Year 11 Mathematical Methods

Student Workbook and Teaching Template

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https://mathofcourse.com/

Contents

1. **Imperative:** Print this pdf document or be prepared to annotate the pdf with a tablet. Some blank spaces for writing are a little small for large writing. If you cannot do either of these annotation options, then write notes on blank paper, noting the relevant position within the typed course notes. As you watch the intructional videos, write notes in the blank spaces. This step is very important.

2. The instructor should write exercises from an appropriate textbook where the text says **Exercises/Homework**.

3. **Optional but highly recommended:** Purchase and use *Mathematica* or obtain it through your institution. We will occasionally use this to display various graphics and verify calculations. All graphics shown in this document were produced with *Mathematica*. You will most likely find it very helpful with your studies. It is a symbolic computation tool which has full programming capabilities. E.g. Try writing

   ```
   Expand[(x+y)^3]
   ```

   then press Shift+Enter or

   ```
   s = 0;
   For[i = 0, i < 6, i++, s = s + i; Print[s]]
   ```

   You can call on *Wolfram alpha* from with in it by beginning a cell with = =.
   If your school has a license, to install this on your machine, visit:
   `wolfram.com/siteinfo/`
   Get *Mathematica* Desktop.
   Create a Wolfram ID, and download and install the software.

1 Year 11 Mathematical Methods

Year 11. Under Construction

1.1 Term 1

Under Construction

1.2 Term 2

Under Construction

1.3.1 Limits and the First Derivative

Let $f(x)$ be a function that is defined near a real number a. If the function $f(x)$ is the real number L near a, then we write

$$\lim_{x \longrightarrow a} f(x) = L.$$

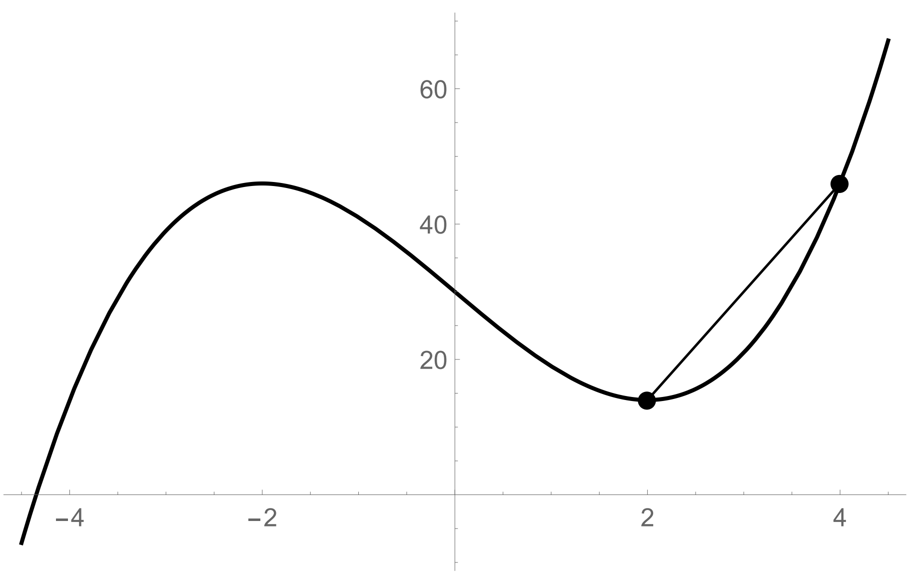
We say, the limit as x approaches a of $f(x)$ is L.

More formally, the real number L is the limit of the sequence $a_1, a_2, \ldots$ if and only if for every real number $\varepsilon > 0$, there exists a natural number N such that for all $n > N$, we have $|a_n - L| < \varepsilon$.

Example 1.1 *The function* $f(x) = \frac{x^3-1}{x-1}$ *is not defined when* $x = 1$. $f(x)$ *is defined for all real* x *except for* $x = 1$ *and hence defined for* x *near* 1. *Calculate* $\lim_{x \longrightarrow 1} f(x)$.

