Year 10 Mathematical Methods

Student Workbook and Teaching Template

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https://mathofcourse.com/

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1. **Imperative:** Print this pdf document or be prepared to annotate the pdf with a tablet. Some blank spaces for writing are a little small for large writing. If you cannot do either of these annotation options, then write notes on blank paper, noting the relevant position within the typed course notes. As you watch the intructional videos, write notes in the blank spaces. This step is very important.

2. The instructor should write exercises from an appropriate textbook where the text says **Exercises/Homework**.

3. **Optional but highly recommended:** Purchase and use *Mathematica* or obtain it through your institution. We will occasionally use this to display various graphics and verify calculations. All graphics shown in this document were produced with *Mathematica*. You will most likely find it very helpful with your studies. It is a symbolic computation tool which has full programming capabilities. E.g. Try writing

   ```
   Expand[(x+y)^3]
   ```

   then press Shift+Enter or

   ```
   s = 0;
   For[i = 0, i < 6, i++, s = s + i; Print[s]]
   ```

   You can call on *Wolfram alpha* from with in it by beginning a cell with = =.
   If your school has a license, to install this on your machine, visit:
   `wolfram.com/siteinfo/`
   Get *Mathematica* Desktop.
   Create a Wolfram ID, and download and install the software.

1 Year 10 Mathematical Methods

1.1 Term 1

1.1.1 Surds and Index Laws

The decimal expansion of $\sqrt{2}$ does not terminate, nor repeat.

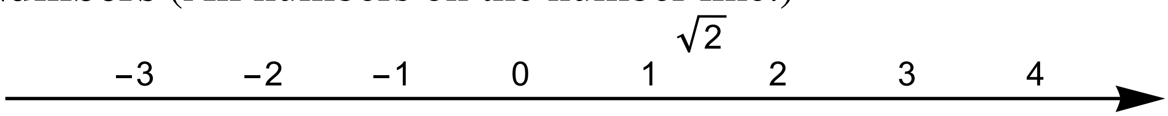
$\sqrt{2} = 1.4142135623730950488016887242096980785696718753769 5\ldots$

There are no whole numbers a, b with $b \neq 0$ and $\sqrt{2} = \frac{a}{b}$. We say $\sqrt{2}$ is **irrational**, meaning not rational.

Next we identify several important sets of numbers and notation for them.

$\mathbb{Z}$ **Integers** This is the set of all whole numbers $\cdots -3, -3, -1, 0, 1, 2, \ldots$.

$\mathbb{R}$ **Real Numbers** (All numbers on the number line.)



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$\mathbb{Q}$ **Rational Numbers** $\left\{\frac{a}{b} \ : \ a, b \in \mathbb{Z}, b \neq 0\right\}$, meaning the set of all fractions a/b, where a and b are elements of ($\in$) the set of integers ($\mathbb{Z}$) and b is non-zero. Note: Integers are also rational numbers ($\mathbb{Z} \subset \mathbb{Q}$).

$\mathbb{R} - \mathbb{Q}$ or $\mathbb{R} \setminus \mathbb{Q}$ **Real Irrational Numbers** (Real numbers that are not rational. e.g. π and $\sqrt{2}$ are real numbers but not rational numbers.)

Note: A real number is rational if and only if it has a repeating decimal expansion or a terminating decimal expansion.

A **surd** is a sum of expressions of the form $a\sqrt[n]{b}$.

Example 1.1 *Which of the following real numbers are surds?*

$$\sqrt{36} \qquad \sqrt{19} \qquad \sqrt{\frac{1}{25}} \qquad \sqrt[3]{21} \qquad 4\pi \qquad \sqrt[3]{1728}$$

Example 1.2 *Simplify the following using the multiplicative property of square roots $\sqrt{ab} = \sqrt{a}\sqrt{b}$.*

$$\sqrt{12} \qquad 3\sqrt{30} \qquad \sqrt{\frac{1}{36}} \qquad 2\sqrt{75} \qquad \frac{3\sqrt{125}}{4} \qquad \sqrt{\frac{15}{81}}$$

Exercises/Homework:

We begin this section with rationalising the denominator of surds. We use the properties

$$\frac{x}{\sqrt{y}} = \frac{x\sqrt{y}}{\sqrt{y}\sqrt{y}} = \frac{x\sqrt{y}}{y},$$

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right) = a-b.$$

Example 1.3 *Rationalise the denominator for the following expressions.*

(a) $\frac{5}{\sqrt{3}}$

(b) $\frac{6\sqrt{5}}{\sqrt{8}}$

(c) $\frac{3\sqrt{6}}{2\sqrt{10}}$

(d) $\frac{4-\sqrt{3}}{\sqrt{15}}$

(e) $\frac{1}{5-\sqrt{3}}$

(f) $\frac{6-\sqrt{5}}{2+\sqrt{8}}$

Exercises/Homework:

We have the following index laws for real numbers a, b, c :

$$a^b a^c = a^{b+c}, \qquad a^b / a^c = a^{b-c}, \text{ for } a \neq 0$$

$$\left(a^b\right)^c = a^{bc},$$

$$(ab)^c = a^c b^c, \qquad (a/b)^c = a^c/b^c \ = \ a^c b^{-c}, \text{ for } b \neq 0$$

$$a^{-1} = \frac{1}{a}, \text{ for } a \neq 0$$

$$a^0 = 1, \qquad 0^0 = 1 \text{ (defined to be 1, but contraversial)}$$

$$\frac{1}{a^{-b}} = a^b, \text{ for } a \neq 0, \qquad a^{-b} = \frac{1}{a^b} \text{ for } a \neq 0.$$

Example 1.4 *Express the following with positive indices*

(a) a^{-3}

(b) $2x^{-3}y^4$

(c) $\frac{4}{y^{-2}}$

(d) $\frac{\left(a^{-3}b\right)^2}{3a^{-1}b^2} \times \frac{b^{-1}}{a}$

(e) $\frac{\left(5a^2b^{-1}\right)^3}{2a^4b^{-2}} \div \frac{b^{-5}}{2a^{-2}}$

Exercises/Homework:

We can write $\sqrt{3} = 3^{1/2}$ and

$$\left(\sqrt{3}\right)^2 = 3^{1/2} \times 3^{1/2} = 3^{\frac{1}{2}+\frac{1}{2}} = 3^1 = 3.$$

This allows us to use index laws to simplify surds. We have the following index laws for real number a and integers m,n:

$$\begin{aligned} a^{m/n} &= \sqrt[n]{m}, \\ a^{1/2} &= \sqrt{a}, \\ a^{1/3} &= \sqrt[3]{a}, \\ a^{1/n} &= \sqrt[n]{a}. \end{aligned}$$

Example 1.5 *Express the following in index form:*

(a) $\sqrt{11}$

(b) $\sqrt{3x^7}$

(c) $2\sqrt[4]{x^9}$

(d) $11\sqrt{7}$

Example 1.6 *Write the following in simplest surd form:*

(a) $12^{1/2}$

(b) $6^{3/2}$

Example 1.7 *Simplify:*

(a) $a^{1/5}a^{3/5}$

(b) $\left(b^2b^3\right)^{\frac{1}{4}}$

(c) $\left(\frac{x^{1/3}}{y^{1/6}}\right)^{1/4}$

Exercises/Homework:

In this section we will learn how to solve equations with one variable.

Example 1.8 *Solve the following equations for x:*

(a) $2x+9=12$

(b) $3(2x+4)=3x$

(c) $\frac{x-1}{3}=2$

(d) $3-\frac{x}{3}=8$

(e) $\frac{3-x}{4}=x-4$

Exercises/Homework:

In this section we learn how to rearrange formulas and substitute values into equations.

Example 1.9 *The volume of a sphere of radius r is given by* $V = \frac{4}{3}\pi r^3$.

(a) *Solve the equation* $V = \frac{4}{3}\pi r^3$ *for r, where r is a real number.*

(b) *If the volume of the sphere is* 42.8 m^3*, find the radius of the sphere.*

Example 1.10 *The area of a rectangular region adjoining a two semi-circle regions on each end of the rectangle is given by* $A = xy + \pi\left(\frac{x}{2}\right)^2$.

(a) *Solve the equation* $A = xy + \pi\left(\frac{x}{2}\right)^2$ *for y in terms of x and A.*

(b) *If* $x = 36$ *m and* $y = 24$ *m, calculate A.*

Exercises/Homework:

Symbols:

$x > 3$ means x is greater than 3.
$x < 3$ means x is less than 3.
$x \geq 3$ means x is greater than or equal to 3.
$x \leq 3$ means x is less than or equal to 3.

Example 1.11 *Sketch the region on the number line corresponding to:*

(a) $\{x \in \mathbb{R} \; : \; x > 3\} = (3, \infty)$.

(b) $\{x \in \mathbb{R} \; : \; x < 3\} = (-\infty, 3)$.

(c) $\{x \in \mathbb{R} \; : \; x \geq 3\} = [3, \infty)$.

(d) $\{x \in \mathbb{R} \; : \; x \leq 3\} = (-\infty, 3]$.

Rules:

- When multiplying an inequality by a negative number, turn the symbol around. ($>$ becomes $<$, $<$ becomes $>$, $\leq$ becomes $\geq$, $\geq$ becomes $\leq$.)
- When inverting both sides of an inequality, turn the symbol around.
- Otherwise, treat solving an inequality like solving an equation.

Example 1.12 $4 > 3$ *but* $-4 \quad\quad -3$ *and* $\frac{1}{4} \quad\quad \frac{1}{3}$.

Example 1.13 *Solve the inequality* $3x - 6 \geq 8$ *for x.*

Example 1.14 *Solve the inequality* $-(4 - 6x) < 2(5 - x)$ *for x.*

Example 1.15 *Solve the inequality* $\frac{x}{4} - \frac{2x}{3} > -7$ *for x.*

Example 1.16 *Solve the inequality* $\frac{5}{3x} > 2$ *for x.*

Exercises/Homework:

The **greatest common divisor** of two integers a and b is written $\gcd(a,b)$. This is the greatest positive integer c such that c divides a and c divides b.

The **least common multiple** of two integers a and b is written $\text{lcm}(a,b)$. This is the least positive integer c such that a divides c and b divides c.

