

Years 10 and 11 Mathematical Methods

Student Workbook and Teaching Template

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Contents

Instructions	ii
1 Year 10 Mathematical Methods	1
1.1 Term 1	1
1.1.1 Surds and Index Laws	1
1.1.2 Arithmetic of Surds	3
1.1.3 Index Laws	4
1.1.4 Fractional Indices	5
1.1.5 Solving Simple Equations in One Variable	7
1.1.6 Substitution	8
1.1.7 Solving Inequalities	9
1.1.8 Linear Equations Involving Fractions	12
1.1.9 Parallel Lines and Perpendicular Lines	14
1.1.10 Distances Between Points and Midpoints of Line Segments	17
1.1.11 Simultaneous Equations by Substitution	20
1.1.12 Simultaneous Equations by Elimination	22
1.1.13 Applications of Simultaneous Equations	24
1.2 Term 2	25
1.2.1 Introduction to Trigonometry	25
1.2.2 Finding Angles in Right Triangles	27
1.2.3 Applications of Trigonometry	29
1.2.4 Directions and Bearings	30
1.2.5 The Unit Circle	32
1.2.6 Exact Surd Values for Trigonometric ratios	34
1.2.7 Expanding Algebraic Expressions	36
1.2.8 Factorizing Polynomials	38
1.2.9 Factorizing Monic Quadratic Polynomials	39
1.2.10 Factorizing Non-monic Quadratics	41
1.2.11 Completing the Square with Quadratics	43
1.2.12 Solving Quadratics by Factorization	46
1.2.13 Solving Quadratics by Completing the Square	47

1.2.14	Solving Quadratics with a Formula	3.	49
1.2.15	Applications of Quadratics		51
1.3	Term 3		53
1.3.1	Time Series		53
1.3.2	Two-variable Data and Scatter Plots		54
1.3.3	Guessing a Line of Best Fit		55
1.3.4	Introduction to Parabolas		57
1.3.5	Sketching Parabolas using Transformations of $y = x^2$		60
1.3.6	Sketching Parabolas using Factorization		63
1.3.7	Sketching Parabolas by Completing the Square		64
1.3.8	Sketching Parabolas using Formulas		66
1.3.9	Applications of Parabolas		69
1.3.10	Introduction to Functions		71
1.3.11	Introduction to Polynomial Functions		73
1.3.12	Expanding Polynomials		74
1.3.13	Polynomial Long Division		75
1.3.14	Quotients and Remainders for Polynomials		77
1.3.15	Roots of Polynomials		79
1.4	Term 4		81
1.4.1	Logarithms		81
1.4.2	Log Scale		83
1.4.3	Log Rules		85
1.4.4	Solving Exponential Equations		87
1.4.5	Probability		88
1.4.6	Sets, Venn Diagrams, and Two-way Tables		90
1.4.7	Probabilities Corresponding to the Union and Intersection of Sets		93
1.4.8	Conditional Probability		95
1.4.9	Two-step Experiments		97
1.4.10	Tree Diagrams		98
1.4.11	Independent Events		100

2	Year 11 Mathematical Methods	104
2.1	Term 1	101
2.2	Term 2	101
2.3	Term 3	102
2.3.1	Limits and the First Derivative	102
2.3.2	Differentiating Polynomials	105
2.3.3	Differentiating kx^{-n} , where $n > 0$	107
2.3.4	Plotting $y = f'(x)$	109
2.3.5	Tangent Lines and Normal Lines	111
2.3.6	Rates of Change and Average Rates of Change	113
2.3.7	Points with Derivative 0	115
2.3.8	Classifying Stationary Points with the First Derivative	117
2.3.9	Optimisation Problems	119
2.3.10	Motion of a Particle	121
2.3.11	The Chain Rule for Differentiation	123
2.3.12	The Derivative of x^a , where a is Rational	125
2.3.13	Using Derivatives in Curve Sketching	127
2.3.14	The Product Rule for Differentiation	129
2.3.15	The Quotient Rule for Differentiation	131
2.4	Term 4	133
2.4.1	Sketching $y = a^x$, $a > 0$	133
2.4.2	Plotting $y = e^x$ and Translations	135
2.4.3	Exponential Equations	139
2.4.4	Logarithms, Log Rules and Solving Equations with Logs	140
2.4.5	Plotting $y = \log_a(x)$ and Translations	143
2.4.6	Differentiating e^x	145
2.4.7	Differentiating $\ln(x)$	148
2.4.8	Equations of Exponential and Log Curves from Geometric Features	150
2.4.9	Differentiating Trigonometric Functions $\sin(x)$, $\cos(x)$, and $\tan(x)$ and the Chain, Product, and Quotient Rules	152
2.4.10	Acceleration of a Particle using the Second Derivative	157
2.4.11	Exercises on Stationary Points	158

2.4.12	The Second Derivative Test	159
2.4.13	Sketching Curves using the Second Derivative	163
2.4.14	Revising Global Maxima and Minima	166
2.4.15	Revising Optimisation	170

3	Books & Notes	173
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1. **Imperative:** Print this pdf document or be prepared to annotate the pdf with a tablet. Some blank spaces for writing are a little small for large writing. If you cannot do either of these annotation options, then write notes on blank paper, noting the relevant position within the typed course notes. As you watch the instructional videos, write notes in the blank spaces. This step is very important.
2. The instructor should write exercises from an appropriate textbook where the text says **Exercises/Homework**.
3. **Optional but highly recommended:** Purchase and use *Mathematica* or obtain it through your institution. We will occasionally use this to display various graphics and verify calculations. All graphics shown in this document were produced with *Mathematica*. You will most likely find it very helpful with your studies. It is a symbolic computation tool which has full programming capabilities. E.g. Try writing

```
Expand[ (x+y) ^3]
```

then press Shift+Enter or

```
s = 0;  
For[i = 0, i < 6, i++, s = s + i; Print[s]]
```

You can call on *Wolfram alpha* from within it by beginning a cell with `==`.

If your school has a license, to install this on your machine, visit:

wolfram.com/siteinfo/

Get *Mathematica* Desktop.

Create a Wolfram ID, and download and install the software.

1.1 Term 1

1.1.1 Surds and Index Laws

The decimal expansion of $\sqrt{2}$ does not terminate, nor repeat.

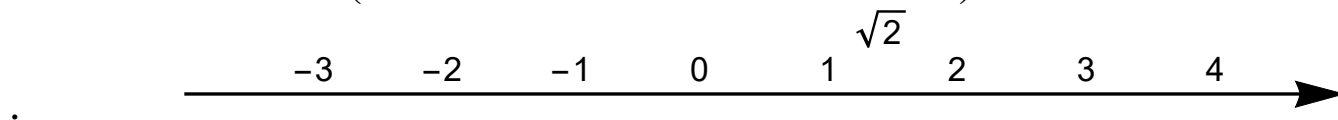
$$\sqrt{2} = 1.41421356237309504880168872420969807856967187537695\dots$$

There are no whole numbers a, b with $b \neq 0$ and $\sqrt{2} = \frac{a}{b}$. We say $\sqrt{2}$ is **irrational**, meaning not rational.

Next we identify several important sets of numbers and notation for them.

$\mathbb{Z}$ Integers This is the set of all whole numbers $\dots -3, -2, -1, 0, 1, 2, \dots$

$\mathbb{R}$ Real Numbers (All numbers on the number line.)



$\mathbb{Q}$ Rational Numbers $\left\{ \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0 \right\}$, meaning the set of all fractions a/b , where a and b are elements of $(\in)$ the set of integers ($\mathbb{Z}$) and b is non-zero. Note: Integers are also rational numbers ($\mathbb{Z} \subset \mathbb{Q}$).

$\mathbb{R} - \mathbb{Q}$ or $\mathbb{R} \setminus \mathbb{Q}$ Real Irrational Numbers (Real numbers that are not rational. e.g. π and $\sqrt{2}$ are real numbers but not rational numbers.)

Note: A real number is rational if and only if it has a repeating decimal expansion or a terminating decimal expansion.

A **surd** is a sum of expressions of the form $a\sqrt[n]{b}$.

2

Example 1.1 Which of the following real numbers are surds?

$$\sqrt{36} \quad \sqrt{19} \quad \sqrt{\frac{1}{25}} \quad \sqrt[3]{21} \quad 4\pi \quad \sqrt[3]{1728}$$

Example 1.2 Simplify the following using the multiplicative property of square roots $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

$$\sqrt{12} \quad 3\sqrt{30} \quad \sqrt{\frac{1}{36}} \quad 2\sqrt{75} \quad \frac{3\sqrt{125}}{4} \quad \sqrt{\frac{15}{81}}$$

Exercises/Homework:

We begin this section with rationalizing the denominator of surds. We use the properties

$$\frac{x}{\sqrt{y}} = \frac{x\sqrt{y}}{\sqrt{y}\sqrt{y}} = \frac{x\sqrt{y}}{y},$$

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b.$$

Example 1.3 *Rationalize the denominator for the following expressions.*

(a) $\frac{5}{\sqrt{3}}$

(b) $\frac{6\sqrt{5}}{\sqrt{8}}$

(c) $\frac{3\sqrt{6}}{2\sqrt{10}}$

(d) $\frac{4-\sqrt{3}}{\sqrt{15}}$

(e) $\frac{1}{5-\sqrt{3}}$

(f) $\frac{6-\sqrt{5}}{2+\sqrt{8}}$

We have the following index laws for real numbers a, b, c :

$$a^b a^c = a^{b+c}, \quad a^b / a^c = a^{b-c}, \text{ for } a \neq 0$$

$$(a^b)^c = a^{bc},$$

$$(ab)^c = a^c b^c, \quad (a/b)^c = a^c / b^c = a^c b^{-c}, \text{ for } b \neq 0$$

$$a^{-1} = \frac{1}{a}, \text{ for } a \neq 0$$

$$a^0 = 1, \quad 0^0 = 1 \text{ (defined to be 1, but contraversial)}$$

$$\frac{1}{a^{-b}} = a^b, \text{ for } a \neq 0, \quad a^{-b} = \frac{1}{a^b} \text{ for } a \neq 0.$$

Example 1.4 *Express the following with positive indices*

(a) a^{-3}

(b) $2x^{-3}y^4$

(c) $\frac{4}{y^{-2}}$

(d) $\frac{(a^{-3}b)^2}{3a^{-1}b^2} \times \frac{b^{-1}}{a}$

(e) $\frac{(5a^2b^{-1})^3}{2a^4b^{-2}} \div \frac{b^{-5}}{2a^{-2}}$

Exercises/Homework:

We can write $\sqrt{3} = 3^{1/2}$ and

$$\left(\sqrt{3}\right)^2 = 3^{1/2} \times 3^{1/2} = 3^{\frac{1}{2}+\frac{1}{2}} = 3^1 = 3.$$

This allows us to use index laws to simplify surds. We have the following index laws for real number a and integers m, n :

$$a^{m/n} = \sqrt[n]{a^m},$$

$$a^{1/2} = \sqrt{a},$$

$$a^{1/3} = \sqrt[3]{a},$$

$$a^{1/n} = \sqrt[n]{a}.$$

Example 1.5 Express the following in index form:

(a) $\sqrt{11}$

(b) $\sqrt{3x^7}$

(c) $2\sqrt[4]{x^9}$

(d) $11\sqrt{7}$

Example 1.6 Write the following in simplest surd form:

(a) $12^{1/2}$

(b) $6^{3/2}$

Example 1.7 Simplify:

(a) $a^{1/5}a^{3/5}$

(b) $(b^2b^3)^{1/4}$

(c) $\left(\frac{x^{1/3}}{y^{1/6}}\right)^{1/4}$

Exercises/Homework:

In this section we will learn how to solve equations with one variable.

Example 1.8 *Solve the following equations for x :*

(a) $2x + 9 = 12$

(b) $3(2x + 4) = 3x$

(c) $\frac{x-1}{3} = 2$

(d) $3 - \frac{x}{3} = 8$

(e) $\frac{3-x}{4} = x - 4$

Exercises/Homework:

In this section we learn how to rearrange formulas and substitute values into equations.

Example 1.9 *The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.*

(a) *Solve the equation $V = \frac{4}{3}\pi r^3$ for r , where r is a real number.*

(b) *If the volume of the sphere is 42.8 m^3 , find the radius of the sphere.*

Example 1.10 *The area of a rectangular region adjoining a two semi-circle regions on each end of the rectangle is given by $A = xy + \pi \left(\frac{x}{2}\right)^2$.*

(a) *Solve the equation $A = xy + \pi \left(\frac{x}{2}\right)^2$ for y in terms of x and A .*

(b) *If $x = 36 \text{ m}$ and $y = 24 \text{ m}$, calculate A .*

Exercises/Homework:

Symbols:

$x > 3$ means x is greater than 3.

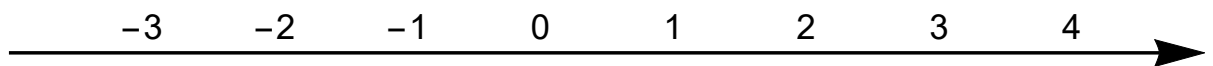
$x < 3$ means x is less than 3.

$x \geq 3$ means x is greater than or equal to 3.

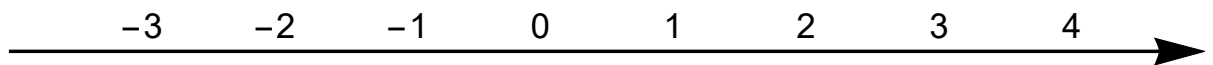
$x \leq 3$ means x is less than or equal to 3.

Example 1.11 *Sketch the region on the number line corresponding to:*

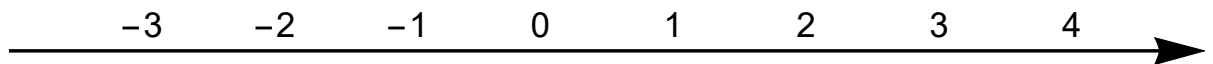
(a) $\{x \in \mathbb{R} : x > 3\} = (3, \infty).$



(b) $\{x \in \mathbb{R} : x < 3\} = (-\infty, 3).$



(c) $\{x \in \mathbb{R} : x \geq 3\} = [3, \infty).$



(d) $\{x \in \mathbb{R} : x \leq 3\} = (-\infty, 3].$



- When multiplying an inequality by a negative number, turn the symbol around. ($>$ becomes $<$, $<$ becomes $>$, $\leq$ becomes $\geq$, $\geq$ becomes $\leq$.)
- When inverting both sides of an inequality, turn the symbol around.
- Otherwise, treat solving an inequality like solving an equation.

Example 1.12 $4 > 3$ but $-4 < -3$ and $\frac{1}{4} > \frac{1}{3}$.

Example 1.13 Solve the inequality $3x - 6 \geq 8$ for x .

Example 1.14 Solve the inequality $-(4 - 6x) < 2(5 - x)$ for x .

Example 1.15 Solve the inequality $\frac{x}{4} - \frac{2x}{3} > -7$ for x .