Theorem 1 *(Properties of Limits) Let a, k_1, k_2 be particular real numbers and $f(x), g(x)$ are functions such that $\lim_{x \longrightarrow a} f(x)$ and $\lim_{x \longrightarrow a} g(x)$ exist. Then:*

- $\lim_{x \longrightarrow a} k_1 f(x) = k_1 \lim_{x \longrightarrow a} f(x)$.
- $\lim_{x \longrightarrow a} (k_1 f(x) + k_2 g(x)) = k_1 \lim_{x \longrightarrow a} f(x) + k_2 \lim_{x \longrightarrow a} g(x)$.
- $\lim_{x \longrightarrow a} (f(x)g(x)) = \left(\lim_{x \longrightarrow a} f(x)\right)\left(\lim_{x \longrightarrow a} g(x)\right)$.
- *If* $\lim_{x \longrightarrow a} g(x) \neq 0$, *then* $\lim_{x \longrightarrow a} \left(\frac{f(x)}{g(x)}\right) = \frac{\left(\lim_{x \longrightarrow a} f(x)\right)}{\left(\lim_{x \longrightarrow a} g(x)\right)}$.
- *If $f(x)$ is defined at $x = a$, then* $\lim_{x \longrightarrow a} f(x) = f(a)$.



The **first derivative** $f'(x)$ or **derivative** for short, is defined by the limit

$$f'(x) = \lim_{h \longrightarrow 0} \frac{1}{h} (f(x+h) - f(x)).$$

This gives a function representing the slope of the function $f(x)$ at any particular x value in the domain of f. To **differentiate** is to find $f'(x)$.

Example 1.2 *Use the definition of the derivative to find* $f'(x)$ *for the function* $f(x) = x^2$.

Example 1.3 *Use the definition of the derivative to find* $f'(x)$ *for the function* $f(x) = x^3 - 4x + 3$.

Exercises/Homework:

Consider the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

We learn how to use rules to differentiate polynomial functions to obtain the polynomial function $f'(x)$.

Let $f(x) = kx^n$, where k is a particular real number and n is a non-negative integer. Using the limit definition of $f'(x)$,

$$\begin{aligned} f'(x) &= \lim_{h \longrightarrow 0} \frac{1}{h}(f(x+h) - f(x)) \\ &= . \qquad\qquad . \\ &= \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

Rule: To differentiate $f(x) = kx^n$, bring down n and subtract 1 from the exponent so that $(kx^n)' = knx^{n-1}$.

Note: Since $\lim_{x\longrightarrow a}(p(x)+q(x)) = \lim_{x\longrightarrow a} p(x) + \lim_{x\longrightarrow a} q(x)$ and the derivative of a constant is 0, the derivative of

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is

$$f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1.$$

Notation: $f'(x) = (f(x))' = \frac{d}{dx}(f(x))$.

Example 1.4 *Let $f(x) = x^3 + 3x - 7$. Calculate $f'(x)$ using both the limit definition and the rules for differentiating polynomials.*

Example 1.5 *Let $f(x) = -3x^5 + 2x^3 - 12x^2 + 14x - 32$. Calculate $f'(x)$ using rules for differentiating polynomials.*

Exercises/Homework:

Let $f(x) = kx^{-n}$, where n is a positive integer and k is a particular non-zero real number. Using the limit definition,

$$
\begin{aligned}
f'(x) &= \lim_{h\longrightarrow} \frac{1}{h}\left(f(x+h) - f(x)\right), \\
&= . \qquad\qquad . \\
&= \\
&= \\
&= \\
&= \\
&= \\
&=
\end{aligned}
$$

Theorem 2 *Let $f(x) = kx^n$, where k and n are real numbers.*

- *If $n = 0$, then $f'(x) = 0$. (The derivative of a constant is zero.)*
- *If $n \neq 0$, then $f'(x) = nkx^{n-1}$.*

Example 1.6 *Differentiate the function* $f(x) = 2x^3 - \frac{5}{x} + 14x - 8.$

Example 1.7 *Differentiate the function* $f(x) = -12x^5 + 3x^2 + \frac{2}{x^2} - \frac{1}{x}.$

Exercises/Homework:

We plot $y = f'(x)$ for functions $f(x)$ and answer questions on the sign of the first derivative of a function.