Theorem 1 *For any two positive integers a and b,*

$$ab = \gcd(a,b)lcm(a,b).$$

The greatest common divisor and least common multiple can be calculated efficiently using the Euclidean algorithm.

Example 1.17 *Calculate:*

(a) 4×6

(b) $\gcd(4,6)$

(c) $\frac{4\times 6}{\gcd(4,6)}$

(c) $lcm(4,6)$

(d) $\frac{1}{4} + \frac{1}{6}$

Example 1.18 *Calculate lcm*$(12,18)$ *and use it to simplify* $\frac{x+4}{12}+\frac{x-6}{18}$.

Example 1.19 *Calculate lcm*$(24,6)$ *and use it to simplify* $\frac{x-3}{6}+\frac{5x-6}{24}$.

Exercises/Homework:

Two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are **parallel** if $m_1 = m_2$.

Two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are **perpendicular** if $m_1m_2 = -1$ (or equivalently, $m_2 = -\frac{1}{m_1}$).

Theorem 2 *In Euclidean space:*

- *Two lines intersect in one point if and only if they are not parallel.*
- *Lines have either one intersection or infinitely many intersections (they are the same line).*
- *There is a unique line passing through two points.*

Example 1.20 *Decide whether the two lines* $y = -9x - 3$ *and* $y = \frac{1}{9}x + 2$ *are parallel, perpendicular, or neither.*

Example 1.21 *Decide whether the two lines* $y = -\frac{1}{3}x + 1$ *and* $3y + x = 2$ *are parallel, perpendicular, or neither.*

Example 1.22 *Decide whether the two lines* $y = \frac{1}{6}x + 4$ *and* $6y + x = 3$ *are parallel, perpendicular, or neither.*

Example 1.23 *Find the equation of the line that is parallel to* $y = -5x + 8$ *and passes through the point* $(2, -3)$.

Example 1.24 *Find the equation of the line that is perpendicular to* $y = -5x + 8$ *and passes through the point* $(-4, -2)$.

To decide whether two lines are parallel, perpendicular, or neither:

Step 1 Put both lines in standard form $y = mx + c$ and hence identify slopes m_1 and m_2.

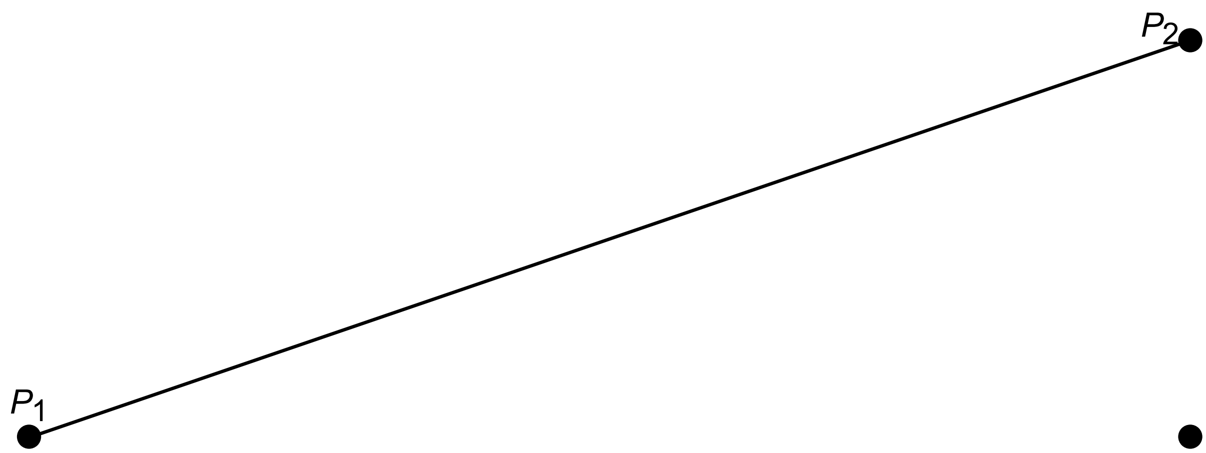
Step 2 If $m_1 = m_2$, then the lines are parallel;

Step 3 Otherwise: if $m_1 m_2 = -1$, then the lines are perpendicular;

Step 4 Otherwise: the lines are neither parallel nor perpendicular.

Exercises/Homework:

Consider the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. The distance between P_1 and P_2 is obtained by Pythagoras' theorem $a^2 + b^2 = c^2$, where $a = |x_1 - x_2|$, $b = |y_1 - y_2|$, and c is the distance between P_1 and P_2.



The formula for the distance between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$c = D(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Example 1.25 *Find the distance between the points* $(0,4)$ *and* $(-2,6)$.

The **midpoint** of the line segment connecting the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is given by

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Example 1.26 *Find the midpoint of the line segment connecting the points* $(0,4)$ *and* $(-2,6)$.

Example 1.27 *Find real numbers* a *and* b *such that the midpoint of* $(2a,a)$ *and* $(3,b)$ *is* $(4,-4)$.

.

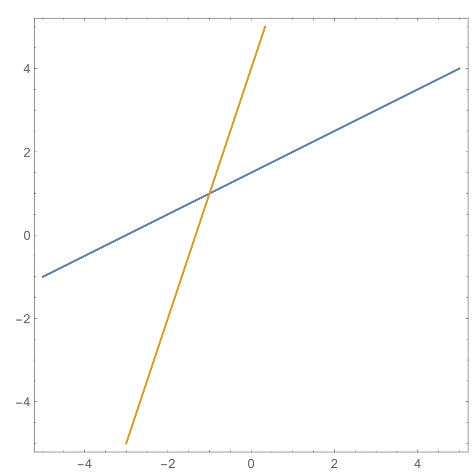
Example 1.28 *The distance between the points* $(-4,1)$ *and* $(6,a)$ *is* $4\sqrt{21}$. *Find a.*

Exercises/Homework:

Given two simultaneous linear equations that do not represent parallel lines, we learn to find the point of their intersection by substitution. That is, we solve one equation for a variable, say y, and then substitute that into the other equation and solve for the other letter, say x.

Example 1.29 *Solve the simultaneous system of linear equations*

$$\begin{aligned} 2x - 4y &= -6, \\ y &= 3x + 4. \end{aligned}$$



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Example 1.30 *Solve the simultaneous system of linear equations*

$$\begin{aligned} y &= 8x-1, \\ y &= 8x+2 \end{aligned}$$

if possible.

Example 1.31 *For which real value of k does the simultaneous system of linear equations*

$$\begin{aligned} y &= -3x-2, \\ y &= kx+6 \end{aligned}$$

(a) *have no solution?*

(b) *have one solution?*

(c) *have infinitely many solutions?*

Exercises/Homework:

Given a system of simultaneous linear equations, solving the system by elimination applies the following procedure. We multiply each equation by a number such that the coefficients of one of the variables (the coefficient of the same letter) becomes the same or of opposite sign. We then add or subtract equations so that that variable vanishes. Finally, we solve for the other variable.

Example 1.32 *Solve the system of equations*

$$\begin{aligned} 2x - 6y &= 8, \\ 3x + 4y &= 10 \end{aligned}$$

by elimination.

Example 1.33 *Solve the system of equations*

$$\begin{aligned} x+2y &= 4, \\ 2x+9y &= 12 \end{aligned}$$

by elimination.

Exercises/Homework:

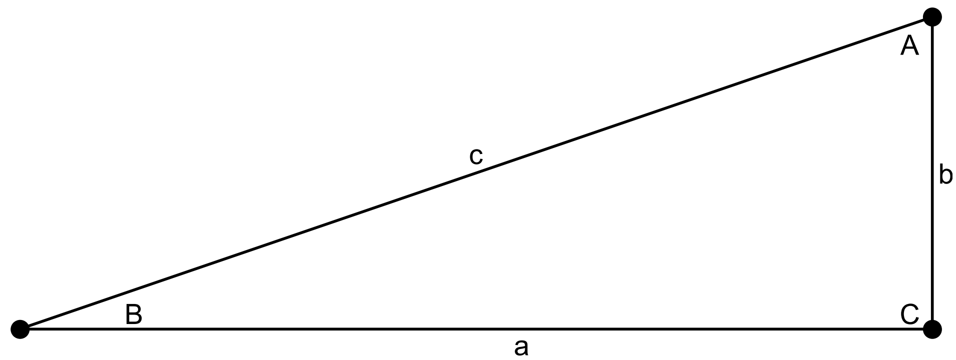
Next we consider applications of simultaneous equations.

Example 1.34 *The sum of the ages of two children Kara and Ben is* 17 *and the difference in their ages is* 5*. If Kara is older than Ben, determine their ages.*

Exercises/Homework:

1.2.1 Introduction to Trigonometry

We learn about the relationship between the angles in a right triangle and the trigonometric ratios sine, cosine and tangent (sin, cos, tan).



The trigonometric ratios sine, cosine and tangent are defined

$$\sin(B) = \frac{\text{opp}}{\text{hyp}},$$
$$\cos(B) = \frac{\text{adj}}{\text{hyp}},$$
$$\tan(B) = \frac{\text{opp}}{\text{adj}}.$$

We have the useful acronym **S**OH**C**AH**T**OA for remembering these trig. ratios.

Recall Pythagoras' theorem:

Theorem 3 *(Pythagoras) Let a, b, c be the lengths of the sides of a right triangle, where $c > a, b$ (c is the hypotenuse). Then $a^2 + b^2 = c^2$.*

Example 1.35 *Show that:*

(a) $\frac{\sin(B)}{\cos(B)} = \tan(B)$,

(b) $\sin(B) = \cos(A)$,

(c) $\cos(B) = \sin(A)$,

(d) $\tan(B) = \frac{1}{\tan(A)}$,

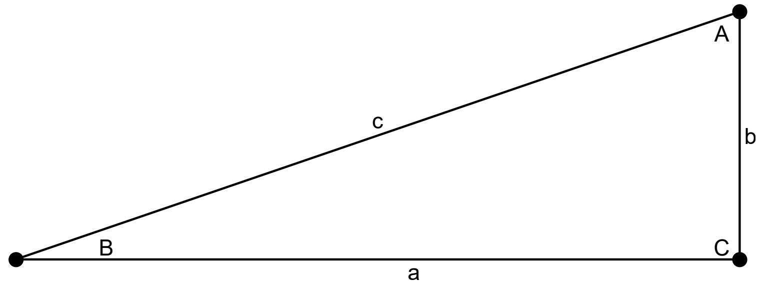
(e) $\cos^2(B) + \sin^2(B) = 1$ by Pythagoras' theorem.

