Multivariate Calculus

Workbook

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https://mathofcourse.com/

Contents

Instructions

1. **Imperative:** Print this pdf document or be prepared to annotate the pdf with a tablet. Some blank spaces for writing are a little small for large writing. If you cannot do either of these annotation options, then write notes on blank paper, noting the relevant position within the typed course notes. As you watch the lecture videos, write notes in the blank spaces. This step is very important.

2. **Optional but highly recommended:** Purchase and use *Mathematica* or obtain it through your institution. We will occasionally use this to display various graphics and verify calculations. All graphics shown in this document was produced with *Mathematica*. You will most likely find it very helpful with your studies. It is a symbolic computation tool which has full programming capabilities. E.g. Try writing

   ```
   Expand[(x+y)^3]
   ```

   then press Shift+Enter or

   ```
   s = 0;
   For[i = 0, i < 6, i++, s = s + i; Print[s]]
   ```

   You can call on *Wolfram alpha* from with in it by beginning a cell with = =.
   If your university has a license, to install this on your machine, visit:
   `wolfram.com/siteinfo/`
   Get *Mathematica* Desktop.
   Create a Wolfram ID, and download and install the software.

1 One-Variable Calculus

Calculus is generally thought of as the study of the rates of change of functions. It goes back to Isaac Newton and Gottfried Leibniz, who each developed tools of calculus independently. To begin our study of calculus, we must recall what a function is.

1.1 Functions, limits, and continuity

A *function* is a rule f for sending elements of a set X to elements of a set Y such that $f(x)$ is uniquely defined for all x in X. For example, if f is a function which takes an element x of the set X and squares it, then mathematically we denote this as follows.

$$f:\, X \longrightarrow Y, \qquad\qquad f(x) = x^2.$$

X is called the *domain* of f and Y is called the *codomain* of f. In Occasionally it is useful to take another point of view. See [9] for the following definition of a function:

Definition 1 *A* function *is a collection of pairs of numbers with the following property: if (a,b) and (a,c) are both in the collection, then $b = c$; in other words, the collection must not contain two different pairs with the same first element.*

Example 1.1 *Show that the collection C of real points (x,y) on the unit circle $x^2 + y^2 = 1$ is not a function.*

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See Figure 1. However, if we define a new rule f by

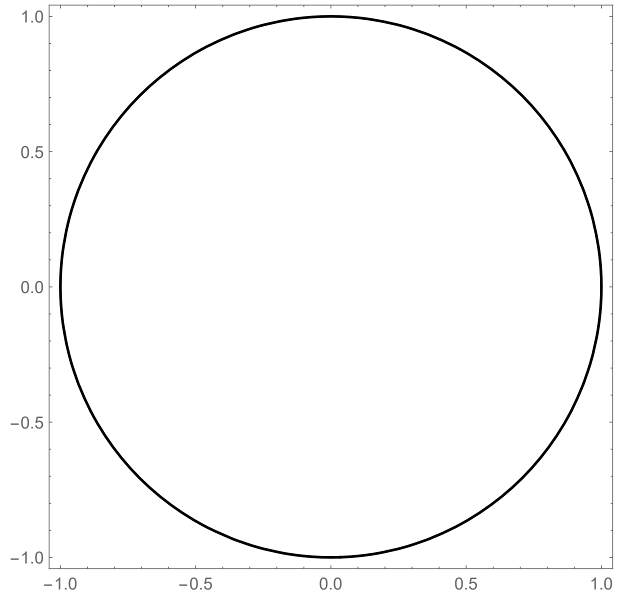


Figure 1: The collection of points (x,y) satisfying $x^2+y^2=1$ shown as a black circle.

$$f:\ [-1,1]\longrightarrow\mathbb{R}, \qquad\qquad f(x)=-\sqrt{1-x^2},$$

then f is a function. See Figure 2.

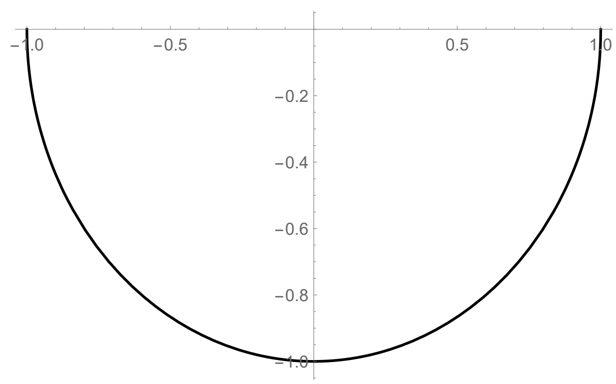


Figure 2: The function $f(x)=-\sqrt{1-x^2}$.

We will generally focus on functions over the set of real numbers $\mathbb{R}$, but functions can be defined on other sets too. For example, $f:\mathbb{Z}\longrightarrow\mathbb{Z}$, by $f(x)=2x$ takes an integer x and sends it to the even integer $2x$. The codomain is the set of integers but the *image* of f (or the *range* of f) is the set of even integers $2\mathbb{Z}$.

To begin our brief introduction to univariate calculus, we must first learn what a limit is. The *limit* as x approaches a of the function $f(x)$, denoted $\lim_{x \longrightarrow a} f(x)$ is the number that is obtained for x close to a but not necessarily equal to a.

According to [9], we have the following definition of the limit.

Definition 2 *The function f approaches the limit ℓ near a means: for every $\varepsilon > 0$ there is some $\delta > 0$ such that, for all x, if $0 < |x-a| < \delta$, then $|f(x) - \ell| < \varepsilon$.*

However, it may be difficult to conceptualize this; see Figure 5.

An important notion is the study of calculus of functions is that of continuity.

Definition 3 *A function $f : X \longrightarrow Y$ is said to be* continuous *at a in X if $\lim_{x \longrightarrow a} f(x) = f(a)$. f is continuous on X if for all $a \in X$, f is continuous at x. In other words, when you sketch the curve $y = f(x)$, you never need to lift your pencil to do so.*

Example 1.2 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function given by $f(x) = x^3 + 5x^2 - 2$ shown in Figure 3. The function f is continuous at all real x and hence f is continuous on $\mathbb{R}$.*

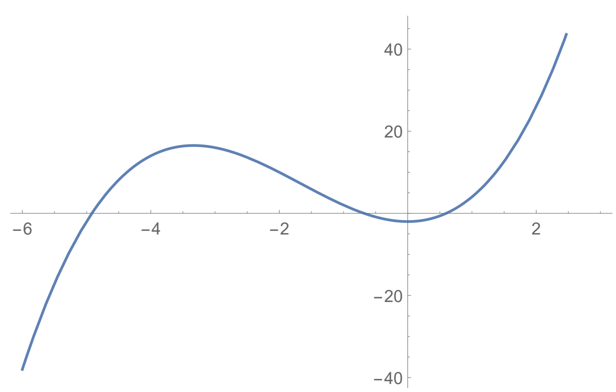


Figure 3: The function $f(x) = x^3 + 5x^2 - 2$.

Example 1.3 *Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ by*

$$f(x)=\begin{cases} x^2+3 & \text{if } x\leq 2\\ 3x+7 & \text{if } x>2. \end{cases}. \tag{1}$$

See Figure 4. The function f is not continuous since $\lim_{x\longrightarrow 2^-} f(x)=7$,

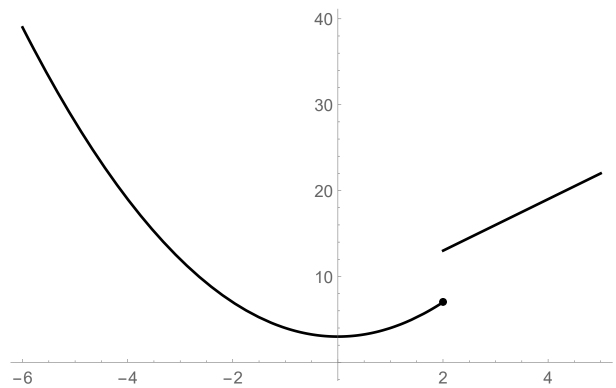


Figure 4: The function given in Equation (1).

$\lim_{x\longrightarrow 2^+} f(x)=13$, $f(2)=7\neq 13$. *In fact, the limit from below and from above are not equal so the limit as x approaches 2 does not even exist.*

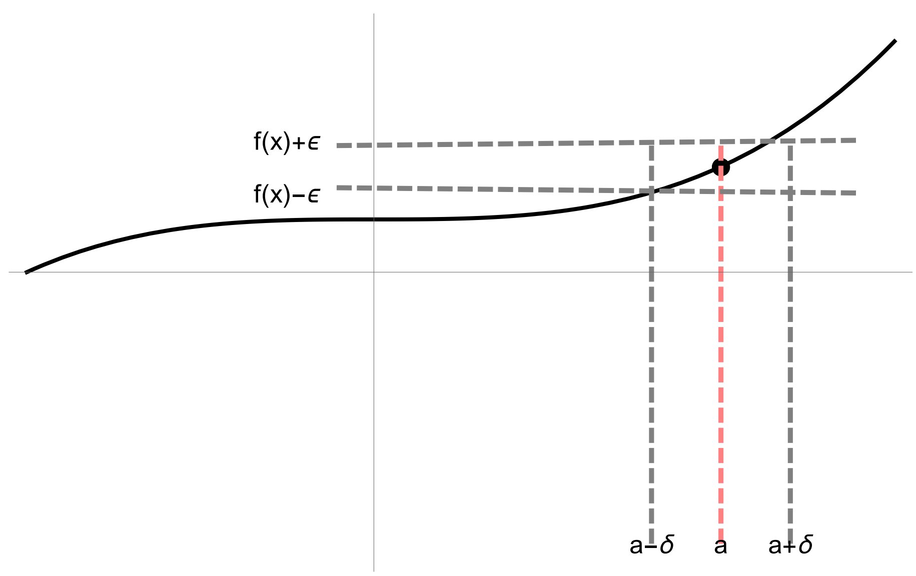


Figure 5: The epsilon-delta definition of continuity at $x=a$.

Spivak [9] gives an epsilon delta definition of continuity:

Definition 4 *$f:\mathbb{R}\longrightarrow\mathbb{R}$ is continuous at a if for all $\varepsilon>0$, there is a $\delta>0$: $|f(x)-f(a)|<\varepsilon$ whenever $|x-a|<\delta$.*

To rephrase this definition, the part between the vertical dashed lines in Figure 5 must be all within the horizontal dashed lines to be called continuous at a.

Example 1.4 *Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = 3x + 2$. Use the epsilon delta definition to show that f is continuous on $\mathbb{R}$.*

1.2 The first derivative

A very important concept in calculus is the derivative of a function.

Definition 5 *If f is a function of x, then the* first derivative *of $f(x)$ is defined as*

$$f'(x) = \lim_{h \longrightarrow 0} \frac{1}{h} \left(f(x+h) - f(x) \right).$$

See Figure 6, which shows the slope of a line segment of a curve $y = f(x)$. As the point on the right gets closer to the point on the left ($h \longrightarrow 0$), the slope of the line segment approaches that of the tangent line to the curve at the point on the left.

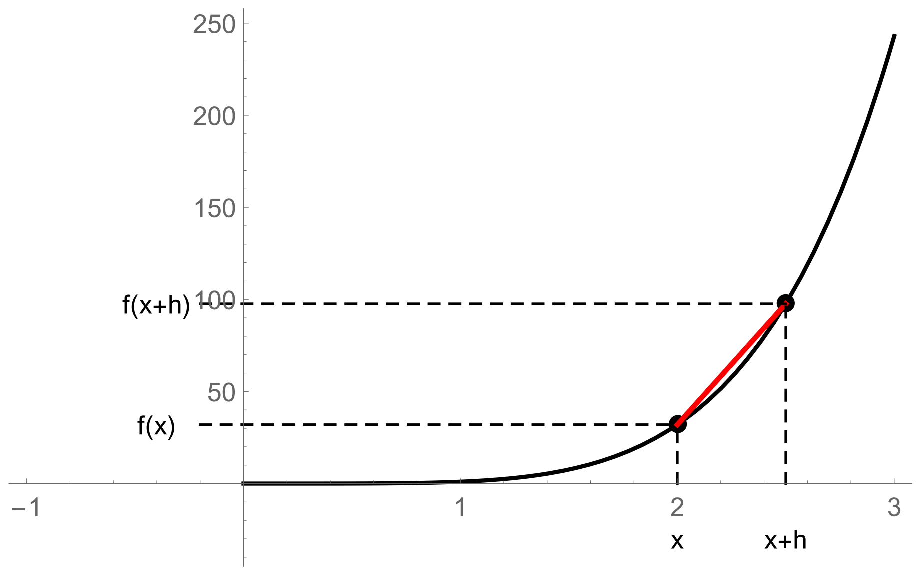


Figure 6: The forward difference approx. of the first derivative when $h = 0.5$ for $f(x) = x^5$.

Example 1.5 *Use the definition of the derivative to calculate $f'(x)$ for the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = x^2 + 3x - 6$.*

The definition of the first derivative can be used to construct a list of rules for differentiation, which we can then use to calculate various derivatives. We will list the most important rules of differentiation below are this is a revision of material learned in previous courses.

- If c is a constant, then $\frac{d(cy)}{dx} = c\frac{dy}{dx}$ or $(cf(x))' = cf'(x)$.
- The derivative of a sum: $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ or $(f(x)+g(x))' = f'(x)+g'(x)$.
- Derivatives to remember: $\frac{dx^n}{dx} = nx^{n-1}$. $\frac{de^x}{dx} = e^x$. $\frac{d\sin(x)}{dx} = \cos(x)$. $\frac{d\cos(x)}{dx} = -\sin(x)$.
- The product rule of differentiation: $\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ or $(f(x)g(x))' = g(x)f'(x) + f(x)g'(x)$.
- The quotient rule of differentiation: $\frac{d(u/v)}{dx} = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)$ or $(f(x)/g(x))' = \frac{1}{(g(x))^2}\left(g(x)f'(x) - f(x)g'(x)\right)$.
- The chain rule of differentiation: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ or $(f(g(x)))' = f'(g(x))g'(x)$.

Example 1.6 *Use use the appropriate rules of differentiation to calculate $f'(x)$ for the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = 9x^6 + 4x^2 + \sin(x)$.*

Example 1.7 *Use use the appropriate rules of differentiation to calculate $f'(x)$ for the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = 2e^{3x^2}\cos(x)$.*

Example 1.8 *Use use the appropriate rules of differentiation to calculate $f'(x)$ for the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = \frac{4x-5}{x^2-2x+4}$.*

Definition 6 *A function* $f : X \longrightarrow Y$ *is said to be* differentiable *at* $x = a$ *if the derivative* $f'(x)$ *exists.*

Definition 7 *A function* $f : X \longrightarrow Y$ *is said to be* differentiable *if the derivative* $f'(x)$ *exists for all* $x \in X$.

A useful consequence of the definition is the following statement.

Theorem 1 *If the function* $f : X \longrightarrow Y$ *is differentiable at* $x = a$, *then* $f'(x)$ *is continuous on* a.

Example 1.9 *consider the function*

$$f(x) = \begin{cases} |x-1| & \text{if } x \neq 1, \\ a & \text{if } x = 1. \end{cases}$$

For which values of a *is* $f(x)$ *continuous at* $x = 1$? *for these values of* a, *if* $f(x)$ *differentiable at* $x = 1$?

1.3 Integration

Definition 8 *An* anti-derivative *or* integral *of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. The* indefinite integral *of a function $f(x)$ is the set of all functions $F(x)$ such that $F'(x) = f(x)$. We denote*

$$F(x) = \int f(x)\,dx$$

and we add an arbitrary constant c to show an indefinite integral.

We have several rules of integration which will make the process easier.

- If c is a constant, then $\int cf(x)\,dx = c\int f(x)\,dx$.
- The integral of a sum $\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$.
- Integrals to remember: If $n \neq -1$, then $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$. $\int x^{-1}\,dx = \log(x) + c$. $\int e^x\,dx = e^x + c$. $\int \sin(x)\,dx = -\cos(x) + c$. $\int \cos(x)\,dx = \sin(x) + c$. $\int \cos^{-2}(x)\,dx = \tan(x) + c$.
- Integration by parts: $\int u\,dv = uv - \int v\,du$.
- A *definite integral* or *Riemann integral*, denoted $\int_a^b f(x)\,dx$, is defined as the limit
$$\int_a^b f(x)\,dx = \lim_{n\longrightarrow\infty} \sum_{j=1}^{n} f(x_j)\,\Delta x.$$
It gives the area bounded by $a \leq x \leq b$, the function $f(x)$ and the line $y = 0$, where $y = f(x)$.
- We write $\int_a^b f(x)\,dx = [\int f(x)\,dx]_a^b$.

Example 1.10 *Calculate $\int x^2 + 3x\,dx$.*

Example 1.11 *Calculate* $\int \log(x)\, dx$ *by parts.*

Example 1.12 *Calculate* $\int \frac{2x-3}{x^2-3x+5}\,dx.$

Example 1.13 *Calculate* $\int \frac{1}{\sqrt{1-x^2}}\,dx$.

Example 1.14 *Calculate* $\int \frac{1}{\sqrt{1+x^2}}\,dx$.

Theorem 2 *The* **fundamental theorem of calculus** *(FTC) states that if $f(x)$ is continuous over $a \le x \le b$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is an anti-derivative of $f(x)$.

Example 1.15 *Consider the function $f(t) = \int_0^{t^3} \left(1 - e^{-2x^2}\right) dx$, $t \ge 0$. Use the FTC to find $f'(t)$ and show that is is increasing when $t \ge 0$.*

2 Partial Derivatives

2.1 Real functions of several variables

You should be familiar with functions f of one variable. The tools of calculus were useful because you could:

- Sketch the graph $y = f(x)$ of f.
- Find the minima and maxima of f.
- Analyze the slope of f by calculating f'.
- Find approximations of f using Taylor series.
- Find solutions to $f(x) = 0$.

Many familiar formulas are essentially just functions of more than one variable. For example, the volume V of a box is a function of its width, height and depth: $V(w,h,d) = whd$, and the profile of a vibrating string is a function of time and position along the string: $f(x,t) = A\sin x\cos t$.

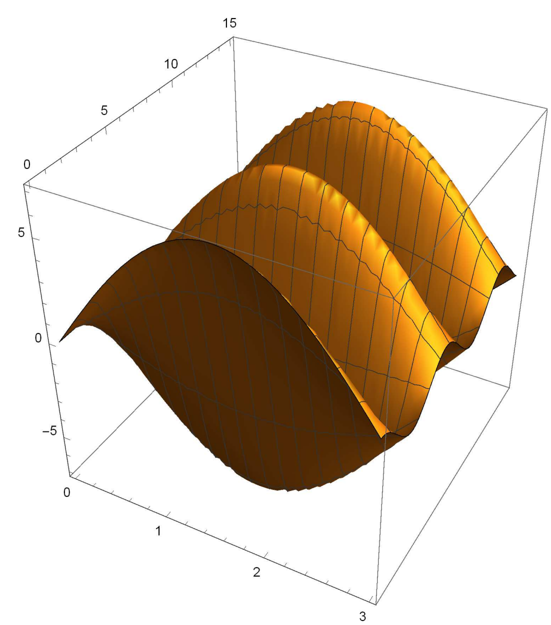


Figure 7: Vibration of a string, $f(x,t) = A\sin x\cos t$, where $t \in [0,15]$ seconds, $x \in [0,3]$ m, f is amplitude, in mm.

Consider the volume of a cylinder as a function of two variables:

$$V(r,\ell) = \pi r^2 \ell.$$

We can visualise this function by graphing V in terms of ℓ and r.

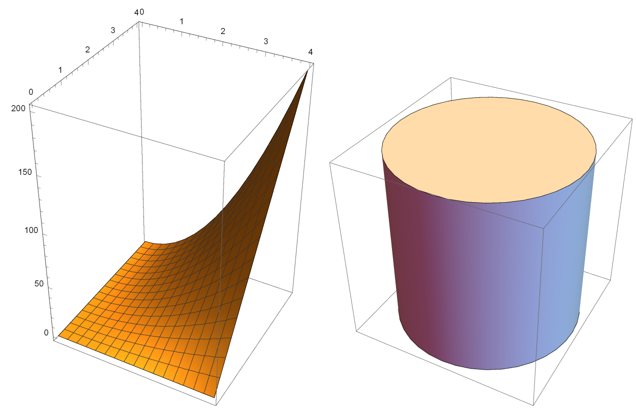


Figure 8: The volume of a cylinder, $V = (r,\ell) = \pi r^2 \ell$.

The equation of a plane can be expressed using scalars or vectors. In this section we will sketch planes in $\mathbb{R}^3$ and determine their scalar equations.

In $\mathbb{R}^3$, we usually take z pointing upwards and the (x,y)-plane to be horizontal. The x and y coordinates give the position on the ground and z gives the height.

Horizontal planes

The x- and y-axes lie in the horizontal plane $z = 0$. All other horizontal planes are parallel to $z = 0$ and are given by the equation $z = c$.

Example 2.1 *Sketch the horizontal plane $z = 2$.*

.

Arbitrary planes

The general equation of a plane in $\mathbb{R}^3$ is given by

$$ax + by + cz = d$$

with a, b, c, d fixed real numbers. If the plane is not vertical, i.e., $c \neq 0$, this equation can be rearranged so that z is expressed as a function of x and y:

$$z = F(x, y) = -(a/c)x - (b/c)y + (d/c) = mx + ny + z_0.$$

The easiest way to sketch the plane by hand is to use the *triangle method*: If all of $a, b, c \neq 0$ the plane $ax + by + cz = d$ intercepts each axis at precisely one point. These three points make up a triangle which fixes the plane.