Example 1.8 *Let* $f(x) = 2x^3 - 4x + 5$. *Plot* $y = f'(x)$ *and determine where* $f'(x)$ *is positive, negative, and zero. Plot* $y = f(x)$ *also and consider any turning points in the context of the sign of* $f'(x)$.

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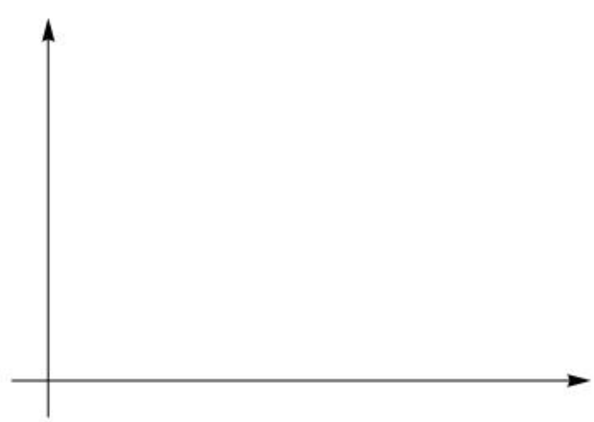

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Example 1.9 *Let* $f(x) = x^2 + 3x - 4$. *Plot* $y = f(x)$ *and* $y = f'(x)$ *on the same graph. State where* $f'(x) > 0$ *and* $f'(x) < 0$.

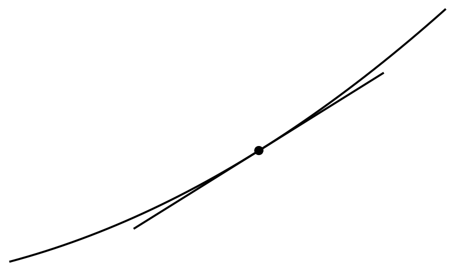
Let a and b be real numbers. We say that $f(x)$ is **increasing** on the interval (a,b) if for all x: $a < x < b$, $f'(x) > 0$. $f(x)$ is **increasing** on the interval $[a,b]$ if for all x: $a \leq x \leq b$, $f'(x) > 0$.

We say that $f(x)$ is **decreasing** on the interval (a,b) if for all x: $a < x < b$, $f'(x) < 0$. $f(x)$ is **decreasing** on the interval $[a,b]$ if for all x: $a \leq x \leq b$, $f'(x) < 0$.

Note: If $f(x)$ is increasing of $[a,b]$, then $f(a) < f(b)$. If $f(x)$ is decreasing of $[a,b]$, then $f(a) > f(b)$.

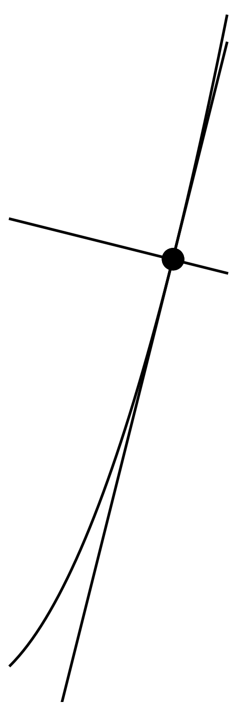
Exercises/Homework:

Let $y = f(x)$ be a curve, where $f(x)$ is a function. The **tangent line** at the point (a,b) is the line which intersects with the curve at exactly the one point (a,b) and has slope equal to the slope of the curve at the point (a,b).



To find the equation of a tangent line at (a,b),

The **normal line** at the point (a,b) is the line perpendicular to the tangent line at the point (a,b) passing through (a,b).