Example 1.36 *Find the side length x opposite an angle of 30° in a right triangle with hypotenuse 8.*

Example 1.37 *Find the hypotenuse x in a right triangle if the triangle has side length 4 adjacent to an angle of 44°.*

Exercises/Homework:

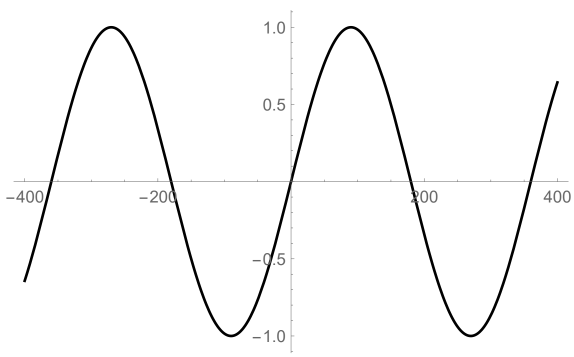
To solve a right triangle for an interior angle we use the inverse functions of sine, cosine and tangent, $(\sin^{-1}, \cos^{-1}, \tan^{-1})$.

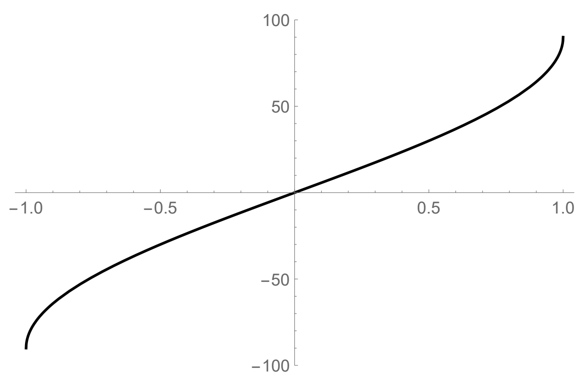


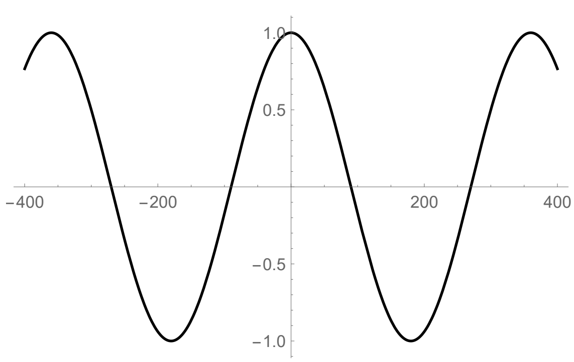
Since $\sin(B) = \frac{b}{c}$, we have $B = \sin^{-1}\left(\frac{b}{c}\right)$. This is also called arcsin.

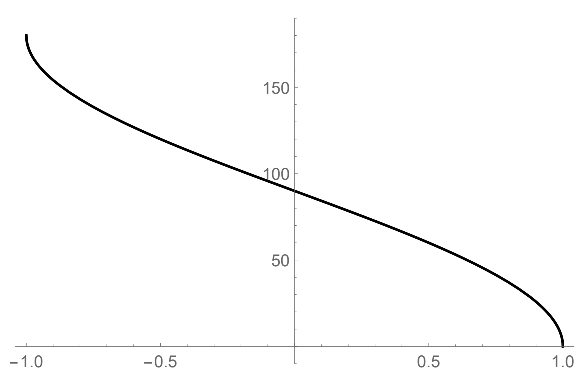
Similarly, $\cos(B) = \frac{a}{c}$ so $B = \cos^{-1}\left(\frac{a}{c}\right)$. This is also called arccos.

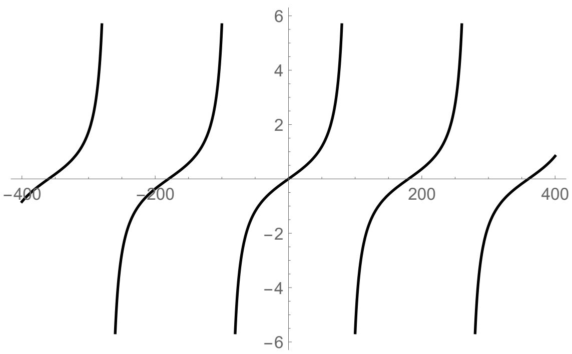
$\tan(B) = \frac{b}{a}$ so $B = \tan^{-1}\left(\frac{b}{a}\right)$. This is also called arctan.

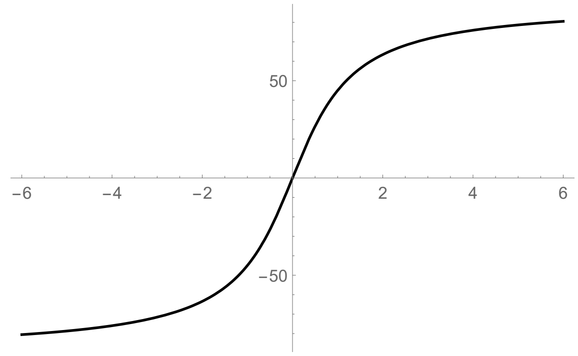












Example 1.38 *A right triangle has hypotenuse of length* 2 *and sides of length* 1 *and* x*. Solve for the angle adjacent to the side of length* x*, and then solve for* x*.*

Exercises/Homework:

We consider some applications of trigonometry.

Example 1.39 *A tower stands x metres high in elevation above the ground. A man standing on the top of a 250 metre tall building looks up to the tower with an elevation angle of* 30° *to the horizontal. The horizontal distance between the man and the tower is* 420 *metres. Calculate the elevation x of the tower.*

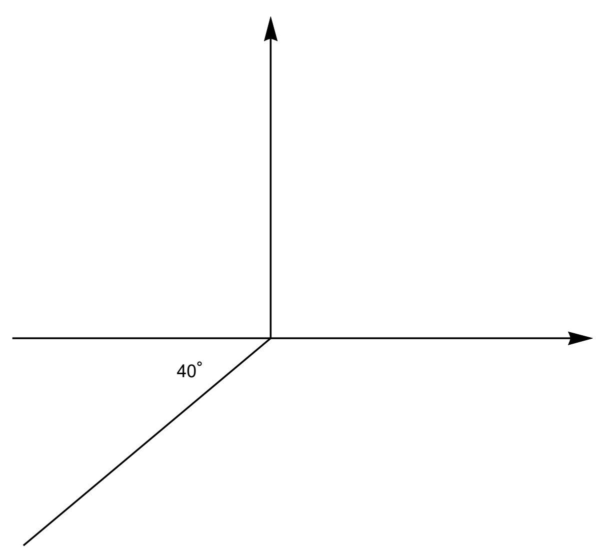
Exercises/Homework:

True Bearings $(^\circ T)$ are measured **clockwise from North**.

.

Recall that the mathematical convention is to measure angles from the positive end of the x-axis counter-clockwise.

Example 1.40 *Consider the points O, A, B show below. If the line segment OA makes an angle of* 40° *South of West, what is the true bearing of the point A from O?*



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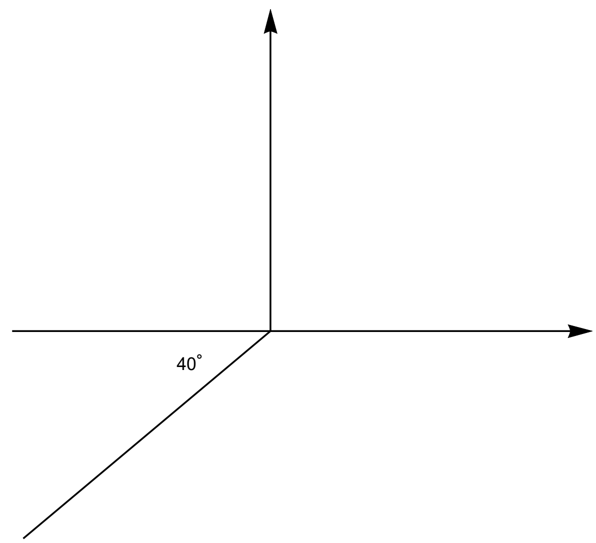
Example 1.41 *Consider the points* O, A, B *show below. If the line segment OA makes an angle of* $40°$ *South of West, what is the true bearing of the point O from* A*?*



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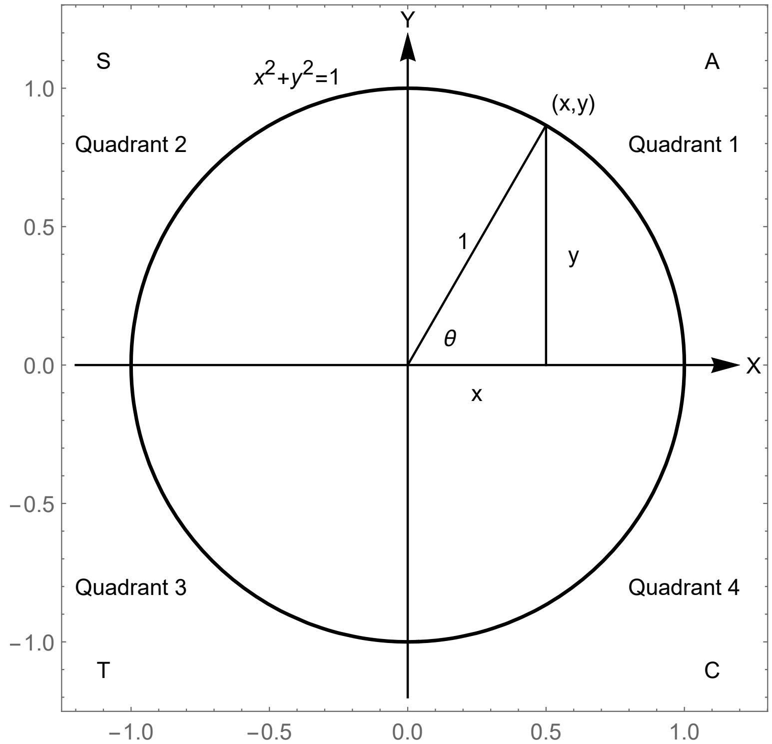
Example 1.42 *A boat travels North-East for* 5 *km followed by a true bearing of* $20°$ *for* 10 *km. Find the true bearing of the boat from the original position.*

Exercises/Homework:



Recall

$$\begin{aligned}\sin(\theta) &= \frac{\text{opp}}{\text{hyp}} = \frac{y}{1},\\ \cos(\theta) &= \frac{\text{adj}}{\text{hyp}} = \frac{x}{1},\\ \tan(\theta) &= \frac{\text{opp}}{\text{adj}} = \frac{y}{x}.\end{aligned}$$

For any point (x,y) on the unit circle $x^2+y^2=1$, there is an angle θ such that $(x,y)=(\cos(\theta),\sin(\theta))$. Since $x^2+y^2=1$ we again have $\cos^2(\theta)+\sin^2(\theta)=1$.

