}. \end{aligned}$$

Display the elements of the sets A and B in the circles shown in the Venn diagram below. Where the two circles meet, put the elements that are in both A and B.



The **Cardinality** of a finite set A is the number of elements in the set, written $|A|$ or sometimes $n(A)$. For example, with A, B as in Example 1.115, $|A| = 6$ and $|B| = 8$.

If the set A has $|A| = 0$, then we say A is the **empty set** or **null set** and we write $A = \emptyset$.

The **intersection** of two sets A and B is the set consisting of those elements that are in both A and B, denoted $A \cap B$.

The **union** of two sets A and B is the set consisting of those elements that are in A or in B, denoted $A \cup B$.

When discussing probability we will use sets and events interchangeably.

Two events A and B are **mutually exclusive** if $A \cap B = \emptyset$ or equivalently $|A \cap B| = 0$.

The **complement of an event** A, denoted A', is the opposite event to A: $A' = \mathscr{U} - A$, meaning the sample space without those elements belonging to A. We have $P(A') = 1 - P(A)$.

Example 1.116 *A six-sided die is rolled. Compare the probability of getting a number greater than* 2 *to the probability of getting a number less than or equal to* 2.

A **two-way table** is a table that displays the number of elements in the sets $A \cap B$, $A' \cap B$, $A \cap B'$, and $A' \cap B'$ in the following form:

	A	A'	
B	$\|A \cap B\|$	$\|A' \cap B\|$	$\|A \cap B\| + \|A' \cap B\|$
B'	$\|A \cap B'\|$	$\|A' \cap B'\|$	$\|A \cap B'\| + \|A' \cap B'\|$
	$\|A \cap B\| + \|A \cap B'\|$	$\|A' \cap B\| + \|A' \cap B'\|$	n

where $n = |A \cap B| + |A \cap B'| + |A \cap B'| + |A' \cap B'|$.

Example 1.117 *For breakfast a class of* 25 *students eat apples or bananas or nothing. Let A be the set of students who eat apples and B be the set of students who eat bananas.* 10 *students eat apples,* 18 *students eat bananas,* 8 *students eat apples and bananas,* 5 *students eat neither apples nor bananas,* 10 *students eat bananas but not apples. Construct a two-way table representing this information and find the probability that a student eats apples but not bananas.*

Exercises/Homework:

Theorem 10 (Addition rule) *Let A and B be two events. Then*

$$
\begin{aligned}
|A\cap B| + |A\cup B| &= |A| + |B|, \\
P(A\cap B) + P(A\cup B) &= P(A) + P(B).
\end{aligned}
$$

Example 1.118 *A card is randomly chosen from a 52-card deck of cards.*

(a) *What is $|\mathscr{U}|$, the cardinality of the sample space?*

(b) *Let A be the event 'the card is a number 2 through 10 inclusive' and B be the event 'the card is a spade'. What is $|A|$, $|B|$, $|A\cap B|$, $|A\cup B|$?*

(c) *Let A be the event 'the card is a number 2 through 10 inclusive' and B be the event 'the card is a spade'. What is $P(A)$, $P(B)$, $P(A\cap B)$, $P(A\cup B)$?*

(d) *Verify that the addition rule holds.*

Example 1.119 *Let A and B be two events with* $P(A) = 0.25$, $P(B) = 0.9$, $P(A \cap B) = 0.2$.

(a) *Calculate* $P(A \cup B)$.

(b) *Calculate* $P(A' \cap B)$.

Example 1.120 *Let A and B be two events with* $P(A) = 0.4$, $P(B) = 0.8$, $P(A \cup B) = 0.98$.

(a) *Calculate* $P(A \cap B)$.

(b) *Calculate* $P(A \cap B')$.

Exercises/Homework:

Conditional probability is the probability of an event A given that another event B has already occurred. This is denoted $P(A|B)$ and we have

Theorem 11 (Multiplication rule of conditional probability)

$$\begin{aligned} P(A \cap B) &= P(A|B)P(B), \\ &= P(B|A)P(A). \end{aligned}$$

Example 1.121 20 *people attend a house-warming party.* 12 *attendees brought flowers,* 9 *attendees brought chocolate cake, and* 4 *attendees brought both flowers and chocolate cake.*

(a) *Calculate the probability that an attendee brought flowers or chocolate cake.*

(b) *Calculate the probability that an attendee brought chocolate cake given that they brought flowers.*

(c) *Represent the information in a two-way table.*

Example 1.122 *A bag of marbles contains 3 red marbles and 2 blue marbles. Calculate the probability that a second chosen marble is red given that the first chosen marble was red and not replaced in the bag before drawing the second marble.*

Exercises/Homework:

A **two-step experiment** is a probability experiment with two trials in which the sample space is a set of pairs (x, y), where x is the outcome of the first trial and y is the outcome of the second trial. A two-step experiment can have $x \in X$, $y \in Y$, where it is possible that $X = Y$ but not necessarily so.

Assuming $X = Y$, if the two-step experiment occurs with replacement, then (x, x) is an element of the sample space. If the two step experiment occurs without replacement, then (x, x) is not an element of the sample space.

A **sample-space table** is a table containing elements of the sample space arranged so that the pair (x, y) is in the i-th row and j-th column of the table, where x is the first outcome of the event and y is the second outcome of the event. Furthermore, all elements in the j-th column contain the same first event and all elements in the i-th row contain the same second event.

Example 1.123 *Two letters are chosen randomly from the letters of the word CAR in a two-step experiment. Construct a sample-space table:*

(a) *with replacement.*

(b) *without replacement.*

	C	A	R
C			
A			
R			

	C	A	R
C			
A			
R			

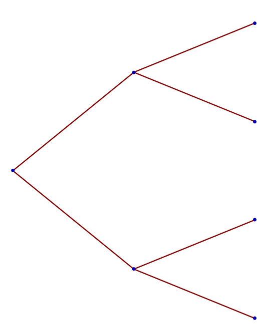
Exercises/Homework:

In mathematics a **tree** is a graph that connects vertices with edges (branches) so that each pair of vertices in connected by exactly one path. A tree has no cycles and has n vertices and $n-1$ edges.

Example 1.124 *Draw a tree with* 6 *vertices.*

A **tree diagram** in probability is a tree in which the edges are labeled with probabilities and the vertices are labeled with outcomes of a probability experiment with multiple steps. The vertices of a tree diagram are set out in columns that correspond to the steps of a multi-step experiment. A path that includes a vertex in each step corresponds to an element of the sample space. The probability of an event is obtained by multiplying the probabilities on the edges of such a path.

Example 1.125 *Consider the following tree diagram on flipping a coin twice. Label the tree diagram with H and T. Label the edges with the appropriate probabilities. Calculate the probability of two heads.*



Example 1.126 *A bag contains* 5 *white marbles and* 3 *black marbles. In a two-step experiment, two marbles are chosen randomly without replacement. Draw a tree diagram and calculate the probability of selecting two white marbles.*

Exercises/Homework:

Two events are **independent events** if the probability of a one event cannot influence the probability of the other event.

For example, in flipping a coin twice, the first and second coin tosses are independent events.

Drawing two coloured marbles from a bag without replacement are not independent events.

If Event A and Event B are independent events, then $P(A|B) = P(A)$ and $P(B|A) = P(B)$. Furthermore, since $P(A|B)P(B) = P(A \cap B)$, we have $P(A \cap B) = P(A)P(B)$.

Example 1.127 *A six-sided die is tossed twice. If A is the event the first toss gave a* 3 *and B is the event the second toss gave a* 4*, calculate:*

(a) $P(A)$,

(b) $P(B)$,

(c) $P(B|A)$,

(d) $P(A \cap B)$.

Exercises/Homework:

2 Year 11 Mathematical Methods

2.1 Term 1

2.1.1 Mathematical Language and Sets

We begin by getting familiar with sets, important notation for sets, interval notation, and relevant examples of these.

A **set** is a collection of objects (usually numbers) in which the elements (members) of the set have no order and no elements are repeated.

A set can have finitely many elements or infinitely many elements.

Example 2.1 *Let* $S = \{1, 5, -4, 3, 91\}$ *and* $C = \{3, 4, 8, 1, 3, 8\}$. *S and C are collections and S is a set.*

We have the following notation for special important sets.
: means 'such that'.
$\in$ means 'is an element of'.
$\mathbb{Z} = \{a \ : \ a \text{ is a whole number } \} = \{\cdots -3, -2, -1, 0, 1, 2, \ldots\}$. We read this as the set of all a such that a is a whole number. In other words, the set of integers.
$\mathbb{Q} = \left\{\frac{a}{b} \ : \ a, b \text{ are in } \mathbb{Z} \text{ and } b \neq 0\right\}$. This is called the set of rational numbers.

2.2 Term 2

Under Construction

2.3.1 Limits and the First Derivative

Let $f(x)$ be a function that is defined near a real number a. If the function $f(x)$ is the real number L near a, then we write

$$\lim_{x \longrightarrow a} f(x) = L.$$

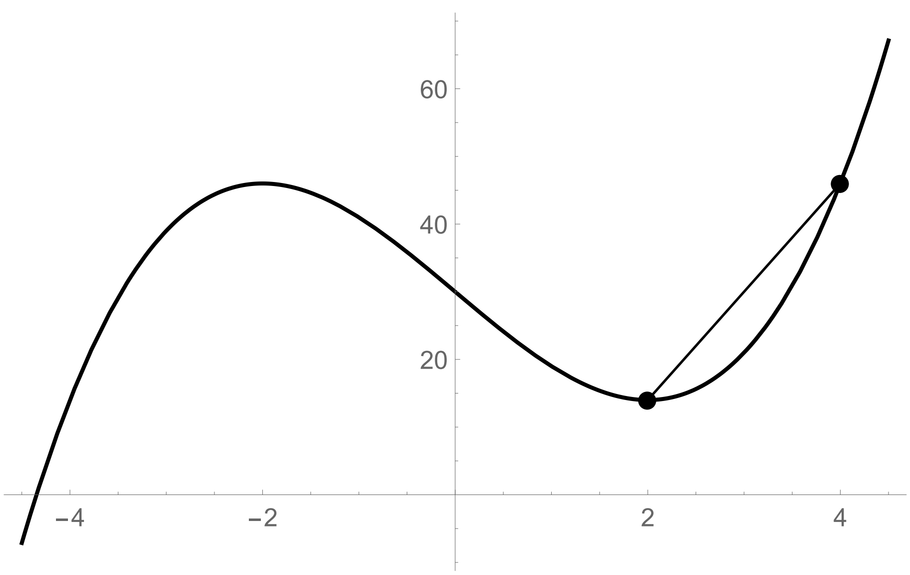
We say, the limit as x approaches a of $f(x)$ is L.

More formally, the real number L is the limit of the sequence $a_1, a_2, \ldots$ if and only if for every real number $\varepsilon > 0$, there exists a natural number N such that for all $n > N$, we have $|a_n - L| < \varepsilon$.

Example 2.2 *The function* $f(x) = \frac{x^3-1}{x-1}$ *is not defined when* $x = 1$. $f(x)$ *is defined for all real* x *except for* $x = 1$ *and hence defined for* x *near* 1. *Calculate* $\lim_{x \longrightarrow 1} f(x)$.

Theorem 12 *(Properties of Limits) Let a, k_1, k_2 be particular real numbers and $f(x), g(x)$ are functions such that $\lim_{x \longrightarrow a} f(x)$ and $\lim_{x \longrightarrow a} g(x)$ exist. Then:*

- $\lim_{x \longrightarrow a} k_1 f(x) = k_1 \lim_{x \longrightarrow a} f(x)$.
- $\lim_{x \longrightarrow a} (k_1 f(x) + k_2 g(x)) = k_1 \lim_{x \longrightarrow a} f(x) + k_2 \lim_{x \longrightarrow a} g(x)$.
- $\lim_{x \longrightarrow a} (f(x) g(x)) = \left(\lim_{x \longrightarrow a} f(x) \right) \left(\lim_{x \longrightarrow a} g(x) \right)$.
- *If* $\lim_{x \longrightarrow a} g(x) \neq 0$, *then* $\lim_{x \longrightarrow a} \left(\frac{f(x)}{g(x)} \right) = \frac{\left(\lim_{x \longrightarrow a} f(x) \right)}{\left(\lim_{x \longrightarrow a} g(x) \right)}$.
- *If $f(x)$ is defined at $x = a$, then* $\lim_{x \longrightarrow a} f(x) = f(a)$.



The **first derivative** $f'(x)$ or **derivative** for short, is defined by the limit

$$f'(x) = \lim_{h \longrightarrow 0} \frac{1}{h} \left(f(x+h) - f(x) \right).$$

This gives a function representing the slope of the function $f(x)$ at any particular x value in the domain of f. To **differentiate** is to find $f'(x)$.

Example 2.3 *Use the definition of the derivative to find* $f'(x)$ *for the function* $f(x) = x^2$.

Example 2.4 *Use the definition of the derivative to find* $f'(x)$ *for the function* $f(x) = x^3 - 4x + 3$.

Exercises/Homework:

Consider the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

We learn how to use rules to differentiate polynomial functions to obtain the polynomial function $f'(x)$.

Let $f(x) = kx^n$, where k is a particular real number and n is a non-negative integer. Using the limit definition of $f'(x)$,

$$\begin{aligned} f'(x) &= \lim_{h \longrightarrow 0} \frac{1}{h} (f(x+h) - f(x)) \\ &= . \qquad . \\ &= \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

Rule: To differentiate $f(x) = kx^n$, bring down n and subtract 1 from the exponent so that $(kx^n)' = knx^{n-1}$.

Note: Since $\lim_{x \longrightarrow a} (p(x)+q(x)) = \lim_{x \longrightarrow a} p(x) + \lim_{x \longrightarrow a} q(x)$ and the derivative of a constant is 0, the derivative of

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is

$$f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1.$$

Notation: $f'(x) = (f(x))' = \frac{d}{dx}(f(x))$.