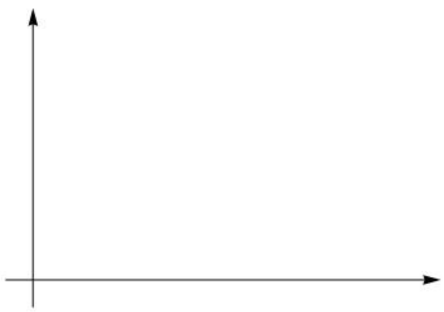
Recall that the lines $y = mx + c$ and $y = -\frac{1}{m}x + d$ are perpendicular.

Example 1.10 *Let* $y = f(x) = 3x^2 - 4x + 4$. *Calculate the tangent line and the normal line at the point* $(1,3)$.

Exercises/Homework:

The **average rate of change** of $y = f(x)$ over the interval $[a,b]$ is

$$m_{\text{av.}} = \frac{f(b)-f(a)}{b-a}.$$



The **instantaneous rate of change** of $f(x)$ at the point where $x = a$ is $f'(a)$.

If $f'(a)$ is positive, then the function is **increasing** at $x = a$.
If $f'(a)$ is negative, then the function is **decreasing** at $x = a$.

Example 1.11 *Let* $y = -4x^2 + 5x - 12$. *Determine the average rate of change of* $f(x)$ *over the interval* $[-1,2]$.

Example 1.12 *The position of a particle is given by*

$$y = f(t) = 5t^3 - 2t + 6,$$

where t is time in seconds and y is the position in metres. The velocity of the particle is $v = f'(t)$. Determine the velocity after 1 *second,* 2 *seconds,* 3 *seconds.*

Exercises/Homework:

Let $y = f(x)$, where $f(x)$ is a function of x. The point (a, b) is a **stationary point** if $f'(a) = 0$, or equivalently, $\left.\frac{dy}{dx}\right|_{x=a} = 0$.

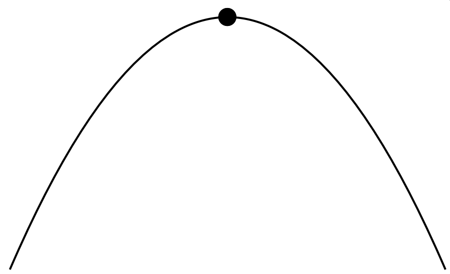
Example 1.13 *Find all stationary points of the curve* $y = 2x^3 - 15x + 8$.

Example 1.14 *The curve* $y = ax^2 + bx + c$*, where* $a \neq 0$*,* a, b, c *are particular real numbers, has one stationary point* $(1, 2)$*. Show that* $y = (c-2)(x-1)^2 + 2$*.*

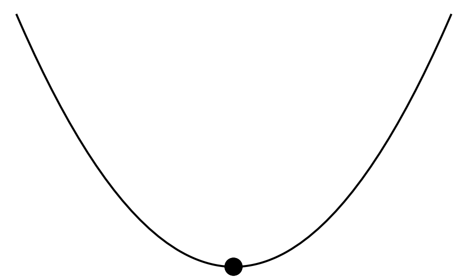
Exercises/Homework:

Recall that the point (a,b) of the curve $y = f(x)$ is a stationary point if $f'(a) = 0$.

The stationary point (a,b) is called a **local maximum** if $f'(x) > 0$ for $x = a - \varepsilon$ (immediately left of (a,b)), where ε is an arbitrarily small positive real number, and $f'(x) < 0$ for $x = a + \varepsilon$ (immediately right of (a,b)).



The stationary point (a,b) is called a **local minimum** if $f'(x) < 0$ for $x = a - \varepsilon$ (immediately left of (a,b)), where ε is an arbitrarily small positive real number, and $f'(x) > 0$ for $x = a + \varepsilon$ (immediately right of (a,b)).



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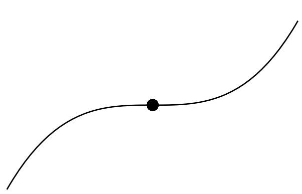
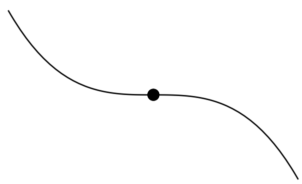
Such stationary points are called **turning points**.