The acronym ASTC refers to the following:

For an angle θ in Quadrant 1, **All** $\sin(\theta),\cos(\theta),\tan(\theta)>0$.

For an angle θ in Quadrant 2, **Only Sine,** $\sin(\theta)>0$.

For an angle θ in Quadrant 3, **Only Tan,** $\tan(\theta)>0$.

For an angle θ in Quadrant 4, **Only Cos,** $\cos(\theta)>0$.

A **reference angle** is an angle α with $0 \leq \alpha < 90^\circ$ such that $\theta = 180^\circ \pm \alpha$, $\theta = 360^\circ - \alpha$, or $\theta = \alpha$. For example, if $\theta = 290^\circ$, then the reference angle $\alpha = 70^\circ$ so that $\theta = 360^\circ - \alpha$.

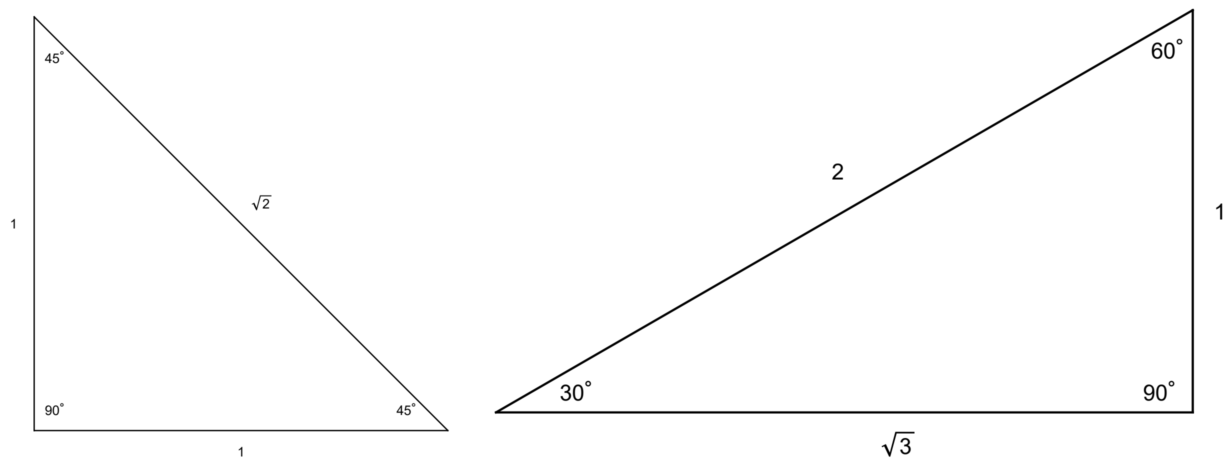
Let α be the reference angle.

- If θ is in Quadrant 1, then $\theta = \alpha$.
- If θ is in Quadrant 2, then $\theta = 180^\circ - \alpha$.
- If θ is in Quadrant 3, then $\theta = 180^\circ + \alpha$.
- If θ is in Quadrant 4, then $\theta = 360^\circ - \alpha$.

Example 1.43 *Calculate* $\cos(320^\circ)$ *and* $\sin(320^\circ)$ *by considering the reference angle.*

Exercises/Homework:

Memorise the following useful triangles:



These two triangles give exact surd values for the trigonometric ratios of angles 45°, 30°, and 60°.

We have

$$\cos(45^\circ) = \qquad , \qquad \cos(30^\circ) = \qquad , \qquad \cos(60^\circ) = \qquad ,$$

$$\sin(45^\circ) = \qquad , \qquad \sin(30^\circ) = \qquad , \qquad \sin(60^\circ) = \qquad ,$$

$$\tan(45^\circ) = \qquad , \qquad \tan(30^\circ) = \qquad , \qquad \tan(60^\circ) = \qquad .$$

The angle addition formulas:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta),$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta),$$

give additional exact values.

Example 1.44 *Calculate* $\cos(15^\circ)$ *and* $\sin(15^\circ)$ *using the above triangles and the angle addition formulas.*

Example 1.45 *Calculate the exact surd value of* $\cos(150^\circ)$.

Example 1.46 *Find all angles* θ *with* $0 \leq \theta < 360^\circ$ *such that* $\cos(\theta) = -\frac{\sqrt{3}}{2}$.

Exercises/Homework:

Like terms are terms of a polynomial with the same letters to the same powers.
Example: $4xy^2$ and $-3xy^2$ **ARE** like terms.
Example: $4x^2y$ and $-3xy^2$ are **NOT** like terms.
Example: x^2 and x are **NOT** like terms.
Example: x and 12 are **NOT** like terms.

We use the **distributive law** to expand brackets. This means multiplication distributes over addition:

$$x(y+z) = xy + xz, \qquad (x+y)z = xz + yz.$$

Notice that $2(3+5) = 2 \times 8 = 16$.
Also $2(3+5) = 2 \times 3 + 2 \times 5 = 6 + 10 = 16$.

The following are all consequences of the distributive law:

$$\begin{aligned} a(b+c) &= \\ a(b-c) &= \\ (a+b)(c+d) &= \\ &= \\ &= \end{aligned}$$

$$\begin{aligned}(a+b+c)(d+e+f) &= \\ \\ (a+b)^2 &= \\ &= \\ &= \\ &= \\ (a+b)(c+d)(e+f) &= \\ &= \\ \\ &= \end{aligned}$$

Example 1.47 *Expand* $(x-4)(x+8)$

Example 1.48 *Expand* $(2x-6)(3x+7)$

Exercises/Homework:

Factorizing a polynomial is the process of expressing the polynomial as a product of polynomials.

For example, $x^2 - 25 = (x+5)(x-5)$ since $a^2 - b^2 = (a+b)(a-b)$.
Similarly, $x^2 - 12 = x^2 - \sqrt{12}^2 = \left(x + \sqrt{12}\right)\left(x - \sqrt{12}\right)$.

Example 1.49 *Factorise* $3x^2 - 18x$.

Example 1.50 *Factorise* $x^2 + 8x + 15$.

Example 1.51 *Factorise* $x(x+3) - 12(x-3)$.

Exercises/Homework:

A **monic** polynomial in one variable has leading coefficient equal to 1. That is, a polynomial in the variable x has coefficient of x^n, where n is greatest, being 1.

x^2+3x+8 is monic. $3x^2-4x+12$ is not monic.

A **quadratic** polynomial in one variable is a polynomial of the form

$$ax^2+bx+c,$$

where a,b,c are specific numbers. Quadratic refers to the greatest exponent being equal to 2.

To factorise x^2+bx+c, where b,c are specific integers, we seek to find integers p,q such that

$$(x+p)(x+q)=x^2+(p+q)x+pq=x^2+bx+c$$

so that

$$c=pq, \qquad b=p+q.$$

Step 1 If $c=0$, put $x^2+bx+c=x(x+b)$. Otherwise:

Step 2 If $b=0$, $x^2+bx+c=(x+\sqrt{c})(x-\sqrt{c})$. Otherwise:

Step 3 List all of the divisor pairs (s,t) of the absolute value of c up to their order: $(1,|c|),\ldots$.

Step 4 If $c>0$, determine which pair (s,t) satisfies $s+t=|b|$. If $b>0$, put $x^2+bx+c=(x+s)(x+t)$. If $b<0$, put $x^2+bx+c=(x-s)(x-t)$.

Step 5 If $c<0$, determine which pair (s,t) satisfies $s-t=\pm b$. If $b>0$, put $x^2+bx+c=(x+s)(x-t)$, where $s>t$. If $b<0$, put $x^2+bx+c=(x-s)(x+t)$, where $s>t$.

Example 1.52 *Factorise the monic quadratic polynomial* $x^2 - x - 20$.

Example 1.53 *Factorise the monic quadratic polynomial* $x^2 + 9x + 18$.

Example 1.54 *Factorise the monic quadratic polynomial* $x^2 + 5x - 84$.

Exercises/Homework:

We learn how to factorise expressions of the form ax^2+bx+c, where $a \neq 0$ and $a,b,c \in \mathbb{Z}$ (are integers). We demonstrate the procedure with an example.

Example 1.55 *Factorise* $10x^2-13x-3$.
We first list the divisor pairs of $|-3|=3$. *We only have* $(1,3)$.
We list the divisor pairs of the absolute value of the leading coefficient $|10|=10$. *We only have* $(1,10)$ *and* $(2,5)$. *Next we form a multiplication table where we multiply divisors:*

×	*1* *10*	*2* *5*
1		
3		

Since $-3<0$, *we seek a pair of products with a difference of* -13, *examining the differences within diagonals of the table.*

We use these to form the required factors
$10x^2-13x-3 = (\qquad\qquad)(\qquad\qquad)$.

We can expand to check our work.

Example 1.56 *Factorise* $6x^2 - 13x - 28$.

.

$\times$	*1 28*	*2 14*	*4 7*
1 *6*			
2 *3*			

Exercises/Homework:

We show how to factorise a quadratic polynomial in one variable by completing the square. The factors we obtain do not always have integer coefficients.