The plane $x + 2y + z = 4$ intersects the x-axis at $x = 4$, the y-axis at $y = 2$ and the z-axis at $z = 4$.

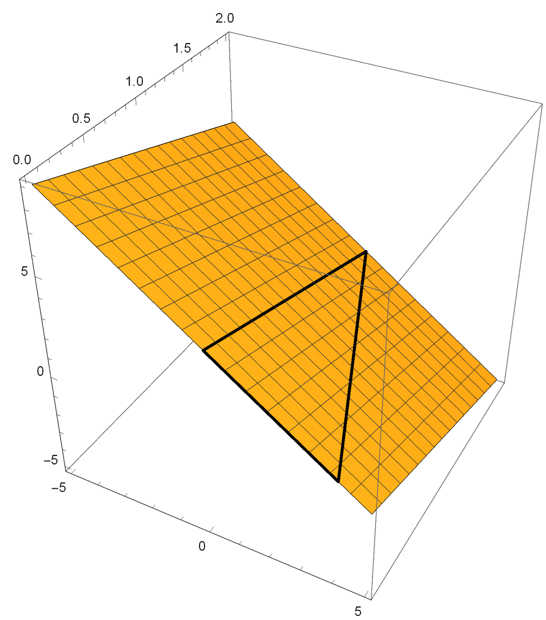


Figure 9: The plane $x + 2y + z = 4$ and lines connecting the points $(4, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 4)$.

The triangle method is based on the simple fact that *any* three points that lie in a plane uniquely determine this plane *provided these three points do not lie on a single straight line*.

It is customary to say *the* equation of a plane, even though it is not unique.

Multiplying the equation of a plane by a nonzero constant gives another equation for the same plane. For example, $x-2y+3z=4$ and $-2x+4y-6z=-8$ are equations of the same plane. The same holds true for lines.

Example 2.2 *Find the equation of the plane through* $(0,0,5)$, $(1,3,2)$ *and* $(0,1,1)$.

Contour diagrams

Geographical maps have curves of constant height above sea level, or curves of constant air pressure (isobars), or curves of constant temperature (isothermals). Drawing contours is an effective method of representing a 3-dimensional surface in two dimensions. We now look at functions f of two variables. A **contour** is a curve corresponding to the equation $z=f(x,y)=C$.

Consider the surface $z=f(x,y)=x^2+y^2$ sliced by horizontal planes $z=0$, $z=1$, $z=2,\ldots$

Plane	Contour	Description
$z=0$	$x^2+y^2=0$	$x=0, y=0$
$z=1$	$x^2+y^2=1$	Circle radius 1
$z=2$	$x^2+y^2=2$	Circle radius $\sqrt{2}$
$z=3$	$x^2+y^2=3$	Circle radius $\sqrt{3}$
$z=4$	$x^2+y^2=4$	Circle radius 2

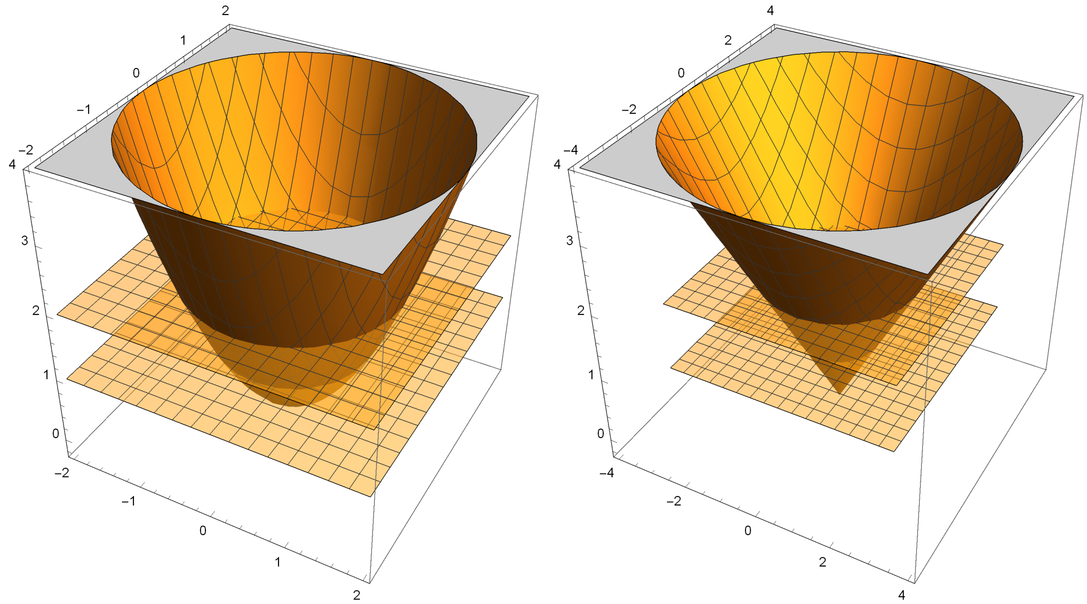


Figure 10: Left: A potential well: $z = f(x,y) = x^2 + y^2$. Right: A Cone: $z = f(x,y) = \sqrt{x^2+y^2}$.

Note that as the radius increases, the contours are more closely spaced.

Example 2.3 *Draw a contour diagram of f given by*

$$f(x,y) = \sqrt{x^2+y^2}.$$

If horizontal planes are equally spaced, say $z = 0, c, 2c, 3c, \ldots$, it is not hard to visualise the surface from its contour diagram. Spread-out contours mean the surface is quite flat and closely spaced ones imply a steep climb.

Note that the contours of the last two functions were all circles. Such surfaces have circular symmetry. When x and y only appear as $x^2 + y^2$ in the definition of f, then the graph of f has circular symmetry about the z axis. The height z depends only on the radial distance $r = \sqrt{x^2+y^2}$.

$z = f(x,y) = \exp^{-x^2-y^2}$

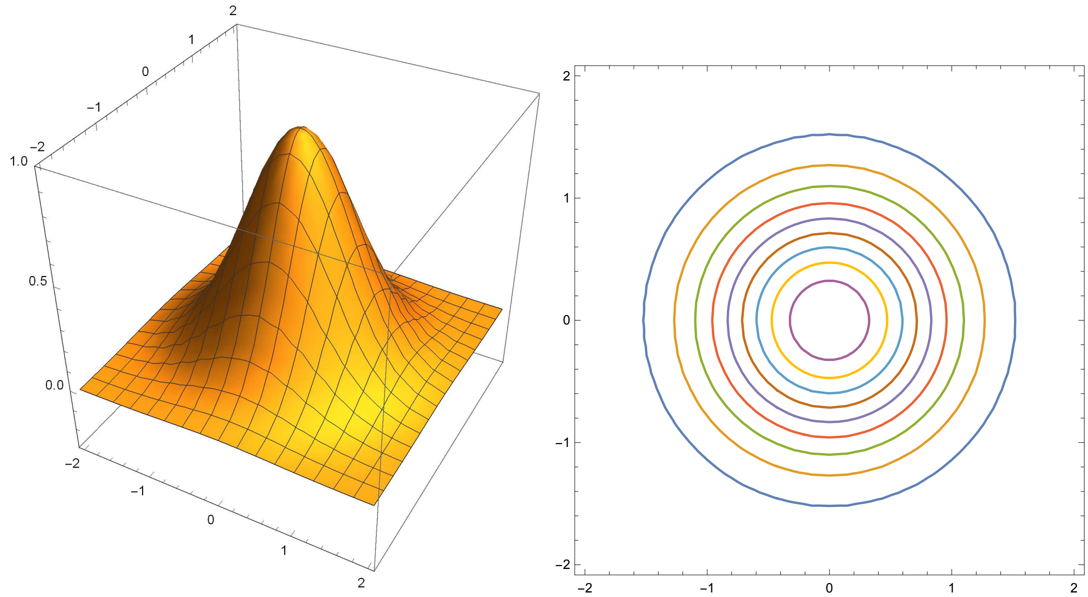


Figure 11: Left: The surface: $z = f(x,y) = e^{-x^2-y^2}$. Right: The contours $z = 0.1n$, $n = 1,2,\ldots 10$.

Example 2.4 *Draw a contour diagram for $z = f(x,y) = x^2 + 4y^2 - 2x + 1$.*

Plane	*Contour*	*Description*
$z = 1$	$(x-1)^2 + 4y^2 = 1$	*ellipse centre* $(1,0)$ *x intercepts:* $x = 0$, $x = 2$ *intersects line* $x = 1$ *at* $y = \pm\frac{1}{2}$
$z = 2$	$(x-1)^2 + 4y^2 = 2$	*ellipse centre* $(1,0)$ *x intercepts:* $x = 1 \pm \sqrt{2}$ *intersects line* $x = 1$ *at* $y = \pm\frac{1}{\sqrt{2}}$
$z = 3$	$(x-1)^2 + 4y^2 = 3$	*ellipse centre* $(1,0)$ *x intercepts* $x = 1 \pm \sqrt{3}$ *intersects line* $x = 1$ *at* $y = \pm\frac{\sqrt{3}}{2}$

A **saddle**, for example $z = x^2 - y^2$, has hyperbolic contours.

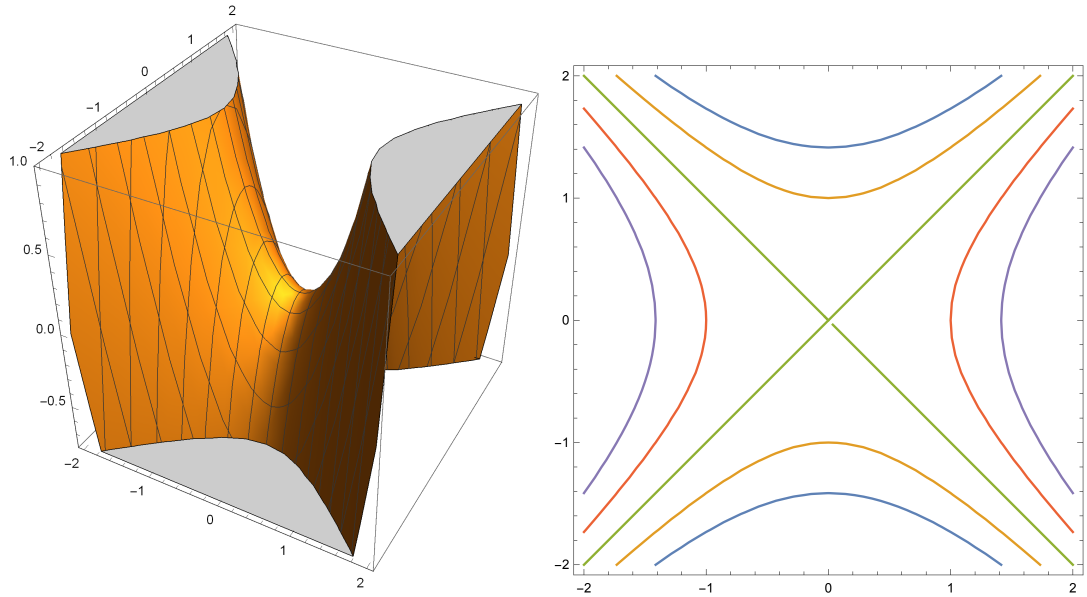


Figure 12: Left: The saddle $z = f(x,y) = x^2 - y^2$. Right: The contours $z = -2, -1, 0, 1, 2$.

$$z = 0:$$

$$z = 1:$$

$$z = 2:$$

$$z = -1:$$

$$z = -2:$$

Example 2.5

Sketch the contour diagram for $z = f(x,y) = 9x^2 - 4y^2 + 2$ *with contours at* $z = -2, 2, 6$ *and 10.*

- *Sketch a contour diagram for* $z = f(x, y) = x^2$ *and use this to sketch the graph of* f.
- *Sketch a contour diagram for* $z = f(x,y) = x - y^2$.
- *Sketch contour diagrams of* $z = f(x,y) = x$ *and* $z = g(x,y) = x + y$.

Note: Contour diagrams of functions whose graphs are planes consist of equidistant parallel lines. (Equidistant lines in $\mathbb{R}^2$are always parallel, unlike $\mathbb{R}^3$)

It is possible to construct the plane itself from its contour diagram provided the contours are labeled. Let the plane be $z = mx + ny + c$. Any point in the plane (x_0, y_0, z_0) gives c:

$$c = z_0 - mx_0 - ny_0.$$

To find m consider moving along the plane in the positive x direction between two contours. Calculate the change in z, Δz, between the two contours, and the change in x, Δx, as you move from one contour line to the next. move in the x direction only, i.e. in a plane $y = C$).

Then $m = \dfrac{\Delta z}{\Delta x}$.
Similarly take a plane $x = C$ and move in the positive y direction to calculate Δy.

Then $n = \dfrac{\Delta z}{\Delta y}$.

Example 2.6 *Find the plane given by the following contour diagram:*

Plane	*Contour*
$z = 0$	$y = -2x + 5$
$z = 1$	$y = -2x + 2$

- *First note that $\Delta z = 1 - 0 = 1$. Moreover, if $y = 0$ then from $2x + y = 3$ and $2x + y = 2$ we obtain $x = 3/2$ and $x = 1$, respectively. So $\Delta x = 1 - 3/2 = -1/2$. It follows that $m = \dfrac{\Delta z}{\Delta x} = -2$.*
- *Similarly, if $x = 0$ then from $2x + y = 3$ and $2x + y = 2$ we obtain $y = 3$ and $y = 2$, respectively. So $\Delta y = 2 - 3 = -1$. It follows that $n = \dfrac{\Delta z}{\Delta y} = -1$.*

- *So the plane is $z = c - 2x - y$. To find c we see that the point $(1,1,0)$ is on the plane $z = 0$ and satisfies $2x + y = 3$. So this point must be on the plane $z = c - 2x - y$. This leads to $c = 3$.*

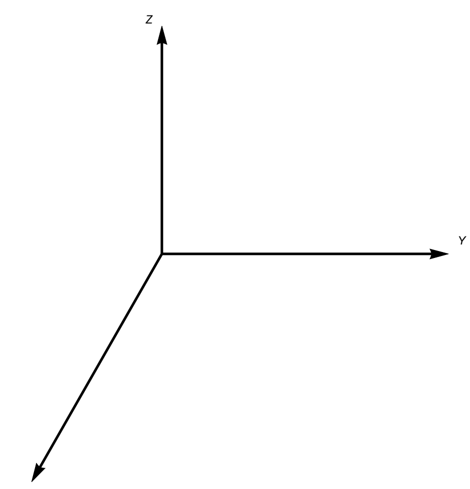
A plane $z = mx + ny + c$ does not have just one slope. It has slope m in the x direction and slope n in the y direction.

To visualise this, imagine starting at $(0,0,c)$ and walking in the (x,z) plane along the line $z = mx + c$ (with slope m). Or walk in the (y,z) plane along the line $z = ny + c$ (with slope n).

Example 2.7 *Find the plane with slope 6 in the x direction and 4 in the y direction which passes through the point $(1,5,4)$.*

$z = mx + ny + c$ with $m = 6$, $n = 4$ and since

$$\begin{aligned} c &= 4 - 6.1 - 4.5, \\ &= -22, \\ \textit{we have} \quad z &= 6x + 4y - 22. \end{aligned}$$



2.2 Partial derivatives and tangent planes

Slope in the x-direction

Consider the surface $z = f(x,y) = 1 - x^2 - y^2$ and the point $P = (1,-1,-1)$ on the surface. Use the "y-is-constant" cross-section through P to find the slope in the x-direction at P. The plane meets the surface $z = f(x,y)$ in the

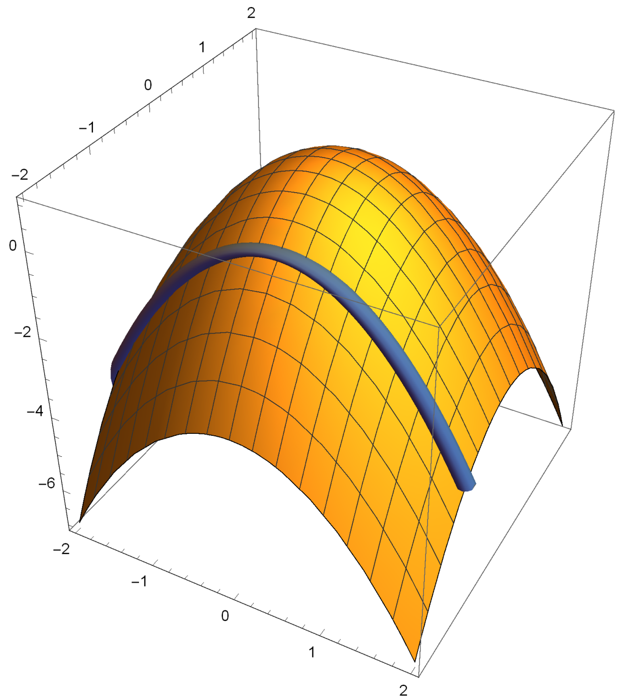


Figure 13: The surface $z = f(x,y) = 1 - x^2 - y^2$ and the intersection of the plane $y = -1$ with the surface.

curve $z = f(x,-1) = g(x)$, say.

Here $g(x) = f(x,-1) = 1 - x^2 - 1$ and $g'(x) = -2x = -2$.

The slope in the x-direction, with y held fixed, is called the partial derivative of f with respect to x at the point (a,b)

$$\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

Slope in the y-direction

Use the "x-is-constant" cross-section to find the slope at $P = (1,-1,-1)$ in the y direction, i.e., where $x = 1$.

When $x = 1$ we have $z = f(1,y) = h(y)$ say, where $h(y) = -y^2$. So $h'(y) = -2y$ and $h'(-1) = -2$.

Similarly, the slope in the y-direction, with x held fixed, is called the partial

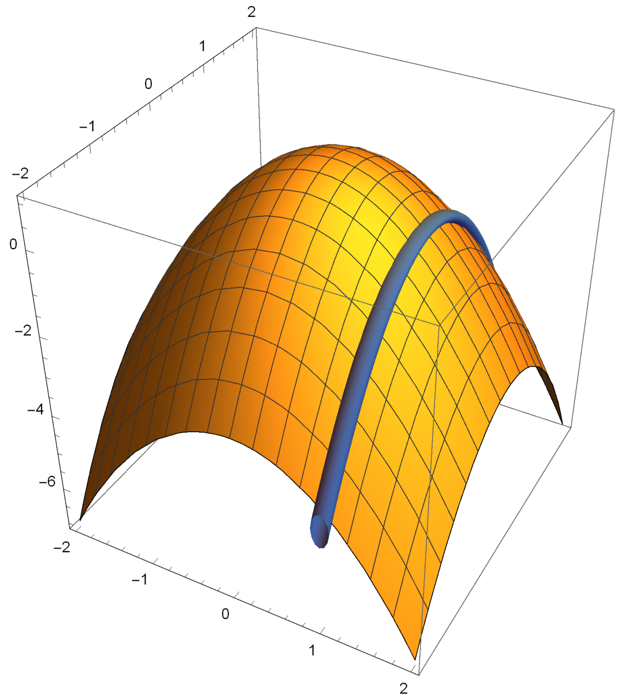


Figure 14: The surface $z = f(x,y) = 1 - x^2 - y^2$ and the intersection of the plane $x = 1$ with the surface.

derivative of f with respect to y at the point (a,b)

$$\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

Definition 9 *A* partial derivative *is a rate of change with respect to a particular variable. For the surface $z = f(x,y)$, it is the slope of a tangent line on a cross-section of the surface. See Figure 51. $f_x = \frac{\partial f}{\partial x}$ is computed by differentiating f with respect to x while treating all other variables as constants.*

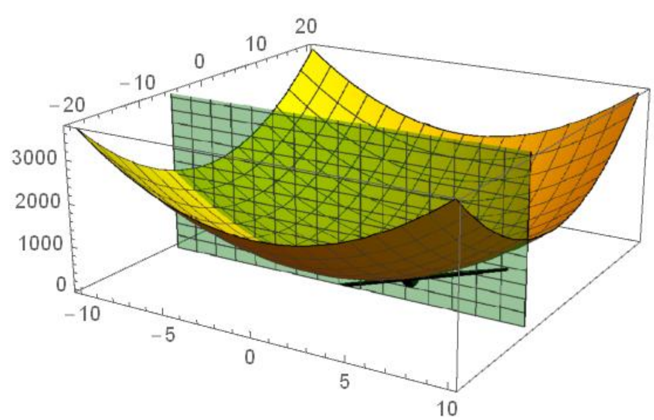


Figure 15: The slope of a tangent line on a cross-section of a surface $z = f(x,y)$.

Normal rules of differentiation apply, we simply think of the variables being held fixed as constants when doing the differentiation.

Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of $f(x,y) = x\sin y + y\cos x$.

Given $f(x,y) = xy^3 + x^2$, find $f_x(1,2)$ and $f_y(1,2)$.

Partial derivatives for $f(x,y,z)$

Example 2.8 *The volume of a box $V(x,y,z) = xyz$.*

If x changes by a small amount, say Δx, denote the corresponding change in V by ΔV. We can easily visualise that $\Delta V = yz\Delta x$.