Example 2.5 *Let $f(x) = x^3 + 3x - 7$. Calculate $f'(x)$ using both the limit definition and the rules for differentiating polynomials.*

Example 2.6 *Let $f(x) = -3x^5 + 2x^3 - 12x^2 + 14x - 32$. Calculate $f'(x)$ using rules for differentiating polynomials.*

Exercises/Homework:

Let $f(x) = kx^{-n}$, where n is a positive integer and k is a particular non-zero real number. Using the limit definition,

$$
\begin{aligned}
f'(x) &= \lim_{h \longrightarrow} \frac{1}{h}\left(f(x+h) - f(x)\right), \\
&= . \qquad . \\
&= \\
&= \\
&= \\
&= \\
&= \\
&=
\end{aligned}
$$

Theorem 13 *Let $f(x) = kx^n$, where k and n are real numbers.*

- *If $n = 0$, then $f'(x) = 0$. (The derivative of a constant is zero.)*
- *If $n \neq 0$, then $f'(x) = nkx^{n-1}$.*

Example 2.7 *Differentiate the function* $f(x) = 2x^3 - \frac{5}{x} + 14x - 8.$

Example 2.8 *Differentiate the function* $f(x) = -12x^5 + 3x^2 + \frac{2}{x^2} - \frac{1}{x}.$

Exercises/Homework:

We plot $y = f'(x)$ for functions $f(x)$ and answer questions on the sign of the first derivative of a function.

Example 2.9 *Let* $f(x) = 2x^3 - 4x + 5$. *Plot* $y = f'(x)$ *and determine where* $f'(x)$ *is positive, negative, and zero. Plot* $y = f(x)$ *also and consider any turning points in the context of the sign of* $f'(x)$.

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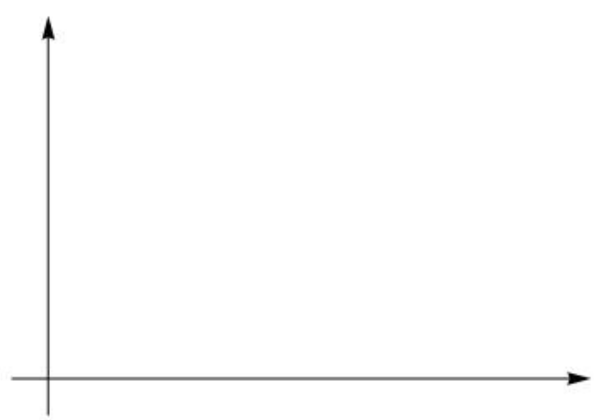

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Example 2.10 *Let* $f(x) = x^2 + 3x - 4$. *Plot* $y = f(x)$ *and* $y = f'(x)$ *on the same graph. State where* $f'(x) > 0$ *and* $f'(x) < 0$.

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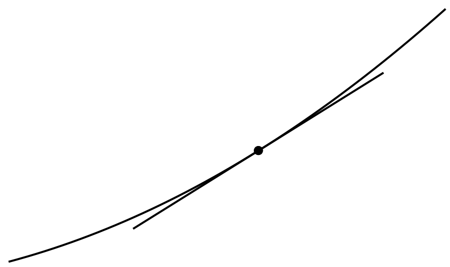
Let a and b be real numbers. We say that $f(x)$ is **increasing** on the interval (a,b) if for all x: $a < x < b$, $f'(x) > 0$. $f(x)$ is **increasing** on the interval $[a,b]$ if for all x: $a \leq x \leq b$, $f'(x) > 0$.

We say that $f(x)$ is **decreasing** on the interval (a,b) if for all x: $a < x < b$, $f'(x) < 0$. $f(x)$ is **decreasing** on the interval $[a,b]$ if for all x: $a \leq x \leq b$, $f'(x) < 0$.

Note: If $f(x)$ is increasing of $[a,b]$, then $f(a) < f(b)$. If $f(x)$ is decreasing of $[a,b]$, then $f(a) > f(b)$.

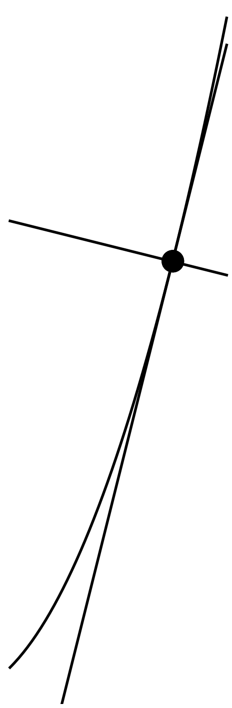
Exercises/Homework:

Let $y = f(x)$ be a curve, where $f(x)$ is a function. The **tangent line** at the point (a,b) is the line which intersects with the curve at exactly the one point (a,b) and has slope equal to the slope of the curve at the point (a,b).



To find the equation of a tangent line at (a,b),

The **normal line** at the point (a,b) is the line perpendicular to the tangent line at the point (a,b) passing through (a,b).



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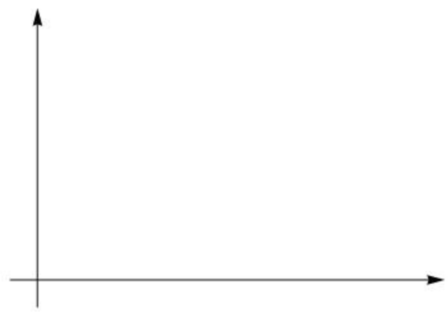
Recall that the lines $y = mx + c$ and $y = -\frac{1}{m}x + d$ are perpendicular.

Example 2.11 *Let* $y = f(x) = 3x^2 - 4x + 4$. *Calculate the tangent line and the normal line at the point* $(1,3)$.

Exercises/Homework:

The **average rate of change** of $y = f(x)$ over the interval $[a,b]$ is

$$m_{\text{av.}} = \frac{f(b)-f(a)}{b-a}.$$



The **instantaneous rate of change** of $f(x)$ at the point where $x = a$ is $f'(a)$.

If $f'(a)$ is positive, then the function is **increasing** at $x = a$.
If $f'(a)$ is negative, then the function is **decreasing** at $x = a$.

Example 2.12 *Let* $y = -4x^2 + 5x - 12$. *Determine the average rate of change of* $f(x)$ *over the interval* $[-1,2]$.

Example 2.13 *The position of a particle is given by*

$$y = f(t) = 5t^3 - 2t + 6,$$

where t is time in seconds and y is the position in metres. The velocity of the particle is $v = f'(t)$. Determine the velocity after 1 second, 2 seconds, 3 seconds.

Exercises/Homework:

Let $y = f(x)$, where $f(x)$ is a function of x. The point (a,b) is a **stationary point** if $f'(a) = 0$, or equivalently, $\left.\frac{dy}{dx}\right|_{x=a} = 0$.

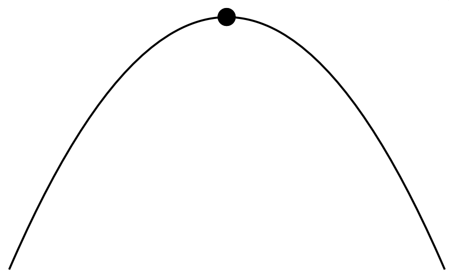
Example 2.14 *Find all stationary points of the curve* $y = 2x^3 - 15x + 8$.

Example 2.15 *The curve* $y = ax^2 + bx + c$*, where* $a \neq 0$*,* a, b, c *are particular real numbers, has one stationary point* $(1,2)$*. Show that* $y = (c-2)(x-1)^2 + 2$*.*

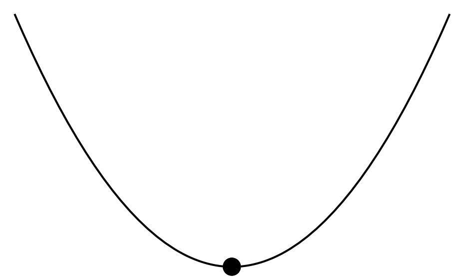
Exercises/Homework:

Recall that the point (a,b) of the curve $y = f(x)$ is a stationary point if $f'(a) = 0$.

The stationary point (a,b) is called a **local maximum** if $f'(x) > 0$ for $x = a - \varepsilon$ (immediately left of (a,b)), where ε is an arbitrarily small positive real number, and $f'(x) < 0$ for $x = a + \varepsilon$ (immediately right of (a,b)).



The stationary point (a,b) is called a **local minimum** if $f'(x) < 0$ for $x = a - \varepsilon$ (immediately left of (a,b)), where ε is an arbitrarily small positive real number, and $f'(x) > 0$ for $x = a + \varepsilon$ (immediately right of (a,b)).



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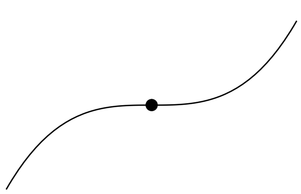
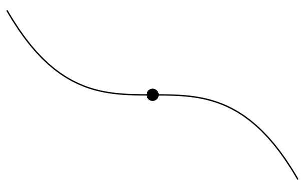
Such stationary points are called **turning points**.

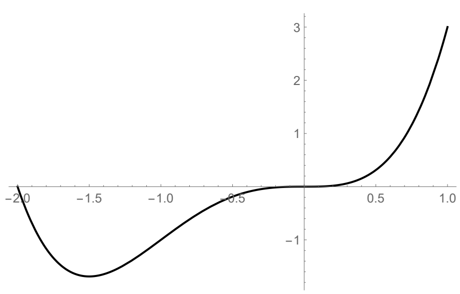
A stationary point (a,b) such that $f'(x) > 0$ for $x = a - \varepsilon$ (immediately left of (a,b)), where ε is an arbitrarily small positive real number, and $f'(x) > 0$ for $x = a + \varepsilon$ (immediately right of (a,b)) is called a **stationary inflection point**.



A stationary point (a,b) such that $f'(x) < 0$ for $x = a - \varepsilon$ (immediately left of (a,b)), where ε is an arbitrarily small positive real number, and $f'(x) < 0$ for $x = a + \varepsilon$ (immediately right of (a,b)) is called a **stationary inflection point**.



Example 2.16 *The curve $y = x^4 + 2x^3$ has a local minimum and an inflection point. Find them and reflect on the definitions of local minimum and inflection point.*

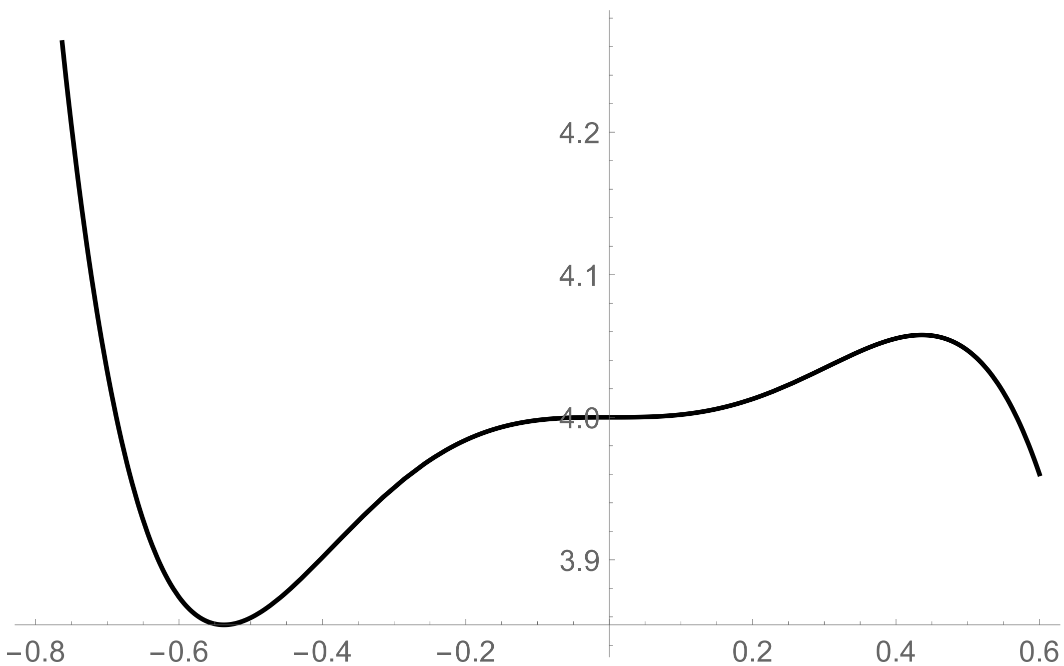


Exercises/Homework:

Let $f:[a,b]\longrightarrow\mathbb{R}$ be a function and let $y=f(x)$ be a curve/ Suppose the curve has local maxima points (p_1,q_1), (p_2,q_2), $\ldots,(p_n,q_n)$. A **global maximum point** (u,v) is a point such that $a\le u\le b$ and

$$v\ge f(a),f(b),f(p_1),f(p_2),\ldots,f(p_n).$$

In other words, the point in the domain with the greatest y-value.



The **global minimum point** is defined analogously; a point in the domain with the least y-value.

Example 2.17 *Visually inspect the curve above to identify local maxima, local minima, inflection points, and the global maximum and the global minimum on the domain* $[-0.8,0.6]$.

Example 2.18 *Find the global maximum and global minimum of the curve* $y = f(x) = x^4 + 2x^3 - 6x^2 - 12x$ *over the interval* $[-3, 3]$.

.

Exercises/Homework:

The **position** of a particle or object is a point $(t, x(t))$ or $(t, x(t), y(t))$, $(x(t), y(t))$, etc. which depends on time. If the object moves on a linear trajectory, then we can express the position x as a function of time as position $= x = x(t)$.

Example 2.19 *A ball is tossed vertically upwards at* $t = 0$ *seconds from* 2 *metres above the ground. The elevation (position) of the ball in metres at* t *seconds is given by* $y(t) = -5t^2 + 4t + 2$*,* $t \geq 0$*. Plot* y *versus* t *with* t *on the horizontal axis.*

The **instantaneous velocity** or just **velocity** of a particle with position $x(t)$ is $v = x'(t) = \frac{dx}{dt}$, the first derivative of the position $x(t)$ with respect to t.