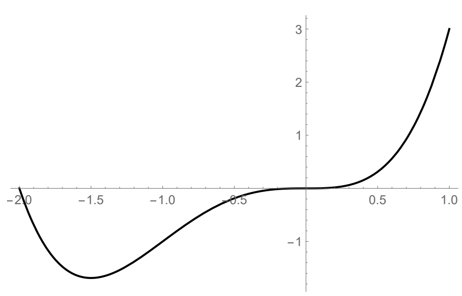
A stationary point (a,b) such that $f'(x) > 0$ for $x = a - \varepsilon$ (immediately left of (a,b)), where ε is an arbitrarily small positive real number, and $f'(x) > 0$ for $x = a + \varepsilon$ (immediately right of (a,b)) is called an **inflection point**.



A stationary point (a,b) such that $f'(x) < 0$ for $x = a - \varepsilon$ (immediately left of (a,b)), where ε is an arbitrarily small positive real number, and $f'(x) < 0$ for $x = a + \varepsilon$ (immediately right of (a,b)) is called an **inflection point**.



Example 1.15 *The curve* $y = x^4 + 2x^3$ *has a local minimum and an inflection point. Find them and reflect on the definitions of local minimum and inflection point.*

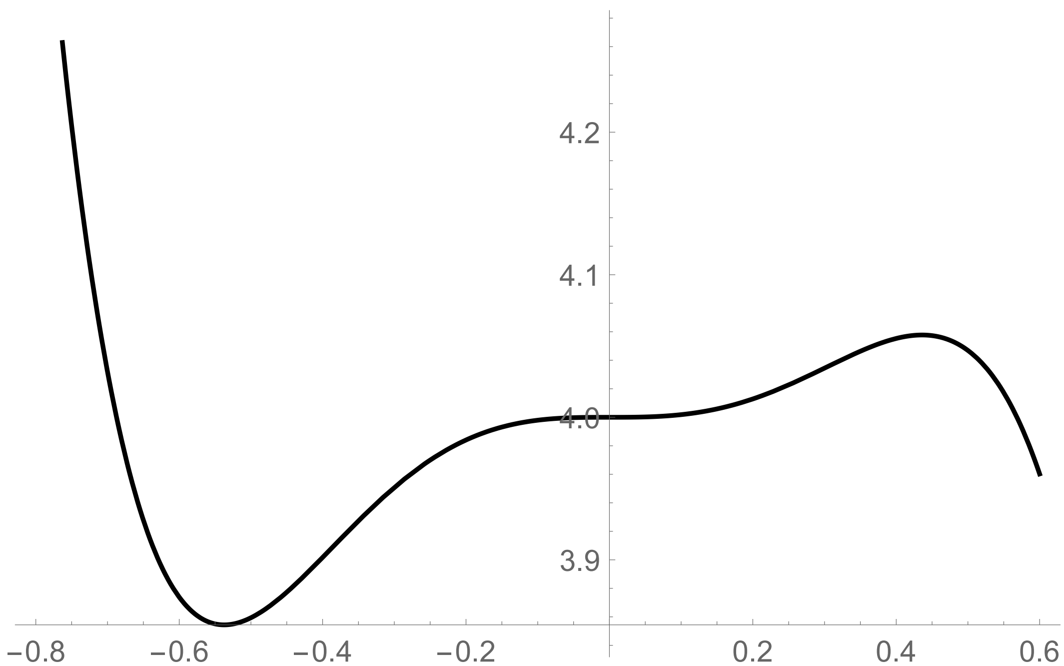


Exercises/Homework:

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a function and let $y = f(x)$ be a curve/ Suppose the curve has local maxima points (p_1, q_1), (p_2, q_2), $\ldots, (p_n, q_n)$. The **global maximum point** (u, v) is the unique point such that $a \leq u \leq b$ and

$$v \geq f(a), f(b), f(p_1), f(p_2), \ldots, f(p_n).$$

In other words, the point in the domain with the greatest y-value.



The **global minimum point** is defined analogously; the point in the domain with the least y-value.