Therefore,

$$\frac{\Delta V}{\Delta x} = yz.$$

Letting $\Delta x \to 0$ we have $\frac{\partial V}{\partial x} = yz$.
For partial derivatives only one independent variable changes and all other independent variables remain fixed.

Higher order derivatives

The second order partial derivatives of f, if they exist, are written as

$$f_{xx} = \frac{\partial^2 f}{dx^2}, \qquad f_{yx} = \frac{d^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2}, \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

Example 2.9 *Returning to the example where* $f(x,y) = x\sin y + y\cos x$*, calculate all of the second order partial derivatives of* f *and show that* $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

2.3 Multivariate chain rule

Example 2.10 *Suppose the radius of a cylinder decreases at a rate of* $r'(t) = -2$ *cm/s. How fast is the volume decreasing when* $r = 1$ *cm and* $h = 2$ *cm?*

Volume of a cylinder is given by $V = \pi r^2 h$*. Since* h *is constant, it follows that*

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 2\pi rh\frac{dr}{dt} = 2\pi 100.200.(-2) = -80,000\pi \ cm^3/sec.$$

Given $f(x,y)$ with x and y functions of t. Then

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))}{\Delta t}.$$

Now if f is smooth and Δf is small, we can relate it to Δx and Δy through the linear approximation

$$\Delta f \simeq f_x \Delta x + f_y \Delta y.$$

Hence

$$\frac{\Delta f}{\Delta t} \simeq f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t}.$$

Now we let $\Delta t \to 0$, and provided $x(t)$ and $y(t)$ are smooth,

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt},$$

which is the chain rule for $f(x(t), y(t))$.

Example 2.11 *Continuing the previous example, suppose that not just the radius but also the height h is decreasing:* $\frac{dh}{dt} = -1$ *cm/s. What is the rate of change in volume?*

The chain rule can be extended to any number of dimensions.

Example 2.12 *If* $V(a(t)), b(t), c(t)) = abc$ *is the volume of a box then find* $\dfrac{dV}{dt}$.

2.4 Second partial derivatives

Second partial derivatives satisfy the following:

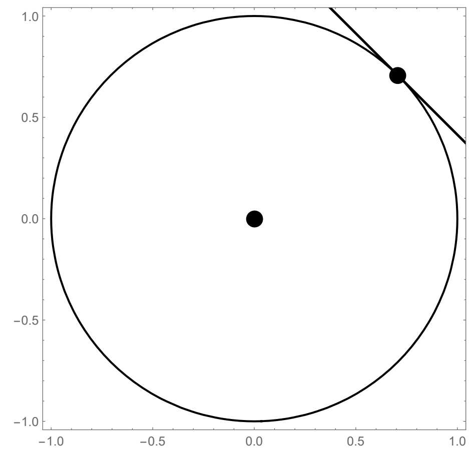
- $$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right),$$
  $$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \qquad f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).$$

- Clairaut's Theorem: If f and it's 1st and 2nd partial derivatives are defined and continuous, then

$$f_{xy} = f_{yx}.$$

2.5 Implicit Differentiation

Consider the implicitly defined curve $x^2+y^2=1$, the unit circle with centre $(0,0)$ and radius 1.



To calculate the slope of the curve at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, we have two ways to proceed. It is possible using univariate calculus, but since we do not have a function, some care is needed. We can set out the two functions:

$$y_1(x) = \sqrt{1-x^2}, \qquad\qquad y_2(x) = -\sqrt{1-x^2}.$$

Taking derivatives,

$$y_1'(x) = \qquad\qquad, \qquad\qquad y_2'(x) = \qquad\qquad.$$

Then

$$y_1'\left(\frac{1}{\sqrt{2}}\right) = \qquad\qquad, \qquad\qquad y_2'\left(\frac{1}{\sqrt{2}}\right) = \qquad\qquad.$$

Alternatively, we let $F(x,y) = x^2+y^2-1$ and use the formula

$$\frac{dy}{dx} = \frac{-F_x}{F_y}.$$

If $F(x,y,z)=0$ is a function of 3 variables, and that F gives z implicitly defined by $z=f(x,y)$, then using the chain rule,

$$\frac{\partial F}{\partial x}=0=\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}.$$

We have

$$\frac{\partial x}{\partial x}=1,\frac{\partial y}{\partial x}=0.$$

We get

$$F_x+F_z\frac{\partial z}{\partial x}=0.$$

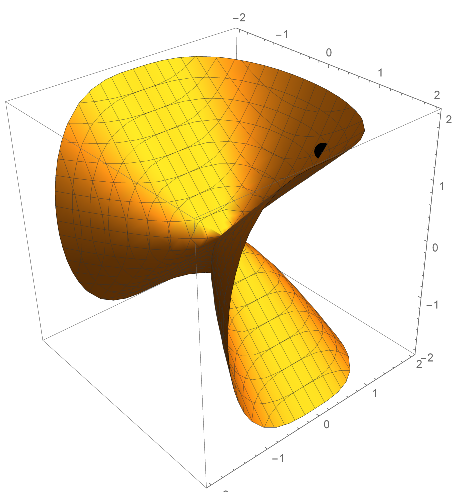
Hence

$$\frac{\partial z}{\partial x}=\frac{-F_x}{F_z}.$$

Similarly

$$\frac{\partial z}{\partial y}=\frac{-F_y}{F_z}.$$

Example 2.13 *Consider the surface $x^2+y^3=z^2$. Calculate $\frac{\partial z}{\partial x}$ at the point $\left(1,1,\sqrt{2}\right)$.*



3 Gradient Vector and Directional Derivative

3.1 Gradient vector

The *gradient vector* or simply *gradient* of f is a vector with the partial derivatives as components. If $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, then

$$\operatorname{grad} f = \nabla f = (f_x,\, f_y) = f_x\mathbf{i} + f_y\mathbf{j}.$$

Example 3.1 *Find the gradient of $f(x,y) = x^2 - 3(y-1)^2 + 3$.*

$\nabla f = 2x\mathbf{i} - 6(y-1)\mathbf{j}$.

Similarly, for a function $f(x,y,z,w)$ of four variables,

$$grad\, f = \nabla f = (f_x, f_y, f_z, f_w)\,.$$

In general we have the following definition.

Definition 10 *Let f be a function of x_1, x_2, $\ldots, x_n$. Then the vector ∇f is given by*

$$\nabla f = (f_{x_1}, f_{x_2}, \ldots, f_{x_n}) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

∇f is called the gradient vector *and we say grad f when we refer to ∇f.*

Definition 11 *The directional derivative of the function f at the point P in the direction of the vector $\mathbf{v}$ is given by*

$$f_{\mathbf{v}}(P) = \nabla f(P) \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

Example 3.2 *Find $f_{(1,-1)}(0,1)$ for $f(x,y) = x - x^2y^2 + y$.*

We have expressed the directional derivative as a definition. However, this is actually a consequence of the following:

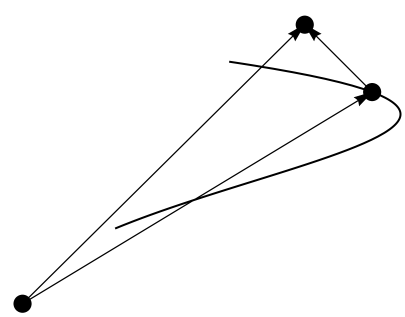
Let $\hat{\mathbf{v}} = (a,b)$ be a unit vector and $P = (x_0, y_0)$ be a point, where we have omitted the z-coordinate z_0. Then the directional derivative is defined

$$f_{\hat{\mathbf{v}}} = \lim_{h \longrightarrow 0} \frac{1}{h} \left(f\left(x_0 + ha, y_0 + hb\right) - f\left(x_0, y_0\right) \right).$$

Notice that the point $Q = (x_0 + ha, y_0 + hb)$ satisfies the vector equation

$$\overrightarrow{OQ} = (x_0 + ha, y_0 + hb) = \overrightarrow{OP} + h\hat{\mathbf{v}}.$$

so the expression on the right hand side of the limit in our definition is of the form Rise/Run. See the diagram below showing a contour of the surface $z = f(x,y)$ showing points $O = (0,0)$, $P = (x_0, y_0)$, $Q = (x_0 + ha, y_0 + hb)$, and vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{PQ}$.



Now let $g(h) = f\left(x_0 + ha, y_0 + hb\right)$. We see that

$$\begin{aligned} f_{\hat{\mathbf{v}}} &= \lim_{h \longrightarrow 0} \frac{1}{h} \left(g(h) - g(0) \right), \\ &= \end{aligned}$$

The directional derivative $f_{\mathbf{v}}(P)$ gives the rate of change of f at P in the direction of the vector $\mathbf{v}$.

Example 3.3 *Consider the function*

$$f(x,y) = 5x^4 + 4x^2y - xy^3 - x + 4y^4.$$

Let

$$P = (0.15, 0.42, 0.0036858)$$

be a point on the surface $z = f(x,y)$. The gradient vector is

$$\nabla f = \left(20x^3 + 8xy - y^3 - 1, 4x^2 - 3xy^2 + 16y^3\right).$$

Evaluating ∇f at the point P, we have

$$\nabla f(P) = (-0.502588, 1.196028) = -0.502588\mathbf{i} + 1.196028\mathbf{j}.$$

See Figures 16 and 17.

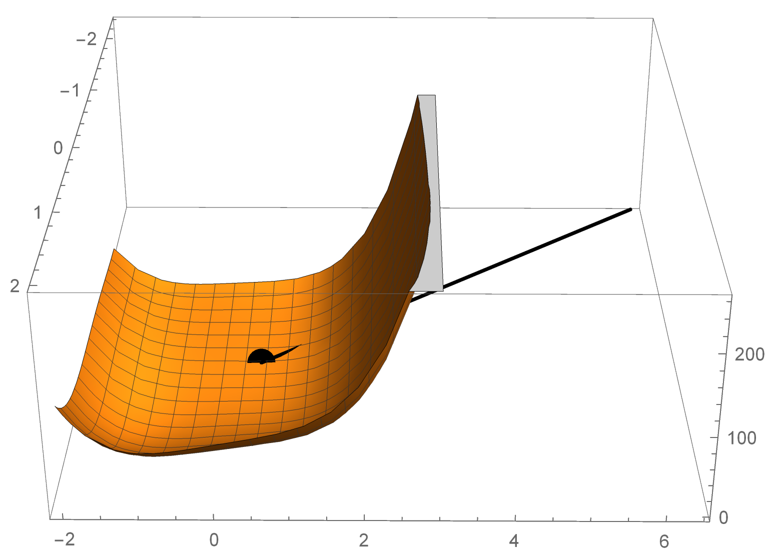


Figure 16: The point P on the surface $z = f(x,y)$ showing $\nabla f(P)$ as a black line emanating from the point P.

Theorem 3 *∇f is a vector whose direction indicates the direction in which the maximum rate of change of f is achieved.*

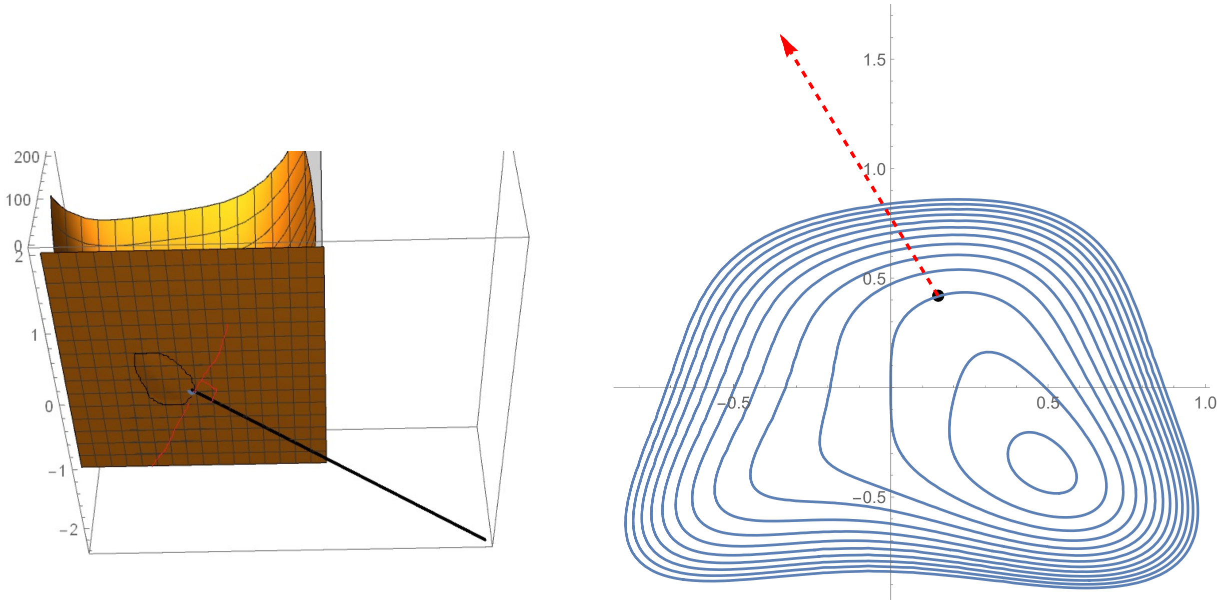


Figure 17: The point P on the surface $z = f(x,y)$ showing $\nabla f(P)$ as a black line emanating from the point P, together with a contour, showing that $\nabla f(P)$ is perpendicular to the contour.

Reiterating what we have said (for $z = f(x,y)$), two important properties of the gradient of a function are:

The gradient $\nabla f(a,b)$ is **perpendicular to the contour line through** (a,b) and points in the direction of increasing f. In fact, the direction and magnitude of **steepest slope** at (a,b) are given by $\nabla f(a,b)$ and $\|\nabla f(a,b)\|$.

We can understand these two facts by considering the value of $\cos\theta$ in

$$f_{\mathbf{v}} = \nabla f \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} = \|\nabla f\| \cos\theta.$$

The directional derivative or slope in the direction of the vector $\mathbf{v} = \nabla f(P)$

is

$$\begin{aligned}
f_{\mathbf{v}}(P) &= \nabla f(P) \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|}, \\
&= \nabla f(P) \cdot \frac{\nabla f(P)}{\|\nabla f(P)\|}, \\
&= \frac{1}{\|\nabla f(P)\|} \nabla f(P) \cdot \nabla f(P), \\
&= \frac{1}{\|\nabla f(P)\|} \|\nabla f(P)\|^2, \\
&= \|\nabla f(P)\|.
\end{aligned}$$

We see that the slope of $z = f(x,y)$ is $\|\nabla f(P)\|$ in the direction of the vector $\nabla f(P)$. The maximum value of $\cos(\theta)$ is 1 so we infer that since $f_{\mathbf{v}}(P) = \|\nabla f(P)\|$ when $\mathbf{v} = \nabla f(P)$, it is maximum in the direction of $\mathbf{v} = \nabla f(P)$.

Example 3.4 *Find the directional derivative of* $g(x,y) = e^{x^2} \cos y$ *at* $(1, \pi)$ *in the direction* $-3\mathbf{i} + 4\mathbf{j}$.

.

Example 3.5 *Consider the contour diagram of a plane*

$$z = f(x,y) = mx + ny + c.$$

For $n \neq 0$ the contours have slope $-m/n$ in the xy-plane.

The gradient vector, $m\mathbf{i}+n\mathbf{j}$ ($m \neq 0$), has slope n/m and so is perpendicular to the contours. It points in the direction of increasing f. In fact, the direction in which it points is the direction of greatest slope.

Example 3.6 *Consider $f(x,y) = x^2+y^2$. The contours are circles centered at the origin, and $\nabla f = 2x\mathbf{i}+2y\mathbf{j}$ points radially out. Once again, ∇f points in the direction of greatest slope, perpendicular to the contour lines.*

Example 3.7 *$T(x,y) = 20-4x^2-y^2$ describes the temperature on the surface of a metal plate. In which direction away from the point $(2,-3)$ does the temperature increase most rapidly? In which directions away from the point $(2,-3)$ is the temperature not changing?*

.

3.2 Using the gradient vector to find max/min

Consider the function $f(x,y) = x^2 + y^2$. See Figure 18. We see that the minimum value 0 of the function occurs when $x = 0 = y$. We can find where this occurs using the gradient vector since if there is a local maximum or minimum point P on the surface, then $\nabla f(P) = \mathbf{0}$. This is analogous to seeking a local maximum or minimum point of a function of one variable $y = f(x)$; we must solve the equation $f'(x) = 0$ for x and use this x to find the turning point in the function.

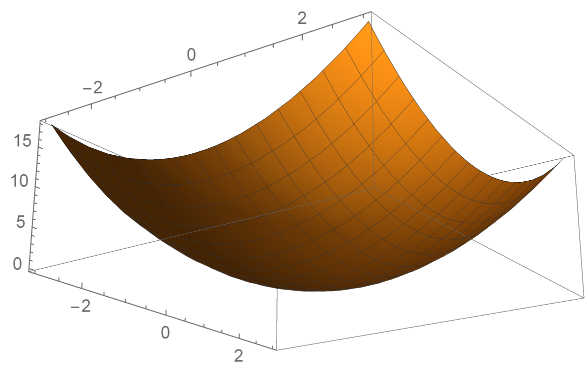


Figure 18: The surface $z = x^2 + y^2$ showing the minimum 0 occurs at the point $(0,0,0)$.

Example 3.8 *Use the gradient vector to find any local maxima or minima of the surface $z = x^2 + y^2$.*

.

A *critical point* (a,b) of a function $f(x,y)$ is a point in which either

$$\nabla f = (0,0)$$

or $f(a,b)$ is not defined - for example at the top of a pyramid or the bottom of an iced-cream cone. In order to classify critical points as maxima or minima, we must introduce the *Hessian matrix* $H_f(a,b)$ given by:

$$H_f(a,b) = \begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{pmatrix}.$$

Theorem 4 *Let $f(x,y)$ be a function with continuous partial derivatives at the point (a,b). Let $\Delta = \det(H_f(a,b))$ be the determinant of the Hessian matrix. If the point (a,b) is a critical point, then it can be classified as follows:*

- *If $f_{xx}(a,b) > 0$ and $\Delta > 0$, then the critical point (a,b) is a local minimum.*
- *If $f_{xx}(a,b) < 0$ and $\Delta > 0$, then the critical point (a,b) is a local maximum.*
- *If $\Delta < 0$, then the critical point (a,b) is a saddle.*
- *If $\Delta = 0$, then the test is inconclusive.*

Example 3.9 *Classify the critical point* $(0,0)$ *of the surface* $z = x^2 + y^2$.

Example 3.10 *Consider the function* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *given by*

$$f(x,y) = x^4 - y^2 + 2x - 3xy + x^3.$$

Find and classify all of the critical points of $f(x,y)$.

See Figure 19.

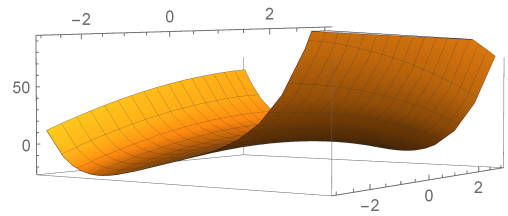


Figure 19: The surface $z = x^4 - y^2 + 2x - 3xy + x^3$ showing a saddle point $\left(-\frac{1}{2}, \frac{3}{4}, -\frac{1}{2}\right)$.

Example 3.11 *Find the global max./min. of the function* $f(x,y) = (3x+3y)^2 + 2$ *on the domain where* $-2 \le x, y \le 2$*.*

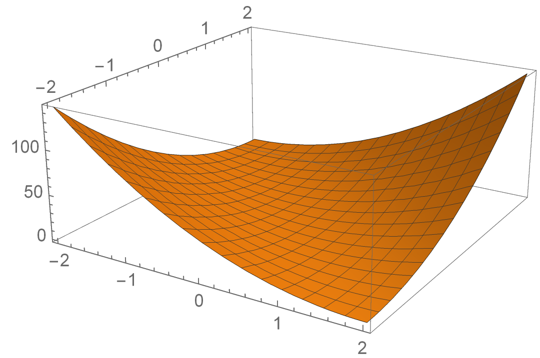


Figure 20: The surface $z = (3x+3y)^2+2$ with a line of critical points along $y = -x$.

3.3 Lagrange multipliers

To find the maximum or minimum of a function $f(x,y)$ subject to a constraint (such as a budget) $g(x,y) = 0$, this occurs when the two gradient vectors ∇f and ∇g are parallel. That is, when there is a real number λ such that

$$\nabla f = \lambda \nabla g.$$

Example 3.12 *A farmer has* 1200 *metres of fence and wishes to enclose a rectangular area. Find the dimensions that maximize the enclosed area using Lagrange multipliers.*

Example 3.13 *Find the maxima/minima of the function*

$$f(x,y) = (3x+3y)^2 + 2$$

subject to the constraint $x^2 + y^2 = 4$.