The **average velocity** of a particle with position $x(t)$ is given by

$$v_{\text{av}} = \frac{x_2 - x_1}{t_2 - t_1}.$$

The **speed** of a particle is the magnitude $|x'(t)|$ (absolute value). The speed of the particle with velocity $(x'(t), y'(t))$ is the magnitude

$$\| (x'(t), y'(t)) \| = \sqrt{(x'(t))^2 + (y'(t))^2}.$$

The **average speed** of a particle with position $x(t)$ is given by $|v_{\text{av}}|$.

Example 2.20 *Calculate the velocity, speed, and average speed of the ball with position* $y(t) = -5t^2 + 4t + 2$, $t \geq 0$ *from vertical launch until it hits the ground again.*

The **instantaneous acceleration** or just **acceleration** of a particle is the first derivative of the velocity of the particle,

$$a(t) = v'(t) = (x'(t))' = x''(t).$$

The **average acceleration** of a particle is $a_{\text{av}} = \frac{v_2 - v_1}{t_2 - t_1}$, where v_1 is the initial velocity and v_2 is the final velocity of the particle.

Exercises/Homework:

Consider the function $y = \left(2x - 4x^3 + 3\right)^5$. Since this is a polynomial function of x, we could expand this before we differentiate the function. The **chain rule** gives us another way to calculate the derivative of a function.

In this example, letting $u = 2x - 4x^3 + 3$,

Let $F(x)$ be a function of x such that there are other functions $g(x)$ and $f(x)$ satisfying

$$F(x) = f(g(x)) = (f \circ g)(x).$$

The **chain rule** states that

$$F'(x) = (f(g(x)))' = f'(g(x)) \cdot g'(x).$$

Alternatively in Leibniz's notation,

$$F'(x) = (f(g(x)))' = \frac{df}{dg}\frac{dg}{dx}$$

or

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx},$$

where y is a function of u and u is a function of x.

Note: $\frac{dy}{dx}$ is **not a fraction** but the fact that the symbols look like fractions is a very useful aide to memory.

Note: For small Δx, $\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$.

Example 2.21 *Let* $y = 3\left(5x^2 - 7x + 2\right)^6$. *Calculate* $\frac{dy}{dx}$.

Example 2.22 *Let* $y = \left(3x^2 - \frac{2}{x^2}\right)^4$. *Calculate* $\frac{dy}{dx}$.

Exercises/Homework:

Recall that to differentiate $y = x^a$, where $a \neq 0$ is a rational number, we have

$$\frac{dy}{dx} = ax^{a-1}.$$

Example 2.23 *Consider the function* $y : \mathbb{R}^{(>0)} \longrightarrow \mathbb{R}$ *by* $y(x) = \sqrt{x}$. *Calculate* $y'(x) = \frac{dy}{dx}$.

Why do we have

$$\frac{d}{dx}(x^a) = ax^{a-1}?$$

If $\frac{dx}{dy} \neq 0$, then $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$, so if $y = x^{\frac{1}{q}}$, where q is a positive integer, then

Example 2.24 *Let* $y = x^{-4/5}$. *Calculate* $\frac{dy}{dx}$.

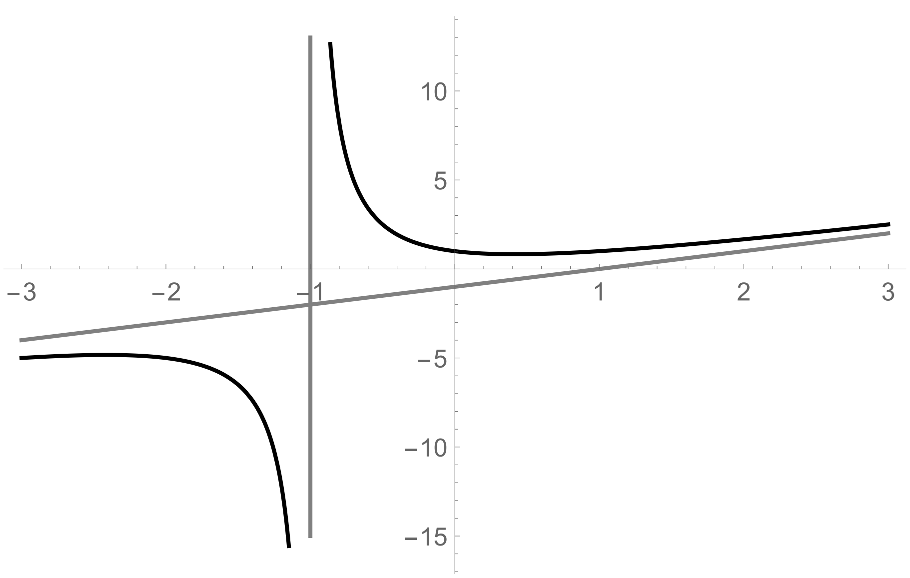
Example 2.25 *Let* $y = \sqrt[3]{x^2+1}$. *Calculate* $\frac{dy}{dx}$ *using the chain rule.*

Exercises/Homework:

The aim of this section is to use the sign of the first derivative to help us to sketch the graph of a curve.

Example 2.26 *Sketch the curve* $y = \frac{x^2+1}{x+1}$.

Continued,



Exercises/Homework:

Suppose that $y(x)$ can be expressed as a product so that $y = uv$ for some functions $u(x)$ and $v(x)$. Then the **product rule** states that

$$\frac{dy}{dx} = \frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} = vu'(x) + uv'(x).$$

Example 2.27 *Calculate* $\frac{dy}{dx}$ *for* $y = x^3(2x-7)^4$ *using the product rule.*

Example 2.28 *Calculate* $\frac{dy}{dx}$ *for* $y = x^2\left(x + \frac{1}{x}\right)$ *using the product rule.*

Example 2.29 *Calculate* $\frac{dy}{dx}$ *for* $y = (x+1)\sqrt{x^2-1}$ *using the product rule and the chain rule.*

Exercises/Homework:

Let $y = \frac{u}{v} = uv^{-1} = uw$, where $w = v^{-1}$.

By the product rule, The rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)$$

is called the **quotient rule**.

Example 2.30 *Use the quotient rule to differentiate* $y = \frac{x+2}{3x+4}$.

Example 2.31 *Use the quotient rule and chain rule to differentiate* $y = \frac{2x+5}{(x-6)^{1/2}}$.

Exercises/Homework:

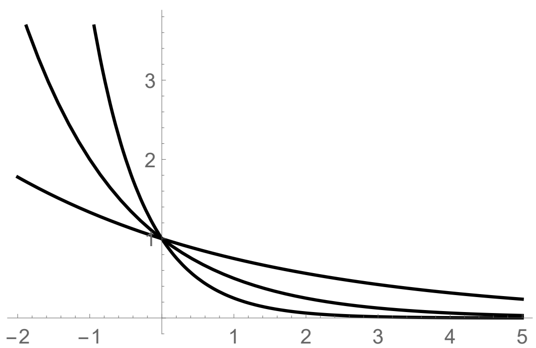
2.4.1 Sketching $y = a^x$, $a > 0$

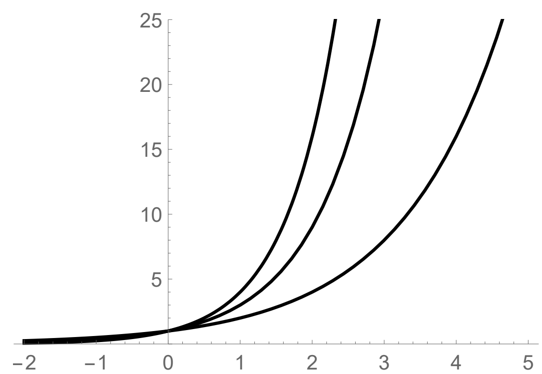
If $y = a^x$, where $0 < a < 1$, then letting $b = \frac{1}{a}$ so $a = \frac{1}{b}$, we have $y = a^x = \frac{1}{b^x}$, where $b > 1$. In this case we have **exponential decay** since as $x \longrightarrow \infty$, $\frac{1}{b^x} \longrightarrow 0$.

When $x = 0$, $y = 1$ so we have the point $(0,1)$.

When $x < 0$, $y > 1$. If $x = -1$, then $y = \frac{1}{a}$ so we have the point $\left(-1, \frac{1}{a}\right)$.

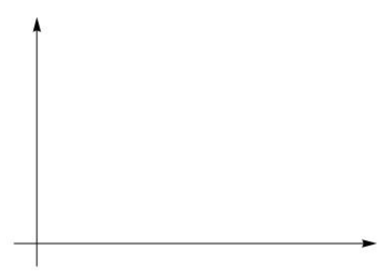
When $x = 1$, $y = a$, so we have the point $(1,a)$.





If $y = a^x$, where $a > 1$, then we have **exponential growth**. As $x \longrightarrow -\infty$, $y \longrightarrow 0$. When $x = 0$, $y = 1$ so we have the point $(0,1)$.

Example 2.32 *Sketch $y = f(x) = 0.1^x$, give the domain and range of $f(x)$, and state the asymptotes.*



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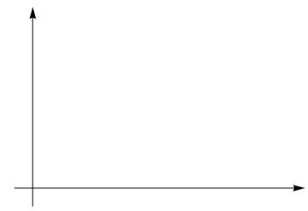
Example 2.33 *Sketch $y = f(x) = 5^x$, give the domain and range of $f(x)$, and state the asymptotes.*



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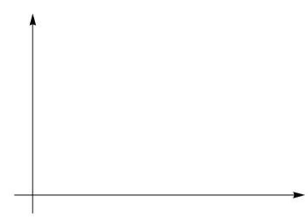
Applying the translation $x \longmapsto X + h$, $y \longmapsto Y + k$ to the equation $Y = a^X$ gives the equation $y = a^{x-h} + k$.

Example 2.34 *Sketch $y = f(x) = 3^{x+1} + 1$, give the domain and range of $f(x)$, and state the asymptotes.*



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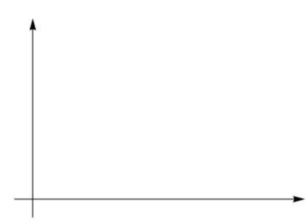
Example 2.35 *Sketch $y = f(x) = 2^{2x} + 3$, give the domain and range of $f(x)$, and state the asymptotes.*



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Exercises/Homework:

The irrational number $e = 2.71828\ldots$ is an important mathematical constant defined as the limit as $n \longrightarrow \infty$ of rational numbers $\{a_n\}$, where

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

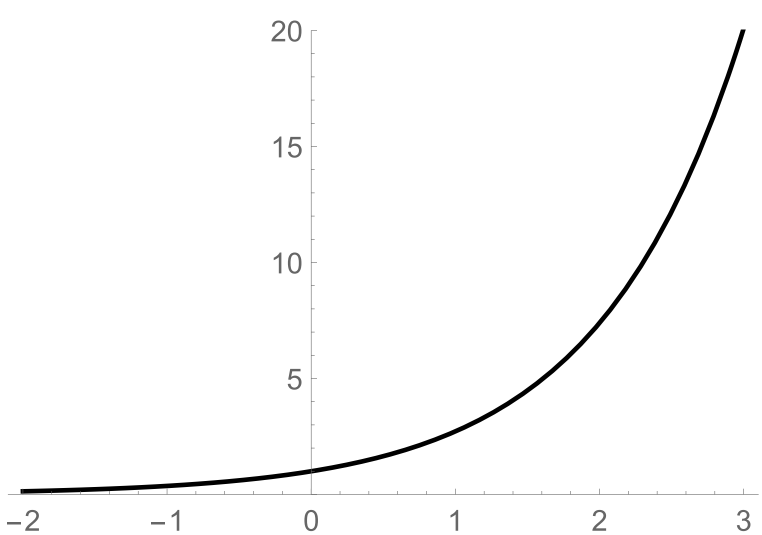
The first few terms of the sequence starting with $n = 1$ are $a_1 = 2$, $a_2 = \left(1 + \frac{1}{2}\right)^2 = \frac{9}{4} = 2.25$, $a_3 = \left(1 + \frac{1}{3}\right)^3 = \frac{64}{27} \approx 2.37$. As $n \longrightarrow \infty$, the sequence converges to the constant e. Another way to obtain e is through the infinite series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

The constants e, π, and $i = \sqrt{-1}$ are related via the formula

$$e^{\pi i} = -1.$$

To plot the curve $y = e^x$, when $x = 0$, $y = 1$ so we have the point $(0,1)$.
When $x = -1$, $y = \frac{1}{e} \approx 0.37$ so we have the point $(-1, 0.37)$.
When $x = 1$, $y = e \approx 2.72$ so we have the point $(1, 2.72)$.
When $x = 2$, $y = e^2 \approx 7.39$ so we have the point $(2, 7.39)$.
When $x = 3$, $y = e^3 \approx 20.09$ so we have the point $(3, 20.09)$.
As $x \longrightarrow \infty$, we have $y \longrightarrow \infty$,
As $x \longrightarrow -\infty$, we have $y \longrightarrow 0$.



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Example 2.36 *Sketch* $y = f(x) = e^{x+2} + 4$, *give the domain and range of* $f(x)$, *and state the asymptotes.*

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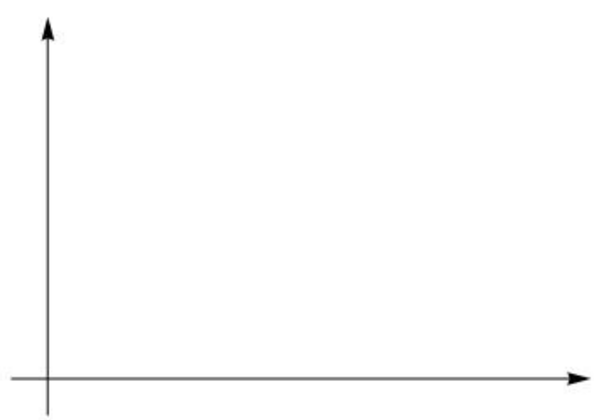
Note that since $a = e^{\log_e(a)}$, we can understand $y = a^x$ as $y = e^{x\log_e(a)}$.