Example 1.16 Solve the inequality $\frac{5}{3x} > 2$ for x .

The **greatest common divisor** of two integers a and b is written $\gcd(a, b)$. This is the greatest positive integer c such that c divides a and c divides b .

The **least common multiple** of two integers a and b is written $\text{lcm}(a, b)$. This is the least positive integer c such that a divides c and b divides c .

Theorem 1 *For any two positive integers a and b ,*

$$ab = \gcd(a, b)\text{lcm}(a, b).$$

The greatest common divisor and least common multiple can be calculated efficiently using the Euclidean algorithm.

Example 1.17 *Calculate:*

(a) 4×6

(b) $\gcd(4, 6)$

(c) $\frac{4 \times 6}{\gcd(4, 6)}$

(c) $\text{lcm}(4, 6)$

(d) $\frac{1}{4} + \frac{1}{6}$

Example 1.18 Calculate $\text{lcm}(12, 18)$ and use it to simplify $\frac{x+4}{12} + \frac{x-6}{18}$.

Example 1.19 Calculate $\text{lcm}(24, 6)$ and use it to simplify $\frac{x-3}{6} + \frac{5x-6}{24}$.

Exercises/Homework:

Two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are **parallel** if $m_1 = m_2$.

Two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are **perpendicular** if $m_1m_2 = -1$ (or equivalently, $m_2 = -\frac{1}{m_1}$).

Theorem 2 *In Euclidean space:*

- *Two lines intersect in one point if and only if they are not parallel.*
- *Lines have either one intersection or infinitely many intersections (they are the same line).*
- *There is a unique line passing through two points.*

Example 1.20 *Decide whether the two lines $y = -9x - 3$ and $y = \frac{1}{9}x + 2$ are parallel, perpendicular, or neither.*

Example 1.21 *Decide whether the two lines $y = -\frac{1}{3}x + 1$ and $3y + x = 2$ are parallel, perpendicular, or neither.*

Example 1.22 *Decide whether the two lines $y = \frac{1}{6}x + 4$ and $6y + x = 3$ are parallel, perpendicular, or neither.*

Example 1.23 Find the equation of the line that is parallel to $y = -5x + 8$ and passes through the point $(2, -3)$.

Example 1.24 Find the equation of the line that is perpendicular to $y = -5x + 8$ and passes through the point $(-4, -2)$.

To decide whether two lines are parallel, perpendicular, or neither:

Step 1 Put both lines in standard form $y = mx + c$ and hence identify slopes m_1 and m_2 .

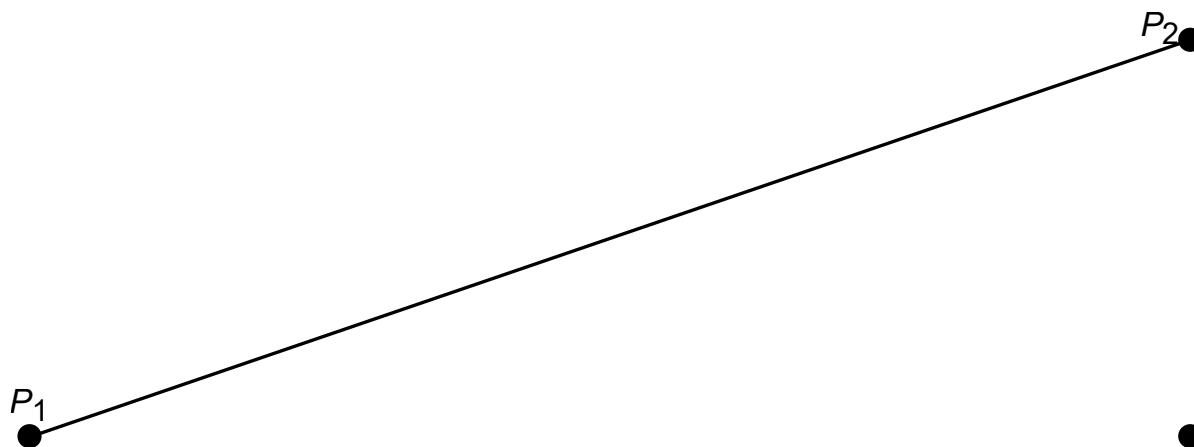
Step 2 If $m_1 = m_2$, then the lines are parallel;

Step 3 Otherwise: if $m_1 m_2 = -1$, then the lines are perpendicular;

Step 4 Otherwise: the lines are neither parallel nor perpendicular.

Exercises/Homework:

Consider the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. The distance between P_1 and P_2 is obtained by Pythagoras' theorem $a^2 + b^2 = c^2$, where $a = |x_1 - x_2|$, $b = |y_1 - y_2|$, and c is the distance between P_1 and P_2 .



The formula for the distance between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$c = D(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Example 1.25 Find the distance between the points $(0, 4)$ and $(-2, 6)$.

The **midpoint** of the line segment connecting

the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is given by

18

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Example 1.26 Find the midpoint of the line segment connecting the points $(0, 4)$ and $(-2, 6)$.

Example 1.27 Find real numbers a and b such that the midpoint of $(2a, a)$ and $(3, b)$ is $(4, -4)$.

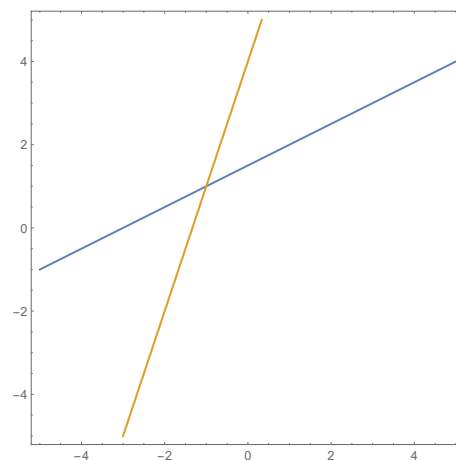
Example 1.28 *The distance between the points $(-4, 1)$ and $(6, a)$ is $4\sqrt{21}$. Find a .*

Exercises/Homework:

Given two simultaneous linear equations that do not represent parallel lines, we learn to find the point of their intersection by substitution. That is, we solve one equation for a variable, say y , and then substitute that into the other equation and solve for the other letter, say x .

Example 1.29 *Solve the simultaneous system of linear equations*

$$\begin{aligned}2x - 4y &= -6, \\ y &= 3x + 4.\end{aligned}$$



Example 1.30 *Solve the simultaneous system of linear equations*

$$y = 8x - 1,$$

$$y = 8x + 2$$

if possible.

Example 1.31 *For which real value of k does the simultaneous system of linear equations*

$$y = -3x - 2,$$

$$y = kx + 6$$

(a) *have no solution?*

(b) *have one solution?*

(c) *have infinitely many solutions?*

Exercises/Homework:

Given a system of simultaneous linear equations, solving the system by elimination applies the following procedure. We multiply each equation by a number such that the coefficients of one of the variables (the coefficient of the same letter) becomes the same or of opposite sign. We then add or subtract equations so that that variable vanishes. Finally, we solve for the other variable.

Example 1.32 *Solve the system of equations*

$$2x - 6y = 8,$$

$$3x + 4y = 10$$

by elimination.

Example 1.33 *Solve the system of equations*

$$\begin{aligned}x + 2y &= 4, \\ 2x + 9y &= 12\end{aligned}$$

by elimination.

Exercises/Homework:

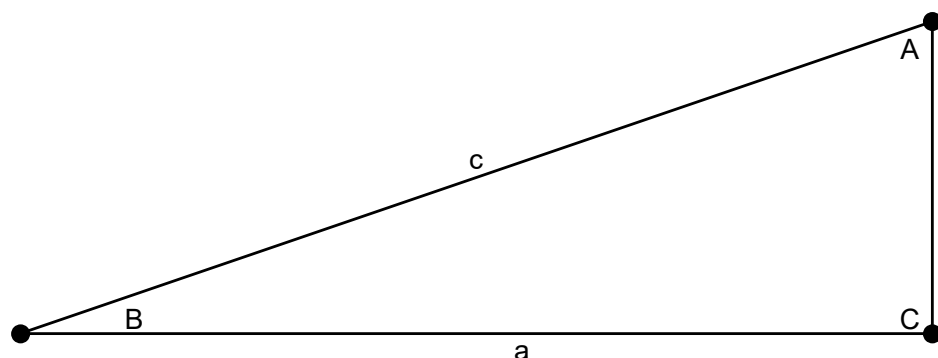
Next we consider applications of simultaneous equations.

Example 1.34 *The sum of the ages of two children Kara and Ben is 17 and the difference in their ages is 5. If Kara is older than Ben, determine their ages.*

Exercises/Homework:

1.2.1 Introduction to Trigonometry

We learn about the relationship between the angles in a right triangle and the trigonometric ratios sine, cosine and tangent (sin, cos, tan).



The trigonometric ratios sine, cosine and tangent are defined

$$\begin{aligned}\sin(B) &= \frac{\text{opp}}{\text{hyp}}, \\ \cos(B) &= \frac{\text{adj}}{\text{hyp}}, \\ \tan(B) &= \frac{\text{opp}}{\text{adj}}.\end{aligned}$$

We have the useful acronym **SOHCAHTOA** for remembering these trig. ratios.

Recall Pythagoras' theorem:

Theorem 3 (Pythagoras) *Let a, b, c be the lengths of the sides of a right triangle, where $c > a, b$ (c is the hypotenuse). Then $a^2 + b^2 = c^2$.*

Example 1.35 *Show that:*

(a) $\frac{\sin(B)}{\cos(B)} = \tan(B),$

(b) $\sin(B) = \cos(A),$

(c) $\cos(B) = \sin(A),$

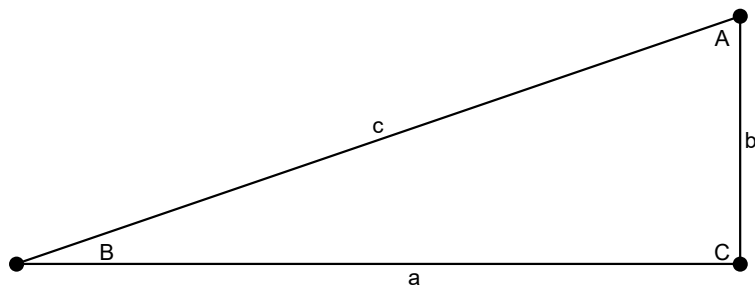
(d) $\tan(B) = \frac{1}{\tan(A)},$

(e) $\cos^2(B) + \sin^2(B) = 1$ by Pythagoras' theorem.

Example 1.36 Find the side length x opposite an angle of 30° in a right triangle with hypotenuse 8.

Example 1.37 Find the hypotenuse x in a right triangle if the triangle has side length 4 adjacent to an angle of 44° .

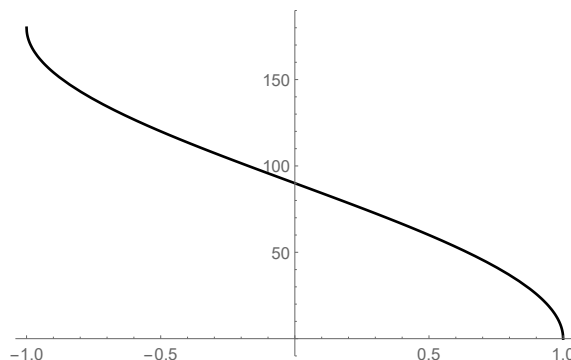
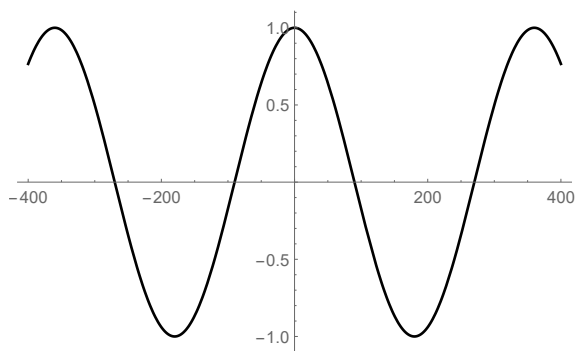
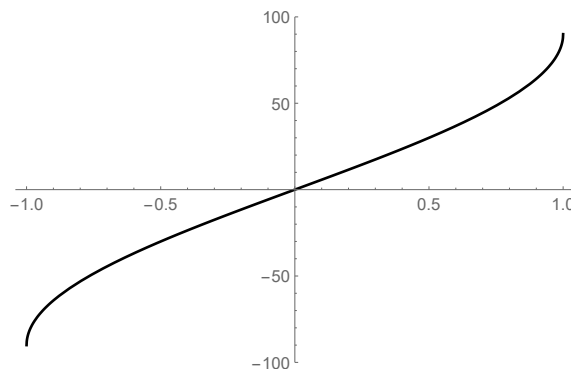
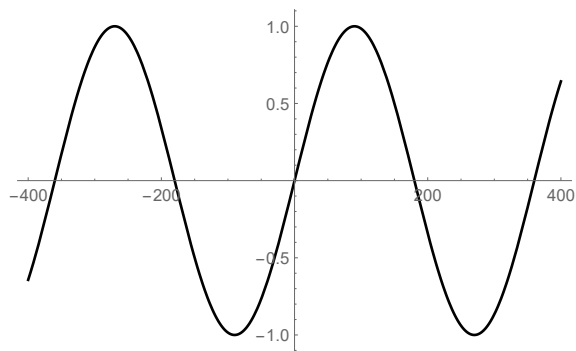
To solve a right triangle for an interior angle we use the inverse functions of sine, cosine and tangent, $(\sin^{-1}, \cos^{-1}, \tan^{-1})$.

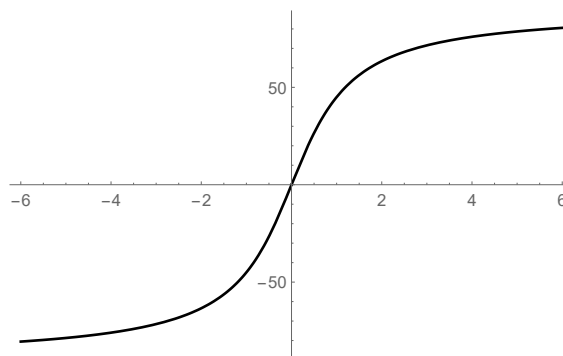
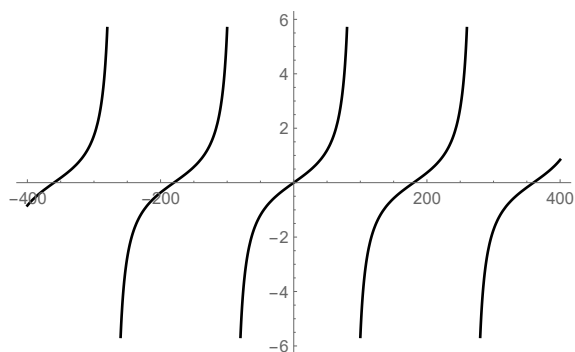


Since $\sin(B) = \frac{b}{c}$, we have $B = \sin^{-1}\left(\frac{b}{c}\right)$. This is also called arcsin.