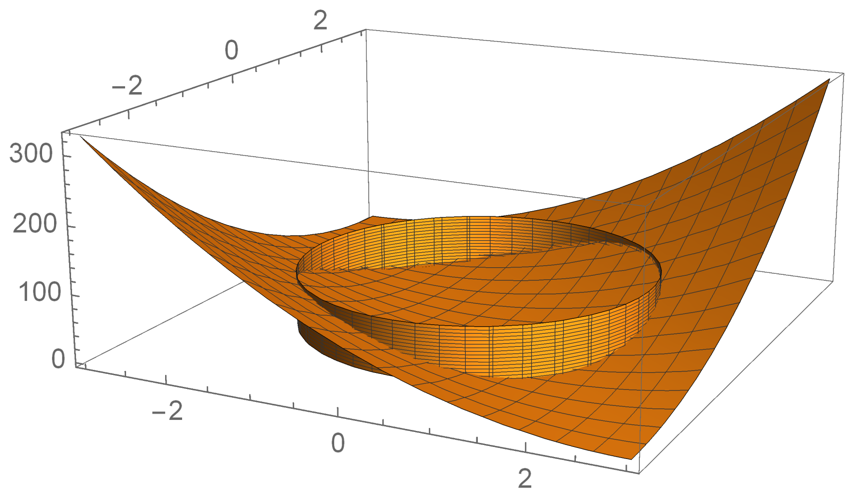


Figure 21: The surface $z = (3x+3y)^2 + 2$ subject to the constraint $x^2 + y^2 = 4$.

4 Divergence and Curl

4.1 Vector fields

A *vector field* is a mathematical concept that describes a set of vectors assigned to each point in a given region of space or a manifold. In other words, a vector field is a collection of vectors that vary in direction and magnitude across a domain.

Vector fields are used in many areas of physics, engineering, and mathematics, including fluid dynamics, electromagnetism, and differential geometry. They are represented graphically by arrows, with the length and direction of each arrow indicating the magnitude and direction of the vector at a particular point in the domain.

Vector fields can be described in both 2-dimensional and 3-dimensional space, and they can be either continuous or discontinuous. Some common examples of vector fields include the velocity field of a fluid, the electric field around a charged particle, and the gravitational field around a massive object.

Example 4.1 *Consider the vector field* $\mathbf{F} = (y, -x)$*. This might represent the velocity of fluid particles at a particular position in space. See Figure 22.*

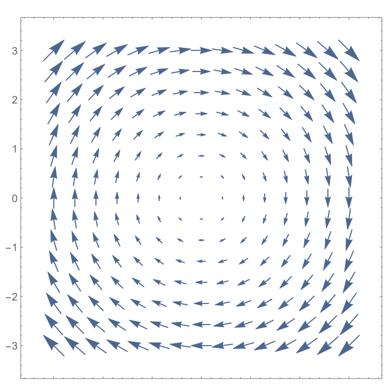


Figure 22: The vector field $\mathbf{F} = (y, -x)$.

4.2 Divergence of a vector field

The divergence of a vector field is a scalar quantity that measures the extent to which the vector field flows out or in at a particular point. It is a measure of how much the vector field is spreading out or contracting around a given point in space.

Definition 12 *Let* $\mathbf{F} = (P, Q, R)$ *be a vector field on* $\mathbb{R}^3$*. Recall that*

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

With this notation we have grad $f = \nabla f = (f_x, f_y, f_z)$*, and we define the* divergence of the vector field $\mathbf{F}$ *as*

$$div\ \mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z.$$

If the divergence of a vector field is positive at a point, it means that the vector field is flowing out from that point. If it is negative, the vector field is flowing in towards that point. If the divergence is zero, it means that the vector field is not spreading out or contracting at that point.

Example 4.2 *Consider the vector field* $\mathbf{F} = \left(3x^2 - 5y, 2 + y\right)$*. Determine the region in the* x, y *plane where* $div(\mathbf{F}) > 0$ *and interpret.*

.

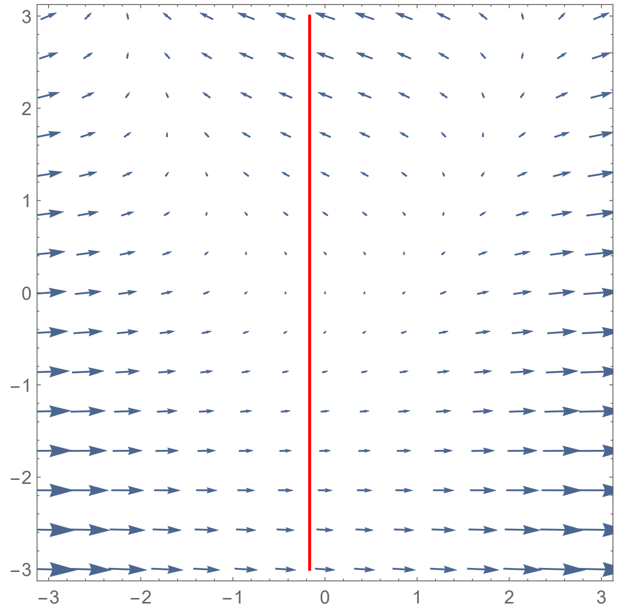


Figure 23: To the right of the red line we have $\mathrm{div}(\mathbf{F}) > 0$ and to the left of the red line $\mathrm{div}(\mathbf{F}) < 0$.

Example 4.3 *Consider the vector field* $\mathbf{F} = \left(x^2y + xy^3 - 5y, xy - y^2 + 2\right)$. *Determine the region in the* x, y *plane where* $div(\mathbf{F}) > 0$ *and interpret.*

In Figure 24 this is shown in green.

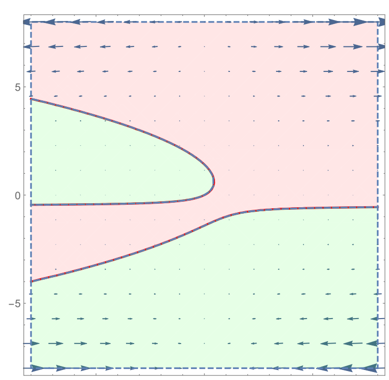


Figure 24: The region where $\mathrm{div}(\mathbf{F}) > 0$ is shown in red.

Definition 13 *We say that the vector field* $\mathbf{F}$ *is* **incompressible** *if div* $\mathbf{F} = 0$ *everywhere.*

Physically, this means that the vector field is not changing its volume at any point in space.

In fluid dynamics, an incompressible fluid is one whose density does not change with respect to changes in pressure or temperature. This means that if a fluid is incompressible, its volume will remain constant even if it is subjected to external forces or pressure changes.

If div $\mathbf{F} = 0$ everywhere, then the flow of the vector field is conserved, and no fluid is created or destroyed within the domain.

Incompressible vector fields are used in various areas of physics and engineering, including fluid mechanics, aerodynamics, and heat transfer. In these applications, the incompressibility condition is often imposed as a simplifying assumption to simplify the governing equations of the problem and make it easier to solve.

Example 4.4 *Recall Example 4.1 where we had* $\mathbf{F} = (y, -x)$. *Show that the vector field* $\mathbf{F}$ *is incompressible.*

Example 4.5 *Let*

$$\mathbf{F}(x,y) = \left(y^2 - x^2, 2xy\right)$$

be the velocity of a fluid at the point (x,y). *Show that no fluid is created or destroyed within the domain. See Figure 25.*

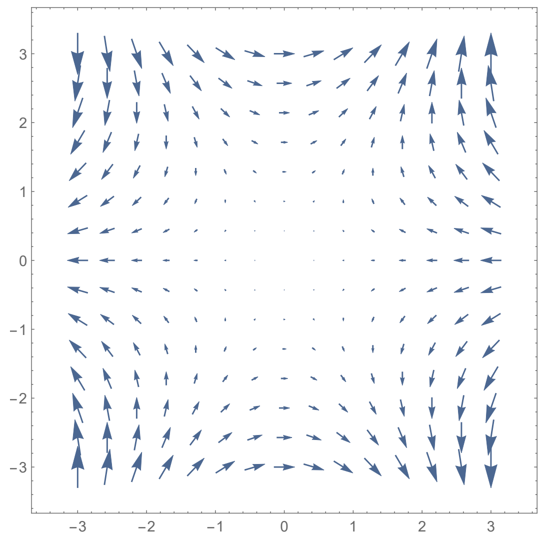


Figure 25: An example of an incompressible fluid.

4.3 Conservative vector fields

A conservative vector field $\mathbf{F}$ is one in which the work done by the field on an object moving along any closed path is zero. This means that the path integral of the vector field around any closed loop is zero. In other words, the value of the line integral of the vector field over a closed curve depends only on the endpoints of the curve, and not on the shape of the curve itself.

Assume the force $\mathbf{F}(x,y,z)$ applied to an object depends on the position (x,y,z), and the force moves the object along a curve C. The the work done in moving the object is given by the equation

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt.$$

If the work done in moving the object from A to B is independent of the path taken, then we say that $\mathbf{F}$ is *conservative*.

Theorem 5 *If the vector field* $\mathbf{F}$ *is the gradient of some function* f. *i.e. there exists* $f(x,y,z)$ *such that* $\mathbf{F} = \nabla f$ *(we say* $\mathbf{F}$ *is a gradient field), then* $\mathbf{F}$ *is* **conservative**. *This is known as the fundamental theorem for line integrals.*

Theorem 6 *If the vector field* $\mathbf{F} = (P,Q)$ *is conservative, then* $P_y = Q_x$.

Example 4.6 *Let* $\mathbf{F} = (x, y)$ *be a vector field. See Figure 26. Find an expression for the work done in moving a particle from point A to point B. Show that* $\mathbf{F}$ *is conservative and find a potential function* $f(x, y)$ *such that* $\mathbf{F} = \nabla f$.

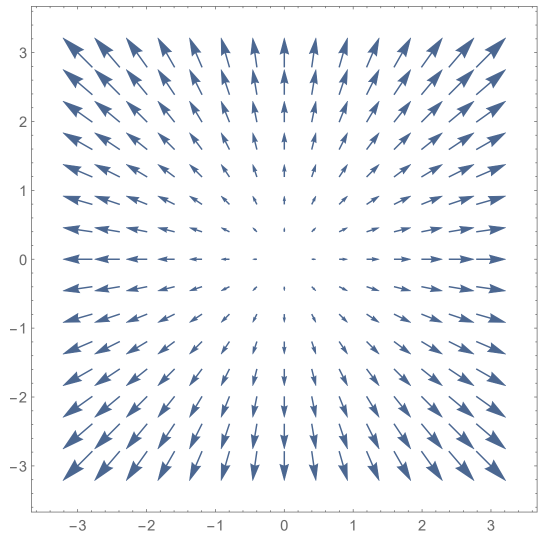


Figure 26: An example of a conservative vector field.

Example 4.7 *Show that the vector field* $\mathbf{F} = \left(x, y^2\right)$ *is conservative.*

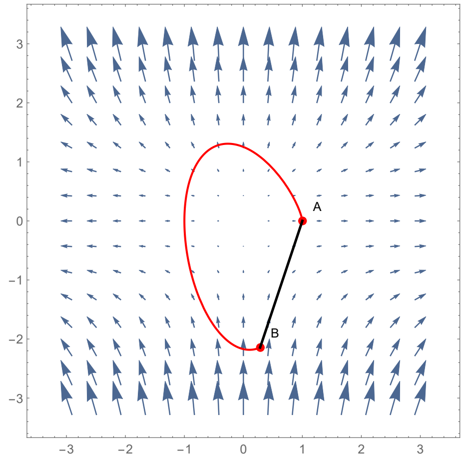


Figure 27: In a conservative vector field the work done in moving an object from A to B is path independent. It would be the same if the red path was taken as it would be if the black path was taken.

Example 4.8 *Find the work done by the electric field*

$$\mathbf{F} = \frac{1}{x^2+y^2+z^2}(x,y,z)$$

from the point $(1,0,0)$ *to the point* $(1,1,1)$.

4.4 The curl of a vector field

The curl of a vector field is a mathematical operation that produces another vector field. It describes the tendency of the original vector field to rotate around a given point in space. More precisely, the curl measures the amount of "circulation" or "rotation" of a vector field at each point in space.

Definition 14 *The* **curl of a vector field** $\mathbf{F} = (P, Q, R)$ *is defined*

$$curl\,\mathbf{F} = \nabla \times \mathbf{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y)\,.$$

Physically, the curl of a vector field describes the tendency of a fluid or electromagnetic field to rotate around a given point. For example, the curl of a velocity field describes the tendency of a fluid to swirl around a point, while the curl of an electromagnetic field describes the tendency of the field to rotate around a charged particle. The curl vector will point in the direction of the axis of spin according to the right hand rule.

Example 4.9 *Let* $\mathbf{F} = (y, -x, 1)$. *See Figure 22 where this shows a contour of the 3D vector field. Calculate curl* $\mathbf{F}$.

Definition 15 *We say that the vector field* $\mathbf{F}$ *is* **irrotational** *if curl* $\mathbf{F} = \mathbf{0}$ *everywhere.*

Example 4.10 *Let $\mathbf{F} = (x, y, 1)$ be a vector field. See Figure 26. Show that $\mathbf{F}$ is irrotational.*

We have the following important results on vector fields.

Theorem 7 *If $f(x,y,z)$ has continuous 2nd partials, then curl$(\nabla f) = \mathbf{0}$.*

We put $\nabla f = (f_x, f_y, f_z) = (P, Q, R)$ and calculate curl(∇f).

$$\begin{aligned} \text{curl}(\nabla f) &= (R_y - Q_z, P_z - R_x, Q_x - P_y), \\ &= \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right), \\ &= (0, 0, 0) \end{aligned}$$

by Clairaut's theorem since $f(x,y,z)$ has continuous 2nd partials.

Theorem 8 *If $\mathbf{F}$ is conservative, then curl $(\mathbf{F}) = \mathbf{0}$. Conversely, if $\mathbf{F}$, defined on $\mathbb{R}^3$, has continuous partial derivatives, and curl $(\mathbf{F}) = \mathbf{0}$, then $\mathbf{F}$ is conservative. If $\mathbf{F}$ is defined on another domain U, then the converse holds when U is simply connected.*

The above result is a more general statement than Theorem 6.

Theorem 9 *If $\mathbf{F}$ has continuous 2nd partial derivatives, then*

$$\textit{div}\ (\textit{curl}\ (\mathbf{F})) = 0.$$

4.5 Green's theorem

Green's theorem is a fundamental result in vector calculus that relates the line integral of a vector field around a simple closed curve C to a double integral of the curl of the same vector field over the region enclosed by C. **First go through Section 6.1 if needed.** This can be stated as follows.

Theorem 10 *(Stokes, Green's thm resp.) Let $\mathbf{F} = (P, Q, R)$ be a vector field, where P, Q, R are continuous in D and P_y, Q_x, etc., are continuous in S.*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_S curl\,(\mathbf{F}) \cdot \mathbf{n}\, dA,$$

where $\mathbf{n}$ is a unit normal vector to S.

If $\mathbf{F} = (P, Q)$, then in the x, y plane,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D Q_x - P_y\, dx\, dy,$$

where C is a simple closed curve in the plane, oriented counterclockwise, D is the region enclosed by C, and $d\mathbf{r}$ is an infinitesimal vector tangent to C. The double integral is taken over the region D and dA is an infinitesimal area element in the plane.

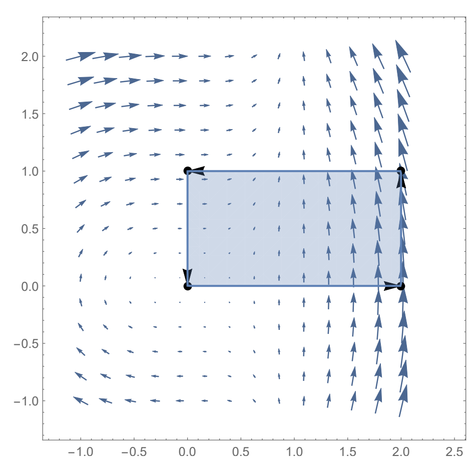


Figure 28: Using Green's theorem to calculate work done along the counter-clockwise path shown.

Example 4.11 *Use Green's theorem to calculate the work done by the force field* $\mathbf{F} = \left(y - xy, x^2\right)$ *in moving an object counterclockwise around a rectangle with vertices* $(0,0)$, $(2,0)$, $(2,1)$, $(0,1)$.

The region D enclosed by the closed curve C is

$$D = \{(x,y) \,:\, 0 \le x \le 2, 0 \le y \le 1\}.$$

See Figure 28 and Example 6.7.

Example 4.12 *Use Green's theorem to evaluate the line integral*

$$\oint_C x^2y^2\,dx + 4xy^3\,dy,$$

counterclockwise along the closed curve C *consisting of a triangle with vertices* $(0,0)$, $(1,3)$, *and* $(0,3)$.

5 Double and Triple Integrals

In this section we will consider double and triple integrals and their meaning. Consider the following motivational example.

Example 5.1 *Let*

$$f(x,y) = \frac{xy^2}{x^2+1} \tag{2}$$

and let

$$\mathscr{R} = \{(x,y)\ :\ 0 \le x \le 1, -3 \le y \le 3\}.$$

We aim to calculate the volume of the region shown in yellow in Figure 29 below. We will complete the calculation in Example 5.2.

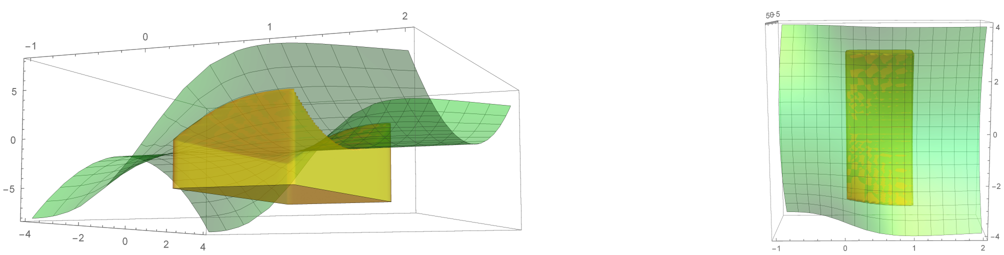


Figure 29: The surface $z = f(x,y)$ given by (2) and the region $\mathscr{R}$.

This volume is given by the double integral

$$V = \iint\limits_{\mathscr{R}} f(x,y)\,dA,$$

Since $f(x,y)$ is continuous in $\mathscr{R}$, we can use Fubini's theorem below to calculate the volume by integrating twice - two definite integrals treated partially, as in perhaps $\int_c^d \int_a^b f(x,y)\partial x \partial y$ is arguably better notation.

Theorem 11 *[Fubini] If f is continuous on the rectangle*

$$R = \{(x,y)\ :\ a \le x \le b,\ c \le y \le d\},$$

then

$$\iint\limits_{D} f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Example 5.2 *Calculate the volume V shown in Figure 29 in two ways using Fubini's theorem.*

At this point we have a very simple integral to complete. Finishing the problem requires that we let $u = x^2 + 1$ and complete the integral by substitution.

According to Theorem 11, we can also calculate the integral first with respect to x.

Theorem 12 *If $f(x,y)$ is continuous on the rectangle*

$$R = \{(x,y)\ :\ a \le x \le b,\ c \le y \le d\},$$

and there exist function g,h such that $f(x,y) = g(x)h(y)$, then

$$\iint_{\mathscr{R}} f(x,y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy.$$

To see this, Fubini's theorem shows that

$$\begin{aligned}\iint_{\mathscr{R}} f(x,y)\,dA &= \int_c^d \int_a^b g(x)h(y)\,dx\,dy,\\ &= \int_c^d h(y)\left(\int_a^b g(x)\,dx\right)dy,\\ &= \left(\int_a^b g(x)\,dx\right)\int_c^d h(y)\,dy.\end{aligned}$$

Example 5.3 *Let* $f(x,y) = x^2 \sin(y)$ *and*

$$\mathscr{R} = \left\{ (x,y) \ : \ 0 \le x \le 1, \ 0 \le y \le \frac{\pi}{2} \right\}.$$

Calculate $\iint_{\mathscr{R}} f(x,y)\, dA$.

5.1 Volume integrals over non-rectangular regions

Consider the two regions of the x,y plane shown in Figure 30.

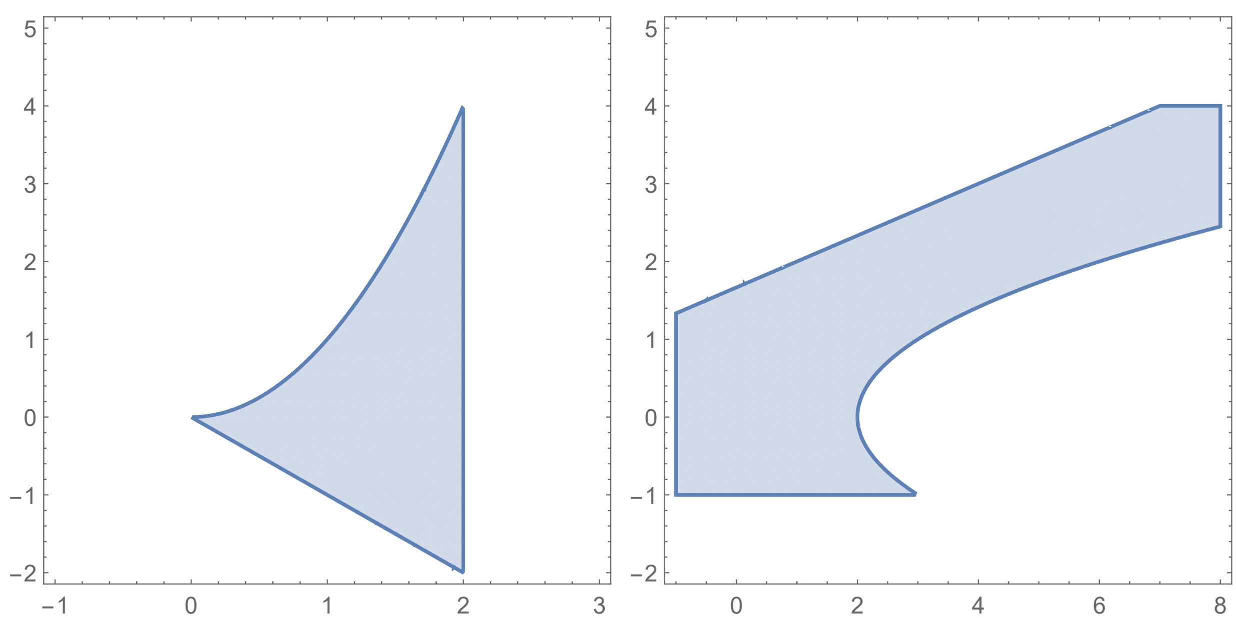


Figure 30: Two regions R_1 and R_2 of different types.