Example 2.37 *Sketch* $y = f(x) = 3^x$ *by finding a real number* k *such that* $y = e^{kx}$.



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Interest in a bank account may be compounded daily, weekly, monthly, annually, or continuously, depending on the terms of the account. Let r be the interest rate per year (per annum) and compounding occurs n times per year. Let P_0 be the initial deposit (principal) and let P be the deposit after some time has elapsed. After t years,

$$P = P_0 \left(1 + \frac{r}{n}\right)^{nt}.$$

Notice that we can rewrite P as

$$P = P_0 \left(\left(1 + \frac{1}{(n/r)}\right)^{n/r}\right)^{rt}.$$

If we let $N = \frac{n}{r}$, then

$$P = P_0 \left(\left(1 + \frac{1}{N}\right)^{N}\right)^{rt}.$$

As $n \longrightarrow \infty$, $N \longrightarrow \infty$ so

$$\begin{aligned} \lim_{N \longrightarrow \infty} P &= \\ \\ &= \\ \\ &= \end{aligned}$$

Hence we obtain the formula for continuously compounding interest.

Example 2.38 *A bank offers an interest rate of* 2% *per annum. An initial deposit of* $ 100 *is is made.*

(a) *If compounded weekly, how many dollars are in the account after 10 year if no deposits or withdrawals are made?*

(b) *If compounded continuously, how many dollars are in the account after 10 year if no deposits or withdrawals are made?*

Exercises/Homework:

Recall the exponent rules for real numbers a, b, c:

$$a^b a^c = a^{b+c}, \qquad a^b/a^c = a^{b-c}, \text{ for } a \neq 0$$

$$\left(a^b\right)^c = a^{bc},$$

$$(ab)^c = a^c b^c, \qquad (a/b)^c = a^c/b^c \;=\; a^c b^{-c}, \text{ for } b \neq 0$$

$$a^{-1} = \frac{1}{a}, \text{ for } a \neq 0$$

$$a^0 = 1, \qquad 0^0 = 1 \text{ (defined to be 1, but contraversial)}$$

$$\frac{1}{a^{-b}} = a^b, \text{ for } a \neq 0, \qquad a^{-b} = \frac{1}{a^b} \text{ for } a \neq 0.$$

To solve an exponential equation like that given in the example below, we first change all of the bases into the same base. Next, once we have the same base, we equate the exponents and solve the resulting equation.

Example 2.39 *Solve the exponential equation* $27 * 5 - 2n = 9^{3n+4}$ *for n.*

Example 2.40 *Solve the exponential equation* $4^x = 6 - 2^x$ *for x.*

Exercises/Homework:

Recall that for real numbers a,b,c with $b,c>0$, $a^b=c$ is equivalent to $b=\log_a(c)$. If $a=e=2.71828\ldots$, then we have $y=e^x$ is equivalent to $x=\log_e(y)$, which is sometimes written as $x=\ln(y)$. We call this the **natural logarithm**.

Recall the following log rules:

$$
\begin{aligned}
\log_a(xy) &= \log_a(x)+\log_a(y), && \text{from } a^b a^c = a^{b+c},\\
\log_a(x/y) &= \log_a(x)-\log_a(y), && \text{from } a^b/a^c = a^{b-c},\\
\log_a(x^n) &= n\log_a(x), && \text{from } \left(a^b\right)^c = a^{bc},\\
\log_a(b) &= \frac{\log_c(b)}{\log_c(a)}, && \text{from } (c^y)^x = c^{xy},\\
\log_a(1/b) &= -\log_a(b), && \text{from } \frac{1}{b}=b^{-1}, \log_a(x^n)=n\log_a(x),\\
b &= a^{\log_a(b)}, && \text{by letting } x=\log_a(b),\\
\log_a(a) &= 1, (a\neq 1) && \text{from } a^1=a,\\
\log_a(1) &= 0, && \text{from } a^0=1.
\end{aligned}
$$

Example 2.41 *Simplify* $\log_2(8)-\log_2(12)$.

Example 2.42 *Solve* $\log_4(2x) = 2$ *for x.*

Example 2.43 *Solve* $\log_2(3x+2) = 6$ *for x.*

Example 2.44 *Solve* $\ln(4x+4) + \ln(3x-2) = 2\ln(x-1)$ *for x.*

Example 2.45 *Solve* $\log_4(8^x) = 3x + 1$ *for x.*

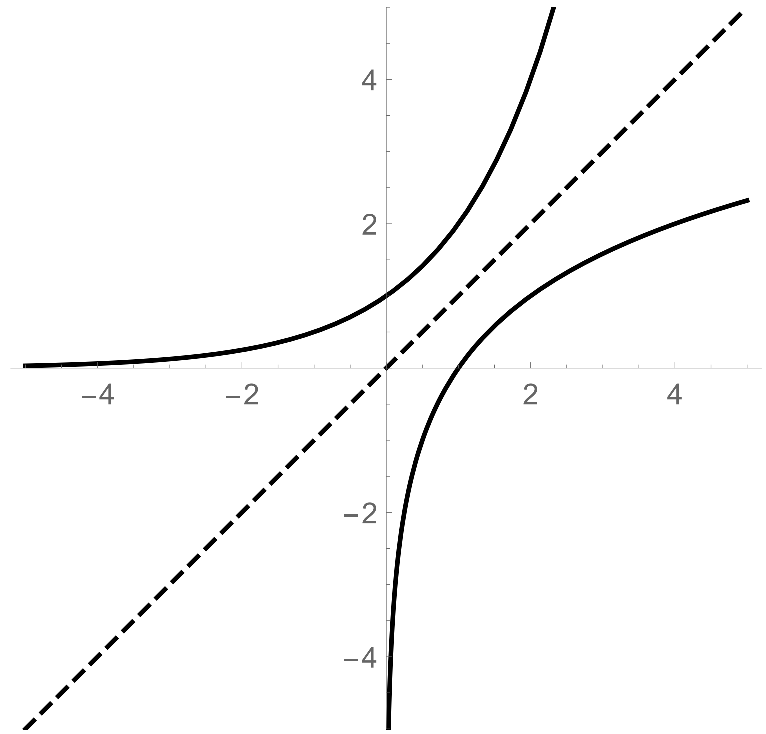
Example 2.46 *Solve* $\log_3(9x - 8) = 2\log_3(x)$ *for x.*

Example 2.47 *Solve* $4\log_x(2) = \log_2(x)$ *for x.*

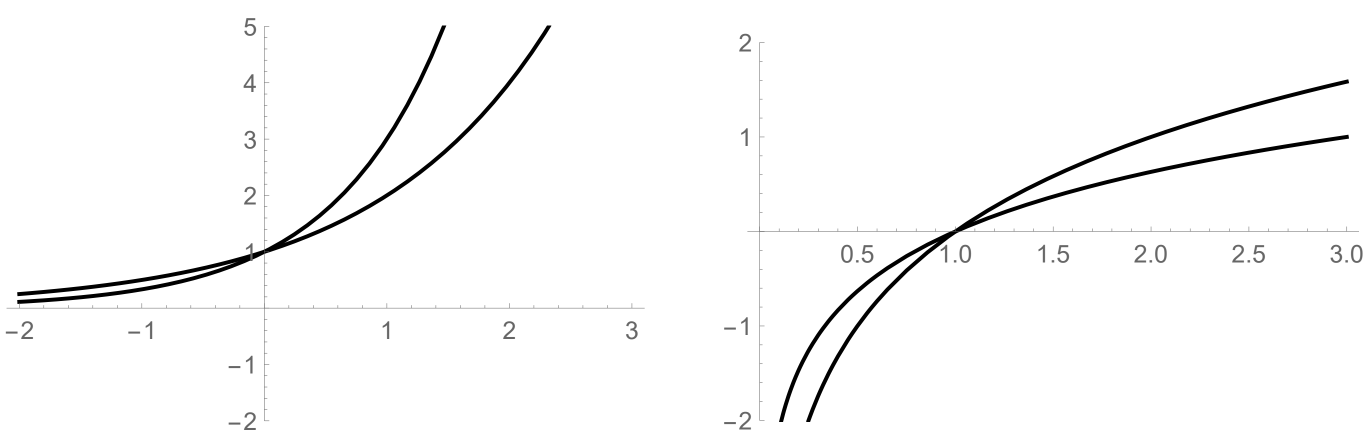
Exercises/Homework:

To understand how to plot the curve $y = \log_a(x)$, we consider the equivalent equation $x = a^y$. The linear transformation $x \longmapsto Y$, $y \longmapsto X$ gives $Y = a^X$ and this transformation is a reflection about the line $y = x$.

$y = 2^x$ and $y = \log_2(x)$:



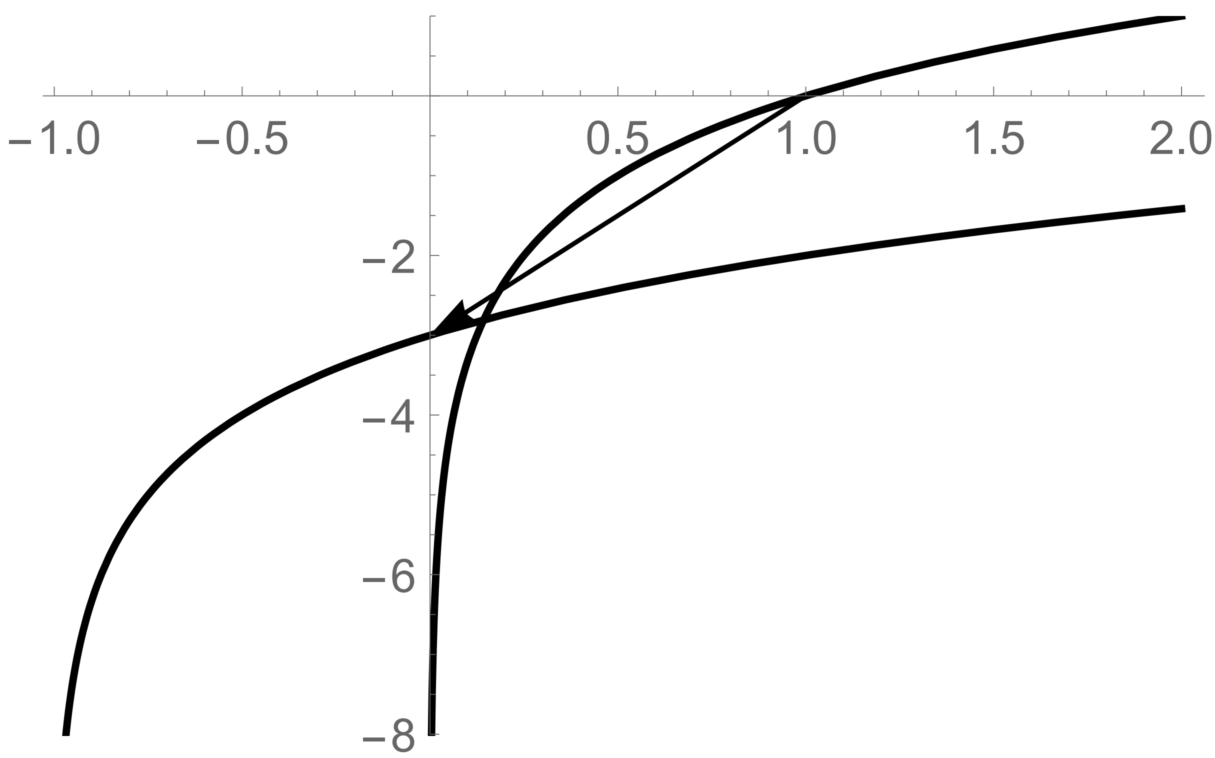
Consider the relationship between the curves $y = 2^x$ and $y = 3^x$ and the relationship between the curves $y = \log_2 x$ and $y = \log_3(x)$.



If $a > 0$ and $a \neq 1$, then the domain D of the $y = f(x) = \log_a(x)$ is the set of positive real numbers. In interval notation $D = (0, \infty) = \{x \in \mathbb{R} \, : \, x > 0\}$. The range of $f(x)$ is $\mathbb{R} = (-\infty, \infty)$.

Consider the curve $Y = \log_a(X)$ under the translation $x \longmapsto X + h$, $y \longmapsto Y + k$. We get $y = k + \log_a(x - h)$.

Example 2.48 *Consider the relationship between* $Y = \log_2(X)$ *and* $y = -3 + \log_2(x+1)$.



What is the translation that relates these curves? State the domains of these curves. Find the two corresponding exponential equations and determine the translation between them.

Exercises/Homework:

Recall that if $y = f(x)$, then

$$\frac{dy}{dx} = f'(x) = \lim_{h \longrightarrow 0} \frac{1}{h} \left(f(x+h) - f(x) \right). \tag{1}$$

Example 2.49 *Let $y = e^x$. Use the definition of the derivative, Equation* (1)*, together with the limit identity $\lim_{a \longrightarrow 0} \frac{e^a - 1}{a} = 1$ to calculate $\frac{dy}{dx}$.*

It follows that

$$\frac{de^x}{dx} = e^x. \tag{2}$$

Example 2.50 *Recall that $a = e^{\ln(x)}$. Use this and the chain rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ to calculate the derivative of $y = a^x$.*

Example 2.51 *Calculate the derivative of* $y = 3^x$.

Example 2.52 *Calculate the derivative of* $y = e^{-2x} + e^{3x}$.

Example 2.53 *Calculate the derivative of* $y = e^{x^2+4x-2}$.

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Example 2.54 *Calculate the derivative of* $y = \left(x^2 + e^{x^2}\right)^3$ *and use it to calculate the tangent line to the curve* $y = \left(x^2 + e^{x^2}\right)^3$ *at the point* $(0,1)$.

Exercises/Homework:

Recall that $x = e^y$ is equivalent to $y = \ln(x)$. We will learn how to differentiate $\ln(x)$ and related examples.