Example 1.16 *Visually inspect the curve above to identify local maxima, local minima, inflection points, and the global maximum and the global minimum on the domain* $[-0.8, 0.6]$.

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Example 1.17 *Find the global maximum and global minimum of the curve* $y = f(x) = x^4 + 2x^3 - 6x^2 - 12x$ *over the interval* $[-3, 3]$.

Exercises/Homework:

The **position** of a particle or object is a point $(t, x(t))$ or $(t, x(t), y(t))$, $(x(t), y(t))$, etc. which depends on time. If the object moves on a linear trajectory, then we can express the position x as a function of time as position $= x = x(t)$.

Example 1.18 *A ball is tossed vertically upwards at $t = 0$ seconds from 2 metres above the ground. The elevation (position) of the ball in metres at t seconds is given by $y(t) = -5t^2 + 4t + 2$, $t \geq 0$. Plot y versus t with t on the horizontal axis.*

The **instantaneous velocity** or just **velocity** of a particle with position $x(t)$ is $v = x'(t) = \frac{dx}{dt}$, the first derivative of the position $x(t)$ with respect to t.

The **average velocity** of a particle with position $x(t)$ is given by

$$v_{\text{av}} = \frac{x_2 - x_1}{t_2 - t_1}.$$

The **speed** of a particle is the magnitude $|x'(t)|$ (absolute value). The speed of the particle with velocity $(x'(t), y'(t))$ is the magnitude

$$\| (x'(t), y'(t)) \| = \sqrt{(x'(t))^2 + (y'(t))^2}.$$

The **average speed** of a particle with position $x(t)$ is given by $|v_{\text{av}}|$.

Example 1.19 *Calculate the velocity, speed, and average speed of the ball with position* $y(t) = -5t^2 + 4t + 2$, $t \geq 0$ *from vertical launch until it hits the ground again.*

The **instantaneous acceleration** or just **acceleration** of a particle is the first derivative of the velocity of the particle,

$$a(t) = v'(t) = (x'(t))' = x''(t).$$

The **average acceleration** of a particle is $a_{\text{av}} = \frac{v_2 - v_1}{t_2 - t_1}$, where v_1 is the initial velocity and v_2 is the final velocity of the particle.

Exercises/Homework:

Consider the function $y = \left(2x - 4x^3 + 3\right)^5$. Since this is a polynomial function of x, we could expand this before we differentiate the function. The **chain rule** gives us another way to calculate the derivative of a function.

In this example, letting $u = 2x - 4x^3 + 3$,

Let $F(x)$ be a function of x such that there are other functions $g(x)$ and $f(x)$ satisfying

$$F(x) = f(g(x)) = (f \circ g)(x).$$

The **chain rule** states that

$$F'(x) = (f(g(x)))' = f'(g(x)) \cdot g'(x).$$

Alternatively in Leibniz's notation,

$$F'(x) = (f(g(x)))' = \frac{df}{dg}\frac{dg}{dx}$$

or

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx},$$

where y is a function of u and u is a function of x.

Note: $\frac{dy}{dx}$ is **not a fraction** but the fact that the symbols look like fractions is a very useful aide to memory.

Note: For small Δx, $\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$.

Example 1.20 *Let* $y = 3\left(5x^2 - 7x + 2\right)^6$. *Calculate* $\frac{dy}{dx}$.

Example 1.21 *Let* $y = \left(3x^2 - \frac{2}{x^2}\right)^4$. *Calculate* $\frac{dy}{dx}$.

Exercises/Homework:

Recall that to differentiate $y = x^a$, where $a \neq 0$ is a rational number, we have

$$\frac{dy}{dx} = ax^{a-1}.$$

Example 1.22 *Consider the function $y : \mathbb{R}^{(>0)} \longrightarrow \mathbb{R}$ by $y(x) = \sqrt{x}$. Calculate $y'(x) = \frac{dy}{dx}$.*

Why do we have

$$\frac{d}{dx}(x^a) = ax^{a-1}?$$

If $\frac{dx}{dy} \neq 0$, then $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$, so if $y = x^{\frac{1}{q}}$, where q is a positive integer, then

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Example 1.23 *Let* $y = x^{-4/5}$. *Calculate* $\frac{dy}{dx}$.