Example 1.57 *Factorise* $3x^2+19x+20$ *by completing the square.*

In our first step, we write ax^2+bx+c *as* $a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$ *since* $a \neq 0$. *In other words, we factor out the leading coefficient of the polynomial so that inside the brackets we have a monic quadratic polynomial. We have:*

$$3x^2+19x+20 = 3\left(x^2+\tfrac{19}{3}x+\tfrac{20}{3}\right).$$

Next we calculate $\frac{1}{2}$ *of the coefficient of x in the monic quadratic inside the brackets.*

$$\tfrac{1}{2}\tfrac{19}{3} = \tfrac{19}{6}.$$

We place this inside a square: $\left(x+\frac{19}{6}\right)^2$. *Since the expansion of* $\left(x+\frac{19}{6}\right)^2$ *contains* $\left(\frac{19}{6}\right)^2$ *which is not in the original quadratic* $3x^2+19x+20$, *we must subtract* $\left(\frac{19}{6}\right)^2$ *from our new expression so that we get an equal expression.*

$$
\begin{aligned}
3x^2+19x+20 &= 3\left(x^2+\frac{19}{3}x+\frac{20}{3}\right), \\
&= 3\left(\left(x+\frac{19}{6}\right)^2-\left(\frac{19}{6}\right)^2+\frac{20}{3}\right).
\end{aligned}
$$

Now we tidy the remaining terms. Since $-\left(\frac{19}{6}\right)^2+\frac{20}{3}=-\frac{121}{36}$, we have

$$
\begin{aligned}
3x^2+19x+20 &= 3\left(\left(x+\frac{19}{6}\right)^2-\frac{121}{36}\right), \\
&= 3\left(\left(x+\frac{19}{6}\right)^2-\left(\frac{11}{6}\right)^2\right).
\end{aligned}
$$

We have completed the square but it remains to use the property $a^2-b^2=(a+b)(a-b)$ to factorize the quadratic polynomial.

$$
\begin{aligned}
3x^2+19x+20 &= 3\left(x+\frac{19}{6}+\frac{11}{6}\right)\left(x+\frac{19}{6}-\frac{11}{6}\right), \\
&= 3\left(x+5\right)\left(x+\frac{8}{6}\right), \\
&= \left(x+5\right)\left(3x+4\right).
\end{aligned}
$$

.

Example 1.58 *Factorise* $6x^2 + 5x - 56$ *by completing the square.*

.

Exercises/Homework:

We aim to solve equations of the form $ax^2+bx+c=0$ for x, where a,b,c are specific numbers, by first factorising the quadratic expression.

Assuming there are real numbers p,q,r,s such that

$$ax^2+bx+c=(px+q)(rx+s)=0,$$

then we get $px+q=0$ or $rx+s=0$ so that $x=-\frac{q}{p}$ or $x=-\frac{-s}{r}$.

Notice that since $a=pr$ and $a\neq 0$, we have $p,r\neq 0$.

We can find real numbers p,q,r,s such that $ax^2+bx+c=(px+q)(rx+s)$ when $b^2-4ac\geq 0$.

Example 1.59 *Solve the equation $2x^2+3x-27=0$ by factorisation.*

Exercises/Homework:

To solve a quadratic equation by completing the square, we first complete the square, writing $ax^2+bx+c=P^2-Q^2$, where $P=p_1x+p_2$ is a linear polynomial in x and Q is a number.

Since $P^2-Q^2=(P+Q)(P-Q)=0$, we have $P+Q=0$ or $P-Q=0$ and hence solve these equations for x.

Example 1.60 *Solve the equation* $2x^2+3x-27=0$ *by completing the square.*

Example 1.61 *Solve the equation* $15x^2 + 28x - 32 = 0$ *by completing the square.*

Exercises/Homework:

Assume that $ax^2+bx+c=0$, where $a \neq 0$. Factoring a from the expression on the left, we have

$$a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=0.$$

Half of $\frac{b}{a}$ is $\frac{b}{2a}$ so that

Simplifying,

Simplifying again,

.

We call $\Delta = b^2-4ac$ the **discriminant**.

Solving the equation for x,

We have the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 1.62 *Solve the equation* $15x^2 + 28x - 32 = 0$ *by using the quadratic formula.*

Exercises/Homework:

We learn to translate a word problem into a quadratic equation, solve the equation, and write the solution.

Example 1.63 *Mike is* 5 *years younger than Nina. The product of their ages is* 266*. How old are Mike and Nina?*

Example 1.64 *Margaret wants to fence an area of her yard adjacent to the house for a dog yard with three sides of a rectangle consisting of fencing and one side of the rectangle being the side of the house. Margaret has only* 128 *metres of fencing material. What are the dimension of the rectangle that give maximum area to the dog yard?*

Exercises/Homework:

1.3.1 Time Series

A **time-series** is a sequence of points in which the consecutive differences in the independent variable is constant, and the independent variable represents time. When plotted, line segments connect consecutive points of a time-series.

Example 1.65 $S = \{t_n, s_n\} = \{(1.2, -7.8), (1.4, -6.4), (1.6, -4.1), (1.8, -5.3)\}$ *is an example of a time-series. If*

$$\frac{s_{n+1} - s_n}{t_{n+1} - t_n} = m$$

is constant for all n, then the time-series is linear. Since $t_{n+1} - t_n$ *is constant for all n in a time-series, a linear time-series has* $s_{n+1} - s_n = k$ *(a constant) for all n.*

Example 1.66 *Plot the data* $S = \{(1.2, -7.8), (1.4, -6.4), (1.6, -4.1), (1.8, -5.3)\}$*, where the dependent variable represents a percentage change in share price over that time interval.*

Exercises/Homework:

A **bivariate data set** is a set of points (x_n, y_n) relating the dependent variable Y to the independent variable X.

A **scatter plot** is a plot of points in a bivariate data set, where the independent variable is shown on the horizontal axis and the dependent variable is shown on the vertical axis.

An **outlier** is a point of a bivariate data set which is deemed to be isolated from other points of the data set.

A bivariate data set has **correlation** if the points closely fit a line. Correlations are described as **strong correlation** or **weak correlation** depending on how well they fit a line. The data has **positive correlation** if the slope of the line of best fit is positive. The data has **negative correlation** if the slope of the line of best fit is negative.

Example 1.67 *Consider the bivariate data set* $S = \{(1,3),(4,2),(5,1),(6,0),(9.-2)\}$. *Draw a scatter plot and describe any correlation observed.*

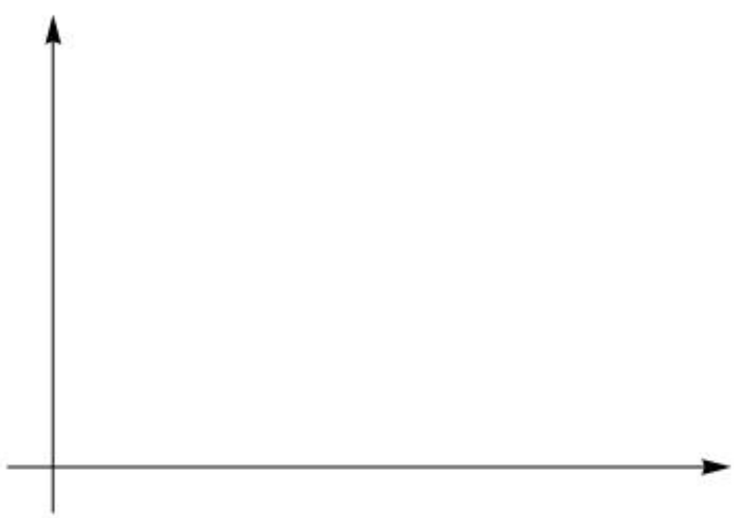
Exercises/Homework:

Given a bivariate data set S, we can guess a line of best fit by choosing two points (x_1, y_1) and (x_2, y_2) and hence a line connecting them such that half of the points in the data set S are above the line and half of the points in the data set S are below the line. The points (x_1, y_1) and (x_2, y_2) do not need to be in S.

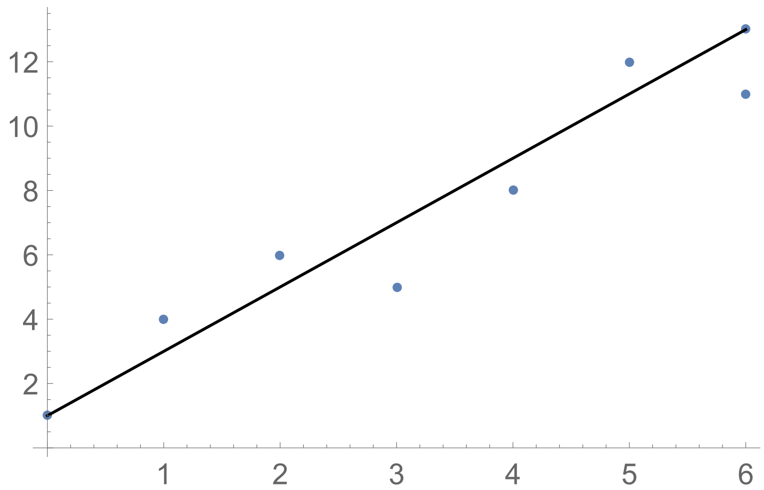
Recall that the slope of the line passing through two points (x_1, y_1) and (x_2, y_2) is given by $Y = mX + c$, where $m = \frac{y_2 - y_1}{x_2 - x_1}$ and $c = y_1 - mx_1$ or $c = y_2 - mx_2$. These c values are the same.

The construction of points in the data range with a line of best fit is called **interpolation**. The construction of points outside the data range with a line of best fit is called **extrapolation**.