Similarly, $\cos(B) = \frac{a}{c}$ so $B = \cos^{-1}\left(\frac{a}{c}\right)$. This is also called arccos.

$\tan(B) = \frac{b}{a}$ so $B = \tan^{-1}\left(\frac{b}{a}\right)$. This is also called arctan.





Example 1.38 *A right triangle has hypotenuse of length 2 and sides of length 1 and x . Solve for the angle adjacent to the side of length x , and then solve for x .*

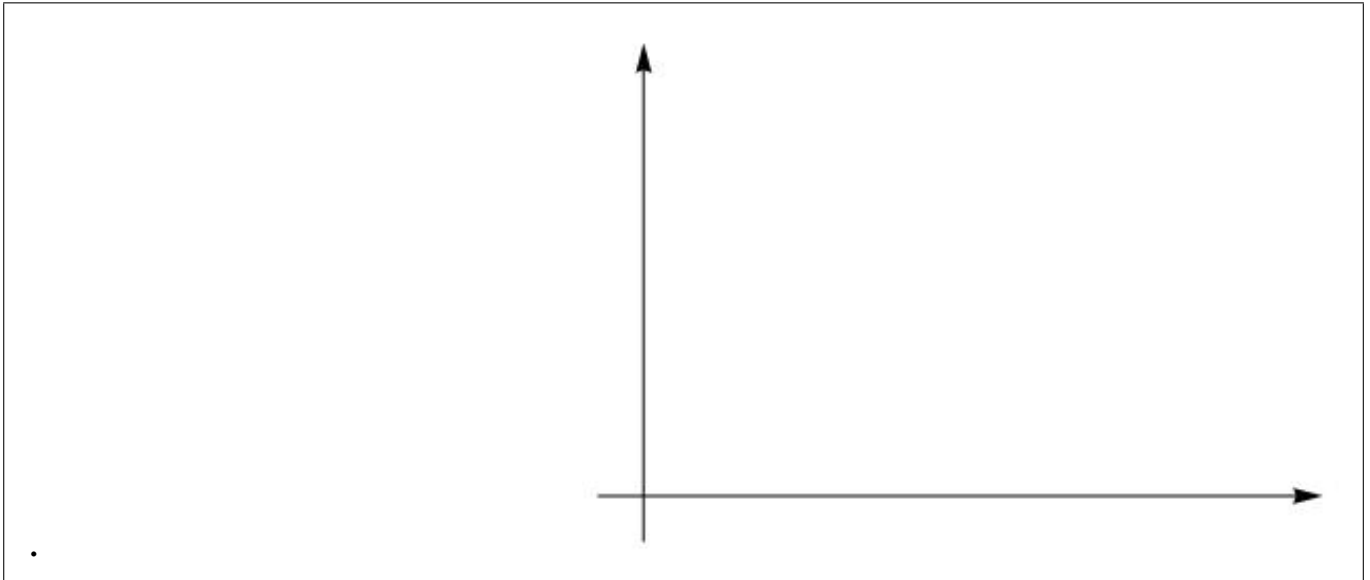
Exercises/Homework:

We consider some applications of trigonometry.

Example 1.39 *A tower stands x metres high in elevation above the ground. A man standing on the top of a 250 metre tall building looks up to the tower with an elevation angle of 30° to the horizontal. The horizontal distance between the man and the tower is 420 metres. Calculate the elevation x of the tower.*

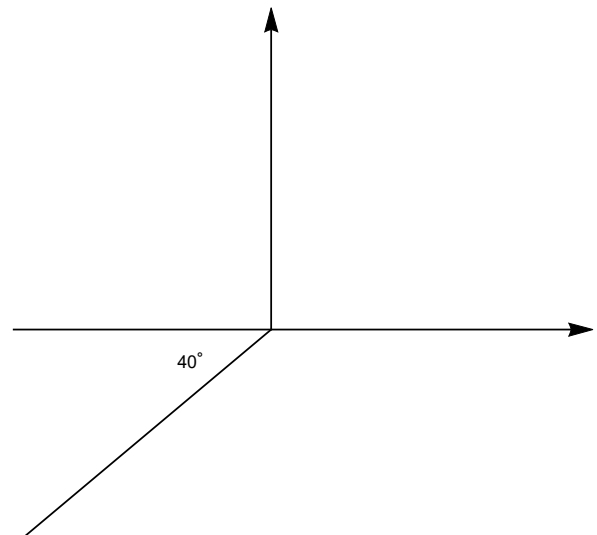
Exercises/Homework:

True Bearings ($^{\circ}T$) are measured **clockwise from North**.

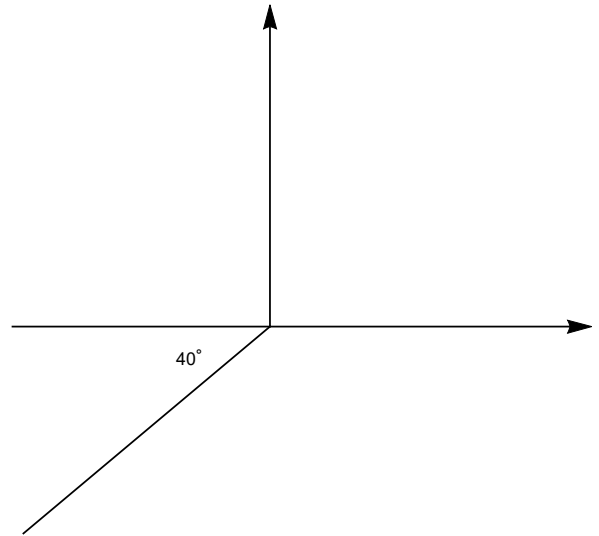


Recall that the mathematical convention is to measure angles from the positive end of the x -axis counter-clockwise.

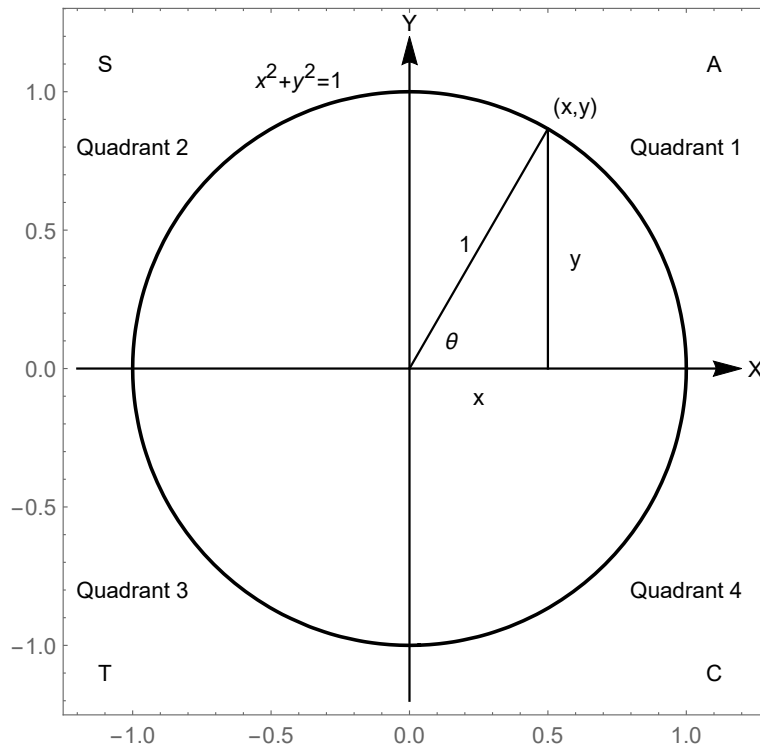
Example 1.40 Consider the points O, A, B show below. If the line segment OA makes an angle of 40° South of West, what is the true bearing of the point A from O ?



Example 1.41 Consider the points O, A, B show below. If the line segment OA makes an angle of 40° South of West, what is the true bearing of the point O from A ?



Example 1.42 A boat travels North-East for 5 km followed by a true bearing of 20° for 10 km. Find the true bearing of the boat from the original position.



Recall

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1},$$

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1},$$

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}.$$

For any point (x, y) on the unit circle $x^2 + y^2 = 1$, there is an angle θ such that $(x, y) = (\cos(\theta), \sin(\theta))$. Since $x^2 + y^2 = 1$ we again have $\cos^2(\theta) + \sin^2(\theta) = 1$.

The acronym ASTC refers to the following:

For an angle θ in Quadrant 1, **All** $\sin(\theta), \cos(\theta), \tan(\theta) > 0$.

For an angle θ in Quadrant 2, **Only Sine**, $\sin(\theta) > 0$.

For an angle θ in Quadrant 3, **Only Tan**, $\tan(\theta) > 0$.

For an angle θ in Quadrant 4, **Only Cos**, $\cos(\theta) > 0$.

A **reference angle** is an angle α with $0 \leq \alpha < 90^\circ$ such that $\theta = 180^\circ \pm \alpha$, $\theta = 360^\circ - \alpha$, or $\theta = \alpha$. For example, if $\theta = 290^\circ$, then the reference angle $\alpha = 70^\circ$ so that $\theta = 360^\circ - \alpha$.

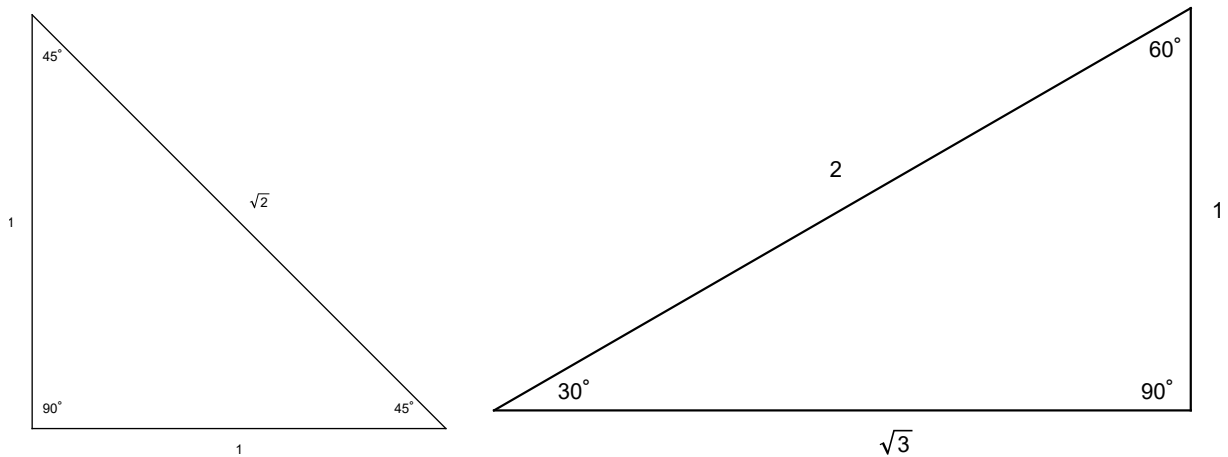
Let α be the reference angle.

- If θ is in Quadrant 1, then $\theta = \alpha$.
- If θ is in Quadrant 2, then $\theta = 180^\circ - \alpha$.
- If θ is in Quadrant 3, then $\theta = 180^\circ + \alpha$.
- If θ is in Quadrant 4, then $\theta = 360^\circ - \alpha$.

Example 1.43 Calculate $\cos(320^\circ)$ and $\sin(320^\circ)$ by considering the reference angle.

Exercises/Homework:

Memorize the following useful triangles:



These two triangles give exact surd values for the trigonometric ratios of angles 45° , 30° , and 60° .

We have

$$\cos(45^\circ) = \quad, \quad \cos(30^\circ) = \quad, \quad \cos(60^\circ) = \quad,$$

$$\sin(45^\circ) = \quad, \quad \sin(30^\circ) = \quad, \quad \sin(60^\circ) = \quad,$$

$$\tan(45^\circ) = \quad, \quad \tan(30^\circ) = \quad, \quad \tan(60^\circ) = \quad.$$

The angle addition formulas:

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta),$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta),$$

give additional exact values.

Example 1.44 Calculate $\cos(15^\circ)$ and $\sin(15^\circ)$ using the above triangles and the angle addition formulas.

Example 1.45 Calculate the exact surd value of $\cos(150^\circ)$.

Example 1.46 Find all angles θ with $0 \leq \theta < 360^\circ$ such that $\cos(\theta) = -\frac{\sqrt{3}}{2}$.

Like terms are terms of a polynomial with the same letters to the same powers.

Example: $4xy^2$ and $-3xy^2$ **ARE** like terms.

Example: $4x^2y$ and $-3xy^2$ are **NOT** like terms.

Example: x^2 and x are **NOT** like terms.

Example: x and 12 are **NOT** like terms.

We use the **distributive law** to expand brackets. This means multiplication distributes over addition:

$$x(y + z) = xy + xz, \quad (x + y)z = xz + yz.$$

Notice that $2(3 + 5) = 2 \times 8 = 16$.

Also $2(3 + 5) = 2 \times 3 + 2 \times 5 = 6 + 10 = 16$.

The following are all consequences of the distributive law:

$$a(b + c) =$$

$$a(b - c) =$$

$$(a + b)(c + d) =$$

$$=$$

$$=$$

$$(a + b + c)(d + e + f) =$$

$$(a + b)^2 =$$

$$=$$

$$=$$

$$=$$

$$(a + b)(c + d)(e + f) =$$

$$=$$

$$=$$

Example 1.47 Expand $(x - 4)(x + 8)$

Example 1.48 Expand $(2x - 6)(3x + 7)$

Exercises/Homework:

Factorizing a polynomial is the process of expressing the polynomial as a product of polynomials.

For example, $x^2 - 25 = (x + 5)(x - 5)$ since $a^2 - b^2 = (a + b)(a - b)$.
Similarly, $x^2 - 12 = x^2 - \sqrt{12}^2 = (x + \sqrt{12})(x - \sqrt{12})$.

Example 1.49 Factorize $3x^2 - 18x$.

Example 1.50 Factorize $x^2 + 8x + 15$.

Example 1.51 Factorize $x(x + 3) - 12(x - 3)$.

Exercises/Homework:

A **monic** polynomial in one variable has leading coefficient equal to 1. That is, a polynomial in the variable x has coefficient of x^n , where n is greatest, being 1.

$x^2 + 3x + 8$ is monic. $3x^2 - 4x + 12$ is not monic.

A **quadratic** polynomial in one variable is a polynomial of the form

$$ax^2 + bx + c,$$

where a, b, c are specific numbers. Quadratic refers to the greatest exponent being equal to 2.

To factorize $x^2 + bx + c$, where b, c are specific integers, we seek to find integers p, q such that

$$(x + p)(x + q) = x^2 + (p + q)x + pq = x^2 + bx + c$$

so that

$$c = pq, \quad b = p + q.$$

Step 1 If $c = 0$, put $x^2 + bx + c = x(x + b)$. Otherwise:

Step 2 If $b = 0$, $x^2 + bx + c = (x + \sqrt{c})(x - \sqrt{c})$. Otherwise:

Step 3 List all of the divisor pairs (s, t) of the absolute value of c up to their order: $(1, |c|), \dots$

Step 4 If $c > 0$, determine which pair (s, t) satisfies $s + t = |b|$. If $b > 0$, put $x^2 + bx + c = (x + s)(x + t)$. If $b < 0$, put $x^2 + bx + c = (x - s)(x - t)$.