Example 2.55 *Let* $y = \ln(x)$. *Calculate the derivative of* $\ln(x)$ *using* $\frac{de^x}{dx} = e^x$, *together with the fact that if* $\frac{dx}{dy} \neq 0$, *then* $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

We have

$$\frac{d}{dx}\ln(x) = \frac{1}{x}. \tag{3}$$

Example 2.56 *Calculate the derivative* $\frac{d}{dx}\ln(4x+3)$ *using the chain rule.*

Example 2.57 *Calculate the derivative* $\frac{d}{dx}\ln\left(xe^{x}\right)$ *without using the product rule.*

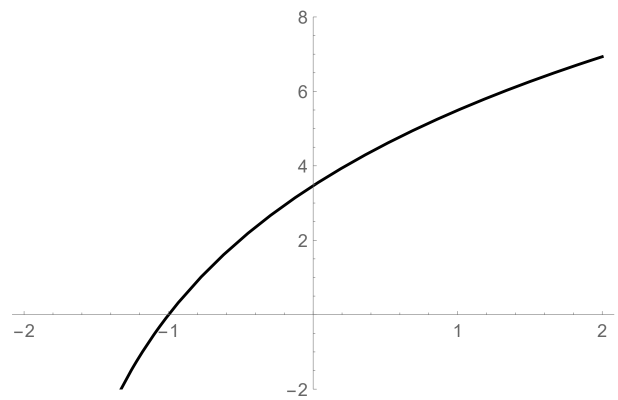
Example 2.58 *Use the chain rule to differentiate* $y = \left(\ln\left(2x+x^{2}\right)\right)^{2}$.

Example 2.59 *Calculate the derivative of* $y = \log_{2}(3x)$.

Exercises/Homework:

In this section we determine the equations of exponential curves and logarithmic curves given certain geometric information such as points on the curve and/or asymptotes.

Example 2.60 *The curve shown below is of the form* $y = k\ln(x+c)$ *and has points* $(-1,0)$ *and* $(0,\ln(32))$. *Find c and k and hence determine the equation of the curve.*



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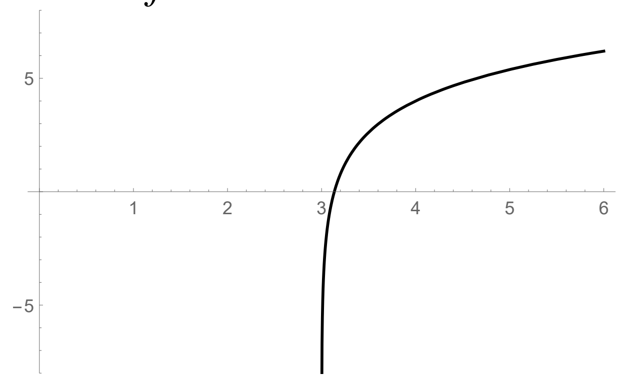
Example 2.61 *The curve shown below is of the form* $y = k\ln(x+c)+d$ *and has the points* $(4,4)$ *and* $\left(3+\frac{1}{\sqrt{e}},0\right)$, *and asymptote* $x = 3$. *Find* c, d *and* k *and hence determine the equation of the curve.*



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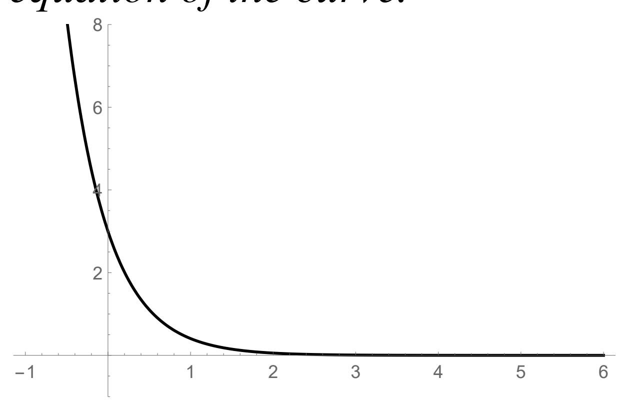
Example 2.62 *The curve shown below is of the form* $y = ce^{kx}$ *and has the point* $(0,3)$ *and the tangent line at the point* $(0,3)$ *is* $y = -6x+3$. *Find* c *and* k *and hence determine the equation of the curve.*



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Exercises/Homework:

We derive the derivatives of $\sin(x)$, $\cos(x)$, and $\tan(x)$ calculate the derivatives of other trigonometric functions.

Example 2.63 *Using the definition of the derivative*

$$f'(x) = \lim_{h \longrightarrow 0} \frac{1}{h}(f(x+h) - f(x))$$

together with the angle addition formulas

$$\begin{aligned} \sin(a+b) &= \cos(a)\sin(b) + \cos(b)\sin(a), \\ \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b), \end{aligned}$$

and the limits:

$$\lim_{a \longrightarrow 0} \frac{\sin(a)}{a} = 1, \qquad \lim_{a \longrightarrow 0} \frac{\cos(a) - 1}{a} = 0,$$

calculate the derivatives of $\sin(x)$ *and* $\cos(x)$.

Continued...

We have

$$\frac{d}{dx}\sin(x) = \cos(x), \tag{4}$$

$$\frac{d}{dx}\cos(x) = -\sin(x). \tag{5}$$

Example 2.64 *Using the quotient rule* $\frac{d(u/v)}{dx} = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)$ *and* $\tan(x) = \frac{\sin(x)}{\cos(x)}$*, calculate the derivative of* $\tan(x)$

Example 2.65 *Calculate the derivative of* $\cos\left(3x^2 - 2\right)$ *using the chain rule.*

Example 2.66 *Calculate the derivative of* $\sin^2(2x+2)$ *using the chain rule.*

Example 2.67 *Calculate the derivative of* $\cos(x)\sin(x)$ *using the product rule* $\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$.

Example 2.68 *Calculate the derivative of* $\frac{\cos(x)}{e^x}$ *using the quotient rule* $\frac{d(u/v)}{dx} = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)$.

Example 2.69 *Calculate the derivative of* $\frac{\sin\left(x^2\right)}{\cos(e^x)}$ *using the quotient rule* $\frac{d(u/v)}{dx} = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)$ *and the chain rule.*

Exercises/Homework:

The **second derivative** of a function $y = f(x)$ is the derivative of the first derivative $f'(x)$ or $\frac{dy}{dx}$. The second derivative is written $f''(x)$ or $f^{(2)}(x)$ or $\frac{d^2y}{dx^2}$, which is $\frac{d}{dx}\left(\frac{dy}{dx}\right)$.

Example 2.70 *Calculate the second derivative of* $y = f(x) = \sin(x) + \sqrt{x}$.

Let $s(t)$ be a scalar function of time t that represents the position of a particle moving in a straight line. Then $v(t) = s'(t)$ is the speed of the particle at time t and $a(t) = v'(t) = s''(t)$ is the acceleration of the particle at time t.

Example 2.71 *A particle moving in a straight line has position given by the point* $(s(t), 0)$, *where* $s(t) = \cos(t) + e^{2t}$. *Calculate the velocity of the particle, the initial velocity of the particle, and the acceleration of the particle at time* t.

Exercises/Homework:

Recall that the point $P = (x_0, y_0)$ of the curve $y = f(x)$ is a stationary point if $f'(x_0) = 0$.

Example 2.72 *Find all stationary points of the curve* $y = f(x) = \frac{1}{2}e^{2x} - 5e^{x} + 4x$.

Example 2.73 *The curve* $y = f(x) = x^4 + ax + b$ *has a stationary point* $(-1, 0)$. *Find* a *and* b. *Are there any other stationary points with real coordinates?*

Exercises/Homework:

Recall that a stationary point $P_0 = (x_0, y_0)$ of a continuous curve $y = f(x)$ is a point with $f'(x_0) = 0$. If the sign of the first derivative changes at P_0, then the point P_0 is a maximum or minimum, which can be classified using the first derivative test. The second derivative test is sometimes easier to use than the first derivative test.

Theorem 14 (Second Derivative Test) *Let $y = f(x)$, where $f(x)$ is continuous and $f'(x)$ and $f''(x)$ exist at the point $P_0 = (x_0, y_0)$.*

- *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then the point P_0 is a local minimum.*
- *If $f'(x_0) = 0$ and $f''(x_0) < 0$, then the point P_0 is a local maximum.*
- *If $f'(x_0) = 0$ and $f''(x_0) = 0$, then the test is inconclusive; P_0 could be an inflection point or and undulation point.*

Example 2.74 *Find and classify all stationary points of the curve $y = x^2$ using the second derivative test.*

An **inflection point** $P_0 = (x_0, y_0)$ of a continuous curve $y = f(x)$ is a point with a change in the curvature of the curve at the point P_0, that is a change in the sign of the second derivative so that $f''(x_0 + \varepsilon)$ and $f''(x_0 - \varepsilon)$ have opposite signs for all real $\varepsilon > 0$ and sufficiently small.

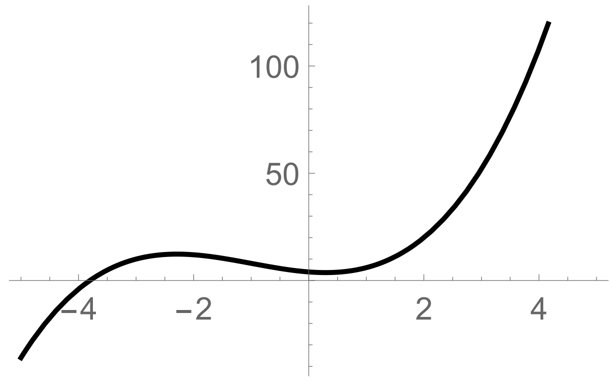
An **undulation point** $P_0 = (x_0, y_0)$ of a continuous curve $y = f(x)$ is a point with $f''(x_0) = 0$ such that there is no change of sign of the second derivative at P_0.

Theorem 15 (Inflection implies f'' is zero) *If $P_0 = (x_0, y_0)$ is an inflection point of the curve $y = f(x)$, then $f''(x_0) = 0$.*

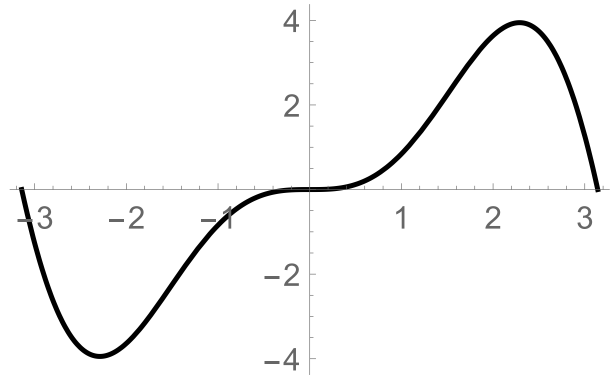
Example 2.75 *Find and classify all stationary points of the curve $y = x^3$. Identify all inflection points of the curve.*

Example 2.76 *Find and classify all stationary points of the curve $y = x^4$. Is the point $(0,0)$ an inflection point?*

Example 2.77 *Find and classify all stationary points of the curve* $y = f(x) = x^3 + 3x^2 - 2x + 4$. *Identify all inflection points of the curve.*



Example 2.78 *Find and classify all stationary points of the curve* $y = f(x) = x^2 \sin(x)$ *defined on the domain* $D = [-\pi, \pi]$. *Identify all inflection points of the curve in D.*



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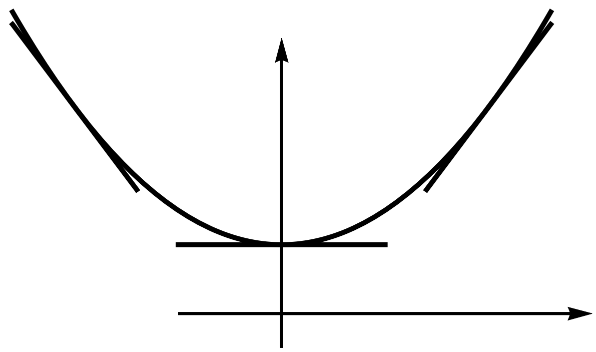
Exercises/Homework:

The curve $y = f(x)$ is **concave up** if $f''(x) > 0$. This means that $f'(x)$ is increasing as x increases.



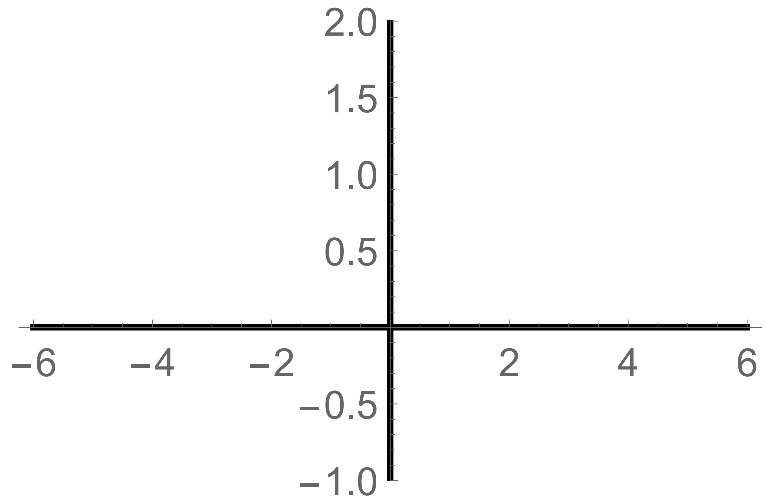
The curve $y = f(x)$ is **concave down** if $f''(x) < 0$. This means that $f'(x)$ is decreasing as x increases.

When sketching the curve $y = f(x)$ defined on the domain D, where $f(x)$ is a function of x, we first calculate the following details:

- The y-intercept. Note that there is only one since f is a function.
- The x-intercept(s).
- Local maxima and local minima points.
- Global maxima and minima points on the domain D.
- Any inflection points.
- Any asymptotes.
- Regions of D in which the function $f(x)$ is concave up.
- Regions of D in which the function $f(x)$ is concave down.

Example 2.79 *Sketch the curve* $y = f(x) = x^4 - 32x - 12$ *on the domain* $D = [-3, 3]$.