Example 1.68 *Consider the bivariate data set* $S = \{(1,4),(2,6),(3,5),(4,8),(5,12),(6,11)\}$. *Draw a scatter plot, guess a line of best fit, and describe any correlation observed.*



Guessing the points $(0,1)$ and $(6,13)$ to find the equation of a line of best fit,



Exercises/Homework:

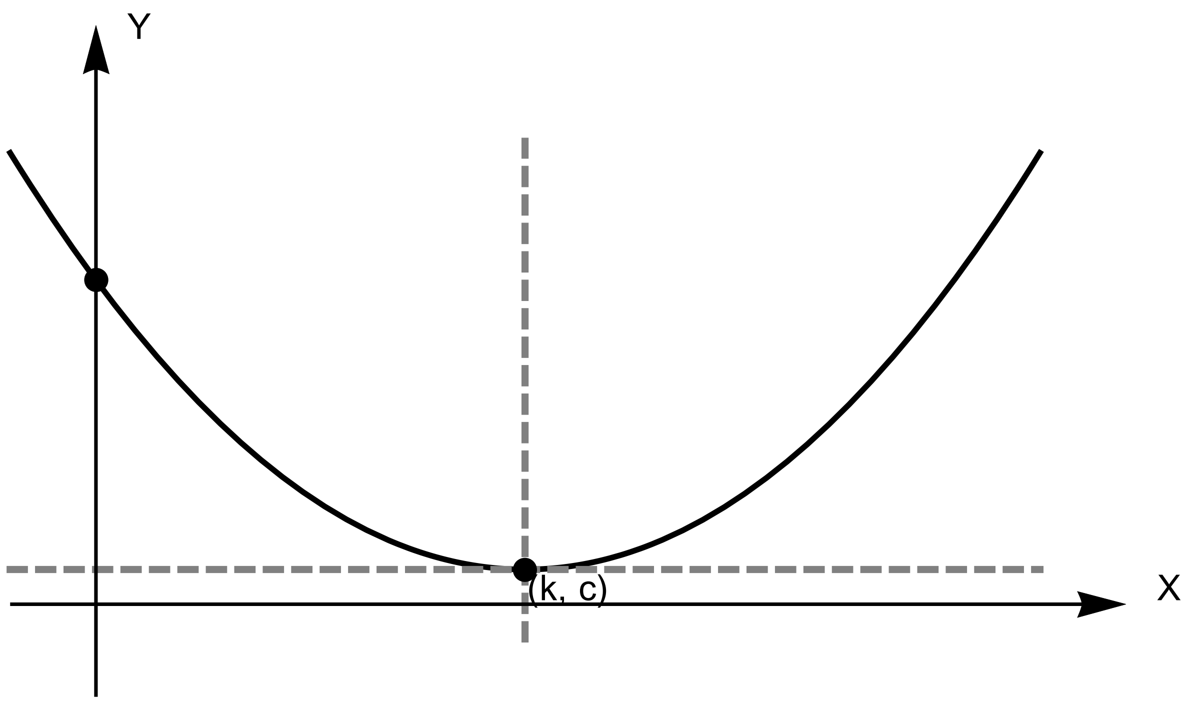
We study parabolas with an equation of the form $y = (x-k)^2 + c$ or $y = -(x-k)^2 + c$, where c and k are particular real numbers.

Note that we can use these techniques more generally since

$$y = (x-k)^2 + c = x^2 - 2kx + \left(k^2 + c\right).$$

Since $(x-k)^2 \geq 0$ and $(x-k)^2 = 0$ precisely when $x = k$, we see that $y = (x-k)^2 + c \geq c$ and $y = c$ precisely when $x = k$. This means that the point (k, c) is the **minimum turning point** of the parabola $y = (x-k)^2 + c$.

Similarly, (k, c) is the **maximum turning point** of the parabola $y = -(x-k)^2 + c$.



Features of $y = (x-k)^2 + c$:

$x - k = 0$ or $x = k$ is the **axis of symmetry**.

When $x = 0$, we have $y = k^2 + c$ so the point $\left(0, k^2 + c\right)$ is the **y-intercept**, which is the intersection with the y-axis.

When $y = 0$, $(x-k)^2 + c = 0$ so $(x-k)^2 = -c$. If $c \leq 0$, then $-c \geq 0$ so that $x - k = \pm\sqrt{-c}$ and we have $x = k \pm \sqrt{-c}$ so that the points $\left(k - \sqrt{-c}, 0\right)$ and $\left(k + \sqrt{-c}, 0\right)$ are the **x-intercepts**, the points of intersection with the x-axis.

Example 1.69 *Consider the parabola shown below. Find the minimum turning point, x-intercepts, the y-intercept, and the axis of symmetry.*

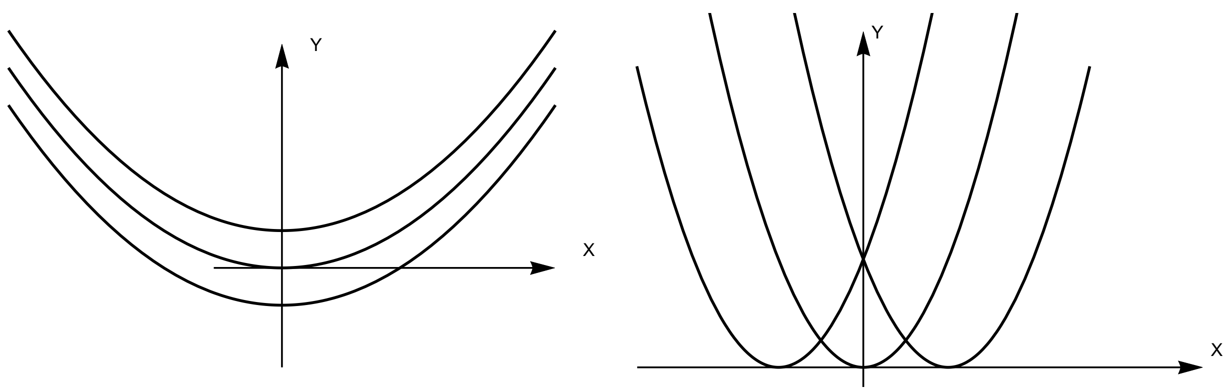
.

Example 1.70 *Sketch the parabola* $y = -(x+4)^2 - 6$*, identifying the maximum turning point, x-intercepts (if real), the y-intercept, and the axis of symmetry.*

Exercises/Homework:

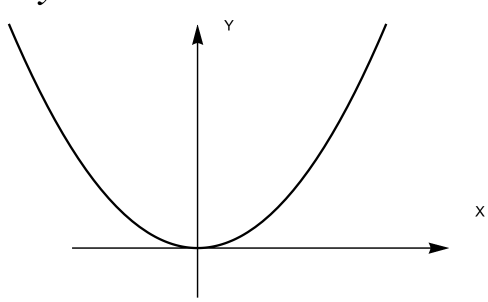
We discuss sketching parabolas of the form $y = a(x-h)^2 + k$ from the point of view of transformations of the parabola $y = x^2$.

Consider up/down and left/right translations of the parabola $y = x^2$.



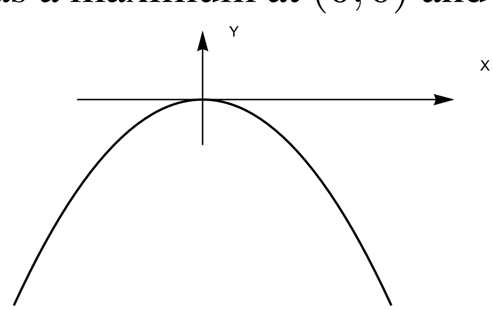
$y = ax^2$ is a **dilation** of $y = x^2$ by a.

If $a > 0$, then the parabola $y = ax^2$ has a minimum at $(0,0)$ and we say the



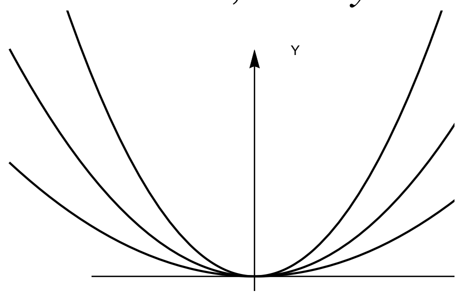
parabola is **concave up**.

If $a < 0$, then the parabola $y = ax^2$ has a maximum at $(0,0)$ and we say the



parabola is **concave down**.

If the dilation has $a > 1$, then $y = x^2$ is stretched upwards to become $y = ax^2$. If the dilation has $1 > a > 0$, then $y = x^2$ is compressed downwards to become $y = ax^2$.



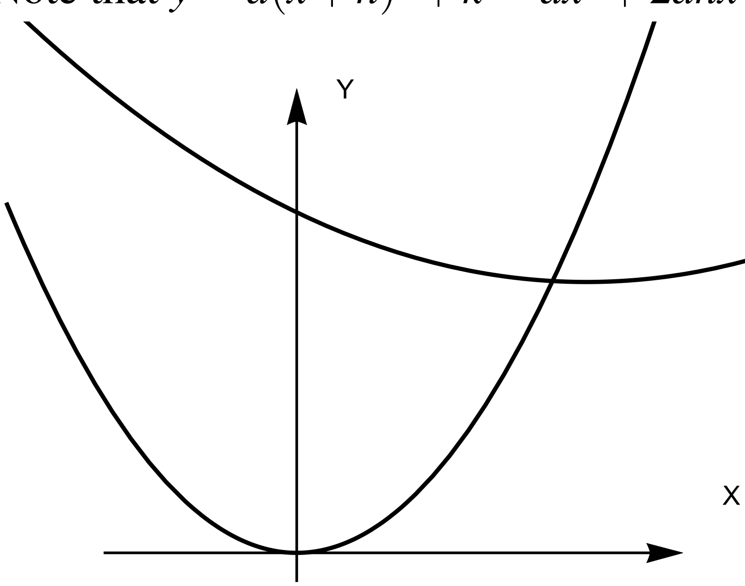
$y = (x - h)^2$ is a **right translation** of $y = x^2$, h units right.

$y = (x + h)^2$ is a **left translation** of $y = x^2$, h units left.