On the left of Figure 30 we have a region in which x is between two constants and y is between two functions of x,

$$R_1 = \{(x,y) \ : \ 0 \leq x \leq 2, -x \leq y \leq x^2\}.$$

On the right of Figure 30 we have a region in which y is between two constants and x is between two functions of y. The goal of this section is to be able to calculate the volume under the surface $z = f(x,y)$ and bounded by such a region. These regions are examples of non-rectangular regions and require more care when we need to swap the order of integration.

Remark 5.1 *If the region over which we integrate $f(x,y)$ is of the form*

$$R_1 = \{(x,y) \ : \ a \leq x \leq b, g(x) \leq y \leq h(x)\},$$

then put dy on the left in the volume calculation:

$$V = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\, dy\, dx.$$

If the region over which we integrate $f(x,y)$ is of the form

$$R_2 = \{(x,y) \ : \ g(y) \leq x \leq h(y), c \leq y \leq d\},$$

then put dx on the left in the volume calculation:

$$V = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\, dx\, dy.$$

These are like what we see in Figure 30 respectively.

Consider the following examples.

Example 5.4 *Calculate the volume V bounded by the surface*

$$z = f(x,y) = xy,$$

the x,y plane, and the triangular region R shown in Figure 31 between $x=1$, $y=1$, and $y=-\frac{1}{3}x+\frac{7}{3}$.

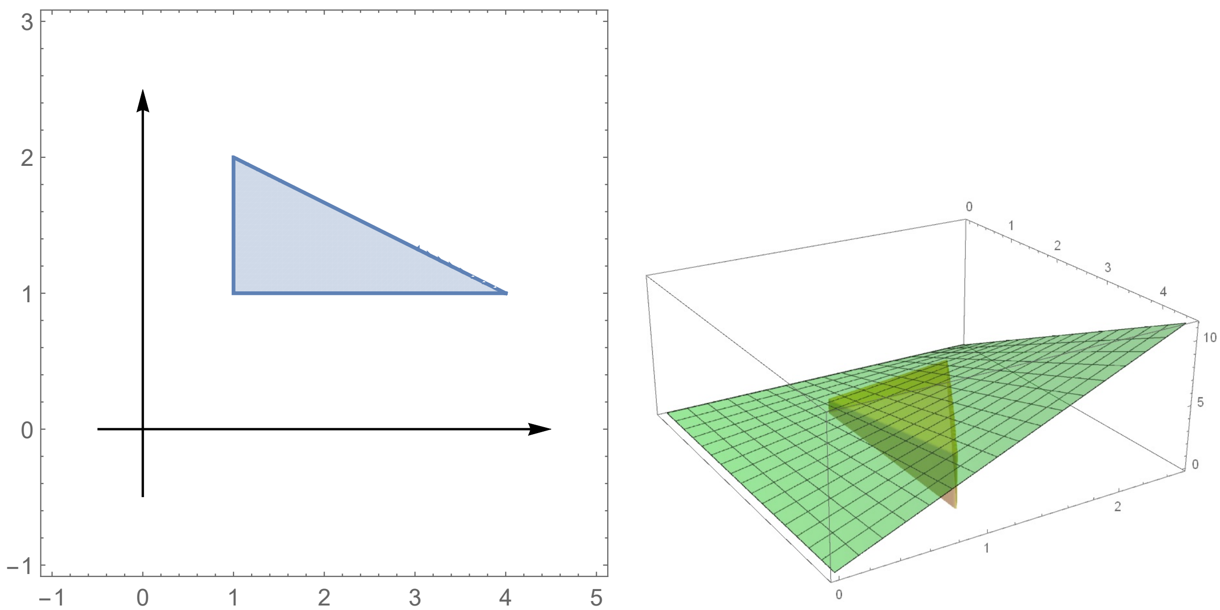


Figure 31: The region R in the x,y plane.

We can express the region R in two ways as

$$\begin{aligned} R &= \left\{(x,y) \;:\; 1 \le x \le 4, 1 \le y \le -\frac{1}{3}x+\frac{7}{3}\right\}, \\ &= \{(x,y) \;:\; 1 \le x \le 7-3y, 1 \le y \le 2\}. \end{aligned}$$

To calculate the volume V, we have:

Reversing the order of the integrals,

Example 5.5 *Let*

$$R = \{(x,y) \ : \ 0 \le x \le 2, -x \le y \le x^2\}$$

be the region shown on the left of Figure 30. Calculate the volume bounded by the surface $z = f(x,y) = x^2 + y^2$, the x,y plane, and the region R.

To swap the order of integration, we must break the region into a disjoint union:

$$R = \{(x,y) \ : \ \sqrt{y} \le x \le 2, 0 \le y \le 4\} \cup \{(x,y) \ : \ -y \le x \le 2, -2 \le y \le 0\}.$$

See Figure 32.

We have

.

$$\begin{aligned} V &= \int_0^4 \int_{\sqrt{y}}^2 x^2 + y^2 \, dx \, dy + \int_{-2}^0 \int_{-y}^2 x^2 + y^2 \, dx \, dy, \\ &= \end{aligned}$$

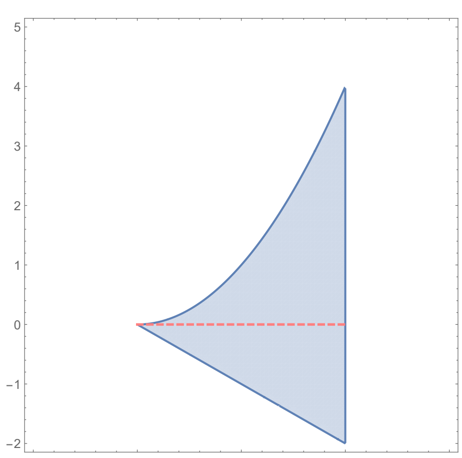


Figure 32: The region R in the x,y plane.

5.2 Change of variables and regions in polar coordinates

Occasionally it is easier to calculate $\iint_R f(x,y)\,dA$ using coordinates other that x and y since the shape of the region R may be better suited to polar coordinates for example.

Theorem 13 *Let $x = x(u,v)$ and $y = y(u,v)$ be functions of u and v, $\mathbf{F} = (x,y)$ a vector, and let*

$$J_{\mathbf{F}} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

be the Jacobian matrix. Then

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f\left(x(u,v), y(u,v)\right) \det\left(J_{\mathbf{F}}\right) du\,dv, \tag{3}$$

where R^* is the region corresponding to R in the new frame of reference u,v.

It follows that:

Corollary 5.1 *If the rectangular coordinates (x,y) correspond to the polar coordinates (r,θ), then*

$$\begin{aligned} x &= r\cos(\theta), \\ y &= r\sin(\theta), \end{aligned}$$

and

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f\left(r\cos(\theta), r\sin(\theta)\right) r\,dr\,d\theta. \tag{4}$$

To see that this follows from Theorem 13,

$$\det\left(J_{\mathbf{F}}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r$$

since $\cos^2(\theta) + \sin^2(\theta) = 1$.

Example 5.6 *Calculate the volume of the quarter cylinder of height* 1 *and radius* 1 *in multiple ways.*

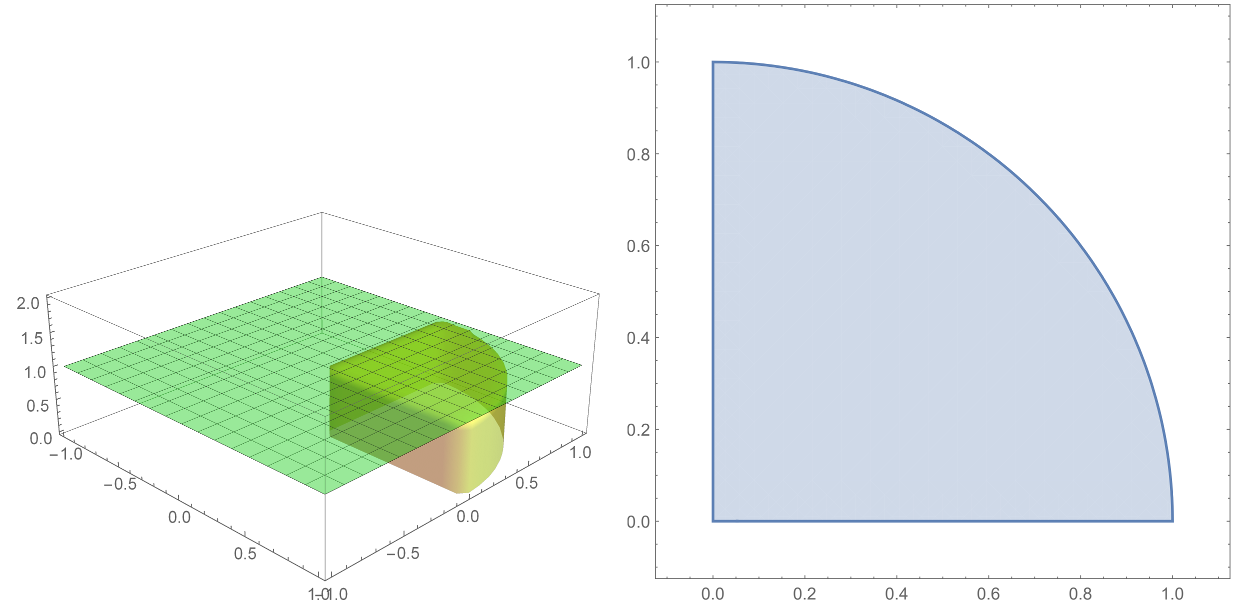


Figure 33: Left: The quarter cylinder of height 1 and radius 1 and the surface $z = 1$. Right: The region R in the x, y plane.

Example 5.7 *Consider the region shown in Figure 34. With the change of variables $u = x+y$, $v = x-y$, calculate $\iint_R x^2+y^2\,dA$ where R is the region shaded in Figure 34.*

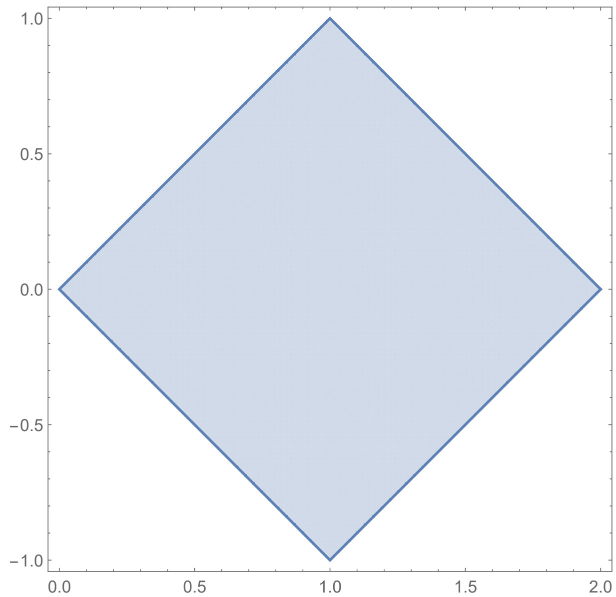


Figure 34: The region R in the x,y plane.

The boundaries of the region R are $y = -x$ so $u = 0$, $y = -x+2$ so $u = 2$, $y = x$ so $v = 0$, and $y = x-2$ so $v = 2$. Solving the change of variable equations for x and y, we have

$$\begin{aligned} x(u,v) &= \\ y(u,v) &= \end{aligned}$$

Taking the determinant of the Jacobian matrix,

$$\begin{aligned} \det\left(J_{\mathbf{F}}\right) &= \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right|, \\ &= \end{aligned}$$

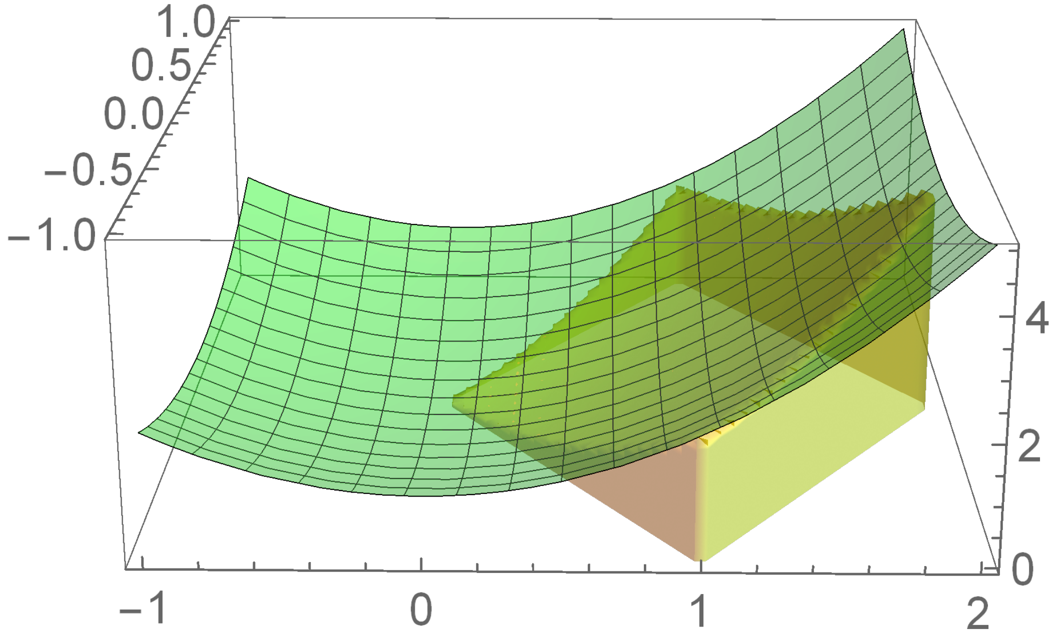


Figure 35: The region R in the x, y plane below the surface $z = x^2 + y^2$.

$$\iint_R x^2 + y^2 \, dA \;=\; \qquad\qquad\qquad\qquad\qquad\qquad .$$

5.3 Triple integrals

We will consider two examples and then some important results on triple integrals.

Example 5.8 *Let*

$$\mathscr{B} = \{(x,y,z)\ :\ 0 \leq x \leq 2, -2 \leq y \leq 3, 0 \leq z \leq 1\}.$$

See Figure 36.

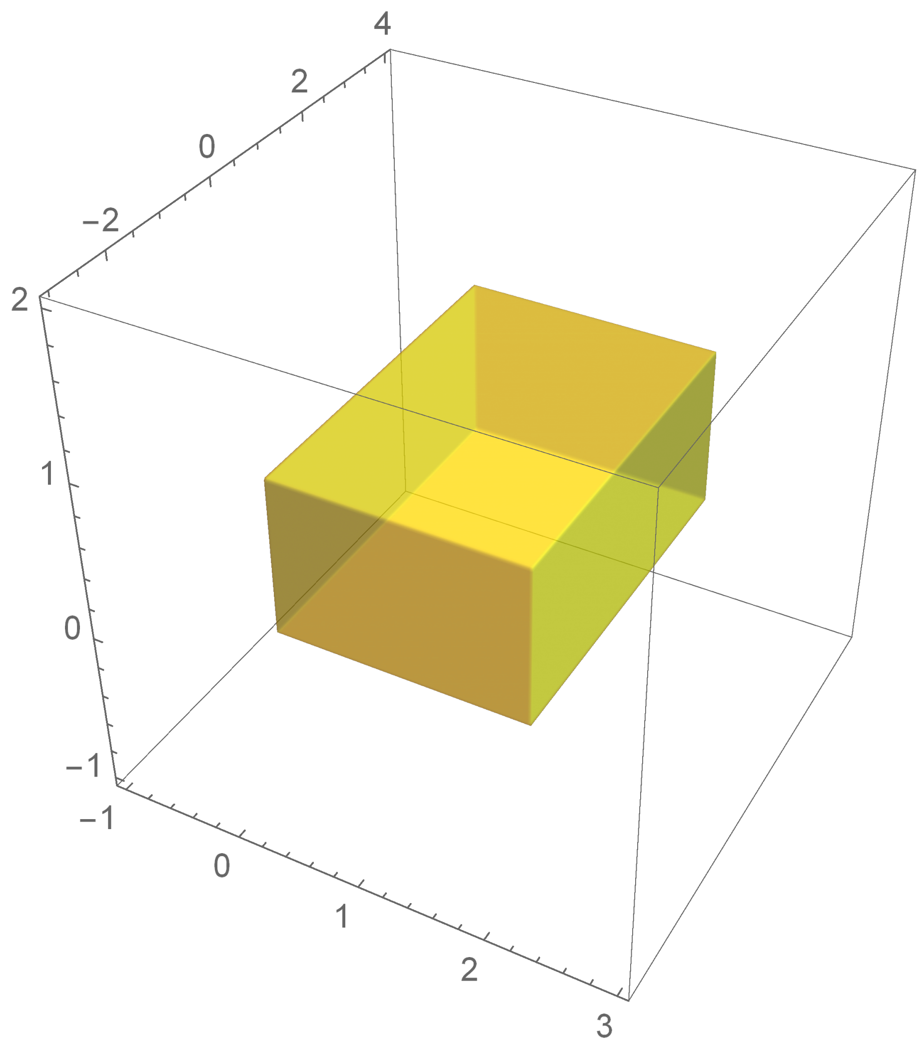


Figure 36: The region $\mathscr{B}$ in $\mathbb{R}^3$.

Calculate $\iiint_{\mathscr{B}} x^3 y z^2 \, dV$.

$$\iiint_{\mathscr{B}} x^3 y z^2 \, dV \quad = \qquad\qquad\qquad\qquad\qquad\qquad .$$

Example 5.9 *Let*

$$\mathscr{B} = \{(x,y,z) \ : \ 0 \leq x \leq 3, 0 \leq y \leq x, 0 \leq z \leq x-y\}.$$

See Figure 37. Calculate $\iiint_{\mathscr{B}} 4xy\, dV$.

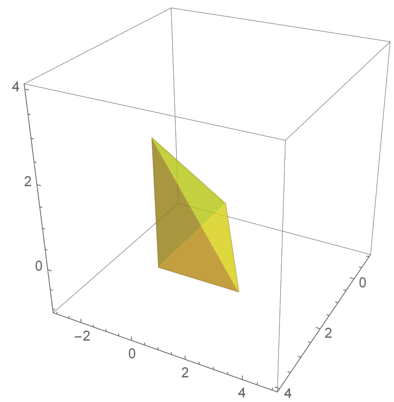


Figure 37: The region $\mathscr{B}$ in $\mathbb{R}^3$.

$$\iiint\limits_{\mathscr{B}} 4xy\, dV \quad = \qquad\qquad\qquad\qquad\qquad\qquad .$$

5.4 Flux and Gauss' divergence theorem

To introduce Gauss' divergence theorem, we first introduce the concept of flux. Flux quantifies a vector field passing through a curve or surface. In the plane, the net outward flux of the vector field $\mathbf{F}$ is given by the line integral

$$\text{Flux} = \int_C \mathbf{F} \cdot \mathbf{n} \, dS,$$

where $\mathbf{n}$ is normal to the curve C pointing perpendicular from the curve or surface. See Figure 38. **First go through Section 6.1 if needed.**

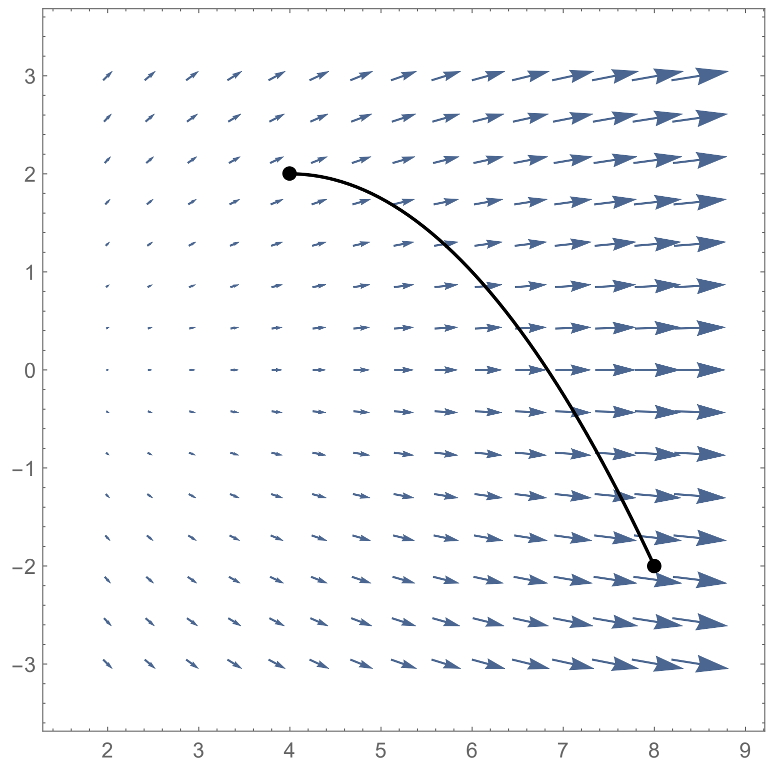


Figure 38: Flux measures the 'density' of the vector field passing through a curve or surface. In this case, C is the curve shown in black.