Example 2.80 *Sketch the curve* $y = f(x) = -\frac{1}{720}x^4 + \frac{1}{24}x^2 + \frac{1}{x^2} - \frac{1}{2}$



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Exercises/Homework:

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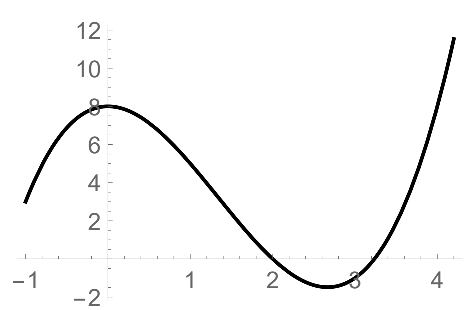
Recall the definition of global maximima and minima from Section 2.3.9. In this section we will present the concept again.

Let $y = f(x)$ be a continuous function on the domain $D = [a,b]$. If for all x in the domain D, $f(x) \leq v = f(u)$, then the point (u,v) is a **global maximum** point.

To find a global maximum point, we find and classify all of the local maximum points and compare their y coordinates to that of the boundaries of the domain, $f(a)$ and $f(b)$. Among those, a point with the greatest y-value corresponds to a global maximum.

Let $y = f(x)$ be a continuous function on the domain $D = [a,b]$. If for all x in the domain D, $f(x) \geq v = f(u)$, then the point (u,v) is a **global minimum** point.

To find a global minimum point, we find and classify all of the local minimum points and compare their y coordinates to that of the boundaries of the domain, $f(a)$ and $f(b)$. Among those, a point with the least y-value corresponds to a global minimum.



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Example 2.81 *Find the global maximum and global minimum of* $y = x^4 - 64x^2 + 800$ *on the domain* $D = [-3, 8]$.

Example 2.82 *Find the global maximum and global minimum of* $y = x^6 - 80x^3 + 1800$ *on the domain* $D = [-3, 5]$.

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Example 2.83 *A* 10000 m^3 *enclosure is to be constructed for penguins at a zoo made from expensive glass and inexpensive rock. One vertical side of the enclosure and the horizontal bottom is to be made from rock. Three vertical sides are to be made from rectangular sheets of glass and the top of the enclosure is to be open. Since the glass is expensive, the zoo aims to minimimise the surface area of glass used. The side opposite the vertical rock face of the enclosure must have width equal to* 20 *metres. Find the dimensions of the enclosure that give the least glass costs.*

Exercises/Homework:

In an **optimisation problem**, we seek to find a maximum or a minimum value of a function of (usually multiple variables) subject to some constraint. The constraint might be a budget, or a volume, or some similar relationship between the variables that restricts the possible values of the variables. When we determine the maxima and minima, we classify these often using the second derivative test and we must consider the domain of definition of the relevant functions since we require global maxima or global minima to answer such questions.

The process we will use in our optimisation problems is to use the constraint to eliminate variables and then find maxima or minima of the resulting function of one variable.

Example 2.84 *The product of two positive numbers x and y is 35. Find x and y such that $f(x,y) = 2x + 3y$ is least.*

Example 2.85 *A farmer growing strawberries in a field wishes to find the maximum crop yield per dollar spent on plants. A high plant density reduces the strawberry yield since they do not get enough light and nutrients. If* x *strawberry plants are planted, then* y *strawberries are expected per square metre, where* $y = 2500 - 10000x^{-1/2} - \frac{1}{10}x^2$. *The cost of east strawberry plant is* \$ 3. *If the field is* 1000 m^2, *what is the maximum number of strawberries that can be produced, and what is the production cost of a strawberry st the maximum yield? If the farm is only viable if at least* 20000 *strawberries are produced, find the* x *such that the production cost of a strawberry is least by finding the minimum value of* $C = \frac{3x}{y}$.

Continued...

Exercises/Homework:

3 Year 12 Mathematical Methods

3.1 Term 1

3.1.1 Antiderivatives of Polynomials and Power Functions

Let $f(x)$ be a function of x. In this section we seek to calculate $F(x)$, a function of x, such that $f(x) = F'(x) = \frac{d}{dx}F(x)$. The function $F(x)$ is called an **antiderivative** of $f(x)$.

Since the derivative of a constant is zero, an antiderivative of a given function is not unique, The family of antiderivatives $F(x)+C$ of a given function $f(x)$ is unique, where C represents any constant.

Example 3.1 *Let $f(x) = x = F'(x)$. Find the family of antiderivatives $F(x)$ and verify that their first derivative is in fact x.*

Analogous to $\frac{df}{dx}$, notation for an antiderivative is

$$F(x) = \int f(x)\,dx.$$

Another name for an antiderivative is the **indefinite integral** representing the family of antiderivatives of $f(x)$. When we calculate the indefinite integral $\int f(x)\,dx$, we always put $+C$, where C represents any constant.

Example 3.2 *Calculate the indefinite integral $\int x^2\,dx$.*

Recall that $\frac{d}{dx}(x^m) = mx^{m-1}$ for $m \in \mathbb{R} - \{-1\}$. If $n \neq -1$, then letting $n = m-1$, we have $m = n+1$ so $\frac{d}{dx}x^{n+1} = (n+1)x^n$ and $\frac{d}{dx}\frac{1}{n+1}x^{n+1} = x^n$. This means that we have the rule:

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C.$$

For the case $n = -1$, since $\frac{d}{dx}\ln(x) = \frac{1}{x} = x^{-1}$, we have

$$\int x^{-1} \, dx = \ln(x) + C.$$

We next consider some properties of the definite integral. For a constant k, we have

$$\begin{aligned} \frac{d}{dx}(f(x) + g(x)) &= f'(x) + g'(x), \\ \frac{d}{dx}(kf(x)) &= kf'(x). \end{aligned}$$

It follows that

$$\begin{aligned} \int f(x) + g(x) \, dx &= \int f(x) \, dx + \int g(x) \, dx, \\ \int kf(x) \, dx &= k \int f(x) \, dx. \end{aligned}$$

If k is a constant, then

$$\int k \, dx = kx + C.$$

Example 3.3 *Calculate the indefinite integral* $\int 4x^5 - 2x^3 + 7 \, dx$.

Example 3.4 *Calculate the indefinite integral* $\int 3x^3 + \sqrt{x} + 4x - 6\, dx$.

Example 3.5 *Find* $f(x)$ *satisfying* $f'(x) = 3x^2 + 9x + 2$ *and* $f(0) = 1$.

Exercises/Homework:

Let $u = ax+b$. Then $\int (ax+b)^n\, dx = \int u^n\, dx$. Since $\frac{du}{dx} = \frac{d}{dx}(ax+b) = a$, we have the substitution $dx \longrightarrow \frac{1}{a} du$ so that

$$\begin{aligned}
\int (ax+b)^n\, dx &= \int u^n\, dx, \\
&= \int u^n \frac{1}{a}\, du \\
&= \frac{1}{a}\int u^n\, du \\
&= \frac{1}{a}\frac{1}{n+1}u^{n+1} + C, \\
&= \frac{1}{a(n+1)}(ax+b)^{n+1} + C,
\end{aligned}$$

where C represents any constant.

To verify this, observe that

$$\begin{aligned}
\frac{d}{dx}\frac{1}{a(n+1)}(ax+b)^{n+1} + C &= \\
&= \quad .
\end{aligned}$$

.

Example 3.6 *Calculate* $\int (4x+2)^3\, dx$.

Example 3.7 *Calculate* $\int (4x+2)^{-1}\, dx$.

Example 3.8 *Find $f(x)$ satisfying $f'(x) = \frac{2}{3x+5}$ and $f(1) = 1$.*

Exercises/Homework:

In this section we will learn how to calculate an antiderivative of e^{kx}, where k is a constant.

Since we have

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{k}e^{kx}+C\right) &= \frac{1}{k}\frac{d}{dx}e^{kx}+0, \\ &= \frac{1}{k}ke^{kx}, \\ &= e^{kx}, \end{aligned}$$

We have the rule

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx}+C,$$

where C represents any constant.

Example 3.9 *Calculate* $\int e^{3x}+e^{-5x}+e^{-\frac{1}{4}x}+2\,dx.$

Exercises/Homework:

Recall that by letting $u = ax+b$,

$$\begin{aligned} \frac{d}{dx}\cos(ax+b) &= \frac{d\cos(u)}{du}\frac{du}{dx}, \\ &= -\sin(u)\frac{d}{dx}(ax+b), \\ &= -a\sin(ax+b). \end{aligned}$$

Similarly

$$\frac{d}{dx}\sin(ax+b) = a\cos(ax+b).$$

To integrate these functions, let $u = ax+b$. Then

$$\int \cos(ax+b)\ dx = \int \cos(u)\ dx.$$

$\frac{du}{dx} = a$ so we substitute $dx \longrightarrow \frac{1}{a}du$. Hence

$$\begin{aligned} \int \cos(ax+b)\ dx &= \int \cos(u)\frac{1}{a}\ du, \\ &= \frac{1}{a}\int \cos(u)\ du, \\ &= \frac{1}{a}\sin(u)+C, \\ &= \frac{1}{a}\sin(ax+b)+C. \end{aligned}$$

Similarly,

$$\int \sin(ax+b)\ dx = -\frac{1}{a}\cos(ax+b)+C.$$

.

Example 3.10 *Calculate* $\int \cos(5x+9)\,dx$.

Example 3.11 *Calculate* $f(x)$ *such that* $f'(x) = 3\sin(2x-2)$ *and* $f(1) = 2$.

Exercises/Homework:

Let $f(x)$ be a function of x and let $u = f(x)$. Then

$$\int \frac{f'(x)}{f(x)}\, dx = \int \frac{f'(x)}{u}\, dx.$$

Differentiating, $\frac{du}{dx} = \frac{d}{dx}f(x) = f'(x)$ so we have the substitution $dx \longrightarrow \frac{1}{f'(x)}\, du$. It follows that

$$\begin{aligned}\int \frac{f'(x)}{f(x)}\, dx &= \int \frac{f'(x)}{u}\frac{1}{f'(x)}\, du,\\ &= \int u^{-1}\, du,\\ &= \ln(u) + C,\\ &= \ln(f(x)) + C,\end{aligned}$$

where C represents any constant. We have the rule:

$$\int \frac{f'(x)}{f(x)}\, dx = \ln(f(x)) + C.$$

Example 3.12 *Calculate* $\int \frac{3x^2}{2x^3+5}\, dx.$

.

Example 3.13 *Calculate* $\int \frac{\cos(x)}{\sin(x)}\, dx$.

Example 3.14 *Calculate* $\int xe^{x^2}\, dx$.

Example 3.15 *Calculate* $\int 2\cos(x-3)\sin(x-3)\, dx$.

Example 3.16 *Use the fact that*

$$\frac{d}{du}(u\ln(u)-u) = u\times\frac{1}{u}+\ln(u)-1 = \ln(u)$$

(by the product rule) to calculate $\int \ln(3x)\,dx$.

.

Exercises/Homework:

In this section we solve problems involving rates of change and use definite integrals to complete the solution of such problems.

Example 3.17 *Let* $y'(t) = 2t\sqrt{1-t^2}$ *be the rate of change of the price of tomatoes after* t *days. Calculate the price* $y(t)$ *of tomatoes after* t *days if* $y(0) = 1$.

Exercises/Homework:

Recall that the velocity $v(t)$ of a particle moving in a straight line, acceleration $a(t)$ of a particle moving in a straight line, and displacement $s(t)$ of a particle moving in a straight line satisfy

$$v(t) = s'(t), \qquad a(t) = v'(t) = s''(t).$$

It follows that

$$s(t) = \int v(t)\,dt, \qquad v(t) = \int a(t)\,dt,$$

and so

$$s(t) = \iint a(t)\,dt\,dt.$$

Example 3.18 *A ball is tossed vertically upwards from* 1 *metre above the ground with velocity* $12m/s$. *Assuming that acceleration due to gravity is* $10m/s^2$, *calculate the maximum height of the ball.*

Continued...

Exercises/Homework:

The **probability of an event** A, $P(A)$ is a real number satisfying $0 \leq P(A) \leq 1$. It measures how likely an event is. More likely events have probabilities closer to 1.

We can associate an event A to a set A. For example, if the event A is getting a number less than 3 on a six-sided die, then $A = \{1,2\}$. The **sample space** $\mathscr{U}$ of an event is the set of all possible outcomes.

If a set X is finite, then we use $|X|$ or $n(X)$ to denote the number of elements in X. This is called the **cardinality** of the set X. The probability of the event A is calculated via the cardinality of the set A corresponding to the eventy A. We have

$$P(A) = \frac{|A|}{n(\mathscr{U})},$$

the number of possibilities in the event A divided by the total number of possibilities.

Recall $A \cap B$ is the subset of both A and B consisting of all elements in both A and B.

$A \cup B$ is the set of all elements in A or B so that $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

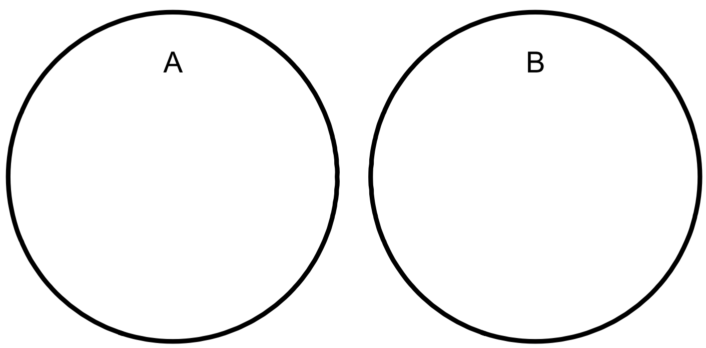
$A - B$ (also written $A \setminus B$) is the set of elements of A that are not in B.

We have $A \subseteq \mathscr{U}$, meaning the event A viewed as a set is a subset of the sample space $\mathscr{U}$.

We say that the events A and B are **equally likely** if $P(A) = P(B)$.