Step 5 If $c < 0$, determine which pair (s, t) satisfies $s - t = \pm b$.

If $b > 0$, put $x^2 + bx + c = (x + s)(x - t)$, where $s > t$. If $b < 0$, put $x^2 + bx + c = (x - s)(x + t)$, where $s > t$.

Example 1.52 Factorize the monic quadratic polynomial $x^2 - x - 20$.

Example 1.53 Factorize the monic quadratic polynomial $x^2 + 9x + 18$.

Example 1.54 Factorize the monic quadratic polynomial $x^2 + 5x - 84$.

Exercises/Homework:

We learn how to factorize expressions of the form $ax^2 + bx + c$, where $a \neq 0$ and $a, b, c \in \mathbb{Z}$ (are integers). We demonstrate the procedure with an example.

Example 1.55 Factorize $10x^2 - 13x - 3$.

We first list the divisor pairs of $|-3| = 3$. We only have $(1, 3)$.

We list the divisor pairs of the absolute value of the leading coefficient $|10| = 10$. We only have $(1, 10)$ and $(2, 5)$. Next we form a multiplication table where we multiply divisors:

$\times$	1 10	2 5
1		
3		

Since $-3 < 0$, we seek a pair of products with a difference of -13 , examining the differences within diagonals of the table.

We use these to form the required factors

$$10x^2 - 13x - 3 = (\quad)(\quad).$$

We can expand to check our work.

Example 1.56 *Factorize* $6x^2 - 13x - 28$.

.

$\times$	$1 \quad 28$	$2 \quad 14$	$4 \quad 7$
1			
6			
2			
3			

We show how to factorize a quadratic polynomial in one variable by completing the square. The factors we obtain do not always have integer coefficients.

Example 1.57 Factorize $3x^2 + 19x + 20$ by completing the square.

In our first step, we write $ax^2 + bx + c$ as $a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ since $a \neq 0$. In other words, we factor out the leading coefficient of the polynomial so that inside the brackets we have a monic quadratic polynomial. We have:

$$3x^2 + 19x + 20 = 3\left(x^2 + \frac{19}{3}x + \frac{20}{3}\right).$$

Next we calculate $\frac{1}{2}$ of the coefficient of x in the monic quadratic inside the brackets.

$$\frac{1}{2} \frac{19}{3} = \frac{19}{6}.$$

We place this inside a square: $\left(x + \frac{19}{6}\right)^2$. Since the expansion of $\left(x + \frac{19}{6}\right)^2$ contains $\left(\frac{19}{6}\right)^2$ which is not in the original quadratic $3x^2 + 19x + 20$, we must subtract $\left(\frac{19}{6}\right)^2$ from our new expression so that we get an equal expression.

$$\begin{aligned}
 3x^2 + 19x + 20 &= 3 \left(x^2 + \frac{19}{3}x + \frac{20}{3} \right), \\
 &= 3 \left(\left(x + \frac{19}{6} \right)^2 - \left(\frac{19}{6} \right)^2 + \frac{20}{3} \right).
 \end{aligned}$$

Now we tidy the remaining terms. Since $-\left(\frac{19}{6}\right)^2 + \frac{20}{3} = -\frac{121}{36}$, we have

$$\begin{aligned}
 3x^2 + 19x + 20 &= 3 \left(\left(x + \frac{19}{6} \right)^2 - \frac{121}{36} \right), \\
 &= 3 \left(\left(x + \frac{19}{6} \right)^2 - \left(\frac{11}{6} \right)^2 \right).
 \end{aligned}$$

We have completed the square but it remains to use the property $a^2 - b^2 = (a + b)(a - b)$ to factorize the quadratic polynomial.

$$\begin{aligned}
 3x^2 + 19x + 20 &= 3 \left(x + \frac{19}{6} + \frac{11}{6} \right) \left(x + \frac{19}{6} - \frac{11}{6} \right), \\
 &= 3(x + 5) \left(x + \frac{8}{6} \right), \\
 &= (x + 5)(3x + 4).
 \end{aligned}$$

Example 1.58 Factorize $6x^2 + 5x - 56$ by completing the square.

Exercises/Homework:

We aim to solve equations of the form $ax^2 + bx + c = 0$ for x , where a, b, c are specific numbers, by first factorizing the quadratic expression.

Assuming there are real numbers p, q, r, s such that

$$ax^2 + bx + c = (px + q)(rx + s) = 0,$$

then we get $px + q = 0$ or $rx + s = 0$ so that $x = -\frac{q}{p}$ or $x = -\frac{s}{r}$.

Notice that since $a = pr$ and $a \neq 0$, we have $p, r \neq 0$.

We can find real numbers p, q, r, s such that $ax^2 + bx + c = (px + q)(rx + s)$ when $b^2 - 4ac \geq 0$.

Example 1.59 Solve the equation $2x^2 + 3x - 27 = 0$ by factorization.

To solve a quadratic equation by completing the square, we first complete the square, writing $ax^2 + bx + c = P^2 - Q^2$, where $P = p_1x + p_2$ is a linear polynomial in x and Q is a number.

Since $P^2 - Q^2 = (P + Q)(P - Q) = 0$, we have $P + Q = 0$ or $P - Q = 0$ and hence solve these equations for x .

Example 1.60 *Solve the equation $2x^2 + 3x - 27 = 0$ by completing the square.*

Example 1.61 *Solve the equation $15x^2 + 28x - 32 = 0$ by completing the square.*

Exercises/Homework:

Assume that $ax^2 + bx + c = 0$, where $a \neq 0$. Factoring a from the expression on the left, we have

$$a \left(x^2 + \frac{b}{a}x + \frac{c}{a} \right) = 0.$$

Half of $\frac{b}{a}$ is $\frac{b}{2a}$ so that

Simplifying,

Simplifying again,

.

We call $\Delta = b^2 - 4ac$ the **discriminant**.

Solving the equation for x ,

We have the **quadratic formula**:

50

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 1.62 *Solve the equation $15x^2 + 28x - 32 = 0$ by using the quadratic formula.*

Exercises/Homework:

We learn to translate a word problem into a quadratic equation, solve the equation, and write the solution.

Example 1.63 *Mike is 5 years younger than Nina. The product of their ages is 266. How old are Mike and Nina?*

Example 1.64 *Margaret wants to fence an area of her yard adjacent to the house for a dog yard with three sides of a rectangle consisting of fencing and one side of the rectangle being the side of the house. Margaret has only 128 metres of fencing material. What are the dimension of the rectangle that give maximum area to the dog yard?*

1.3.1 Time Series

A **time-series** is a sequence of points in which the consecutive differences in the independent variable is constant, and the independent variable represents time. When plotted, line segments connect consecutive points of a time-series.

Example 1.65 $S = \{t_n, s_n\} = \{(1.2, -7.8), (1.4, -6.4), (1.6, -4.1), (1.8, -5.3)\}$ is an example of a time-series. If

$$\frac{s_{n+1} - s_n}{t_{n+1} - t_n} = m$$

is constant for all n , then the time-series is linear. Since $t_{n+1} - t_n$ is constant for all n in a time-series, a linear time-series has $s_{n+1} - s_n = k$ (a constant) for all n .

Example 1.66 Plot the data $S = \{(1.2, -7.8), (1.4, -6.4), (1.6, -4.1), (1.8, -5.3)\}$, where the dependent variable represents a percentage change in share price over that time interval.

A **bivariate data set** is a set of points (x_n, y_n) relating the dependent variable Y to the independent variable X .

A **scatter plot** is a plot of points in a bivariate data set, where the independent variable is shown on the horizontal axis and the dependent variable is shown on the vertical axis.

An **outlier** is a point of a bivariate data set which is deemed to be isolated from other points of the data set.

A bivariate data set has **correlation** if the points closely fit a line. Correlations are described as **strong correlation** or **weak correlation** depending on how well they fit a line. The data has **positive correlation** if the slope of the line of best fit is positive. The data has **negative correlation** if the slope of the line of best fit is negative.

Example 1.67 Consider the bivariate data set $S = \{(1, 3), (4, 2), (5, 1), (6, 0), (9, -2)\}$. Draw a scatter plot and describe any correlation observed.

Exercises/Homework:

Given a bivariate data set S , we can guess a line of best fit by choosing two points (x_1, y_1) and (x_2, y_2) and hence a line connecting them such that half of the points in the data set S are above the line and half of the points in the data set S are below the line. The points (x_1, y_1) and (x_2, y_2) do not need to be in S .

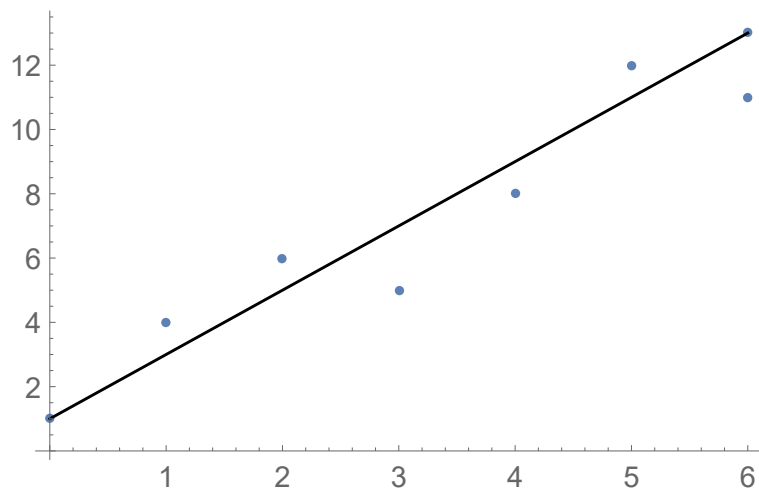
Recall that the slope of the line passing through two points (x_1, y_1) and (x_2, y_2) is given by $Y = mX + c$, where $m = \frac{y_2 - y_1}{x_2 - x_1}$ and $c = y_1 - mx_1$ or $c = y_2 - mx_2$. These c values are the same.

The construction of points in the data range with a line of best fit is called **interpolation**. The construction of points outside the data range with a line of best fit is called **extrapolation**.

Example 1.68 Consider the bivariate data set $S = \{(1, 4), (2, 6), (3, 5), (4, 8), (5, 12), (6, 11)\}$. Draw a scatter plot, guess a line of best fit, and describe any correlation observed.



Guessing the points $(0, 1)$ and $(6, 13)$ to find the equation of a line of best fit,



Exercises/Homework:

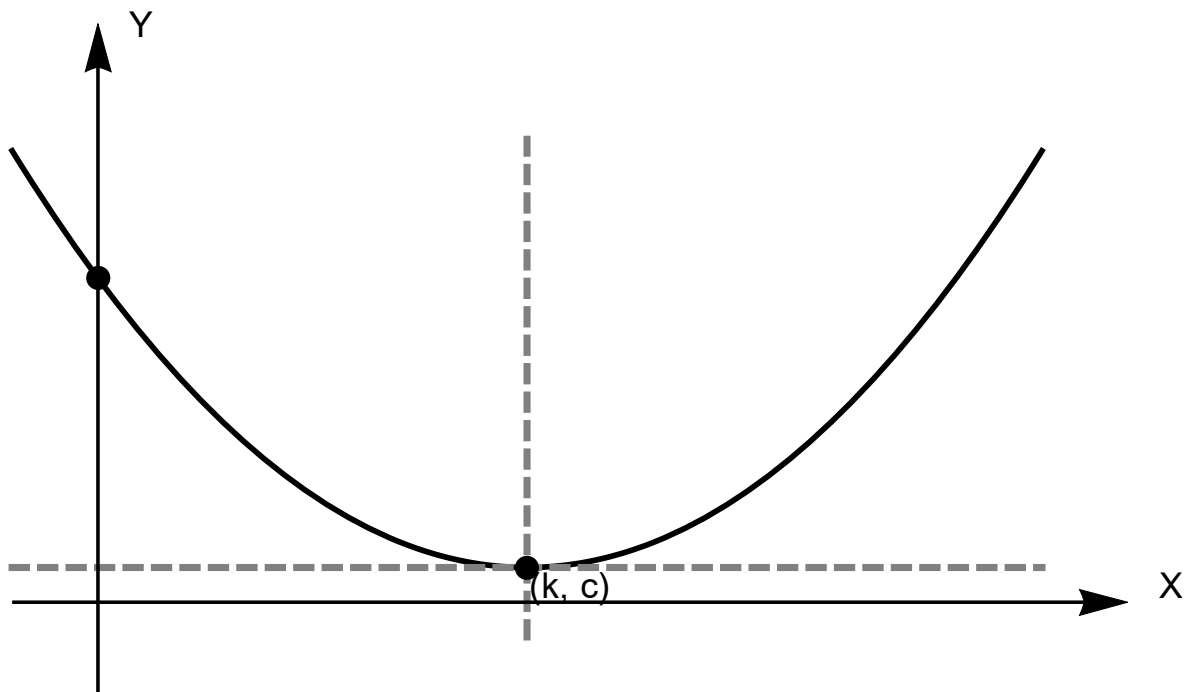
We study parabolas with an equation of the form $y = (x - k)^2 + c$ or $y = -(x - k)^2 + c$, where c and k are particular real numbers.

Note that we can use these techniques more generally since

$$y = (x - k)^2 + c = x^2 - 2kx + (k^2 + c).$$

Since $(x - k)^2 \geq 0$ and $(x - k)^2 = 0$ precisely when $x = k$, we see that $y = (x - k)^2 + c \geq c$ and $y = c$ precisely when $x = k$. This means that the point (k, c) is the **minimum turning point** of the parabola $y = (x - k)^2 + c$.

Similarly, (k, c) is the **maximum turning point** of the parabola $y = -(x - k)^2 + c$.



Features of $y = (x - k)^2 + c$:

$x - k = 0$ or $x = k$ is the **axis of symmetry**.

When $x = 0$, we have $y = k^2 + c$ so the point $(0, k^2 + c)$ is the **y-intercept**, which is the intersection with the y-axis.

When $y = 0$, $(x - k)^2 + c = 0$ so $(x - k)^2 = -c$. If $c \leq 0$, then $-c \geq 0$ so that $x - k = \pm\sqrt{-c}$ and we have $x = k \pm \sqrt{-c}$ so that the points $(k - \sqrt{-c}, 0)$ and $(k + \sqrt{-c}, 0)$ are the **x-intercepts**, the points of intersection with the x-axis.

Example 1.69 Consider the parabola shown below. Find the minimum turning point, x-intercepts, the y-intercept, and the axis of symmetry.