Example 1.24 *Let* $y = \sqrt[3]{x^2+1}$. *Calculate* $\frac{dy}{dx}$ *using the chain rule.*

Exercises/Homework:

The aim of this section is to use the sign of the first derivative to help us to sketch the graph of a curve.

Example 1.25 *Sketch the curve* $y = \frac{x^2+1}{x+1}$.

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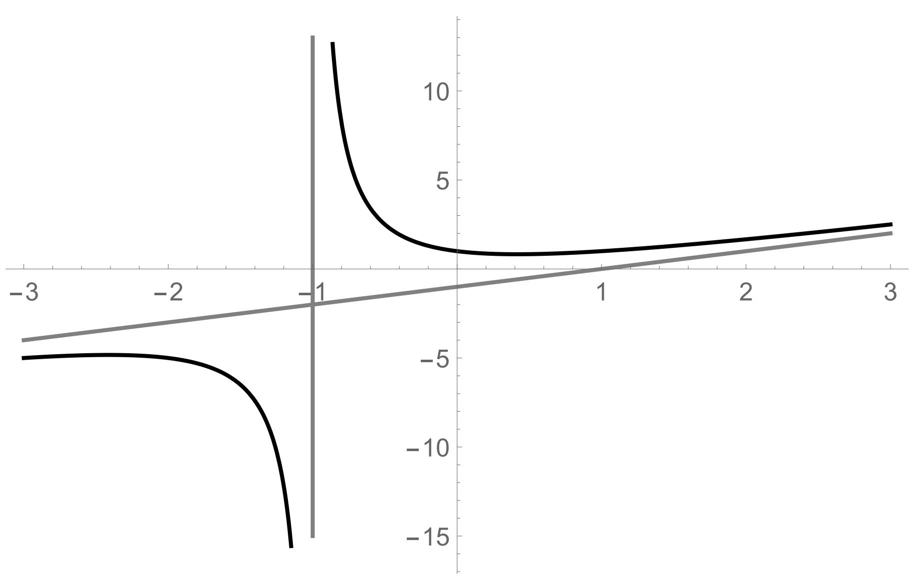

Exercises/Homework:

Suppose that $y(x)$ can be expressed as a product so that $y = uv$ for some functions $u(x)$ and $v(x)$. Then the **product rule** states that

$$\frac{dy}{dx} = \frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} = vu'(x) + uv'(x).$$

Example 1.26 *Calculate* $\frac{dy}{dx}$ *for* $y = x^3(2x-7)^4$ *using the product rule.*

Example 1.27 *Calculate* $\frac{dy}{dx}$ *for* $y = x^2\left(x + \frac{1}{x}\right)$ *using the product rule.*

Example 1.28 *Calculate* $\frac{dy}{dx}$ *for* $y = (x+1)\sqrt{x^2-1}$ *using the product rule and the chain rule.*

Exercises/Homework:

Let $y = \frac{u}{v} = uv^{-1} = uw$, where $w = v^{-1}$.

By the product rule, The rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)$$

is called the **quotient rule**.

Example 1.29 *Use the quotient rule to differentiate* $y = \frac{x+2}{3x+4}$.

Example 1.30 *Use the quotient rule and chain rule to differentiate* $y = \frac{2x+5}{(x-6)^{1/2}}$.

Exercises/Homework:

Under Construction

References

[1] D. Greenwood, S. Woolley, J. Goodman, J. Vaughan, S. Palmer, Essential Mathematics for the Australian Curriculum, 4th Ed. Cambridge, 2024.

[2] Michael Evans, Kay Lipson, Douglas Wallace, David Greenwood, Mathematical Methods Units 1 & 2, Cambridge Senior Mathematics for Queensland, 2nd Ed., 2024.

[3] A. J., Washington. Basic Technical Mathematics with Calculus, SI Version.