Example 5.10 *Calculate the net outward flux of the vector field* $\mathbf{F} = \left(x+y, -x^2-y^2\right)$ *across the line* C *from* $(1,0)$ *to* $(0,1)$.

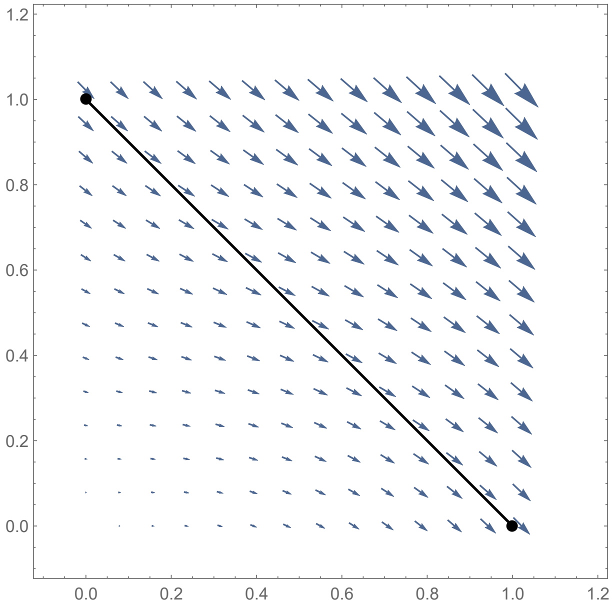


Figure 39: Flux of $\mathbf{F}$ through the line segment from $(1,0)$ to $(0,1)$.

C *is the line* $y = -x+1$. *Parametrizing this line,*

$$C : \{(1-t, t)\ :\ t \in \mathbb{R}, 0 \le t \le 1\}.$$

We let $\mathbf{r}(t) = (1-t, t)$ *so* $\mathbf{r}'(t) = (-1, 1)$. *To calculate* $\mathbf{n}$, *we use the unit tangent vector* $\mathbf{T}$ *to* C *given by* $\mathbf{T} = \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t) = \frac{1}{\sqrt{2}}(-1,1)$ *and take* $\mathbf{n}$ *to be the cross product* $\mathbf{n} = \mathbf{T} \times \hat{\mathbf{k}} = \frac{1}{\sqrt{2}}(1,1)$. *It follows that:*

$$\begin{aligned} Flux &= \int_C \mathbf{F} \cdot \mathbf{n} \, dS, \qquad\qquad\qquad\qquad . \\ &= \end{aligned}$$

Theorem 14 (Gauss' divergence thm) *Assume* $\mathbf{F}(x,y,z)$ *is a vector field that is continuously differentiable in the closed region* $V \subseteq \mathbb{R}^3$. *Let* $\mathbf{n}$ *be a unit normal vector to the surface* S *that is the boundary of* V. *Then the flux of* $\mathbf{F}$ *through the surface* S *is*

$$Flux = \oiint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_V div(\mathbf{F}) \, dV.$$

If there is net flow out of the closed surface S, then integral Flux > 0. If there is net flow into the closed surface, then Flux < 0.

Example 5.11 *Use Gauss' divergence theorem to calculate the flux of the vector field* $\mathbf{F} = (3x, 4y, 5z)$ *over the surface* S *of a sphere of radius* 2 *and centre* $(0,0,0)$. *See Figure 40.*

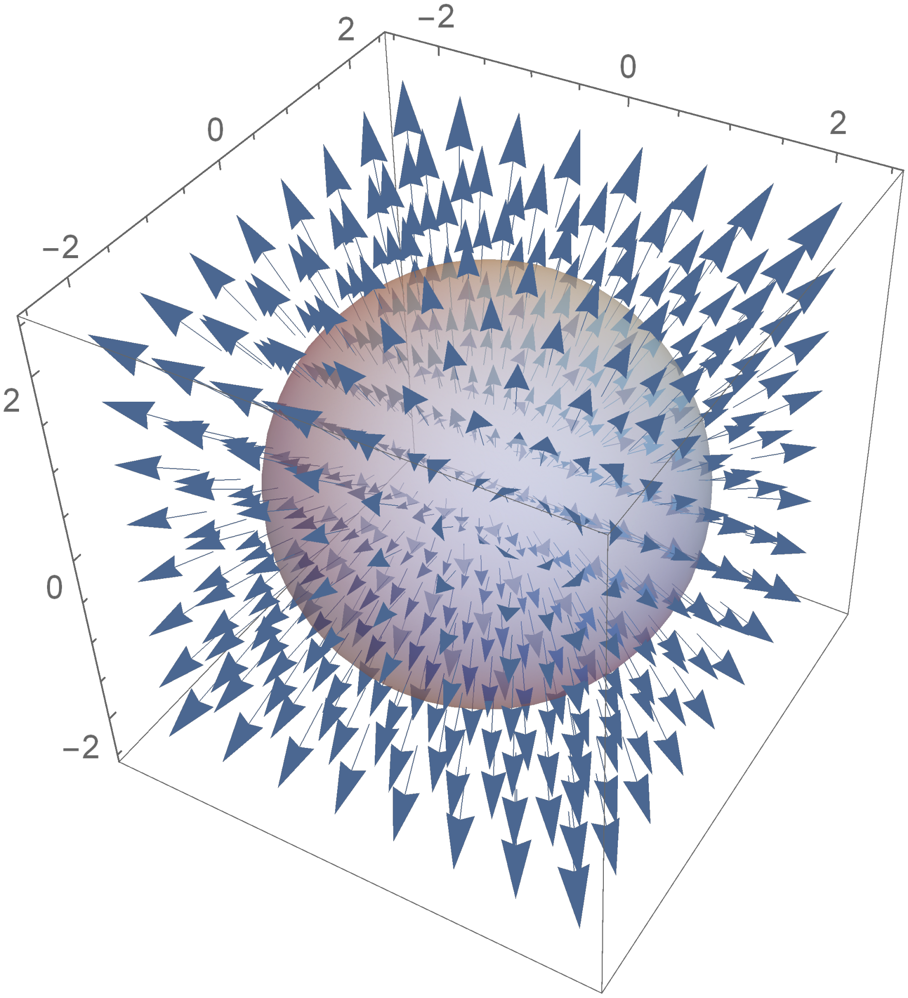


Figure 40: Flux of **F** through the sphere of radius 2.

In spherical coordinates, the inside of the sphere of radius 2 *is*

$$V = \{(r, \theta, \phi) \; : \; 0 \leq r \leq 2, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}.$$

By Gauss' divergence theorem,

$$\begin{aligned} Flux &= \oiint_S \mathbf{F} \cdot \mathbf{n} \, dS, \qquad . \\ &= \end{aligned}$$

5.5 Appendix on spherical coordinates

Let

$$\begin{aligned} x &= r\sin(\phi)\cos(\theta), \\ y &= r\sin(\phi)\sin(\theta), \\ z &= r\cos(\phi). \end{aligned}$$

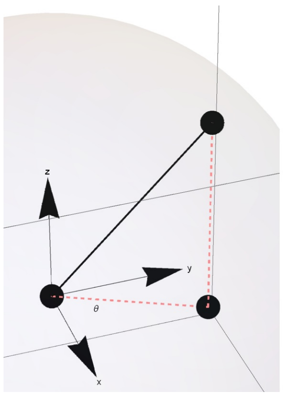


Figure 41: Spherical coordinates.

We can think of θ as the longitude, ϕ as the co-latitude, and r as the distance to the origin.

The Jacobian matrix is

$$\begin{aligned} J_{(x,y,z)} &= \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}, \\ &= \begin{pmatrix} \cos(\theta)\sin(\phi) & r\cos(\theta)\cos(\phi) & -r\sin(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) & r\cos(\phi)\sin(\theta) & r\cos(\theta)\sin(\phi) \\ \cos(\phi) & -r\sin(\phi) & 0 \end{pmatrix}. \end{aligned}$$

The determinant is

$$\begin{aligned} \det\left(J_{(x,y,z)}\right) &= r^2\sin(\phi)\left(\sin^2(\theta)+\cos^2(\theta)\right)\left(\sin^2(\phi)+\cos^2(\phi)\right), \\ &= r^2\sin(\phi). \end{aligned}$$

Consider the scalar equation of the line $\mathscr{L} : y = \frac{1}{2}x + 3$. See Figure 42. The line has slope $\frac{1}{2}$, so we can say that the vector $\mathbf{v} = \left(1, \frac{1}{2}\right) = \frac{1}{2}(2,1)$ is parallel to the line. To write a parametric version of the line $\mathscr{L}$, require a point that is on the line, say, $P = (-4,1)$. Now, using any vector that is parallel to the line $\mathscr{L}$, and forming the vector $\mathbf{u} = \overrightarrow{OP} = (-4,1)$, we can write the vector equation of the line $\mathscr{L}$ as follows:

$$\mathbf{r} = (-4,1) + (2,1)t,\ t \in \mathbb{R}.$$

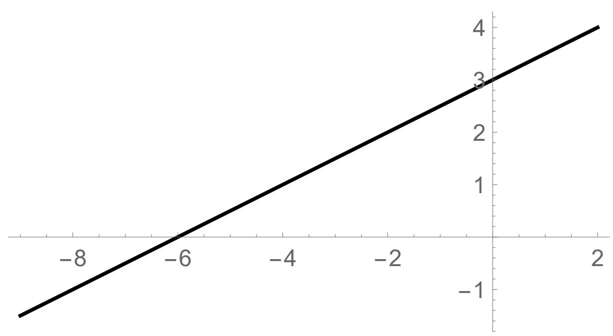


Figure 42: The line $y = \frac{1}{2}x + 3$

With the understanding that the vector $\mathbf{r} = \overrightarrow{OR}$, where $R = (x,y)$ is a point on the line $\mathscr{L}$, we now have an equation which describes all points (x,y) on the line $\mathscr{L}$ using just one letter t. That is,

$$\begin{aligned} (x,y) &= (-4,1) + (2,1)t, \\ &= (2t-4, t+1). \end{aligned}$$

If we choose $t = 0$, then we get the point $(-4,1)$. If we choose $t = 1$, then we get the point $(-2,2)$.

Conversely, if we have the parametrization $(x,y) = (2t-4, t+1)$, and we wish to find the scalar equation of the corresponding line, then we can write

$$\begin{aligned} x &= 2t-4, \\ y &= t+1, \end{aligned}$$

and eliminate t. To do so, we have $t = y-1$ from the second equation. Substituting this into the first equation, $x = 2(y-1)-4 = 2y-6$. Rearranging gives $y = \frac{1}{2}x+3$.

Parametizations are useful in calculating line integrals. However, they can also be of use in the study of integers. Perhaps a well known parametrization of the unit circle $x^2+y^2 = 1$ is $(x,y) = (\cos(t), \sin(t))$. Another parametrization of the unit circle is

$$(x,y) = \left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}\right).$$

The latter parametrization allows use to choose any rational number t and obtain a rational point on the unit circle and hence obtain infinitely many Pythagorean triples. For example, if $t = 2$, then we get the point $\left(\frac{3}{5}, \frac{4}{5}\right)$, which give the Pythagorean triple $(3,4,5)$: $3^2+4^2 = 5^2$. If we choose $t = \frac{3}{2}$, then we get the rational point $\left(\frac{5}{13}, \frac{12}{13}\right)$, which gives the Pythagorean triple $(3,4,5)$: $5^2+12^2 = 13^2$.

6.1 Parametrization of curves

Next we will illustrate how to find a parametrization of a curve.

Example 6.1 *Find a parametrization of the curve $y = x^2$.*

This is easy. We have $(x,y) = \left(x,x^2\right) = \left(t,t^2\right)$, where $t = x$.

Example 6.2 *Consider the parametrization* $(x,y) = \left(t+2, t^2\right)$. *Find the scalar equation of the curve, and then sketch curved arrows showing* $-2 \leq t \leq 2$.

From the equation for x *we have* $t = x - 2$. *Substituting this into the equation for* y *gives* $y = (x-2)^2$. *When* $t = -2$, *we get the point* $(0,4)$. *When* $t = 0$ *we obtain the point* $(2,0)$. *When* $t = 2$ *we have the point* $(4,4)$. *Thus the arrows show* x *increasing with increasing* t. *See Figure 43.*

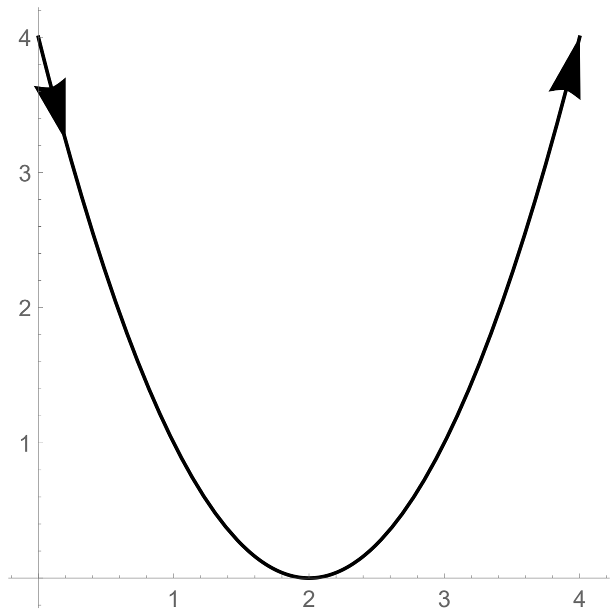


Figure 43: The curve $(x,y) = \left(t+2, t^2\right)$, $-2 \leq t \leq 2$ showing the movement of the point (x,y) as t increases.

The cycloid

$$(x,y) = (r(t - \sin(t)), r(1 - \cos(t))),$$

see Figure 44, is best represented with parametric equations.

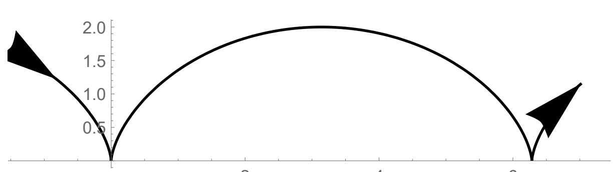


Figure 44: The cycloid with $r = 1$ and $-2 \leq t \leq 8$.

6.2 Parametrization of 3 dimensional objects

To parametrize a surface, we represent the point (x,y,z) on the surface using only two letters. For example, to parametrize the unit sphere $x^2+y^2+z^2=1$, we have

$$(x,y,z) = (\sin(t)\cos(u), \sin(t)\sin(u), \cos(t)).$$

The surface $(x,y,z) = \left(\cos(t)+u, t^2u, t-u\right)$ with $-2 \le t,u \le 2$ is shown in Figure 45.

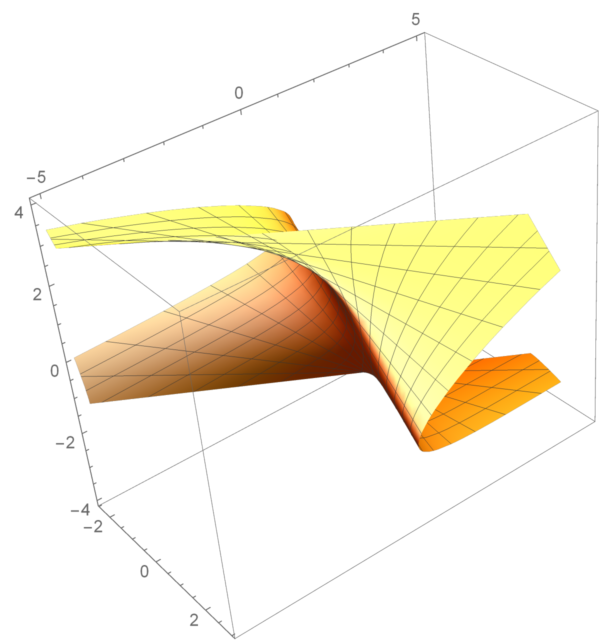


Figure 45: The surface $(x,y,z) = \left(\cos(t)+u, t^2u, t-u\right)$ showing $-2 \le t,u \le 2$.

The following parametric equation represents a curve in 3 dimensions. Notice that we require only one letter to parametrize the curve.

$$(x,y,z) = \left(\sin(t), \cos(t), \frac{1}{10}t\right).$$

This helix can be described by the equations

$$\begin{aligned} x^2+y^2 &= 1, \\ x &= \sin(10z). \end{aligned}$$

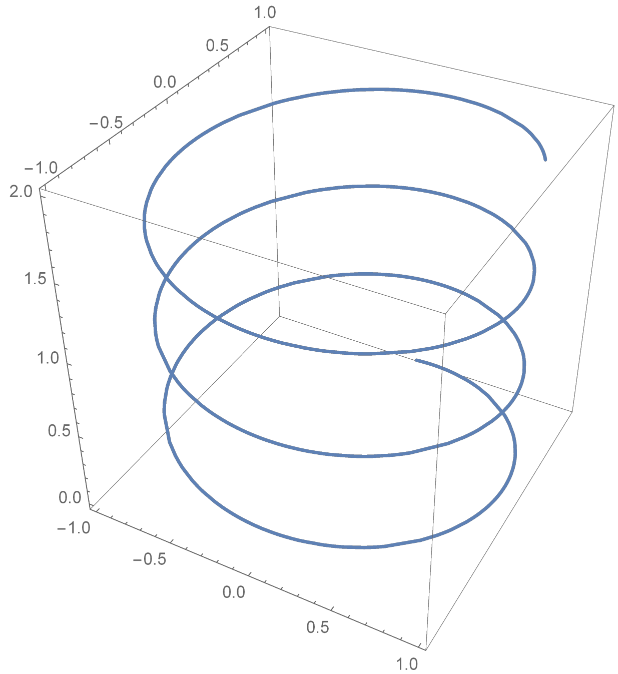


Figure 46: The curve $(x, y, z) = \left(\sin(t), \cos(u), \frac{1}{10}t\right)$ showing $0 \leq t \leq 20$.

6.3 Derivatives of parametrized curves

Example 6.3 *Calculate $\frac{dy}{dx}$ for the parametric equations of a curve given by*

$$x = 8 + t^2, \qquad\qquad y = 4t^2 - 5t^4.$$

6.4 Line Integrals

If $f(x,y)$ is defined on a smooth curve C in $\mathbb{R}^2$, the line integral of f along C is defined

$$\int_C f(x,y)\,dS.$$

Geometrically this represents the area under the surface $z = f(x,y)$ along the curve C. See Figure 47. In other words, the surface area of the surface that follows the path C in the x,y under the surface $z = f(x,y)$.

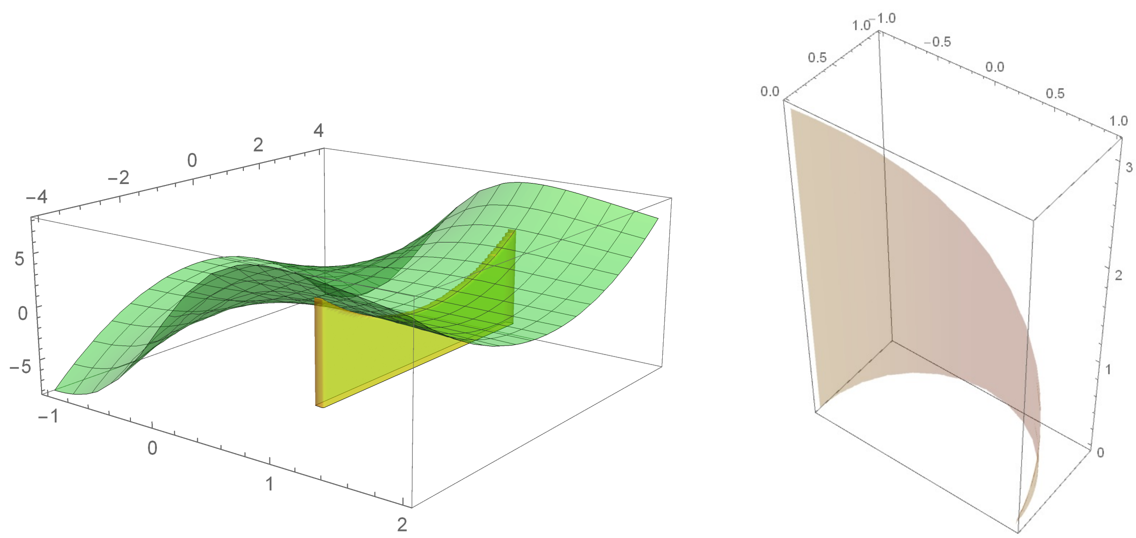


Figure 47: Two examples of surfaces that follow a path C in the x,y plane under the surface $z = f(x,y)$. On the left we have $C: x = 1$. On the the right C is a circular path.

Line integrals have many applications, including the calculate on the arc length ℓ of a curve C, where this is given by the line integral

$$\ell = \int_C 1\,dS.$$

It is easy to calculate this arc length when we first parametrize the curve C.