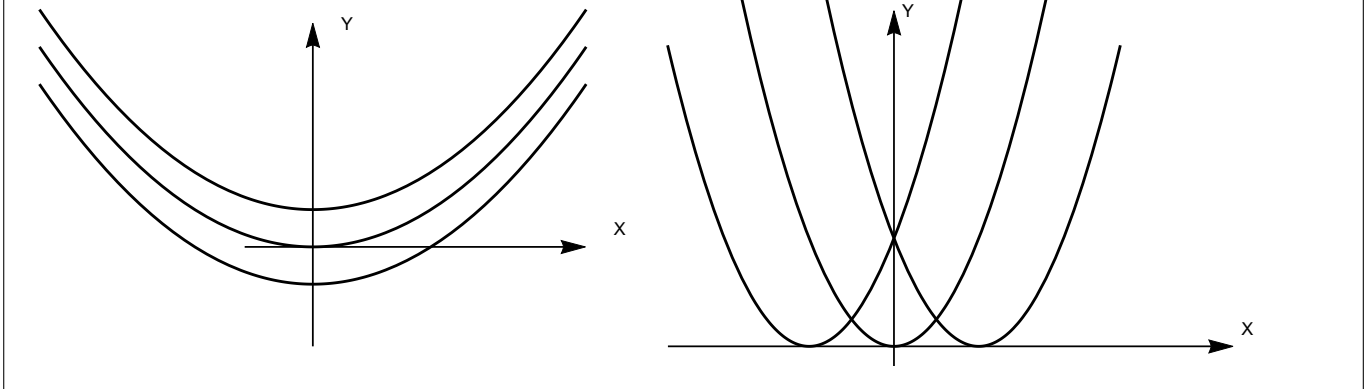


Example 1.70 *Sketch the parabola $y = -(x + 4)^2 - 6$, identifying the maximum turning point, x -intercepts (if real), the y -intercept, and the axis of symmetry.*



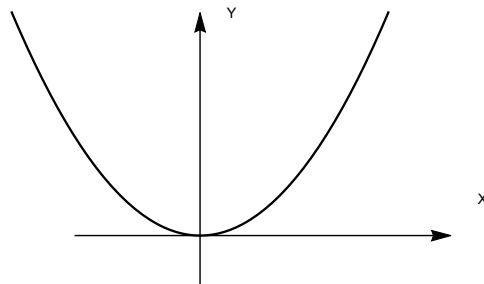
We discuss sketching parabolas of the form $y = a(x - h)^2 + k$ from the point of view of transformations of the parabola $y = x^2$.

Consider up/down and left/right translations of the parabola $y = x^2$.



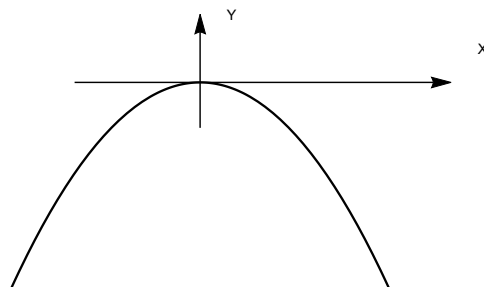
$y = ax^2$ is a **dilation** of $y = x^2$ by a .

If $a > 0$, then the parabola $y = ax^2$ has a minimum at $(0,0)$ and we say the



parabola is **concave up**.

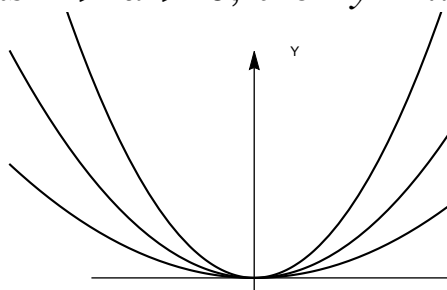
If $a < 0$, then the parabola $y = ax^2$ has a maximum at $(0,0)$ and we say the



parabola is **concave down**.

If the dilation has $a > 1$, then $y = x^2$ is stretched upwards to become $y = ax^2$. If the dilation has $1 > a > 0$, then $y = x^2$ is compressed down-

wards to become $y = ax^2$.



$y = (x - h)^2$ is a **right translation** of $y = x^2$, h units right.

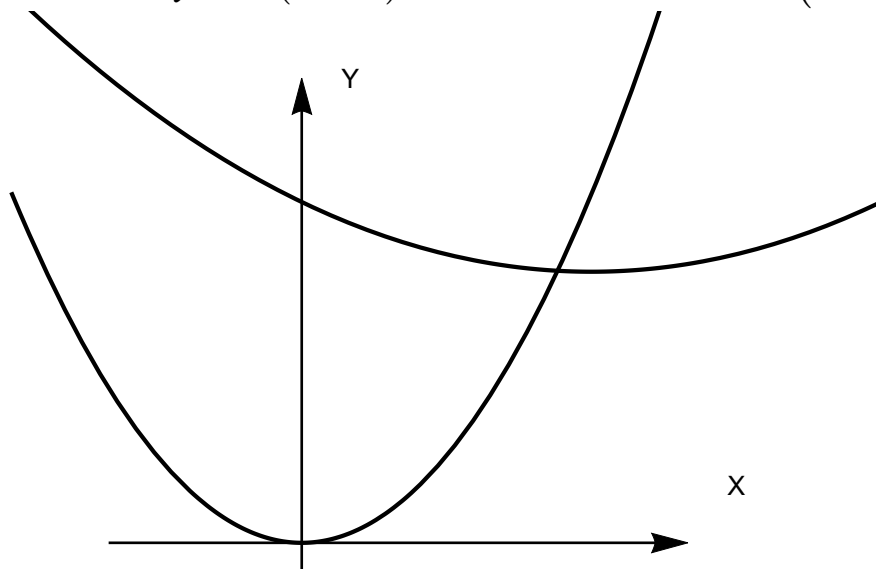
$y = (x + h)^2$ is a **left translation** of $y = x^2$, h units left.

$y = x^2 + k$ is an **upwards translation** of $y = x^2$, k units upwards.

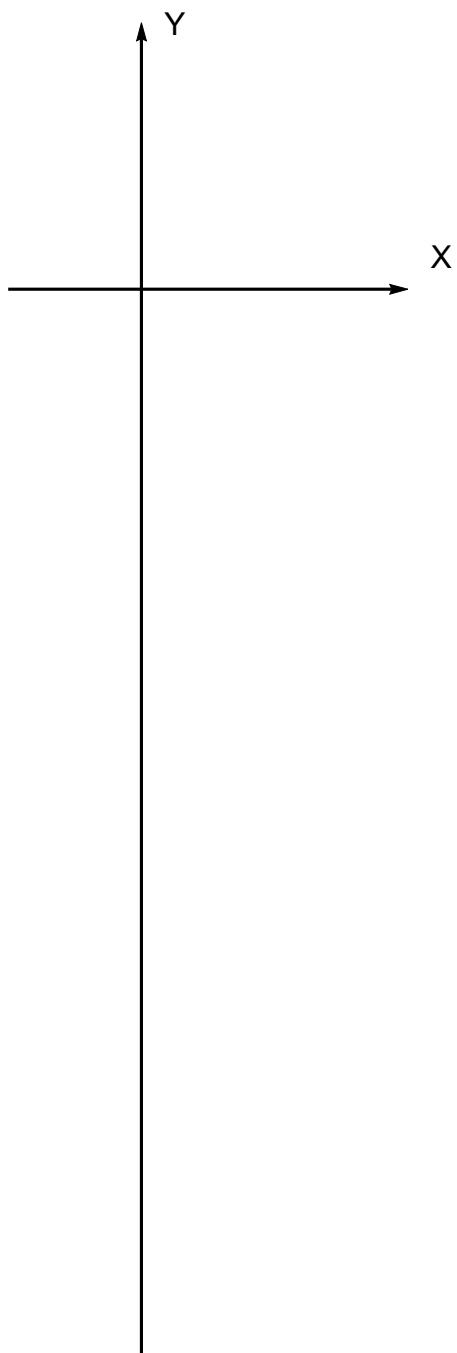
$y = x^2 - k$ is an **downwards translation** of $y = x^2$, k units downwards.

$y = a(x + h)^2 + k$ is a combination of translations and a dilation of $y = x^2$.

Note that $y = a(x + h)^2 + k = ax^2 + 2ahx + (ah^2 + k)$.



Example 1.71 *Describe the transformations that transform $y = x^2$ into $y = -2(x - 3)^2 + 4$ and sketch the parabola $y = -2(x - 3)^2 + 4$.*



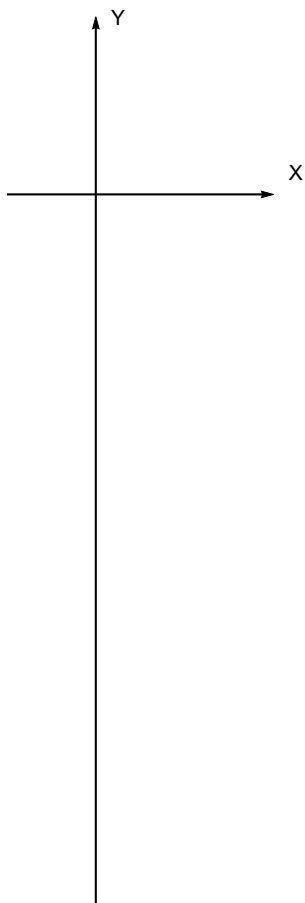
Exercises/Homework:

To sketch $y = ax^2 + bx + c$, with $a \neq 0$, suppose we are able to factorize the right hand side as

$$y = ax^2 + bx + c = (px + q)(rx + s).$$

When $y = 0$ we get the x -intercepts by solving $px + q = 0$ and $rx + s = 0$ so that $x = -\frac{q}{p}$, $x = -\frac{s}{r}$. We have the two x -intercept points $\left(-\frac{q}{p}, 0\right)$ and $\left(-\frac{s}{r}, 0\right)$. When $x = 0$ we have $y = c$ so we have the y -intercept point $(0, c)$.

Example 1.72 Sketch the parabola $y = x^2 + 4x - 21$ by factorization.



Exercises/Homework:

If we have $y = a(x - h)^2 + k$, then expanding,

$$y = a(x - h)^2 + k,$$

$$=$$

$$=$$

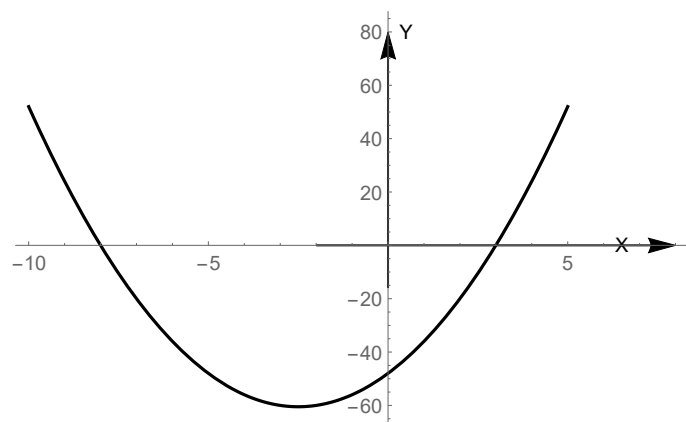
If given $y = ax^2 + bx + c$, then to complete the square is to find h and k such that

$$y = ax^2 + bx + c = a(x - h)^2 + k.$$

We have

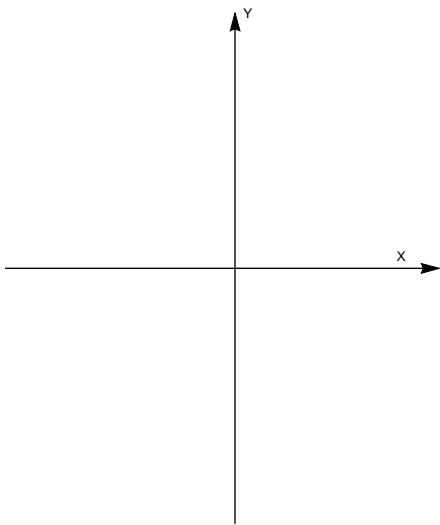
$$h = \quad , \quad k = \quad .$$

Example 1.73 Let $y = 2x^2 + 10x - 48$. Complete the square using the formulas for h and k above.



Example 1.74 Let $y = 2x^2 + 10x - 48$. Complete the square without using formulas.

Example 1.75 Use the completed square
 $y = 2x^2 + 10x - 48 = 2\left(x + \frac{5}{2}\right)^2 - \frac{121}{2}$ to sketch the parabola.



Exercises/Homework:

Let $y = ax^2 + bx + c$.

Completing the square,

$$\begin{aligned} y &= a(\qquad \qquad \qquad), \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

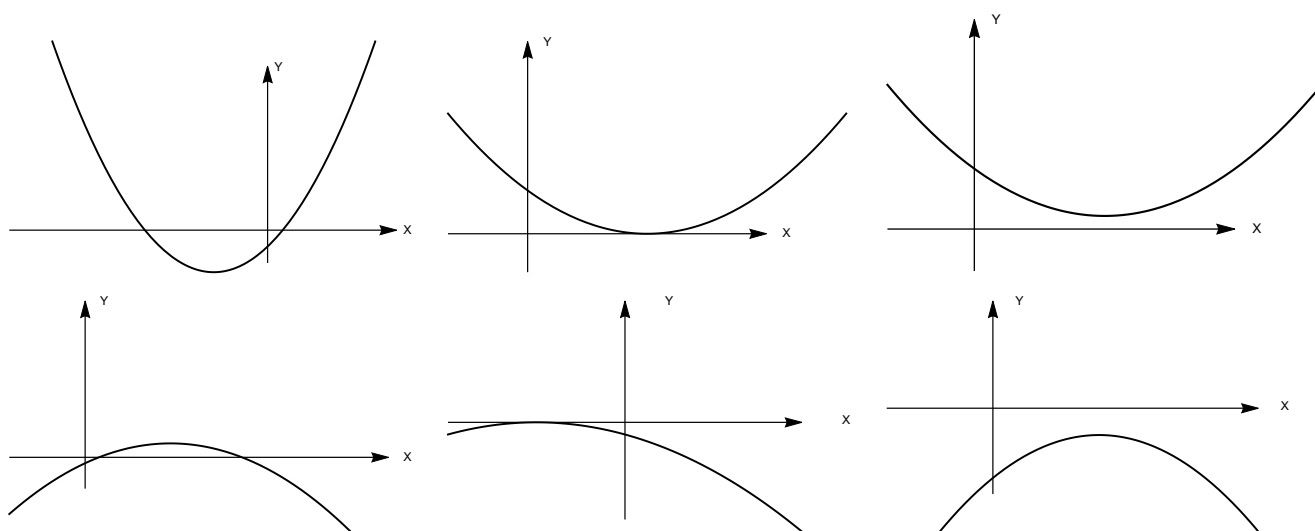
If $y = 0$, then

. This is called the **quadratic formula**.

Let $\Delta = b^2 - 4ac$. This is called the **discriminant** of the quadratic polynomial $ax^2 + bx + c$.

Theorem 4 Let Δ be the discriminant of $f(x) = ax^2 + bx + c$.