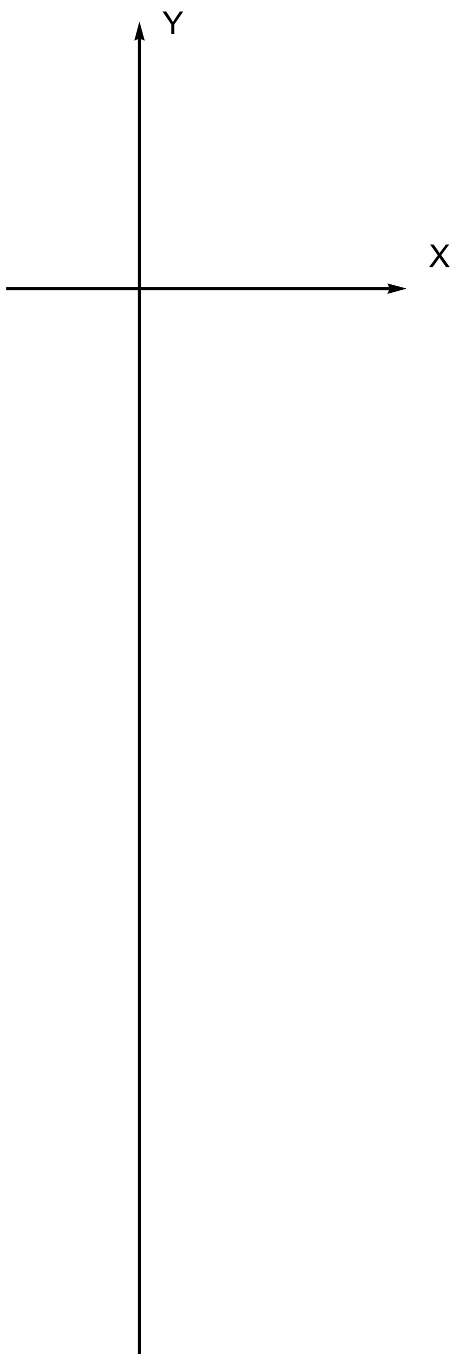
$y = x^2 + k$ is an **upwards translation** of $y = x^2$, k units upwards.

$y = x^2 - k$ is an **downwards translation** of $y = x^2$, k units downwards.

$y = a(x + h)^2 + k$ is a combination of translations and a dilation of $y = x^2$. Note that $y = a(x + h)^2 + k = ax^2 + 2ahx + \left(ah^2 + k\right)$.



Example 1.71 *Describe the transformations that transform $y = x^2$ into $y = -2(x-3)^2 + 4$ and sketch the parabola $y = -2(x-3)^2 + 4$.*



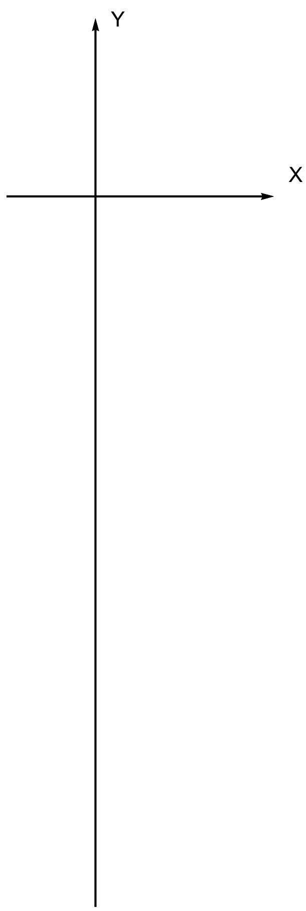
Exercises/Homework:

To sketch $y = ax^2 + bx + c$, with $a \neq 0$, suppose we are able to factorise the right hand side as

$$y = ax^2 + bx + c = (px + q)(rx + s).$$

When $y = 0$ we get the x-intercepts by solving $px + q = 0$ and $rx + s = 0$ so that $x = -\frac{q}{p}$, $x = -\frac{s}{r}$. We have the two x-intercept points $\left(-\frac{q}{p}, 0\right)$ and $\left(-\frac{s}{r}, 0\right)$. When $x = 0$ we have $y = c$ so we have the y-intercept point $(0, c)$.

Example 1.72 *Sketch the parabola* $y = x^2 + 4x - 21$ *by factorisation.*



Exercises/Homework:

If we have $y = a(x-h)^2 + k$, then expanding,

$$y = a(x-h)^2 + k,$$

$$=$$

$$=$$

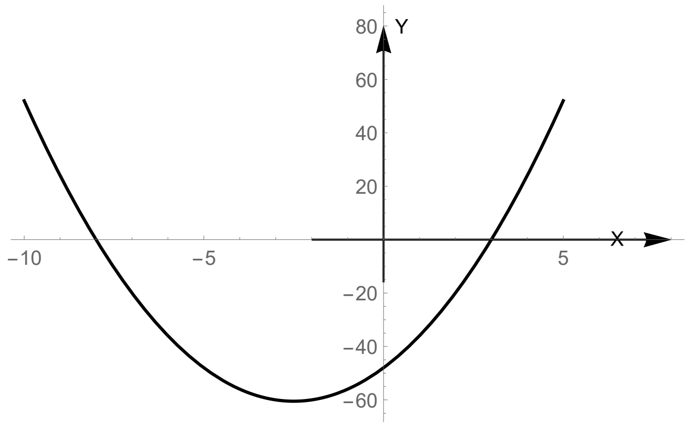
If given $y = ax^2 + bx + c$, then to complete the square is to find h and k such that

$$y = ax^2 + bx + c = a(x-h)^2 + k.$$

We have

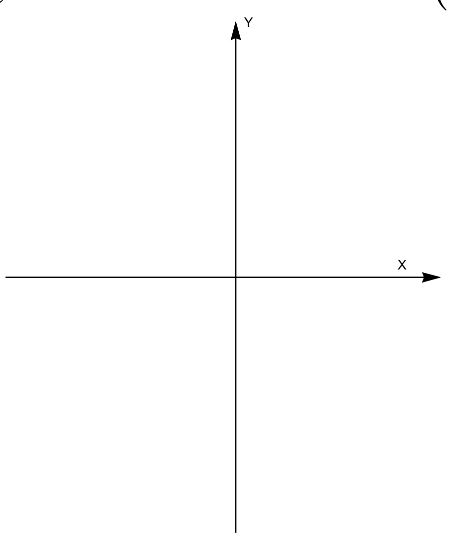
$$h = \qquad , \qquad k = \qquad .$$

Example 1.73 *Let $y = 2x^2 + 10x - 48$. Complete the square using the formulas for h and k above.*



Example 1.74 *Let* $y = 2x^2 + 10x - 48$. *Complete the square without using formulas.*

Example 1.75 *Use the completed square*
$y = 2x^2 + 10x - 48 = 2\left(x + \frac{5}{2}\right)^2 - \frac{121}{2}$ *to sketch the parabola.*



Exercises/Homework:

Let $y = ax^2 + bx + c$.

Completing the square,

$$y = a(\qquad\qquad\qquad\qquad),$$

$$=$$

$$=$$

$$=$$

$$=$$

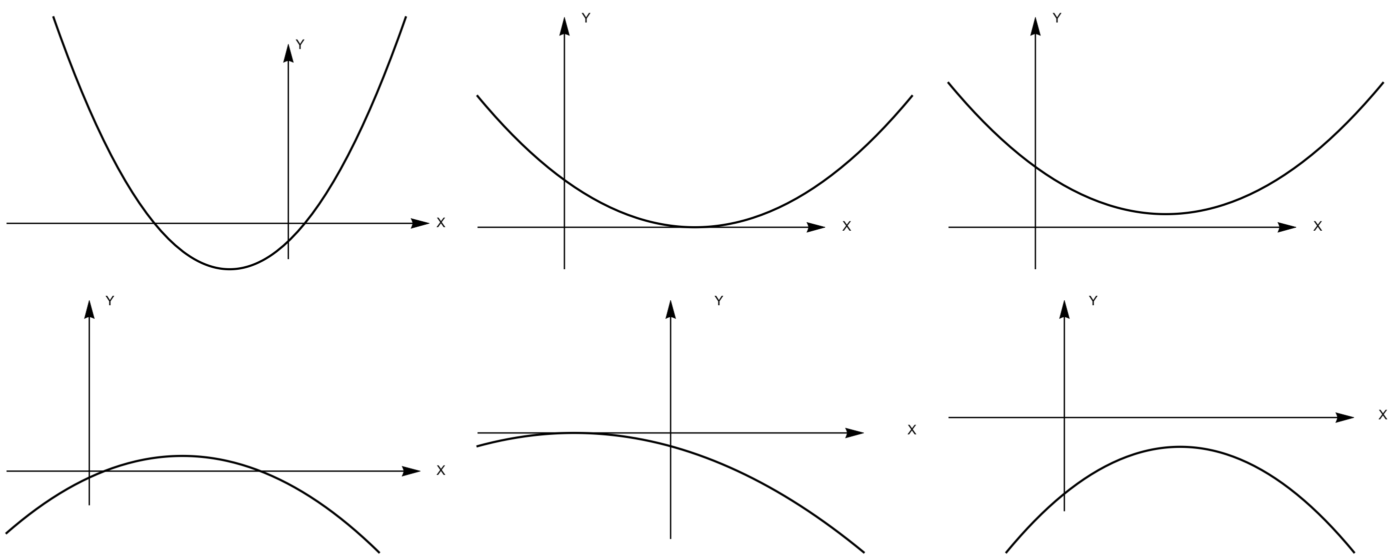
If $y = 0$, then

. This is called the **quadratic formula**.

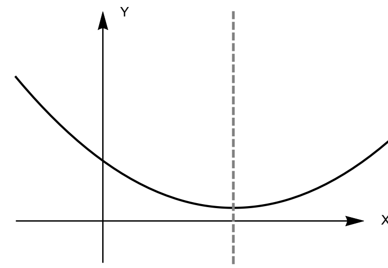
Let $\Delta = b^2 - 4ac$. This is called the **discriminant** of the quadratic polynomial $ax^2 + bx + c$.

Theorem 4 *Let Δ be the discriminant of $f(x) = ax^2 + bx + c$.*

- *If $\Delta > 0$, then $f(x)$ has two distinct real roots; the sketch of the parabola crosses the x-axis.*
- *If $\Delta = 0$, then $f(x)$ has one distinct real root; it is repeated and the quadratic is of the form $f(x) = a(x-h)^2$. The sketch of the parabola touches the x-axis at a tangent.*
- *If $\Delta < 0$, then $f(x)$ has no real roots; there are two non-real roots. The sketch of the parabola is above or below the x-axis entirely.*



$\Delta > 0$, $\qquad$ $\Delta = 0$, $\qquad$ $\Delta < 0$.