$\emptyset$ denotes the empty set $\emptyset = \{\ \}$, the set without any elements, and we have $|\emptyset| = 0$.

The events A and B are **mutually exclusive** if $A \cap B = \emptyset$; equivalently, $P(A \cap B) = 0$ and $P(A \cup B) = P(A) + P(B)$.



The **complement** of an event A viewed as a set, denoted A' is given by $A' = \mathscr{U} - A$ and we have $P(A') = 1 - P(A)$.

The **addition rule** states that for events A and B,

$$P(A \cap B) + P(A \cup B) = P(A) + P(B).$$

We can estimate the probability of an event A by

$$P(A) \approx \frac{\text{number of times event A happened in trials}}{\text{number of trials in the experiment}}.$$

Exercises/Homework:

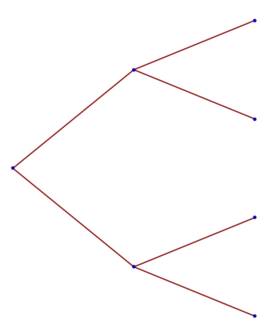
The probability of event A, given that event B has already occurred, $P(A \mid B)$ is called the **conditional probability** of A given B.

If $P(B) \neq 0$, then

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Example 3.19 *A bag of marbles has* 2 *white marbles and* 1 *blue marble. Calculate the probability of drawing a white marble given that a white marble was already drawn and not replaced.*

Tree diagram:



If A and B are two events, then

$$P(A) = P(A \mid B)P(B) + P(A \mid B')P(B').$$

This is a consequence of $P(A) = P(A \cap B) + P(A \cap B')$.

A and B are **independent events** if $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$. In other words, $P(A \mid B)$ does not depend on whether or not B has occurred. If A and B are independent events, then

$$P(A \cap B) = P(A \mid B)P(B) = P(A)P(B).$$

Example 3.20 *A 6 schools in a small town have a total of 845 girls and 830 boys in Year 12. The mathematics courses studied by the students are shown in the following table:*

	Girls	*Boys*	*Total*
Essential	*281*	*265*	*546*
General	*420*	*412*	*832*
Methods	*144*	*153*	*297*
Total	*845*	*830*	*1675*

(a) *Determine the probability that a student is a girl.*

(b) *Determine the probability that a student is a girl studying Mathematical Methods.*

(c) Determine the probability that a student is a girl, given that they are studying Mathematical Methods.

(d) Use the identity $P(A) = P(A \mid B)P(B) + P(A \mid B')P(B')$ to calculate the probability that a student studies Mathematical Methods.

Exercises/Homework:

A **random variable** is a function that assigns a real number to each outcome of a sample space.

A **discrete random variable** is a function that assigns a discrete value to each outcome of a sample space $\mathscr{U}$ with the property that $\mathscr{U}$ has a one-to-one correspondence with the set of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ when $\mathscr{U}$ is infinite.

Example 3.21 *A pair of dice is thrown and we have sample space*

$$\begin{aligned}\mathscr{U} &= \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\ &\quad (2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\dots,(6,6)\}.\end{aligned}$$

We have $|\mathscr{U}| = 36$ *elements of the sample space. The sum of the numbers on the dice is a discrete random variable in the set*

$$X = \{2, 3, 4, 5, 6, \dots, 11, 12\}.$$

A **continuous random variable** is a random variable in which the outcomes are real numbers not restricted to a set of discrete values.

Example 3.22 *A pair of dice is tossed. Calculate the probability that the sum on the dice is equal to* 7.

The **probability distribution** of a discrete random variable is a set consisting of pairs $(x, P(x))$, where x is the random variable and $P(x)$ is the probability that x occurs. This information is sometimes represented in a table. $P(x)$ is called a **discrete probability function** when we have a discrete random variable.

Example 3.23 *Write the probability distribution for the sum on a pair of dice, write this in a table, then display a plot of points* $(x, P(x))$.

The notation $\sum_x P(x)$ refers to the sum of the probabilities of all discrete random variables.

$\sum\limits_{x<b} P(x)$ is the sum of all random variables satisfying $x < b$. Similarly we have $\sum\limits_{x\le b} P(x)$, $\sum\limits_{a<x} P(x)$, $\sum\limits_{a\le x} P(x)$, $\sum\limits_{a\le x\le b} P(x)$, $\sum\limits_{a<x\le b} P(x)$, $\sum\limits_{a\le x<b} P(x)$.

Theorem 16 $\sum_x P(x) = 1$ *for a discrete random variable.*

Example 3.24 *Verify that* $\sum_x P(x) = 1$ *for the sum of a pair of dice.*

Example 3.25 *For a pair of dice, calculate the probability that the sum of the dice is greater than* 3, $\sum_{3<x} P(x)$.

A **uniform discrete random variable** is a discrete random variable such that for all x, $P(X = x) = c$, where c is constant. If $\mathscr{U}$ is finite and $n = |\mathscr{U}|$, then $P(X = x) = \frac{1}{n}$.

Example 3.26 *The toss of a single die is an example of a uniform random variable. The sum of a pair of dice is an example of a non-uniform discrete random variable.*

Exercises/Homework:

The **mean** of a collection A is the average of the elements of A, $\frac{1}{n}\sum_{j=1}^{n} a_j$, where $A = \{a_1, a_2, \ldots, a_n\}$.

The **expected value** of an event in a discrete random variable probability distribution is the sum $E = \sum_x xp(x)$.

Example 3.27 *In a game of chance a pair of dice is tossed. The player receives the dollar value of the sum of the dice. The cost of a game is \$ 4. Calculate the expected value of a game and hence decide whether the game is worth playing.*

Example 3.28 *In a game of chance a pair of dice is tossed. Again, the player receives the dollar value of the sum of the dice. However, this time a game costs \$ k to play. Calculate the least k such that the player loses on average.*

The expected value is an average value or mean value, so we put $\mu = E(X)$. If $f(x)$ is a function of x, then $E(f(x)) = \sum_x f(x)P(x)$.

Example 3.29 *Calculate $E(f(x))$ in terms of $E(X)$, where* $f(x) = 2x - 1$.

In general, $f(E(X)) \neq E(f(X))$. If $f(x)$ is linear, then $f(E(X)) = E(f(X))$. Also,

$$E(X+Y) = E(X) + E(Y).$$

The **variance** of a random variable X is defined

$$\sigma^2 = \mathrm{Var}(X) = E\left((X-\mu)^2\right),$$

where $\mu = E(X)$. The variance measures how closely about the mean the probability distribution is.

$\sigma = \sqrt{\mathrm{Var}(X)}$ is called the **standard deviation** of the random variable X.

.

Example 3.30 *Consider the probability distribution shown in the table below:*

x	1	2	3	4
$P(x)$	0.02	0.25	0.40	0.33

Calculate the mean $\mu = E(X)$ and the standard deviation σ.

Theorem 17 *Let μ and σ be the mean and standard deviation of a discrete random variable X. Then $\sigma^2 = E\left(X^2\right) - \mu^2$.*

Example 3.31 *Use the above theorem to calculate the variance σ^2 for the probability distribution in Example 3.30.*

For a uniform discrete random variable X with a finite sample space of cardinality n, for all x we have $P(x) = \frac{1}{n}$. It follows that

$$E(X) = \sum_x xP(x) = \frac{1}{n}\sum_x x,$$

which is the average of the elements x of the random variable.

The variance satisfies

$$\begin{aligned}\sigma^2 &= E\left(X^2\right) - E(X)^2, \\ &= \left(\sum_x x^2 P(x)\right) - \frac{1}{n^2}\left(\sum_x x\right)^2.\end{aligned}$$

Example 3.32 *Calculate $\mu = E(X)$ and σ^2 for a 12-sided die.*

Exercises/Homework:

A **Bernoulli trial** is a random experiment with two outcomes: success or failure. We use p to denote the probability of a success. Hence the probability of a failure is $1-p$. A Bernoulli trial is independent of previous or successive events.

A **Bernoulli random variable** x is 0 or 1 for failure and success respectively. It follows that the probability distribution of a Bernoulli random variable is $\{(0, p-1), (1, p)\}$.

Example 3.33 *An unfair coin has probability* 0.43 *of landing on heads. Write a table displaying the probability distribution for tossing the coin, given that landing on heads is a success. What is the probability of obtaining two heads in a row in tossing the coin twice?*

x	*0*	*1*
$P(X=x)$		

Recall that the binomial symbol is given by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

The probability of x successes in a sequence of n Bernoulli trials is given by

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}.$$

Recall that $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$ so it follows that

$$1 = ((1-p)+p)^n = \sum_{x=0}^{n} \binom{n}{x} (1-p)^{n-x} p^x.$$

Since the sum of the probabilities is 1, we have a probability distribution. This probability distribution is called a **binomial distribution**.

Example 3.34 *The unfair coin in Example 3.33 above is tossed* 5 *times. Calculate the probability of getting exactly* 2 *heads.*

The **expected value for a binomial distribution** is

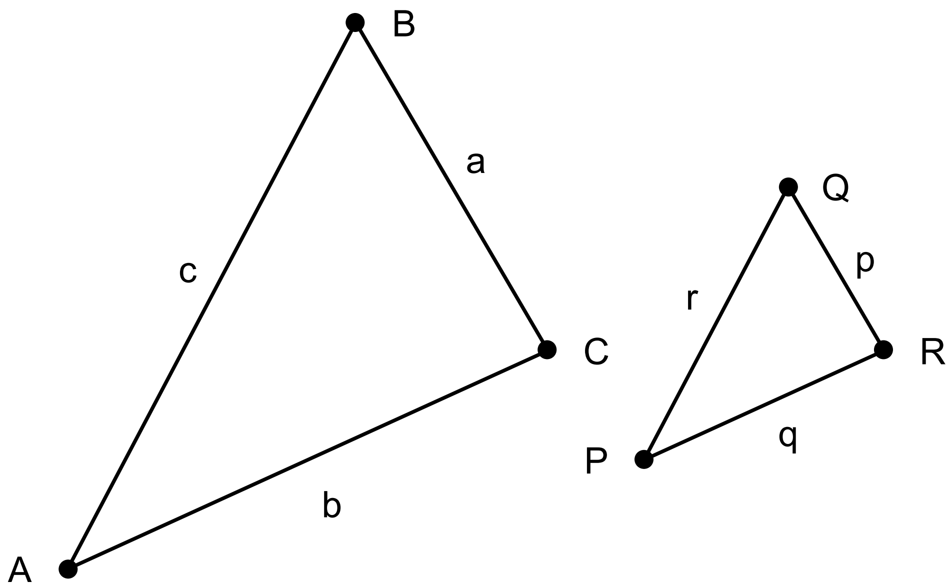
$$\mu = \sum_x xP(x) = \sum_{x=0}^{n} x \binom{n}{x} (1-p)^{n-x} p^x = np,$$

the number of times a success is expected in n trials. If a fair coin is tossed 10 times, then we would expect $10 \times 0.5 = 5$ heads. The variance σ^2 for a binomial distribution is

$$\sigma^2 = E(X^2) - \mu^2 = \left(\sum_{x=0}^{n} x^2 \binom{n}{x} (1-p)^{n-x} p^x \right) - n^2 p^2 = np(1-p).$$

Exercises/Homework:

A **triangle** has three sides and three vertices. Each of the vertices has an interior angle that is less than $180^\circ = \pi$ radians. We will label the vertices and refer to them as angles interchangeably. We will use uppercase letters for vertices and corresponding angles and the same lowercase letter for the side length opposite to that vertex/angle. Two triangles $\triangle ABC$ and $\triangle PQR$ are **congruent** if all interior angles in one triangle are equal to those in the other triangle.



Congruent triangles $\triangle ABC$ and $\triangle PQR$ can be labeled so that:

$$\begin{aligned} \angle A &= \angle P, \\ \angle B &= \angle Q, \\ \angle C &= \angle R. \end{aligned}$$

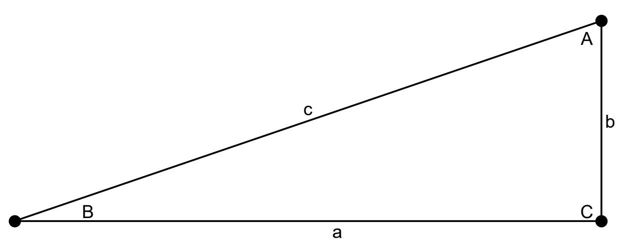
Theorem 18 *Two triangles $\triangle ABC$ and $\triangle PQR$ are congruent if and only if they have side lengths a, b, c and p, q, r respectively such that there is a real number m satisfying*

$$p = am, \qquad q = bm, \qquad r = cm.$$

Note that the side length a is opposite the angle A, b is opposite the angle B, c is opposite the angle C, p is opposite the angle P, q is opposite the angle Q, and r is opposite the angle R. Furthermore, the triangles are labeled so that $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$.

Example 3.35 *Triangles $\triangle ABC$ and $\triangle PQR$ shown in the figure above have vertices at points: $A=(0,0), B=(3,5), C=(5,2)$ and $P=(6,1), Q=(7.5,3.5), R=(8.5,2)$. Determine the scale factor m in Theorem 18.*

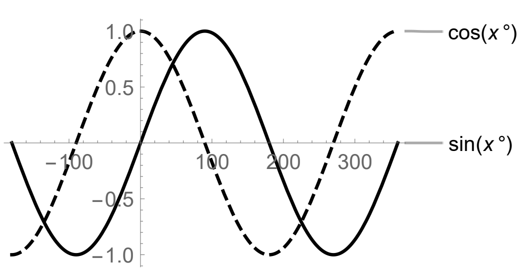
Let $\triangle ABC$ be a right triangle with $\angle C = 90^\circ = \pi$, labeled as shown.