Let $\mathbf{r}(t)$ be a parametrization of the curve C using parameter t. Consider Figure 48. The vector $\mathbf{r}'(t)$ is a vector tangent to the curve C at t. We substitute

$$dS \longrightarrow \|\mathbf{r}'(t)\|\,dt.$$

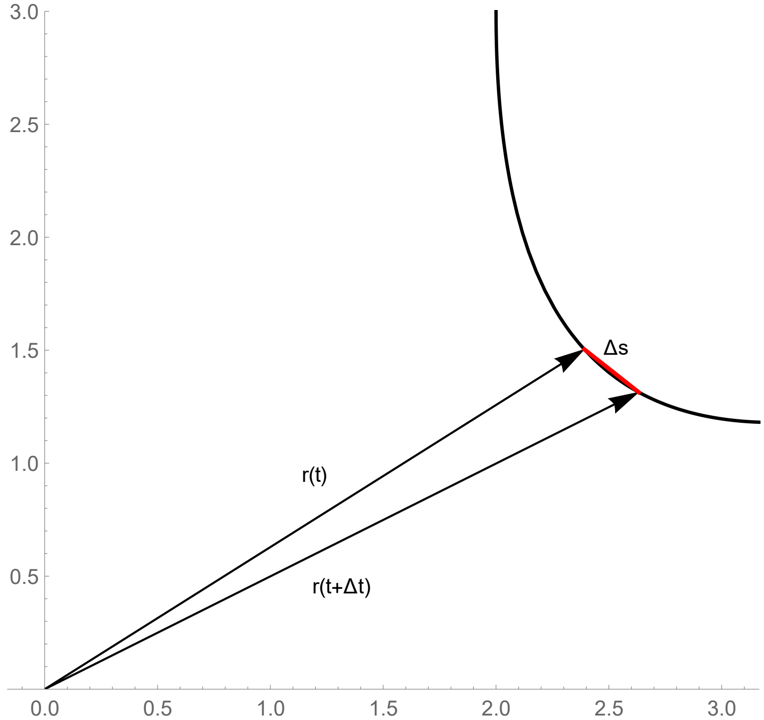


Figure 48: Consideration of ΔS on a parametrized curve C helps to think about substitution of dS in the line integral $\int_C f(x,y)\,dS$

Theorem 15 *Let $\mathbf{r}(t)$ be a parametrization of the curve C using parameter t. The arc length ℓ of the curve C from $t = a$ to $t = b$ is given by*

$$\ell = \int_C 1\,dS = \int_a^b \|\mathbf{r}'(t)\|\,dt.$$

Example 6.4 *Calculate the circumference of the circle $C : x^2 + y^2 = 1$ using the line integral $\int_C 1\, dS$. See Figure 49.*

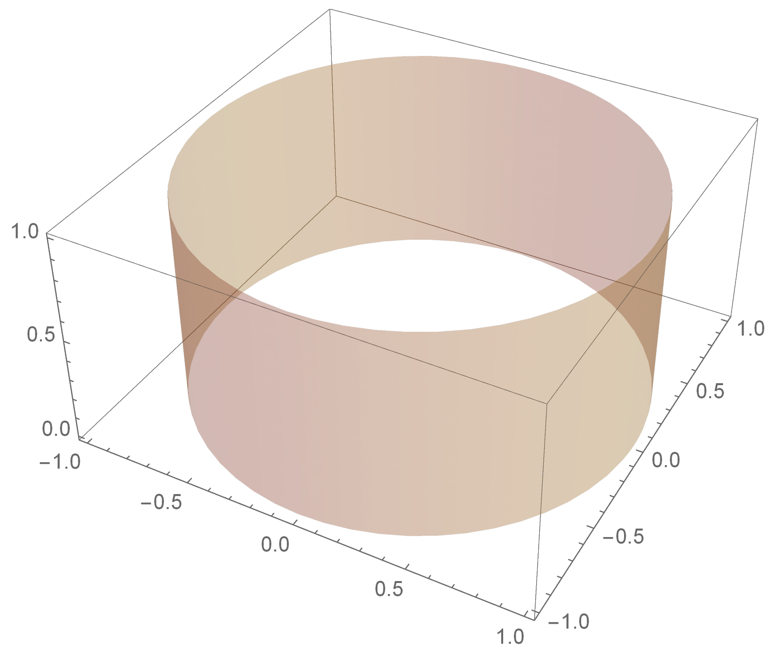


Figure 49: The line integral of $zf(x, y) = 1$ along $C : x^2 + y^2 = 1$ gives the arc length 2π of the unit circle because the bounding surface has height 1.

The unit circle has the following parametrization:

$$C = \{\mathbf{r}(t) = (\cos(t), \sin(t)) \ : \ t \in \mathbb{R}, 0 \le t \le 2\pi\}.$$

Taking the first derivative,

Theorem 16 (Fundamental theorem of line integrals) *If*

$$f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$$

is differentiable on the domain U and the curve C starting at point A and ending at point B is continuous on U, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

Example 6.5 *Consider the vector field* $\mathbf{F} = (x+z, z, x+y)$ *defined on* $\mathbb{R}^3$.

1. *Determine whether* $\mathbf{F}$ *is conservative.*
2. *If so, calculate* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *along path* $C : \mathbf{r}(t) = \left(t, t^2, t^3\right)$, $0 \leq t \leq 1$ *via* $\frac{d\mathbf{r}}{dt}$.
3. *Find a potential function* $f : \nabla f = \mathbf{F}$.
4. *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *again using the fundamental theorem of line integrals.*

Example 6.6 *Evaluate the line integral* $\int_C 3x^2y^2\,dx + 2x^3y\,dy$*, where* C *is the path* $y = x^3 - 3x^2 + 2x$ *from the point* $(0,0)$ *to the point* $(1,0)$.

Example 6.7 *Without using Green's theorem, calculate the work done by the force field* $\mathbf{F} = \left(y - xy, x^2\right)$ *in moving an object counterclockwise around a rectangle with vertices* $(0,0)$*,* $(2,0)$*,* $(2,1)$*,* $(0,1)$*. Compare the result with Example 4.11.*

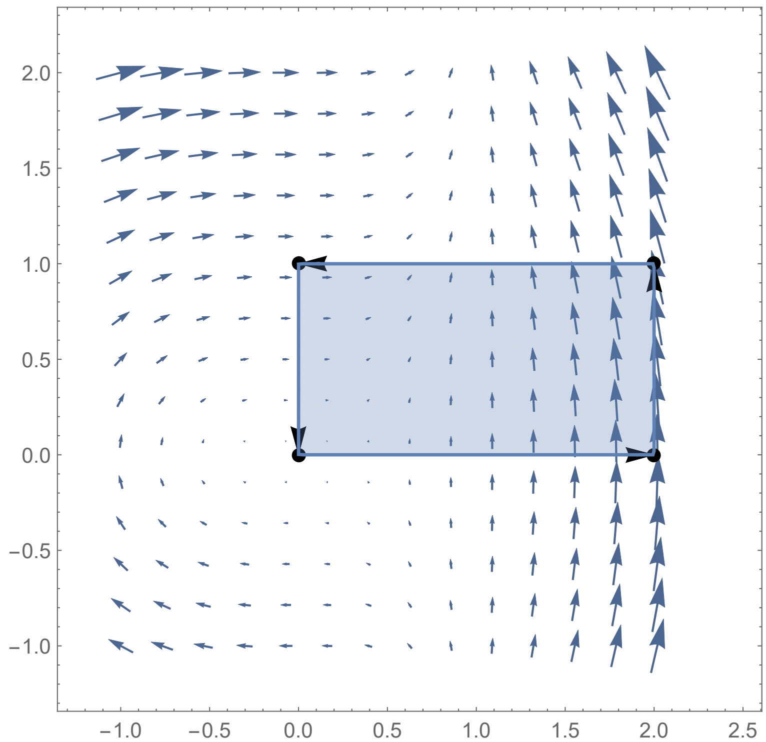


Figure 50: Calculate work done along the counter-clockwise path shown.

We parametrize each of the four paths:

$$
\begin{aligned}
&C_1 : (0,0) \longrightarrow (2,0), && (x,y) = (t,0), && 0 \le t \le 2,\\
&C_2 : (2,0) \longrightarrow (2,1), && (x,y) = (2,t-2), && 2 \le t \le 3,\\
&C_3 : (2,1) \longrightarrow (0,1), && (x,y) = (5-t,1), && 3 \le t \le 5,\\
&C_4 : (0,1) \longrightarrow (0,0), && (x,y) = (0,6-t), && 5 \le t \le 6.
\end{aligned}
$$

For each of these paths, we calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ *and add the result.*

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \qquad .$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \qquad .$$

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \qquad .$$

$$\int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \qquad .$$

It follows that the work done by the force field $\mathbf{F} = \left(y - xy, x^2\right)$ *in moving an object counterclockwise around a rectangle with vertices* $(0,0)$, $(2,0)$, $(2,1)$, $(0,1)$ *is equal to*

$$\begin{aligned} W &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r}, \\ &= . \end{aligned}$$

in agreement with Example 4.11, which was much easier.

6.6 Exam Practice Exercises

Hopefully by now you are able to compute a partial derivative, calculate grad, div, and curl, compute a simple double integral, and parametrize a curve. If you require more practice on those topics, then please review the tutorial problems. A sample of more advanced integration exercises are listed below. Expect between 20 and 30 percent of your exam questions to be similar to those found below.

(1) Calculate $\iint_D DA$, where D is the region in the x, y plane bounded by $y = 2 - x$ and $y = x^2$.

(2) Find the volume under $z = e^{-x^2}$ and above and bounded by the triangle in the x, y plane with vertices $(0,0)$, $(2,0)$, and $(2,2)$.

(3) Evaluate $\iint_D x^2 + y^2\, dA$, where D is the region in the x, y plane bounded by a circle of radius 1, centred at point $(1,2)$. Use polar coordinates.

(4) Evaluate $\iiint_D 6xy\, dV$, where D lies under the plane $z = 1 + x + y$ and above the region in the x, y plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$.

(5) Evaluate $\int_C x\, dx + xy\, dy$, where C is the line $y = 1 - x$, $0 \le x \le 1$.

(a) By using $\mathbf{r}(t) = \left(t^2, 1 - t^2\right)$, $0 \le t \le 1$.

(b) By using $\mathbf{r}(t) = (\sin(t), 1 - \sin(t))$, $0 \le t \le \frac{\pi}{2}$.

(6) Evaluate $\int_C 3x^2y^2\, dx + 2x^3y\, dy$, where $C\ :\ y = x^3 - 3x^2 + 2x$ from $(0,0)$ to $(1,0)$. Use the fundamental theorem for line integrals if possible.

(7) Use Green's theorem to evaluate the line integral

$$\int_C xe^{-2x}\,dx + \left(x^4 + 2x^2y^2\right)\,dy,$$

where C is the positively oriented curve defined by the boundary of the region between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

(8) Use Green's theorem to calculate the net outward flux of the vector

$$\left(x+y, -x^2-y^2\right)$$

across the boundary of the triangle with vertices $(1,0)$, $(0,1)$, and $(-1,0)$.

(9) Find the flux of the vector field

$$\mathbf{F} = \left(x^3+xy^2+xz^2, x^2y+y^3+yz^2, x^2z+y^2z+z^3\right)$$

across the surface of a sphere of radius a, centred at the origin.

(10) Show that the vector field $\mathbf{F} = \left(y^2z^3, 2xyz^3, 3xy^2z^2\right)$ is conservative and find $f(x,y,z)$ such that $\mathbf{F} = \nabla f$.

(11) Find a vector function that represents the elliptic paraboloid $z = x^2+2y^2$.

(12) Describe the curve whose vector equation is

$$\mathbf{r}(t) = (1+t, 2+5t, -1+6t).$$

(13) Sketch the curve with vector equation

$$\mathbf{r}(t) = (\cos(t), \sin(t), t).$$

(14) Evaluate $\int_C \left(2+x^2y\right)\, dS$, where C is the upper half of the unit circle $x^2+y^2=1$.

(15) Evaluate $\int_C 2x\, dS$, where C consists of the arc C_1 of the parabola $y=x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

(16) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = (xy, yz, xz)$ and $C\ :\ x=t, y=t^2, z=t^3$, $0 \le t \le 1$.

(17) Find the work done by the force field $\mathbf{F}(x,y) = \left(x^2, -xy\right)$ in moving a particle along the quarter circle $\mathbf{r}(t) = (\cos(t), \sin(t))$, $0 \le t \le \frac{\pi}{2}$.

6.7 A Calculus Exam

This is intended to be completed in 90 Minutes

This is an open-book, open-notes, and open-tutorial-solutions exam.

Full working must be shown on the pages provided.

Permitted materials: A pocket calculator or graphics calculator.

Mobile phones and laptops are not permitted. Please switch phones off.

Name:

1. Calculate the partial derivatives: $f_x(x,y)$ and $f_y(x,y)$ for the function

$$f(x,y) = log_e\left(x+y+\sqrt{x^2+y^2}\right),$$

where $e \approx 2.71828$ is the base of the natural logarithm. **(10 Marks)**

2. Let

$$f(x,y) = log_e\left(x+y+\sqrt{x^2+y^2}\right).$$

(a) Calculate the slope of $f(x,y)$ at the point $(1,0)$ in the direction of the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. **(5 Marks)**

(b) What is the maximum slope of $f(x,y)$ at the point $(1,0)$?
. **(5 Marks)**

3. Calculate the integral $\int_0^1 \int_{x^2}^{x} (1+2y)\, dy\, dx$. **(10 Marks)**

4. Consider the vector field $\mathbf{F}(x,y) = \left(1 + \frac{x}{\sqrt{x^2+y^2}}, 1 + \frac{y}{\sqrt{x^2+y^2}}\right)$.

(a) Show that $\mathbf{F}$ is conservative. **(5 Marks)**

(b) Find a potential function $f(x,y)$ such that $\mathbf{F} = \nabla f$. **(5 Marks)**

(c) Calculate the work done in moving a particle from the point $(1,0)$ to the point $(2,0)$ along the path C, where C is the line connecting $(1,0)$ and $(2,0)$ by parametrizing C.

. **(5 Marks)**

(d) Calculate the work done in moving a particle from the point $(1,0)$ to the point $(2,0)$ using the fundamental theorem for line integrals.

. **(5 Marks)**

5. Consider the vector field $\mathbf{F} = \left(2x, 2y, z^2\right)$.

(a) Calculate $\operatorname{div}(\mathbf{F})$. **(5 Marks)**

(b) Calculate $\operatorname{curl}(\mathbf{F})$. **(5 Marks)**

(c) Use the divergence theorem to find the flux of the vector field $\mathbf{F}$ over the unit sphere $x^2 + y^2 + z^2 = 1$. You may use the identity $\sin(2x) = 2\sin(x)\cos(x)$.

. **(5 Marks)**

6. Consider the vector field $\mathbf{F} = (P, Q) = (y, 2x)$.

(a) Calculate the partial derivatives P_y and Q_x, and $Q_x - P_y$. **(3 Marks)**

(b) Calculate the dot product $\mathbf{F} \cdot \mathbf{dr}$, where $d\mathbf{r} = (dx, dy)$. **(2 Marks)**

(c) Use Green's theorem to calculate the line integral $\int_C y\,dx + 2x\,dy$, where C is the counter-clockwise path around the unit circle from the point $(1, 0)$ to the point $(1, 0)$. **(5 Marks)**

7 Tutorial Problems

7.1 Problem Set 1

Week 1 Summary.

- If f is a function of x, then the *first derivative* of $f(x)$ is defined as
$$f'(x) = \lim_{h \longrightarrow \infty} \frac{1}{h}\left(f(x+h) - f(x)\right).$$
- If c is a constant, then $\frac{d(cy)}{dx} = c\frac{dy}{dx}$ or $(cf(x))' = cf'(x)$.
- The derivative of a sum: $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ or $(f(x)+g(x))' = f'(x)+g'(x)$.
- Derivatives to remember: $\frac{dx^n}{dx} = nx^{n-1}$. $\frac{de^x}{dx} = e^x$. $\frac{d\sin(x)}{dx} = \cos(x)$. $\frac{d\cos(x)}{dx} = -\sin(x)$.
- The product rule of differentiation: $\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$
or $(f(x)g(x))' = g(x)f'(x) + f(x)g'(x)$.
- The quotient rule of differentiation: $\frac{d(u/v)}{dx} = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)$
or $(f(x)/g(x))' = \frac{1}{(g(x))^2}\left(g(x)f'(x) - f(x)g'(x)\right)$.
- The chain rule of differentiation: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ or $(f(g(x)))' = f'(g(x))g'(x)$.
- An *anti-derivative* or *integral* of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. The *indefinite integral* of a function $f(x)$ is the set of all functions $F(x)$ such that
$F'(x) = f(x)$. We denote $F(x) = \int f(x)\,dx$ and we add an arbitrary constant c to show an indefinite integral.
- If c is a constant, then $\int cf(x)\,dx = c\int f(x)\,dx$.
- The integral of a sum $\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$.
- Integrals to remember: If $n \neq -1$, then $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$. $\int x^{-1}\,dx = \log(x) + c$. $\int e^x\,dx = e^x + c$. $\int \sin(x)\,dx = -\cos(x) + c$. $\int \cos(x)\,dx = \sin(x) + c$. $\int \cos^{-2}(x)\,dx = \tan(x) + c$.
- Integration by parts: $\int u\,dv = uv - \int v\,du$.

- A *definite integral* or *Riemann integral*, denoted $\int_a^b f(x)\,dx$, is defined as the limit

$$\int_a^b f(x)\,dx = \lim_{n\longrightarrow\infty} \sum_{j=1}^{n} f\left(x_j\right)\Delta x.$$

It gives the area bounded by $a \le x \le b$, the function $f(x)$ and the line $y=0$, where $y=f(x)$.

- We write $\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$.

- The *fundamental theorem of calculus* (FTC) states that if $f(x)$ is continuous over $a \le x \le b$, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is an anti-derivative of $f(x)$.

(1) Complete the missing entries in the following table of integrals and derivatives:

Function Integral $(+C)$	Function	Function Derivative
	$f(x) = 2x^2+4$	
	$y = \sqrt[3]{x}$	
	$y = \sin(x)$	
$F(x) = \sin(x) + C$		
	$y = \tan(x)$	
		$y' = e^x$
$F(t) = \log(t) + 1,\ t > 0$		

(2) Use the rules of differentiation to calculate the derivatives of the following functions and then simplify where possible:

(a) $y = 2x\left(x^2+x\right)$.

(b) $f(x) = \frac{\sin(x)}{x^2+4x+2}$.

(c) $f(x) = e^{2x} + \cos(2x)$.

(d) $y = \left(x^4+x^3\right)^2$.

(e) $y = \frac{\sqrt{t}}{e^2}$.

(3) Calculate the value of the following definite integrals:

(a) $\int_2^5 x^2 + 3\,dx$.

(b) $\int_0^{2\pi} \sin(x) + x\,dx$.

(c) $\int_1^e e^t + 1\,dt$.

(d) $\int_{-1}^{1} x^2\,dx$.

(e) $\int_1^\infty \frac{1}{x}\,dx$.

(4) Make quick sketches of the following functions:

(a) $f(x) = x^2 + 2$.

(b) $y = \sin(t) - 1$.

(c) $y = 2e^x$.

(5) Calculate the value of the function $z(x,y)$ for the following:

(a) $z(x,y) = x^2 + 3y^2$ when $x = 2$, $y = 3$.

(b) $z(x,y) = x\sin(y) + y\cos(x)$ when $x = 0$, $y = 1$.

(c) $z(x,y) = 3e^{xy}$ when $x = 2$, $y = \pi$.

(6) Consider the function

$$f(x) = \begin{cases} x^2 + 3x + 1 & \text{if } x \geq 0, \\ 1 - x & \text{if } x < 0. \end{cases}$$

Show that $f(x)$ is continuous but not differentiable.

(7) Consider the function

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is not continuous. Hint: the limit as $(x,y) \longrightarrow (0,0)$ does not exist. If the limit exists, then it must be the same along every path towards $(0,0)$. Try paths $C_1 : y = 0$ and $C_2 : y^2 = x$.

(8) Repeat the above problem but replace f with

$$f(x,y) = \begin{cases} \frac{2x^2+2xy+y^2}{2x^2+3y^2} & \text{if } (x,y) \neq (0,0), \\ \frac{1}{3} & \text{if } (x,y) = (0,0). \end{cases}$$

7.2 Problem Set 2

Week 2 Summary.

- A partial derivative is a rate of change with respect to a particular variable. For the surface $z = f(x,y)$, it is the slope of a tangent line on a cross-section of the surface. See Figure 51. $f_x = \frac{\partial f}{\partial x}$ is computed by differentiating f with respect to x while treating all other variables as constants.

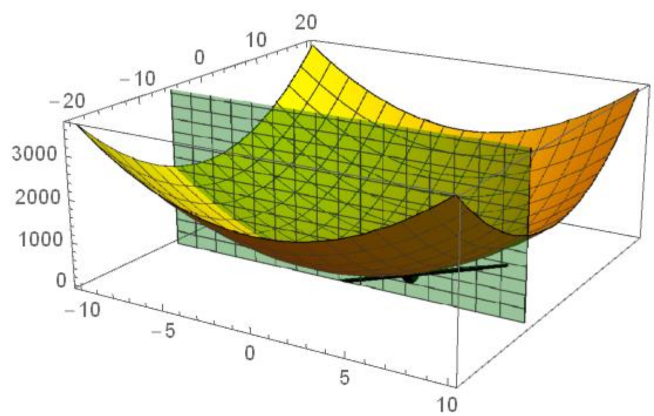


Figure 51: The slope of a tangent line on a cross-section of a surface $z = f(x,y)$.