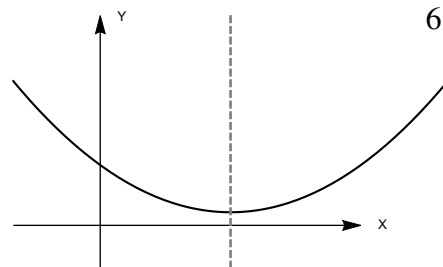
- If $\Delta > 0$, then $f(x)$ has two distinct real roots; the sketch of the parabola crosses the x -axis.
- If $\Delta = 0$, then $f(x)$ has one distinct real root; it is repeated and the quadratic is of the form $f(x) = a(x - h)^2$. The sketch of the parabola touches the x -axis at a tangent.
- If $\Delta < 0$, then $f(x)$ has no real roots; there are two non-real roots. The sketch of the parabola is above or below the x -axis entirely.



$$\Delta > 0,$$

$$\Delta = 0,$$

$$\Delta < 0.$$



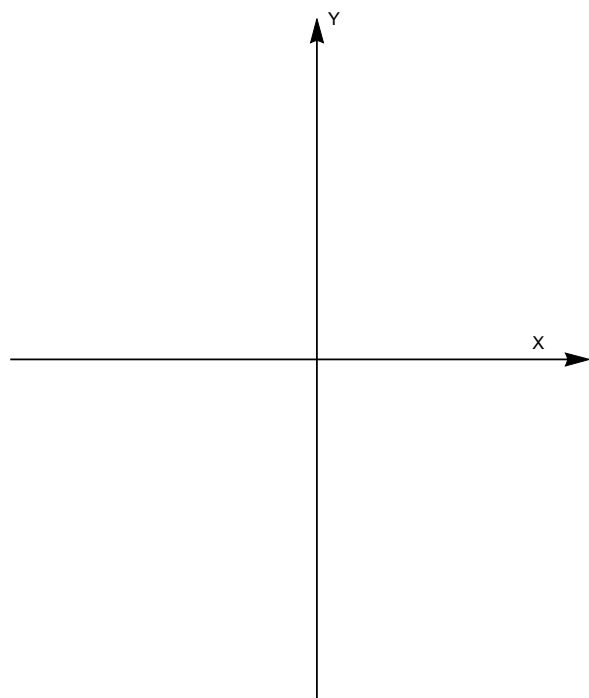
The vertical line $x = \frac{-b}{2a}$ is the **axis of symmetry**.

The point $\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right)$ is the **turning point**; if $a > 0$, then minimum and the parabola is **concave up**, if $a < 0$, then maximum and the parabola is **concave down**.

The y-intercept point is $(0, c)$.

If $\Delta \geq 0$, then the x-intercept point(s) is/are $\left(\frac{-b-\sqrt{\Delta}}{2a}, 0\right), \left(\frac{-b+\sqrt{\Delta}}{2a}, 0\right)$.

Example 1.76 Sketch $y = 3x^3 - 18x - 48$ using the above formulas.



Exercises/Homework:

Example 1.77 *A stone is tossed vertically upwards at 15 m/s and accelerates downwards due to gravity at 10 m/s^2 so that $u = 15$ and $a = -10$. Let y be the height of the stone after t seconds. If the height y is given by*

$$y = ut + \frac{1}{2}at^2,$$

find the maximum height of the stone and sketch the parabola relating y to t .



Example 1.78 *A gardener wants to erect a rectangular fenced area using 50 m of fencing. Determine the dimensions that give maximum area to the fenced region.*

A **function** f is a rule for sending (assigning) elements of a set X to elements of a set Y , and we put $y = f(x)$ or equivalently say (x, y) is in f , such that:

- For all elements x in X , there is a y in Y with $y = f(x)$, meaning f sends x to y .
- If y_1 and y_2 are in Y with $f(x) = y_1$ and $f(x) = y_2$, then we must have $y_1 = y_2$. This means that f is unambiguous and x is never sent to two different elements of Y . Rephrased, this is known as the vertical line test: If a vertical line intersects with the graph of a function in the x, y -plane, then it intersects exactly once.

If f is a function sending elements of X to elements of Y , we write $f : X \longrightarrow Y$ and state the rule for sending elements x to y ; e.g. $f(x) = y^2$. The set X is called the **domain** of the function f and Y is called the **co-domain** of the function f . The **range** of a function f is the subset of Y of elements that we sent to Y , the outputs of f .

Example 1.79 Let $X = \{1, 2, 3\}$ and $Y = \{4, 5, -2, 0\}$. We define the function $f : X \longrightarrow Y$ by $f(1) = 5$, $f(2) = 4$, $f(3) = -2$. f is an example of a function with domain $\{1, 2, 3\}$, co-domain $\{4, 5, -2, 0\}$, and range $\{-2, 4, 5\}$. Notice that the range is a subset of the co-domain.

Example 1.80 Let X be the set of all real numbers x satisfying $-1 \leq x \leq 1$ and let Y be the set of all real numbers. Does the relation $x^2 + y^2 = 1$ correspond to a function?

Example 1.81 $[-1, 1]$ refers to the set all real numbers satisfying $-1 \leq x \leq 1$. $\mathbb{R}$ refers to the set of real numbers. Define $f : [-1, 1] \longrightarrow \mathbb{R}$ by $f(x) = -\sqrt{1-x^2}$. Is f a function? Sketch the relation f .

Example 1.82 Show that $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = x^2$ is a function.

Example 1.83 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be given by $f(x) = 3x^2 - 2x + 4$ be a function. Calculate $f(1)$, $f(0)$, and $f(5)$.

A **polynomial in one variable x with real coefficients** is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where:

- n is a non-negative whole number.
- $a_0, a_1, a_2, \dots, a_n$ are real numbers called **coefficients**.

The **coefficient of x^n** is the real number multiplying by x^n in an expression.

In this section we will refer to polynomials in one variable x with real coefficients simply as **polynomials**.

Note: A polynomial is a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

The **degree** of a polynomial $f(x)$ is the greatest exponent n among all of the terms of the polynomial. We write $\deg(f) = n$.

a_n is called the **leading coefficient** of f , where $\deg(f) = n$ and a_n is the coefficient of $a_n x^n$.

a_0 is called the **constant coefficient** or **constant term** of f . a_0 is the only term of f that is a real number.

Theorem 5 *If $f(x)$ and $g(x)$ are polynomials and k_1, k_2 are real numbers, then $k_1 f(x)$, $k_2 g(x)$, $k_1 f(x) + k_2 g(x)$, $k_1 f(x)g(x)$, $k_1 f(g(x))$, and $k_1 g(f(x))$ are also polynomials.*

Exercises/Homework:

Example 1.84 *Expand the polynomial $f(x) = (x + 2)(x - 3)$ in two ways.*

Example 1.85 *Expand the polynomial $f(x) = (x^2 + 2x - 5)(x^2 + 6x - 2)$.*

Example 1.86 *Expand the polynomial $f(x) = (x + 1)(x - 3)(x - 1)$.*

Exercises/Homework:

Let $f(x)$ and $g(x)$ be polynomials, where $\deg(f) \geq \deg(g)$. We learn how to find polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x)q(x) + r(x),$$

where $\deg(q), \deg(r) \leq \deg(f)$. The polynomial $q(x)$ is called the **quotient** and $r(x)$ is called the **remainder**.

Note: If $f(x) = g(x)q(x) + r(x)$, then $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$.

Example 1.87 Let $f(x) = x^3 - 2x^2 - 2x - 3$ and let $g(x) = x^2 + x + 1$. Use long division to calculate $\frac{f(x)}{g(x)}$.

Example 1.88 Let $f(x) = 2x^3 - 9x^2 + 20x - 7$ and let $g(x) = x^2 - 3x + 4$. Use long division to find polynomials $q(x)$ and $r(x)$ such that $f(x) = g(x)q(x) + r(x)$.

Example 1.89 Given that $x - 1$ is a factor of $f(x) = x^3 - 2x^2 - 5x + 6$, factorize $f(x)$ as a product of 3 linear factors.

In this section we will introduce some important results for polynomial quotients and remainders.

Theorem 6 *Let $f(x)$ be a polynomial. If $f(a) = 0$, then $x - a$ is a factor of $f(x)$.*

If $g(x)$ is a factor of $f(x)$, then we say $g(x)$ **divides** $f(x)$ and equivalently, the remainder $r(x) = 0$ in $f(x) = g(x)q(x) + r(x)$.

Theorem 7 *Let $f(x)$ and $g(x)$ be polynomials with $g(x) \neq 0$ and $\deg(f) \geq \deg(g)$. There exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = g(x)q(x) + r(x)$, where $r(x) = 0$ or $\deg(r) < \deg(g)$.*

Corollary 1.1 (Bézout) *Let $f(x)$ and $x - a$ be polynomials, where a is a real number. Then the quotient and remainder polynomials satisfy $r(x) = f(a)$ and*

$$f(x) = (x - a)q(x) + f(a).$$

Example 1.90 *Let $f(x) = 3x^3 + x^2 - 10x - 8$. Factorize $f(x)$ as a product of 3 linear factors by calculating $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(2)$ and using long division.*

Example 1.91 Show that $x^2 + 2$ divides $x^4 + x^3 + 6x^2 + 2x + 8$.

Example 1.92 Let $f(x) = x^4 + x^3 + 6x^2 + 2x + 8$, $g(x) = x + 1$. Use Corollary 1.1 to find a polynomial $q(x)$ such that $f(x) = (x + 1)q(x) + f(-1)$.

Theorem 8 (Null Factor Law) *Let $f(x)$ be a polynomial.*

If $f(x) = p_1(x)p_2(x)\dots p_n(x) = 0$, where the $p(x)$ are polynomials, then $p_1(x) = 0$ or $p_2(x) = 0$ or $\dots$ $p_n(x) = 0$.

Theorem 9 (Fundamental Theorem of Algebra) *Let $f(x)$ be a polynomial of degree n . Then there are exactly n complex numbers*

$\alpha = a + b\sqrt{-1}$, where a and b are real numbers, and it is possible that $b = 0$ so the α may include real numbers, such that

$$f(x) = k(x - \alpha_1)(x - \alpha_2)\dots(x - \alpha_n),$$

and it is possible that the α coincide. Any non-real roots of $f(x)$ occur in conjugate pairs, meaning if $f(a + b\sqrt{-1}) = 0$, then $f(a - b\sqrt{-1}) = 0$.

Example 1.93 *Find all roots α of $f(x) = 2x^3 + x^2 - 4x - 3$ and reflect on the fundamental theorem of algebra.*

Example 1.94 Find all roots α of $f(x) = x^3 + x^2 + x + 1$ and reflect on the fundamental theorem of algebra.

Exercises/Homework:

1.4.1 Logarithms

$$a^x = b, \quad \text{and} \quad x = \log_a(b)$$

are different rearrangements of the same equation.

If $x = \log_a(b)$, then $a^x = b$. Conversely, if $a^x = b$, then $x = \log_a(b)$.

Example 1.95 Write $x = \log_{10}(1000)$ in exponential form and simplify x numerically.

Example 1.96 Evaluate $\log_3(81)$.

Example 1.97 Evaluate $\log_2(8)$.

Example 1.98 *Use a calculator to evaluate $\log_{10}(6)$.*

Example 1.99 *Evaluate $\log_5(125)$.*

Example 1.100 *Evaluate $\log_6(216)$.*

Exercises/Homework:

Data which shows exponential growth can be understood and modelled by taking the logarithm of the y -coordinate of the data points, plotting x on the horizontal axis and $\log(y)$ on the vertical axis.

Example 1.101 Consider the data set consisting of points (x, y) :

$$S = \{(1, 2), (2, 4), (3, 8), (4, 16), (5, 32), (6, 64)\}.$$

Let $Y = \log_2(y)$. Then $\{(x, Y)\} =$



In similar examples we can find a line of best fit to model such data sets.

Example 1.102 *The following table shows the population P of rabbits in a paddock over t years.*

t	0	1	2	3	4	5
P	4	32	256	2048	16384	131072

- (a) Calculate $\log_{10}(P)$ for the five P values in the table.
- (b) Plot t versus $\log_{10}(P)$.
- (c) Find the relationship between t and $\log_{10}(P)$.
- (d) Use Part (c) to find the relationship between P and t .

The exponent rule $a^b a^c = a^{b+c}$ is equivalent to the log rule $\log_a(x) + \log_a(y) = \log_a(xy)$.

Let $x = a^b$ so

Let $y = a^c$ so

$$a^b a^c =$$

Hence

$$\log_a(xy) = \log_a(x) + \log_a(y).$$

Similarly we have the following log rules:

$$\begin{aligned} \log_a(xy) &= \log_a(x) + \log_a(y), & \text{from } a^b a^c &= a^{b+c}, \\ \log_a(x/y) &= \log_a(x) - \log_a(y), & \text{from } a^b / a^c &= a^{b-c}, \\ \log_a(x^n) &= n \log_a(x), & \text{from } (a^b)^c &= a^{bc}, \\ \log_a(b) &= \frac{\log_c(b)}{\log_c(a)}, & \text{from } (c^y)^x &= c^{xy}, \\ \log_a(1/b) &= -\log_a(b), & \text{from } \frac{1}{b} &= b^{-1}, \log_a(x^n) = n \log_a(x), \\ b &= a^{\log_a(b)}, & \text{by letting } x &= \log_a(b), \\ \log_a(a) &= 1, (a \neq 1) & \text{from } a^1 &= a, \\ \log_a(1) &= 0, & \text{from } a^0 &= 1. \end{aligned}$$

Example 1.103 Simplify $\log_a(3) + \log_a(5)$.

Example 1.104 *Simplify $3 \log_a(2)$.*

Example 1.105 *Calculate $\log_2(1/8)$.*

Example 1.106 *Calculate $\log_6(36) - \log_6(4)$.*

Example 1.107 *Prove that $\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$ using $(c^y)^x = c^{xy}$, where $x = \log_a(b)$, $y = \log_c(a)$, $z = \log_c(b)$.*

Recall that if $a^x = b$, then $x = \log_a(b)$. In this section we use this fact to solve exponential equations.

Example 1.108 Solve $3^x = 9$ for x using logarithms.

Example 1.109 Solve $5^x = 35$ for x using logarithms.

Example 1.110 Solve $12 \times 2^x = 4$ for x using logarithms.

Example 1.111 Solve $3^{2x+5} = 18$ for x using logarithms.

A **trial** is a probability experiment such as tossing a coin or rolling a die or a pair of dice.

The set of all possible outcomes of a probability experiment is called the **sample space**.

An **outcome** of a probability experiment is a possible result such as obtaining a 5 on rolling a die.

A collection of specific outcomes of a probability experiment is called an **event**.

The **probability of an event** x is a real number P (or $P(x)$) with $0 \leq P \leq 1$ that represents and measures the chance or likelihood of the event occurring.

If $P = 0$, then the event cannot occur.

If $P = 1$, then the event must occur.

For example, the probability of obtaining heads when tossing a fair coin is $\frac{1}{2} = 0.5$.