The trigonometric ratios sine, cosine and tangent are defined and satisfy:

$$\begin{aligned}\sin(A) &= \frac{a}{c} = \frac{\text{opp to A}}{\text{hyp}} = \cos(B),\\ \cos(A) &= \frac{b}{c} = \frac{\text{adj to A}}{\text{hyp}} = \sin(B),\\ \tan(A) &= \frac{a}{b} = \frac{\text{opp to A}}{\text{adj to A}} = \frac{\sin(A)}{\cos(A)} = \frac{1}{\tan(B)}.\end{aligned}$$

Recall the phase shift between $\sin(x)$ and $\cos(x)$ and that these are periodic functions of period $360^\circ = 2\pi$:



$$\begin{aligned}\sin(180^\circ - x) &= \sin(x),\\ \cos(180^\circ - x) &= \cos(x),\\ \sin(x+90^\circ) &= \cos(x),\\ \cos(x-90^\circ) &= \sin(x).\end{aligned}$$

Next we recall the following important results.

Theorem 19 *The sum of the interior angles of a triangle is equal to* $180^\circ = \pi$ *radians.*

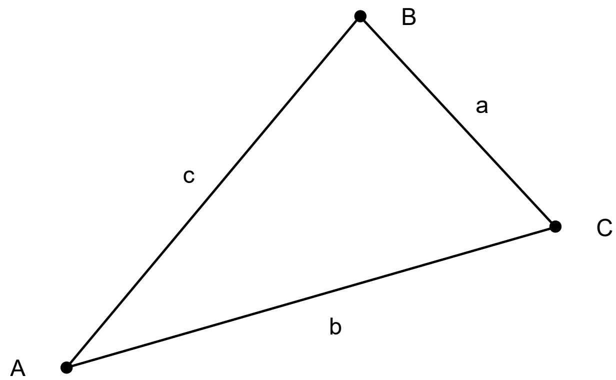
Theorem 20 (Pythagoras) *If* $\triangle ABC$ *is a right triangle with* $C = 90^\circ$ *and hypotenuse* c*, then* $a^2 + b^2 = c^2$.

To **solve a triangle** means to find all interior angles and all side lengths. We use the definitions of sin, cos, and tan to solve a given right triangle.

Example 3.36 *A right triangle has hypotenuse* 8 *cm and another side length equal to* 5 *cm. Solve the triangle.*

Exercises/Homework:

Theorem 21 (Sine Rule) *The triangle shown on the left satisfies all three of the identities below on the right:*



$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)},$$

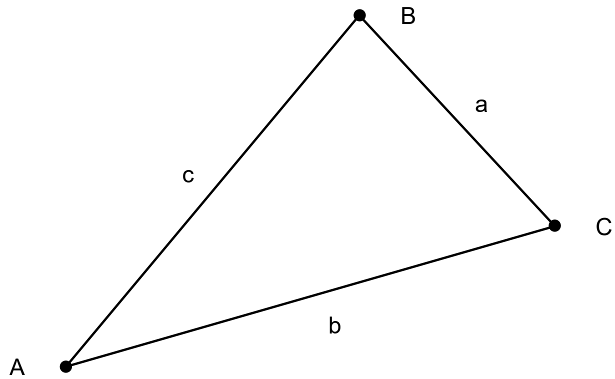
$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c},$$

$$A+B+C = 180^\circ = \pi \textit{ radians}.$$

Example 3.37 *A triangle has an interior angle* 112° *with opposite side length* 9 *m and adjacent side length* 8 *m. Use the sine rule to solve the triangle.*

Exercises/Homework:

Theorem 22 (Cosine Rule) *The triangle shown on the left satisfies all two of the identities below on the right:*



$$a^2 + b^2 = c^2 + 2ab\cos(C),$$

$$A + B + C = 180^\circ = \pi \text{ radians}.$$

Notice that when $C = 90^\circ$, $\cos(90^\circ) = 0$ and we have $a^2 + b^2 = c^2$.

Theorem 23 (Triangle Inequality) *The triangle $\triangle ABC$ satisfies:*

$$a + b > c, \qquad a + c > b, \qquad b + c > a.$$

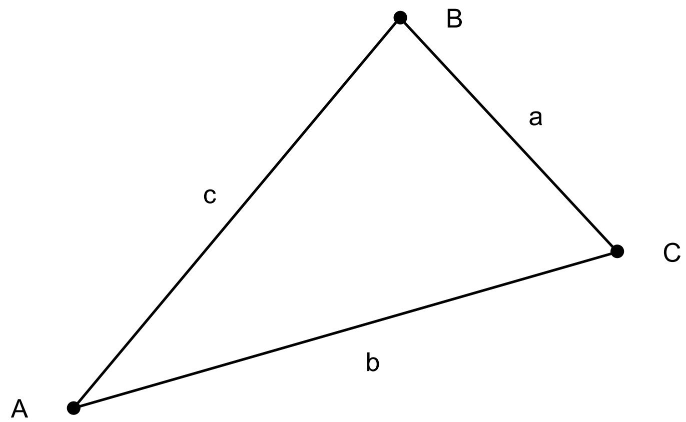
Furthermore, there a triangle of side lengths a, b, c if and only if the three inequalities above are satisfied.

Example 3.38 *Is there a triangle of side lengths $9, 21, 8$?*

Example 3.39 *Solve the triangle with side lengths $7, 9, 15$ if possible.*

Exercises/Homework:

We will use the following triangle as a reference to think about the following results on the area of a triangle.



Theorem 24 *The triangle $\triangle ABC$ has area given by Area* $= \frac{1}{2}bc\sin(A)$.

Note: equivalently, Area $= \frac{1}{2}ac\sin(B)$ and Area $= \frac{1}{2}ab\sin(C)$.

Theorem 25 (Heron's Formula) *The triangle $\triangle ABC$ has area given by Area$^2 = s(s-a)(s-b)(s-c)$, where $s = \frac{1}{2}(a+b+c)$.*

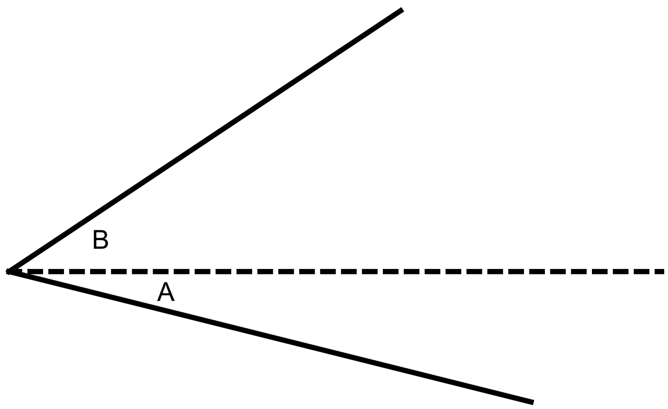
Heron's formula is easy to use and can also be used to confirm calculations obtained via Theorem 24.

Example 3.40 *Calculate the area of the triangle with side lengths 7 m, 9 m, 15 m in two ways by using Area* $= \frac{1}{2}bc\sin(A)$ *and by Heron's formula.*

Exercises/Homework:

3.1.17 Angle of Elevation, Bearings

An **angle of depression** is the angle between an object and a horizontal line where the object is below the horizontal line. See Angle A below.



An **angle of elevation** is the angle between an object and a horizontal line where the object is above the horizontal line. See Angle B above.

Example 3.41 *An airplane has an angle of elevation of* 30° *above sea level from an observer in a boat and is* 4 *km from the observer. Calculate the height of the airplane above sea level.*

A **bearing** is the angle an object makes measured clockwise from North.

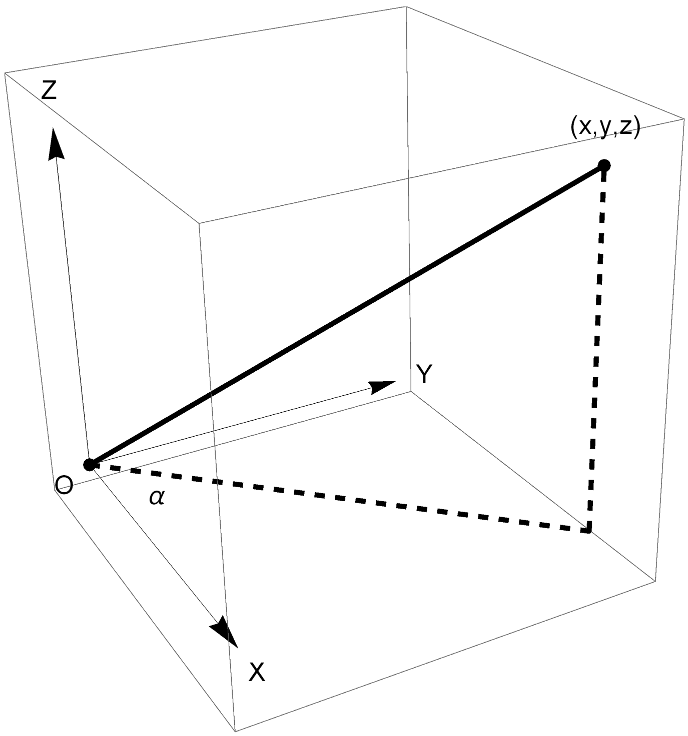
Example 3.42 *Two ships at points A and B are* 30 *km apart. The bearing of the ship at point B from point A is* 55°.

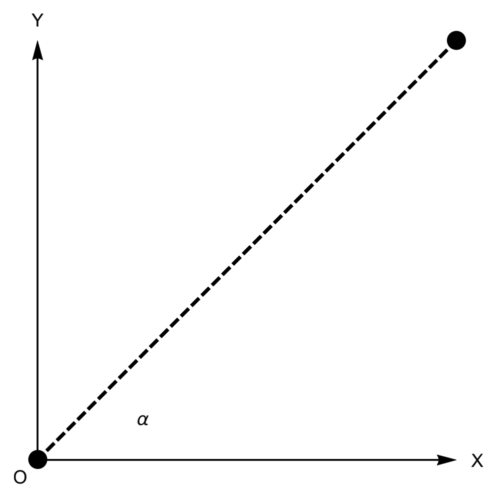
(a) *Calculate the bearing of the ship at point A from point B.*

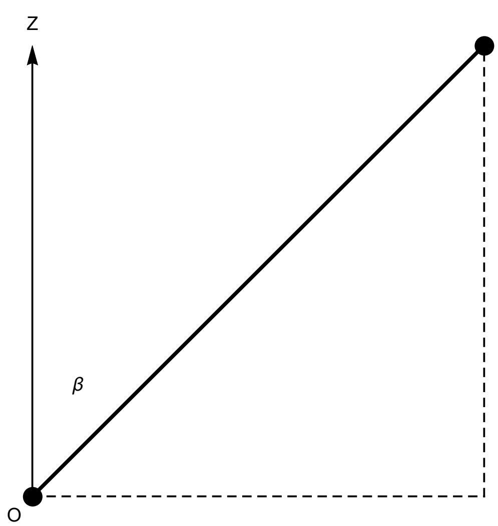
(b) *How far East is B from A?*

Exercises/Homework:

A point (x,y,z) in 3-dimensional space has a distance r from the origin $O=(0,0,0)$ given by $r^2=x^2+y^2+z^2$ and two associated angles α and β shown in the diagram below.





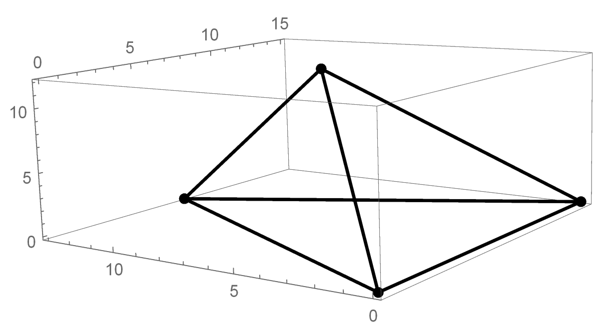


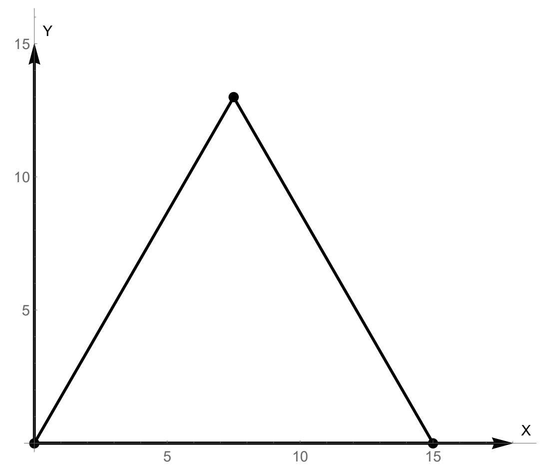
The triple (r,α,β) is the **spherical coordinate** representation of the rectangular coordinate (x,y,z). On the left the plane containing the points O, (x,y,z), and the z-axis illustrates the measurement of the angle β. This β is the angle from the z-axis to the line connecting O and (x,y,z) so that $\cos(\beta)=\frac{z}{r}$. Also, $\tan(\alpha)=\frac{y}{x}$ and $r^2=x^2+y^2+z^2$.

To get rectangular coordinates from spherical coordinates, we have

$$\begin{aligned} x &= r\cos(\alpha)\sin(\beta), \\ y &= r\sin(\alpha)\sin(\beta), \\ z &= r\cos(\beta). \end{aligned}$$

Example 3.43 *A pyramid has height* 12 *m and the base is an equilateral triangle with one corner at the origin. The length of each side of the base of the pyramid is* 15 *m. Calculate the coordinates of the peak of the pyramid in both rectangular and spherical coordinates if the coordinates of corners of the base of the pyramid are* $(0,0,0)$, $(15,0,0)$, $\left(\frac{15}{2}, \frac{15\sqrt{3}}{2}, 0\right)$.





Continued

Exercises/Homework:

3.2 Term 2

Under Construction

3.3 Term 3

Under Construction

3.4 Term 4

Under Construction

References

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[2] Michael Evans, Kay Lipson, Douglas Wallace, David Greenwood, Mathematical Methods Units 1 & 2, Cambridge Senior Mathematics for Queensland, 2nd Ed., 2024.

[3] A. J., Washington. Basic Technical Mathematics with Calculus, SI Version.