- $$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right),$$
  $$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \qquad f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).$$

- Clairaut's Theorem: If f and it's 1st and 2nd partial derivatives are defined and continuous, then
  $$f_{xy} = f_{yx}.$$

The following exercises 1 to 13 are found in Washington [11, S. 29.3], Problems 3-6, 11-13, 21, 25, 26, 29, 31, 35.

For Exercises 1-8 below, compute partial derivatives. Here, $z_x = \frac{\partial z}{\partial x}$ and $z_y = \frac{\partial z}{\partial y}$.

(1) z_x and z_y, where $z = 9x^6 - 3y^5$.

(2) z_x and z_y, where $z = 3x^2y^3 - 3x + 4y$.

(3) z_x and z_y, where $z = x^2e^{5xy}$.

(4) z_x and z_y, where $z = 3y\cos(2x)$.

(5) z_x and z_y, where $z = \left(x^2 + xy^3\right)^4$.

(6) $f_x(x,y)$ and $f_y(x,y)$, where $f(x,y) = \left(2xy - x^2\right)^5$.

(7) z_x and z_y, where $z = \cos(xy)$.

(8) z_x and z_y, where $z = \sin(x) + \cos(xy) - \cos(y)$.

Evaluate the following partial derivatives at the given points in Problems 9 and 10.

(9) $\left.\frac{\partial z}{\partial x}\right|_{(1,-2,-7)}$, where $z = 3xy - x^2$.

(10) $\left.\frac{\partial z}{\partial y}\right|_{(2,\frac{\pi}{2},4)}$, where $z = x^2\cos(4y)$.

Find all second partial derivatives in Problems 11 and 12.

(11) $z = 2xy^3 - 3x^2y$.

(12) $z = \frac{x}{y} + e^x\sin(y)$.

(13) Two resistors R_1 and R_2, placed in parallel, have a combined resistance R_T given by $R_T = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}$. Find $\frac{\partial R_T}{\partial R_1}$.

The following exercises are found in Stewart [10, Chp. 14, pp, 964].

Find the first partial derivatives of the function.

(14) $f(x,y) = x^4 + 5xy^3$.

(15) $f(x,y) = x^2y - 3y^4$.

(16) $f(x,t) = t^2e^{-x}$.

(17) $f(x,t) = \sqrt{3x+4t}$.

(18) Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$: $x^2 + 2y^2 + 3z^2 = 1$.

(19) Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$: $x^2 - y^2 + z^2 - 2z = 4$.

(20) Find all the second partial derivatives. $f(x,y) = x^4y - 2x^3y^2$.

(21) Find all the second partial derivatives. $z = \frac{y}{2x+3y}$.

(22) Calculate f_{xxx} and f_{xyx}, where $f(x,y) = x^4y^2 - x^3y$.

$$
\begin{aligned}
f_x &= 4x^3y^2 - 3x^2y, \\
f_{xx} &= 12x^2y^2 - 6xy, \\
f_{xy} &= 8x^3y - 3x^2, \\
f_{xxx} &= 24xy^2 - 6y, \\
f_{xyx} &= 24x^2y - 6x.
\end{aligned}
$$

7.3 Problem Set 3

Week 3 Summary.

Let f be a function of $x_1, x_2, \ldots, x_n$. Then the vector ∇f is given by

$$\nabla f = (f_{x_1}, f_{x_2}, \ldots, f_{x_n}) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

∇f is a vector whose direction indicates the direction in which the maximum rate of change of f is achieved.

The directional derivative of the function f at the point P in the direction of the vector $\mathbf{v}$ is given by

$$f_{\mathbf{v}} = \nabla f(P) \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

It gives the rate of change of f at P in the direction of the vector $\mathbf{v}$.

The following exercises are found in Stewart [10, S. 14.6], Problems 7-10, 13, 15, 17, 21-23 and 32.

For Exercises 1-4 below, find the gradient of f, evaluate $\nabla f(P)$, and find the rate of change of f at P in the direction of the vector $\mathbf{u}$.

(1) $f(x,y) = x/y$, $P = (2,1)$, $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$.

(2) $f(x,y) = x^2 \log(y)$, $P = (3,1)$, $\mathbf{u} = -\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}$.

(3) $f(x,y,z) = xe^{2yz}$, $P = (3,0,2)$, $\mathbf{u} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$.

(4) $f(x,y,z) = \sqrt{x+yz}$, $P = (1,3,1)$, $\mathbf{u} = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$.

For Exercises 5-7 below, find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

(5) $g(s,t) = s\sqrt{t}$, $P = (2,4)$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$.

(6) $f(x,y,z) = x^2y + y^2z$, $P = (1,2,3)$, $\mathbf{v} = (2,-1,2)$.

(7) $f(x,y,z) = xe^y + ye^z + ze^x$, $(0,0,0)$, $\mathbf{v} = (5,1,-2)$.

For Exercises 8-10 below, find the maximum rate of change of f at the given point and the direction in which it occurs.

(8) $f(x,y) = 4y\sqrt{x}$, $P = (4,1)$.

(9) $f(x,y,z) = \frac{x+y}{z}$, $P = (1,1,-1)$.

(10) $f(x,y) = \sin(xy)$, $P = (1,0)$.

(11) The temperature at a point (x,y,z) is given by

$$T(x,y,z) = 200e^{-x^2-3y^2-9z^2},$$

where T is measured in C° and x,y,z in metres.

1. Find the rate of change of temperature at the point $P = (2,-1,2)$ in the direction toward the point $(3,-3,3)$.
2. In which direction does the temperature increase fastest at P?
3. Find the maximum rate of increase at P.

(12) Find the equation of the tangent plane to $z = f(x,y) = y^2\cos(x+y)$ at the point $\left(\frac{\pi}{2}, 0\right)$.

(13) Find and classify critical points of the function $f(x,y) = 8xy - \frac{1}{4}(x+y)^4$.

(14) Let $f(x,y) = 5x^2 + y^2$.

(a) Find the max/min of f subject to the constraint $x^2 + y^2 = 1$ using Lagrange multipliers.

(b) Find the global max/min of f over the disc $x^2 + y^2 \leq 1$.

7.4 Problem Set 4

Week 4 Summary.
Recall the definitions:

- Let $\mathbf{F} = (P, Q, R)$ be a vector field on $\mathbb{R}^3$.
- $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.
- grad $f = \nabla f = (f_x, f_y, f_z)$.
- div $\mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z$.
- curl $\mathbf{F} = \nabla \times \mathbf{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y)$.
- If the work done in moving the object from A to B is independent of the path taken, then we say that $\mathbf{F}$ is *conservative*. If $\mathbf{F}$ is a vector field that is the gradient of some function f. i.e. given $\mathbf{F}$, there exists $f(x,y,z)$ such that $\mathbf{F} = \nabla f$, then $\mathbf{F}$ is conservative.
- $\mathbf{F}$ is **irrotational** if curl $\mathbf{F} = \mathbf{0}$ everywhere.
- $\mathbf{F}$ is **incompressible** if div $\mathbf{F} = 0$ everywhere.

Recall the theorems:

- If $f(x,y,z)$ has continuous 2nd partials, then curl$(\nabla f) = \mathbf{0}$.
- If $\mathbf{F}$ defined on $\mathbb{R}^3$ has continuous partial derivatives, and curl $(F) = \mathbf{0}$, then $\mathbf{F}$ is conservative. The converse is true defined on some domain U.
- If $\mathbf{F}$ has continuous 2nd partial derivatives, then div (curl $(\mathbf{F})) = 0$.

The following exercises are found in Stewart [10, S. 16.5], Problems 1–4, 9–11, 12(a)–12(g), 13–16, 21, 22, Extra: 23–29.

For Exercises 1-4 below, calculate curl $(\mathbf{F})$ and div $(\mathbf{F})$.

(1) $\mathbf{F} = \left(xy^2z^2, x^2yz^2, x^2y^2z\right)$.

(2) $\mathbf{F} = \left(0, x^3yz^2, y^4z^3\right)$.

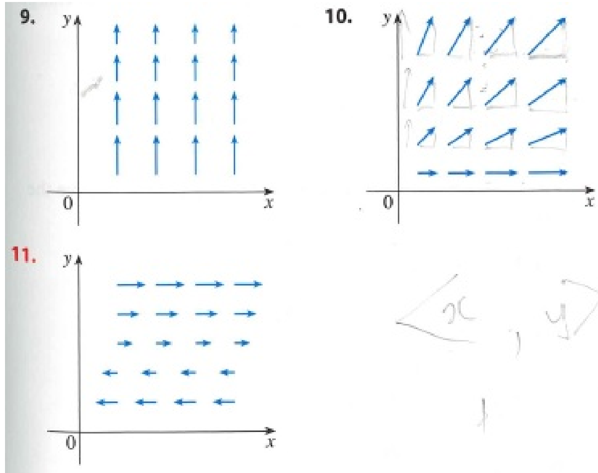


Figure 52: See [10][pp. 1149.]

(3) $\mathbf{F} = \left(xyz, 0, -x^2y\right)$.

(4) $\mathbf{F} = \left(x^2yz, xy^2z, xyz^2\right)$.

For Exercises 5-6 below, refer to Figure 52, 9-11 respectively.

(5) For each of the vector fields, what is the sign of div $(\mathbf{F})$, explain.

(6) Determine whether curl $(F) = \mathbf{0}$. If curl $(F) \neq \mathbf{0}$, which direction does it point?

(7) Let f be a scalar field and $\mathbf{F}$ a vector field. State whether each expression below is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.

(a) curl f.

(b) grad f.

(c) div $\mathbf{F}$.

(d) curl $(\text{grad } f)$.

(e) grad $\mathbf{F}$.

(f) grad $(\text{div } \mathbf{F})$.

(g) div $(\text{grad } f)$.

For Exercises 8-11 below, determine whether or not the vector field is conservative. If so, find f such that $\mathbf{F} = \nabla f$.

(8) $\mathbf{F} = \left(y^2z^3, 2xyz^3, 3xy^2z^2\right)$.

(9) $\mathbf{F} = \left(xyz^4, x^2z^4, 4x^2yz^3\right)$.

(10) $\mathbf{F} = (z\cos y, xz\sin y, x\cos y)$.

(11) $\mathbf{F} = \left(xyz^2, x^2yz^2, x^2y^2z\right)$.

(12) Show that any vector field of the form

$$\mathbf{F}(x,y,z) = (f(x), g(y), h(z)),$$

where f, g, h are differentiable, is irrotational.

(13) Show that any vector field of the form

$$\mathbf{F}(x,y,z) = (f(y,z), g(x,z), h(x,y)),$$

is incompressible.

For the following exercises, assume the following definitions, where $\mathbf{F}$, $\mathbf{F}_1$ and $\mathbf{F}_2$ are vector fields:

$$\begin{aligned}
(f\mathbf{F})(x,y,z) &= f(x,y,z)\mathbf{F}(x,y,z), \\
(\mathbf{F}_1 \cdot \mathbf{F}_2)(x,y,z) &= \mathbf{F}_1(x,y,z) \cdot \mathbf{F}_2(x,y,z), \\
(\mathbf{F}_1 \times \mathbf{F}_2)(x,y,z) &= \mathbf{F}_1(x,y,z) \times \mathbf{F}_2(x,y,z).
\end{aligned}$$

In (14) to (18), prove:

(14) $\operatorname{div}(\mathbf{F}_1 + \mathbf{F}_2) = \operatorname{div}(\mathbf{F}_1) + \operatorname{div}(\mathbf{F}_2)$.

(15) $\operatorname{curl}(\mathbf{F}_1 + \mathbf{F}_2) = \operatorname{curl}(\mathbf{F}_1) + \operatorname{curl}(\mathbf{F}_2)$.

(16) $\operatorname{div}(f\mathbf{F}) = f\ \operatorname{div}(\mathbf{F}) + \mathbf{F} \cdot \nabla f$.

(17) $\operatorname{div}(\mathbf{F}_1 \times \mathbf{F}_2) = \mathbf{F}_2 \cdot \operatorname{curl}(\mathbf{F}_1) - \mathbf{F}_1 \cdot \operatorname{curl}(\mathbf{F}_2)$.

(18) $\operatorname{div}(\nabla f \times \nabla g) = 0$.

(19) Evaluate $\int_C x^4\, dx + xy\, dy$, where C is the triangular curve $(0,0)$ to $(1,0)$, $(1,0)$ to $(0,1)$, and $(0,1)$ to $(0,0)$. See Stuart, 7th Ed., pp. 1110, Example 1.

(20) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \left(-y^2, x, z^2\right)$ and C is the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$, with C oriented counterclockwise from above. See Stuart, 7th Ed., pp. 1148, Example 1.

7.5 Problem Set 5

Week 5 Summary.
Recall the theorem:

- (Fubini) If f is continuous on the rectangle

$$R = \{(x,y) \ : \ a \leq x \leq b, \ c \leq y \leq d\},$$

then

$$\iint\limits_D f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

The following exercises are found in Stewart [10, S. 15.1], Problems 15–18, 37, 38, 40, and Washington [11, S. 29.4] 3–10.

(1) $\int_1^4 \int_0^2 6x^2y - 2x\,dy\,dx.$

(2) $\int_0^1 \int_0^1 (x+y)^2\,dy\,dx.$

(3) $\int_0^1 \int_1^2 \left(x + e^{-y}\right)\,dx\,dy.$

(4) $\int_0^{\pi/6} \int_0^{\pi/2} \sin(x) + \sin(y)\,dy\,dx.$

(5) Find the volume of the solid that lies under the plane $4x + 6y - 2z + 15 = 0$ and above the rectangle $R = \{(x,y) \ : \ -1 \leq x \leq 2, \ -1 \leq y \leq 1\}$.

(6) Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $[-1,1] \times [1,2]$.

(7) Find the volume of the solid enclosed by the surface $z = x^2 + xy^2$ and the planes $z = 0$, $x = 0$, $x = 5$, $y = \pm 2$.

(8) $\int_2^4 \int_0^1 xy^2\,dx\,dy.$

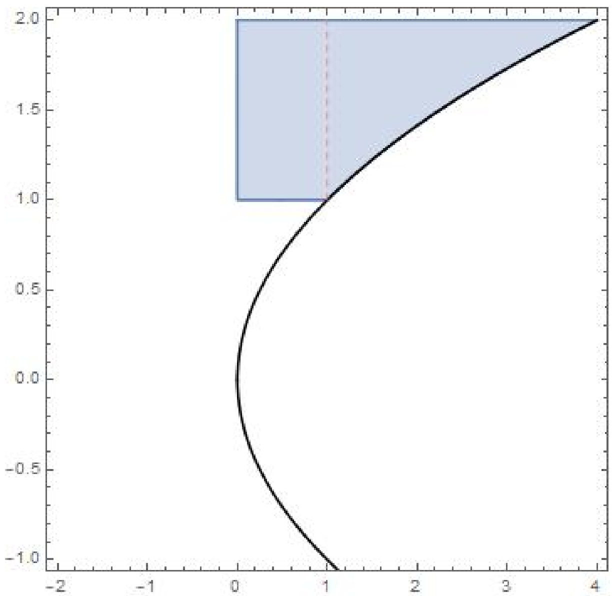


Figure 53: Q.10: Region for $\int_1^2 \int_0^{y^2} xy^2\,dx\,dy$.

(9) $\int_0^2 \int_0^1 \frac{y}{(xy+1)^2}\,dx\,dy$.

(10) $\int_1^2 \int_0^{y^2} xy^2\,dx\,dy$.

(11) $\int_0^4 \int_1^{\sqrt{y}} (x-y)\,dx\,dy$.

(12) $\int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx$.

(13) $\int_4^9 \int_0^x \sqrt{x-y}\,dy\,dx$.

(14) $\int_1^{\pi/6} \int_{\pi/3}^y \sin(x)\,dx\,dy$.

(15) $\int_0^{\sqrt{3}} \int_{x^2/3}^1 4-x^2\,dy\,dx$.

(16) Calculate the net outward flux of the vector field $\mathbf{F} = (x+2y, -3x+y)$ across the quarter circle $x^2+y^2 = 1$, $0 \leq x, y \leq 1$. Hint: parametrize the quarter circle and the normal vector $\mathbf{n}$, and then compute $\int_C \mathbf{F} \cdot \mathbf{n}\,dS$.

(17) Find the flux of the vector field $\mathbf{F} = (z, y, x)$ over the unit sphere $x^2+y^2+z^2 = 1$.

(18) Evaluate $\iint_S \mathbf{F} \cdot dS$, where $\mathbf{F} = \left(xy, y^2+e^{xz^2}, \sin(xy)\right)$, and S is the surface of the region bounded by $z = 1-x^2$ and the planes $z = 0$, $y = 0$, and $y+z = 2$.

7.6 Problem Set 6

Week 6 Summary.

- If $x = f(t)$, $y = g(t)$ are differentiable and $\frac{dx}{dt} \neq 0$, then $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$.
- Area $A = \int_a^b y\,dx = \int_\alpha^\beta g(t) f'(t)\,dt$ by substitution.
- Arc-length

$$\begin{aligned}
L &= \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx, \\
&= \int_{t=\alpha}^{\beta} \sqrt{1 + \left(\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}\right)^2}\,\frac{dx}{dt}\,dt, \\
&= \int_{t=\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt, \\
&= \int_{t=\alpha}^{\beta} \|\mathbf{r}'(t)\|\,dt, \text{where } \mathbf{r}(t) = (x(t), y(t)).
\end{aligned}$$

The following exercises are found in Stewart [10, S. 10.1, 10.2], Problems 1, 2, 5, 7, 8, 14, 17, 24, 45. 1, 3, 5, 7, 11, 37, 41.

(1) Sketch, indicating arrow as t increases.

$$x = 1 - t^2, \qquad y = 2t - t^2, \qquad -1 \le t \le 2.$$

(2) Sketch, indicating arrow as t increases.

$$x = t^3 + t, \qquad y = t^2 + 2, \qquad -2 \le t \le 2.$$

(3) Sketch, indicating arrow as t increases. Find the Cartesian equation of the curve.

$$x = 2t - 1, \qquad y = \frac{1}{2}t + 1.$$

(4) Sketch, indicating arrow as t increases. Find the Cartesian equation of the curve.

$$x = t^2 - 3, \qquad y = t + 2, \qquad -3 \le t \le 3.$$

(5) Sketch, indicating arrow as t increases. Find the Cartesian equation of the curve.

$$x = \sin(t), \qquad y = 1 - \cos(t), \qquad 0 \le t \le 2\pi.$$

(6) Sketch, indicating arrow as t increases. Find the Cartesian equation of the curve.

$$x = e^t, \qquad y = e^{-2t}.$$

(7) Sketch, indicating arrow as t increases. Find the Cartesian equation of the curve.

$$x = e^{2t}, \qquad y = t + 1.$$

(8) Suppose that the position of one particle at time t is given by

$$x_1 = 3\sin(t), \qquad y_1 = 2\cos(t), \qquad 0 \le t \le 2\pi$$

and the position of the second particle is given by

$$x_2 = -3 + \cos(t), \qquad y_2 = 1 + \sin(t), \qquad 0 \le t \le 2\pi.$$

(a) Graph the paths of both particles. How many points of intersection are there?

(b) Are any of these points of intersection collision points? If so, find the collision points.

(c) Describe what happens if the path of the second particle is given by

$$x_2 = 3 + \cos(t), \qquad y_2 = 1 + \sin(t), \qquad 0 \le t \le 2\pi.$$

(9) Find $\frac{dy}{dx}$ for

$$x = \frac{t}{1+t}, y \qquad = \sqrt{1+t}.$$

(10) Find the equation of the tangent line at $t = -1$ for

$$x = t^3 + 1, \qquad y = t^4 + t.$$

(11) Find the equation of the tangent line at $t = -1$ for

$$x = t^4 + 1, \qquad\qquad y = t^4 + t.$$

(12) Find the equation of the tangent line at $(1,3)$ for

$$x = 1 + \log(t), \qquad\qquad y = t^2 + 2.$$

(13) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of t is the curve concave upward?

$$x = t^2 + 1, \qquad\qquad y = t^2 + t.$$

(14) Set up an integral that represents the arc length of the curve, then give a numerical estimate.

$$x = t + e^{-t}, \qquad\qquad y = t - e^{-t}, \qquad\qquad 0 \le t \le 2.$$

(15) Find the exact length of the curve

$$x = 1 + 3t^2, \qquad\qquad y = 4 + 2t^3, \qquad\qquad 0 \le t \le 1.$$

(16) Evaluate $\int_C y^2\,dx + x\,dy$, where

(a) C is the line segment from $(-5,-3)$ to $(0,2)$.

(a) C is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

(17) Find the work done by the force field $\mathbf{F} = \left(x^2, -xy\right)$ in moving a particle along the quarter circle $\mathbf{r}(t) = (\cos(t), \sin(t))$, $0 \le t \le \frac{\pi}{2}$.

(18) Consider the vector field $\mathbf{F} = \left(3 + 2xy, x^2 - 3y^2\right)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$C : \mathbf{r}(t) = \left(e^t \sin(t), e^t \cos(t)\right), \qquad\qquad 0 \le t \le \pi.$$

8 Books & Notes

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