If two events have the same probability (chance of occurring), then we say that the events are **equally likely**.

$$P(A) = \frac{|A|}{n},$$

where n is the number of elements in the sample space and $|A|$ is the number of elements in the event space A .

Example 1.112 *A card is randomly selected from a 52-card pack of playing cards. What is the probability that the card is a spade?*

Example 1.113 *A 6-sided die is rolled. What is the probability that the outcome is greater than 1 and odd?*

Example 1.114 *A letter is randomly chosen from the word INDOOROOPILLY. What is the probability that O is chosen?*

A **set** is a collection of objects (usually numbers) such that there are no repeated elements and there is no order of elements.

The **universal set** $\mathcal{U}$ is the set of all elements we could possibly be referring to. In probability this is the set of all possible outcomes, the sample space.

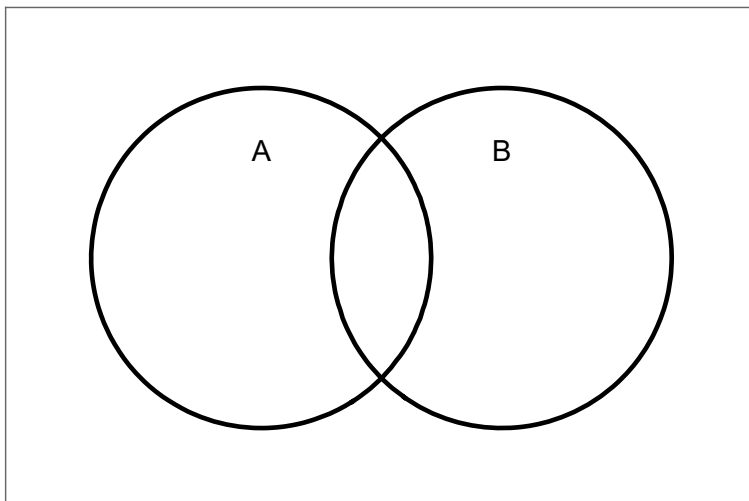
A **Venn diagram** is an illustration with intersecting circles used for displaying sets and their elements, common elements, and the universal set. We often list the elements of sets within the drawn circles. However, sometimes the number of elements are instead written in the circles.

Example 1.115 *Let*

$$A = \{1, 2, -5, 8, 6, 0\},$$

$$B = \{9, 3, 7, 1, 5, -4, 2, 8\}.$$

Display the elements of the sets A and B in the circles shown in the Venn diagram below. Where the two circles meet, put the elements that are in both A and B.



The **Cardinality** of a finite set A is the number of elements in the set, written $|A|$ or sometimes $n(A)$. For example, with A, B as in Example 1.115, $|A| = 6$ and $|B| = 8$.

If the set A has $|A| = 0$, then we say A is the **empty set** or **null set** and we write $A = \emptyset$.

The **intersection** of two sets A and B is the set consisting of those elements that are in both A and B , denoted $A \cap B$.

The **union** of two sets A and B is the set consisting of those elements that are in A or in B , denoted $A \cup B$.

When discussing probability we will use sets and events interchangeably.

Two events A and B are **mutually exclusive** if $A \cap B = \emptyset$ or equivalently $|A \cap B| = 0$.

The **complement of an event** A , denoted A' , is the opposite event to A : $A' = \mathcal{U} - A$, meaning the sample space without those elements belonging to A . We have $P(A') = 1 - P(A)$.

Example 1.116 *A six-sided die is rolled. Compare the probability of getting a number greater than 2 to the probability of getting a number less than or equal to 2.*

A **two-way table** is a table that displays the number of elements in the sets $A \cap B$, $A' \cap B$, $A \cap B'$, and $A' \cap B'$ in the following form:

	A	A'	
B	$ A \cap B $	$ A' \cap B $	$ A \cap B + A' \cap B $
B'	$ A \cap B' $	$ A' \cap B' $	$ A \cap B' + A' \cap B' $
	$ A \cap B + A \cap B' $	$ A' \cap B + A' \cap B' $	n

where $n = |A \cap B| + |A \cap B'| + |A' \cap B| + |A' \cap B'|$.

Example 1.117 For breakfast a class of 25 students eat apples or bananas or nothing. Let A be the set of students who eat apples and B be the set of students who eat bananas. 10 students eat apples, 18 students eat bananas, 8 students eat apples and bananas, 5 students eat neither apples nor bananas, 10 students eat bananas but not apples. Construct a two-way table representing this information and find the probability that a student eats apples but not bananas.

Exercises/Homework:

Theorem 10 (Addition rule) *Let A and B be two events. Then*

$$\begin{aligned} |A \cap B| + |A \cup B| &= |A| + |B|, \\ P(A \cap B) + P(A \cup B) &= P(A) + P(B). \end{aligned}$$

Example 1.118 *A card is randomly chosen from a 52-card deck of cards.*

- (a) *What is $|\mathcal{U}|$, the cardinality of the sample space?*
- (b) *Let A be the event ‘the card is a number 2 through 10 inclusive’ and B be the event ‘the card is a spade’. What is $|A|$, $|B|$, $|A \cap B|$, $|A \cup B|$?*
- (c) *Let A be the event ‘the card is a number 2 through 10 inclusive’ and B be the event ‘the card is a spade’. What is $P(A)$, $P(B)$, $P(A \cap B)$, $P(A \cup B)$?*
- (d) *Verify that the addition rule holds.*

Example 1.119 Let A and B be two events with $P(A) = 0.25$, $P(B) = 0.9$, $P(A \cap B) = 0.2$.

(a) Calculate $P(A \cup B)$.

(b) Calculate $P(A' \cap B)$.

Example 1.120 Let A and B be two events with $P(A) = 0.4$, $P(B) = 0.8$, $P(A \cup B) = 0.98$.

(a) Calculate $P(A \cap B)$.

(b) Calculate $P(A \cap B')$.

Conditional probability is the probability of an event A given that another event B has already occurred. This is denoted $P(A|B)$ and we have

Theorem 11 (Multiplication rule of conditional probability)

$$\begin{aligned}P(A \cap B) &= P(A|B)P(B), \\ &= P(B|A)P(A).\end{aligned}$$

Example 1.121 *20 people attend a house-warming party. 12 attendees brought flowers, 9 attendees brought chocolate cake, and 4 attendees brought both flowers and chocolate cake.*

- (a) *Calculate the probability that an attendee brought flowers or chocolate cake.*
- (b) *Calculate the probability that an attendee brought chocolate cake given that they brought flowers.*
- (c) *Represent the information in a two-way table.*

Example 1.122 *A bag of marbles contains 3 red marbles and 2 blue marbles. Calculate the probability that a second chosen marble is red given that the first chosen marble was red and not replaced in the bag before drawing the second marble.*

A **two-step experiment** is a probability experiment with two trials in which the sample space is a set of pairs (x, y) , where x is the outcome of the first trial and y is the outcome of the second trial. A two-step experiment can have $x \in X, y \in Y$, where it is possible that $X = Y$ but not necessarily so.

Assuming $X = Y$, if the two-step experiment occurs with replacement, then (x, x) is an element of the sample space. If the two step experiment occurs without replacement, then (x, x) is not an element of the sample space.

A **sample-space table** is a table containing elements of the sample space arranged so that the pair (x, y) is in the i -th row and j -th column of the table, where x is the first outcome of the event and y is the second outcome of the event. Furthermore, all elements in the j -th column contain the same first event and all elements in the i -th row contain the same second event.

Example 1.123 *Two letters are chosen randomly from the letters of the word CAR in a two-step experiment. Construct a sample-space table:*

(a) *with replacement.*

(b) *without replacement.*

	<i>C</i>	<i>A</i>	<i>R</i>
<i>C</i>			
<i>A</i>			
<i>R</i>			

	<i>C</i>	<i>A</i>	<i>R</i>
<i>C</i>			
<i>A</i>			
<i>R</i>			

Example 1.126 *A bag contains 5 white marbles and 3 black marbles. In a two-step experiment, two marbles are chosen randomly without replacement. Draw a tree diagram and calculate the probability of selecting two white marbles.*

Exercises/Homework:

Two events are **independent events** if the probability of a one event cannot influence the probability of the other event.

For example, in flipping a coin twice, the first and second coin tosses are independent events.

Drawing two coloured marbles from a bag without replacement are not independent events.

If Event A and Event B are independent events, then $P(A|B) = P(A)$ and $P(B|A) = P(B)$. Furthermore, since $P(A|B)P(B) = P(A \cap B)$, we have $P(A \cap B) = P(A)P(B)$.

Example 1.127 *A six-sided die is tossed twice. If A is the event the first toss gave a 3 and B is the event the second toss gave a 4, calculate:*

- (a) $P(A)$,
- (b) $P(B)$,
- (c) $P(B|A)$,
- (d) $P(A \cap B)$.

2.1 Term 1

Under Construction

2.2 Term 2

Under Construction

2.3.1 Limits and the First Derivative

Let $f(x)$ be a function that is defined near a real number a . If the function $f(x)$ is the real number L near a , then we write

$$\lim_{x \rightarrow a} f(x) = L.$$

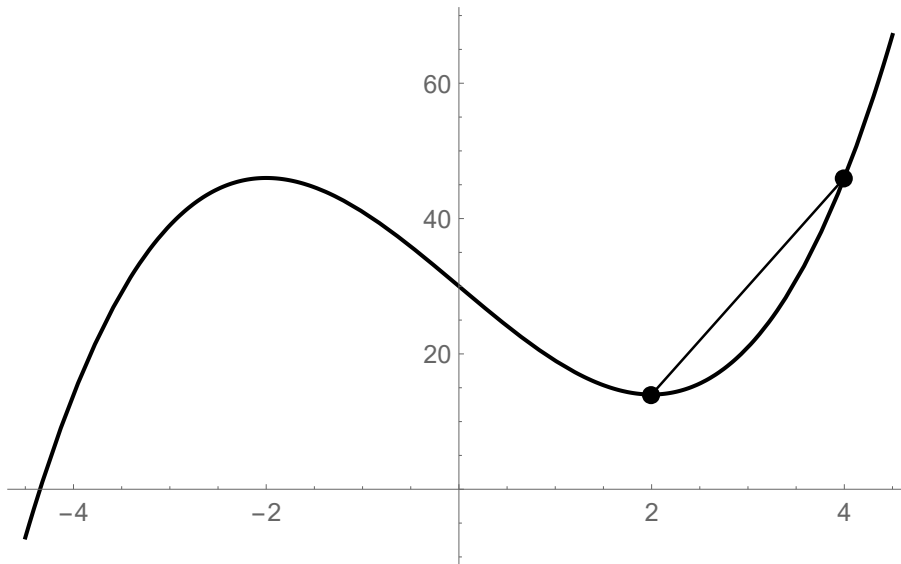
We say, the limit as x approaches a of $f(x)$ is L .

More formally, the real number L is the limit of the sequence $a_1, a_2, \dots$ if and only if for every real number $\varepsilon > 0$, there exists a natural number N such that for all $n > N$, we have $|a_n - L| < \varepsilon$.

Example 2.1 *The function $f(x) = \frac{x^3-1}{x-1}$ is not defined when $x = 1$. $f(x)$ is defined for all real x except for $x = 1$ and hence defined for x near 1. Calculate $\lim_{x \rightarrow 1} f(x)$.*

Theorem 12 (*Properties of Limits*) Let a, k_1, k_2 be particular real numbers and $f(x), g(x)$ are functions such that $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. Then:

- $\lim_{x \rightarrow a} k_1 f(x) = k_1 \lim_{x \rightarrow a} f(x).$
- $\lim_{x \rightarrow a} (k_1 f(x) + k_2 g(x)) = k_1 \lim_{x \rightarrow a} f(x) + k_2 \lim_{x \rightarrow a} g(x).$
- $\lim_{x \rightarrow a} (f(x)g(x)) = \left(\lim_{x \rightarrow a} f(x) \right) \left(\lim_{x \rightarrow a} g(x) \right).$
- If $\lim_{x \rightarrow a} g(x) \neq 0$, then $\lim_{x \rightarrow a} \left(\frac{f(x)}{g(x)} \right) = \frac{\left(\lim_{x \rightarrow a} f(x) \right)}{\left(\lim_{x \rightarrow a} g(x) \right)}.$
- If $f(x)$ is defined at $x = a$, then $\lim_{x \rightarrow a} f(x) = f(a).$



The **first derivative** $f'(x)$ or **derivative** for short, is defined by the limit

$$f'(x) = \lim_{h \rightarrow 0} \frac{1}{h} (f(x+h) - f(x)).$$

This gives a function representing the slope of the function $f(x)$ at any particular x value in the domain of f . To **differentiate** is to find $f'(x)$.

Example 2.2 Use the definition of the derivative to find $f'(x)$ for the function $f(x) = x^2$.

Example 2.3 Use the definition of the derivative to find $f'(x)$ for the function $f(x) = x^3 - 4x + 3$.

Consider the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

We learn how to use rules to differentiate polynomial functions to obtain the polynomial function $f'(x)$.

Let $f(x) = kx^n$, where k is a particular real number and n is a non-negative integer. Using the limit definition of $f'(x)$,

$$f'(x) = \lim_{h \rightarrow 0} \frac{1}{h} (f(x+h) - f(x))$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

Rule: To differentiate $f(x) = kx^n$, bring down n and subtract 1 from the exponent so that $(kx^n)' = knx^{n-1}$.

Note: Since $\lim_{x \rightarrow a} (p(x) + q(x)) = \lim_{x \rightarrow a} p(x) + \lim_{x \rightarrow a} q(x)$ and the derivative of a constant is 0, the derivative of

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is

$$f'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + 2 a_2 x + a_1.$$

Notation: $f'(x) = (f(x))' = \frac{d}{dx}(f(x)).$

Example 2.4 Let $f(x) = x^3 + 3x - 7$. Calculate $f'(x)$ using both the limit definition and the rules for differentiating polynomials.

Example 2.5 Let $f(x) = -3x^5 + 2x^3 - 12x^2 + 14x - 32$. Calculate $f'(x)$ using rules for differentiating polynomials.

Exercises/Homework:

Let $f(x) = kx^{-n}$, where n is a positive integer and k is a particular non-zero real number. Using the limit definition,

$$f'(x) = \lim_{h \rightarrow 0} \frac{1}{h} (f(x+h) - f(x)),$$

$$= \quad .$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

Theorem 13 Let $f(x) = kx^n$, where k and n are real numbers.

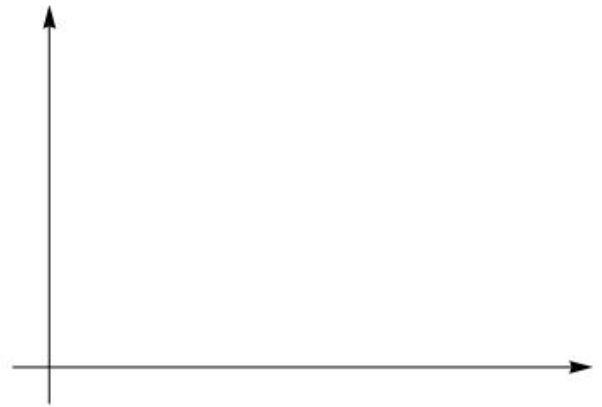
- If $n = 0$, then $f'(x) = 0$. (The derivative of a constant is zero.)
- If $n \neq 0$, then $f'(x) = nkx^{n-1}$.