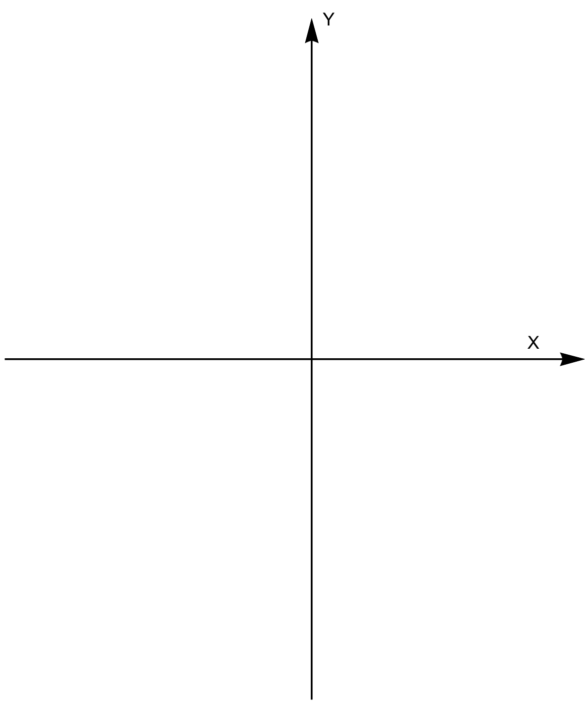
The vertical line $x = \frac{-b}{2a}$ is the **axis of symmetry**.

The point $\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right)$ is the **turning point**; if $a > 0$, then minimum and the parabola is **concave up**, if $a < 0$, then maximum and the parabola is **concave down**.

The y-intercept point is $(0, c)$.

If $\Delta \geq 0$, then the x-intercept point(s) is/are $\left(\frac{-b-\sqrt{\Delta}}{2a}, 0\right)$, $\left(\frac{-b+\sqrt{\Delta}}{2a}, 0\right)$.

Example 1.76 *Sketch* $y = 3x^3 - 18x - 48$ *using the above formulas.*



Exercises/Homework:

Example 1.77 *A stone is tossed vertically upwards at* $15\, m/\, s$ *and acce;erates downwards due to gravity at* $10\, m/\, s^2$ *so that* $u = 15$ *and* $a = -10$. *Let y be the height of the stone after t seconds. If the height y is given by*

$$y = ut + \frac{1}{2}at^2,$$

find the maximum height of the stone and sketch the parabola relating y to t.

Example 1.78 *A gardener wants to erect a rectangular fenced area using* 50 *m of fencing. Determine the dimensions that give maximum area to the fenced region.*

Exercises/Homework:

A **function** f is a rule for sending (assigning) elements of a set X to elements of a set Y, and we put $y = f(x)$ or equivalently say (x, y) is in f, such that:

- For all elements x in X, there is a y in Y with $y = f(x)$, meaning f sends x to y.
- If y_1 and y_2 are in Y with $f(x) = y_1$ and $f(x) = y_2$, then we must have $y_1 = y_2$. This means that f is unambiguous and x is never sent to two different elements of Y. Rephrased, this is known as the vertical line test: If a vertical line intersects with the graph of a function in the x, y-plane, then it intersects exactly once.

If f is a function sending elements of X to elements of Y, we write $f : X \longrightarrow Y$ and state the rule for sending elements x to y; e.g. $f(x) = y^2$. The set X is called the **domain** of the function f and Y is called the **codomain** of the function f. The **range** of a function f is the subset of Y of elements that we sent to Y, the outputs of f.

Example 1.79 *Let* $X = \{1, 2, 3\}$ *and* $Y = \{4, 5, -2, 0\}$. *We define the function* $f : X \longrightarrow Y$ *by* $f(1) = 5$, $f(2) = 4$, $f(3) = -2$. f *is an example of a function with domain* $\{1, 2, 3\}$, *codomain* $\{4, 5, -2, 0\}$, *and range* $\{-2, 4, 5\}$. *Notice that the range is a subset of the codomain.*

Example 1.80 *Let* X *be the set of all real numbers* x *satisfying* $-1 \leq x \leq 1$ *and let* Y *be the set of all real numbers. Does the relation* $x^2 + y^2 = 1$ *correspond to a function?*

Example 1.81 $[-1,1]$ *refers to the set all real numbers satisfying* $-1 \leq x \leq 1$. $\mathbb{R}$ *refers to the set of real numbers. Define* $f : [-1,1] \longrightarrow \mathbb{R}$ *by* $f(x) = -\sqrt{1-x^2}$. *Is* f *a function? Sketch the relation* f.

Example 1.82 *Show that* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *given by* $f(x) = x^2$ *is a function.*

Example 1.83 *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be given by* $f(x) = 3x^2 - 2x + 4$ *be a function. Calculate* $f(1)$, $f(0)$, *and* $f(5)$.

Exercises/Homework:

A **polynomial in one variable** x **with real coefficients** is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where:

- n is a non-negative whole number.
- $a_0, a_1, a_2, \ldots, a_n$ are real numbers called **coefficients**.

The **coefficient of** x^n is the real number multiplying by x^n in an expression.

In this section we will refer to polynomials in one variable x with real coefficients simply as **polynomials**.

Note: A polynomial is a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

The **degree** of a polynomial $f(x)$ is the greatest exponent n among all of the terms of the polynomial. We write $\deg(f) = n$.

a_n is called the **leading coefficient** of f, where $\deg(f) = n$ and a_n is the coefficient of $a_n x^n$.

a_0 is called the **constant coefficient** or **constant term** of f. a_0 is the only term of f that is a real number.

Theorem 5 *If $f(x)$ and $g(x)$ are polynomials and k_1, k_2 are real numbers, then $k_1 f(x)$, $k_2 g(x)$, $k_1 f(x) + k_2 g(x)$, $k_1 f(x) g(x)$, $k_1 f(g(x))$, and $k_1 g(f(x))$ are also polynomials.*

Exercises/Homework:

Example 1.84 *Expand the polynomial* $f(x) = (x+2)(x-3)$ *in two ways.*

Example 1.85 *Expand the polynomial* $f(x) = \left(x^2+2x-5\right)\left(x^2+6x-2\right)$.

Example 1.86 *Expand the polynomial* $f(x) = (x+1)(x-3)(x-1)$.

Exercises/Homework:

Let $f(x)$ and $g(x)$ be polynomials, where $\deg(f) \geq \deg(g)$. We learn how to find polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x)q(x) + r(x),$$

where $\deg(q), \deg(r) \leq \deg(f)$. The polynomial $q(x)$ is called the **quotient** and $r(x)$ is called the **remainder**.

Note: If $f(x) = g(x)q(x) + r(x)$, then $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$.

Example 1.87 *Let* $f(x) = x^3 - 2x^2 - 2x - 3$ *and let* $g(x) = x^2 + x + 1$. *Use long division to calculate* $\frac{f(x)}{g(x)}$.

Example 1.88 *Let* $f(x) = 2x^3 - 9x^2 + 20x - 7$ *and let* $g(x) = x^2 - 3x + 4$. *Use long division to find polynomials* $q(x)$ *and* $r(x)$ *such that* $f(x) = g(x)q(x) + r(x)$.

Example 1.89 *Given that* $x - 1$ *is a factor of* $f(x) = x^3 - 2x^2 - 5x + 6$, *factorise* $f(x)$ *as a product of* 3 *linear factors.*

Exercises/Homework:

In this section we will introduce some important results for polynomial quotients and remainders.

Theorem 6 *Let $f(x)$ be a polynomial. If $f(a) = 0$, then $x - a$ is a factor of $f(x)$.*

If $g(x)$ is a factor of $f(x)$, then we say $g(x)$ **divides** $f(x)$ and equivalently, the remainder $r(x) = 0$ in $f(x) = g(x)q(x) + r(x)$.

Theorem 7 *Let $f(x)$ and $g(x)$ be polynomials with $g(x) \neq 0$ and $deg(f) \geq deg(g)$. There exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = g(x)q(x) + r(x)$, where $r(x) = 0$ or $deg(r) < deg(g)$.*

Corollary 1.1 (Bézout) *Let $f(x)$ and $x - a$ be polynomials, where a is a real number. Then the quotient and remainder polynomials satisfy $r(x) = f(a)$ and*

$$f(x) = (x - a)q(x) + f(a).$$

Example 1.90 *Let $f(x) = 3x^3 + x^2 - 10x - 8$. Factorise $f(x)$ as a product of 3 linear factors by calculating $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(2)$ and using long division.*

Example 1.91 *Show that* x^2+2 *divides* $x^4+x^3+6x^2+2x+8$.

Example 1.92 *Let* $f(x) = x^4+x^3+6x^2+2x+8,\ g(x) = x+1$. *Use Corollary 1.1 to find a polynomial* $q(x)$ *such that*
$f(x) = (x+1)q(x)+f(-1)$.

Exercises/Homework:

Theorem 8 (Null Factor Law) *Let $f(x)$ be a polynomial. If $f(x) = p_1(x)p_2(x)\dots p_n(x) = 0$, where the $p(x)$ are polynomials, then $p_1(x) = 0$ or $p_2(x) = 0$ or $\dots$ $p_n(x) = 0$.*

Theorem 9 (Fundamental Theorem of Algebra) *Let $f(x)$ be a polynomial of degree n. Then there are exactly n complex numbers $\alpha = a + b\sqrt{-1}$, where a and b are real numbers, and it is possible that $b = 0$ so the α may include real numbers, such that*

$$f(x) = k\,(x-\alpha_1)\,(x-\alpha_2)\dots(x-\alpha_n)\,,$$

and it is possible that the α coincide. Any non-real roots of $f(x)$ occur in conjugate pairs, meaning if $f(a+b\sqrt{-1}) = 0$, then $f(a-b\sqrt{-1}) = 0$.

Example 1.93 *Find all roots α of $f(x) = 2x^3 + x^2 - 4x - 3$ and reflect on the fundamental theorem of algebra.*

Example 1.94 *Find all roots* α *of* $f(x) = x^3 + x^2 + x + 1$ *and reflect on the fundamental theorem of algebra.*

Exercises/Homework:

Under Construction

References

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