Example 2.6 Differentiate the function $f(x) = 2x^3 - \frac{5}{x} + 14x - 8$.

Example 2.7 Differentiate the function $f(x) = -12x^5 + 3x^2 + \frac{2}{x^2} - \frac{1}{x}$.

We plot $y = f'(x)$ for functions $f(x)$ and answer questions on the sign of the first derivative of a function.

Example 2.8 Let $f(x) = 2x^3 - 4x + 5$. Plot $y = f'(x)$ and determine where $f'(x)$ is positive, negative, and zero. Plot $y = f(x)$ also and consider any turning points in the context of the sign of $f'(x)$.



Example 2.9 Let $f(x) = x^2 + 3x - 4$. Plot $y = f(x)$ and $y = f'(x)$ on the same graph. State where $f'(x) > 0$ and $f'(x) < 0$.



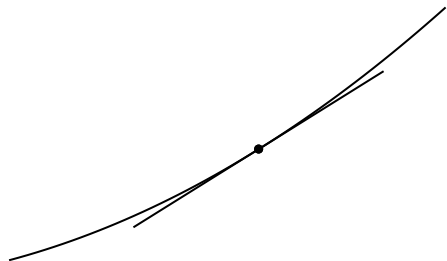
Let a and b be real numbers. We say that $f(x)$ is **increasing** on the interval (a, b) if for all x : $a < x < b$, $f'(x) > 0$. $f(x)$ is **increasing** on the interval $[a, b]$ if for all x : $a \leq x \leq b$, $f'(x) > 0$.

We say that $f(x)$ is **decreasing** on the interval (a, b) if for all x : $a < x < b$, $f'(x) < 0$. $f(x)$ is **decreasing** on the interval $[a, b]$ if for all x : $a \leq x \leq b$, $f'(x) < 0$.

Note: If $f(x)$ is increasing of $[a, b]$, then $f(a) < f(b)$. If $f(x)$ is decreasing of $[a, b]$, then $f(a) > f(b)$.

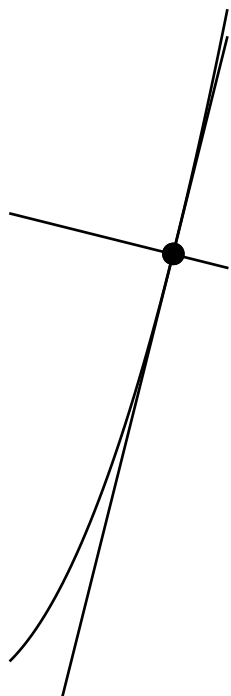
Exercises/Homework:

Let $y = f(x)$ be a curve, where $f(x)$ is a function. The **tangent line** at the point (a, b) is the line which intersects with the curve at exactly the one point (a, b) and has slope equal to the slope of the curve at the point (a, b) .



To find the equation of a tangent line at (a, b) ,

The **normal line** at the point (a, b) is the line perpendicular to the tangent line at the point (a, b) passing through (a, b) .

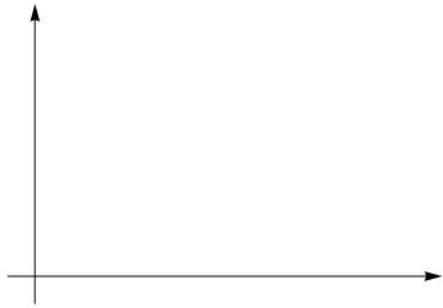


Recall that the lines $y = mx + c$ and $y = -\frac{1}{m}x + d$ are perpendicular.

Example 2.10 Let $y = f(x) = 3x^2 - 4x + 4$. Calculate the tangent line and the normal line at the point $(1, 3)$.

The **average rate of change** of $y = f(x)$ over the interval $[a, b]$ is

$$m_{\text{av.}} = \frac{f(b) - f(a)}{b - a}.$$



The **instantaneous rate of change** of $f(x)$ at the point where $x = a$ is $f'(a)$.

If $f'(a)$ is positive, then the function is **increasing** at $x = a$.

If $f'(a)$ is negative, then the function is **decreasing** at $x = a$.

Example 2.11 Let $y = -4x^2 + 5x - 12$. Determine the average rate of change of $f(x)$ over the interval $[-1, 2]$.

Example 2.12 *The position of a particle is given by*

$$y = f(t) = 5t^3 - 2t + 6,$$

where t is time in seconds and y is the position in metres. The velocity of the particle is $v = f'(t)$. Determine the velocity after 1 second, 2 seconds, 3 seconds.

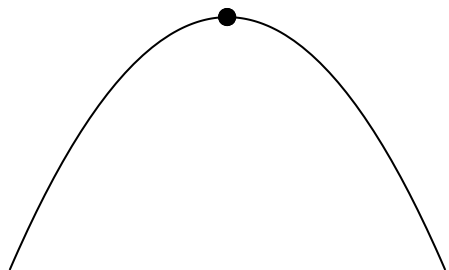
Let $y = f(x)$, where $f(x)$ is a function of x . The point (a, b) is a **stationary point** if $f'(a) = 0$, or equivalently, $\left. \frac{dy}{dx} \right|_{x=a} = 0$.

Example 2.13 Find all stationary points of the curve $y = 2x^3 - 15x + 8$.

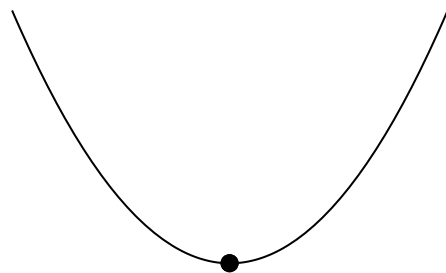
Example 2.14 *The curve $y = ax^2 + bx + c$, where $a \neq 0$, a, b, c are particular real numbers, has one stationary point $(1, 2)$. Show that $y = (c - 2)(x - 1)^2 + 2$.*

Recall that the point (a, b) of the curve $y = f(x)$ is a stationary point if $f'(a) = 0$.

The stationary point (a, b) is called a **local maximum** if $f'(x) > 0$ for $x = a - \varepsilon$ (immediately left of (a, b)), where ε is an arbitrarily small positive real number, and $f'(x) < 0$ for $x = a + \varepsilon$ (immediately right of (a, b)).

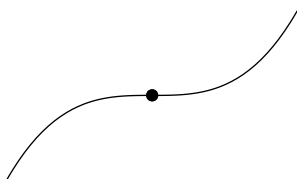


The stationary point (a, b) is called a **local minimum** if $f'(x) < 0$ for $x = a - \varepsilon$ (immediately left of (a, b)), where ε is an arbitrarily small positive real number, and $f'(x) > 0$ for $x = a + \varepsilon$ (immediately right of (a, b)).

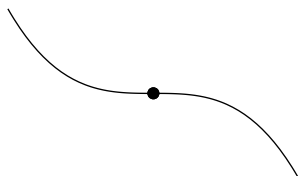


Such stationary points are called **turning points**.

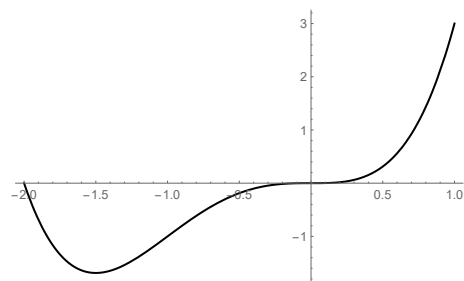
A stationary point (a, b) such that $f'(x) > 0$ for $x = a - \varepsilon$ (immediately left of (a, b)), where ε is an arbitrarily small positive real number, and $f'(x) > 0$ for $x = a + \varepsilon$ (immediately right of (a, b)) is called a **stationary inflection point**.



A stationary point (a, b) such that $f'(x) < 0$ for $x = a - \varepsilon$ (immediately left of (a, b)), where ε is an arbitrarily small positive real number, and $f'(x) < 0$ for $x = a + \varepsilon$ (immediately right of (a, b)) is called a **stationary inflection point**.



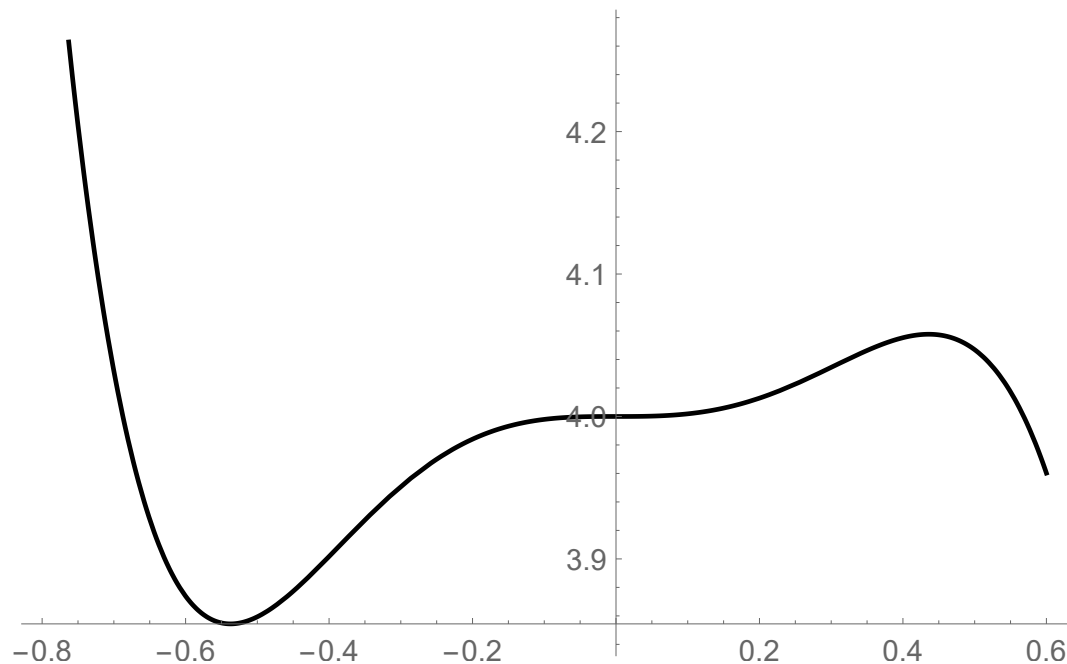
Example 2.15 The curve $y = x^4 + 2x^3$ has a local minimum and an inflection point. Find them and reflect on the definitions of local minimum and inflection point.



Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function and let $y = f(x)$ be a curve/ Suppose the curve has local maxima points $(p_1, q_1), (p_2, q_2), \dots, (p_n, q_n)$. A **global maximum point** (u, v) is a point such that $a \leq u \leq b$ and

$$v \geq f(a), f(b), f(p_1), f(p_2), \dots, f(p_n).$$

In other words, the point in the domain with the greatest y-value.



The **global minimum point** is defined analogously; a point in the domain with the least y-value.

Example 2.16 Visually inspect the curve above to identify local maxima, local minima, inflection points, and the global maximum and the global minimum on the domain $[-0.8, 0.6]$.

Example 2.17 Find the global maximum and global minimum of the curve $y = f(x) = x^4 + 2x^3 - 6x^2 - 12x$ over the interval $[-3, 3]$.

Exercises/Homework:

The **position** of a particle or object is a point $(t, x(t))$ or $(t, x(t), y(t))$, $(x(t), y(t))$, etc. which depends on time. If the object moves on a linear trajectory, then we can express the position x as a function of time as position $= x = x(t)$.

Example 2.18 *A ball is tossed vertically upwards at $t = 0$ seconds from 2 metres above the ground. The elevation (position) of the ball in metres at t seconds is given by $y(t) = -5t^2 + 4t + 2$, $t \geq 0$. Plot y versus t with t on the horizontal axis.*

The **instantaneous velocity** or just **velocity** of a particle with position $x(t)$ is $v = x'(t) = \frac{dx}{dt}$, the first derivative of the position $x(t)$ with respect to t .

The **average velocity** of a particle with position $x(t)$ is given by

$$v_{\text{av}} = \frac{x_2 - x_1}{t_2 - t_1}.$$

The **speed** of a particle is the magnitude $|x'(t)|$ (absolute value). The speed of the particle with velocity $(x'(t), y'(t))$ is the magnitude

$$\|(x'(t), y'(t))\| = \sqrt{(x'(t))^2 + (y'(t))^2}.$$

The **average speed** of a particle with position $x(t)$ is given by $|v_{\text{av}}|$.

Example 2.19 Calculate the velocity, speed, and average speed of the ball with position $y(t) = -5t^2 + 4t + 2$, $t \geq 0$ from vertical launch until it hits the ground again.

The **instantaneous acceleration** or just **acceleration** of a particle is the first derivative of the velocity of the particle,

$$a(t) = v'(t) = (x'(t))' = x''(t).$$

The **average acceleration** of a particle is $a_{\text{av}} = \frac{v_2 - v_1}{t_2 - t_1}$, where v_1 is the initial velocity and v_2 is the final velocity of the particle.

Exercises/Homework:

Consider the function $y = (2x - 4x^3 + 3)^5$. Since this is a polynomial function of x , we could expand this before we differentiate the function. The **chain rule** gives us another way to calculate the derivative of a function.

In this example, letting $u = 2x - 4x^3 + 3$,

Let $F(x)$ be a function of x such that there are other functions $g(x)$ and $f(x)$ satisfying

$$F(x) = f(g(x)) = (f \circ g)(x).$$

The **chain rule** states that

$$F'(x) = (f(g(x)))' = f'(g(x)) \cdot g'(x).$$

Alternatively in Leibniz's notation,

$$F'(x) = (f(g(x)))' = \frac{df}{dg} \frac{dg}{dx}$$

or

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx},$$

where y is a function of u and u is a function of x .

Note: $\frac{dy}{dx}$ is **not a fraction** but the fact that the symbols look like fractions is a very useful aide to memory.

Note: For small Δx , $\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$.

Example 2.20 Let $y = 3(5x^2 - 7x + 2)^6$. Calculate $\frac{dy}{dx}$.

Example 2.21 Let $y = \left(3x^2 - \frac{2}{x^2}\right)^4$. Calculate $\frac{dy}{dx}$.

Recall that to differentiate $y = x^a$, where $a \neq 0$ is a rational number, we have

$$\frac{dy}{dx} = ax^{a-1}.$$

Example 2.22 Consider the function $y: \mathbb{R}^{(>0)} \longrightarrow \mathbb{R}$ by $y(x) = \sqrt{x}$. Calculate $y'(x) = \frac{dy}{dx}$.

Why do we have

$$\frac{d}{dx}(x^a) = ax^{a-1}?$$

If $\frac{dx}{dy} \neq 0$, then $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$, so if $y = x^{\frac{1}{q}}$, where q is a positive integer, then

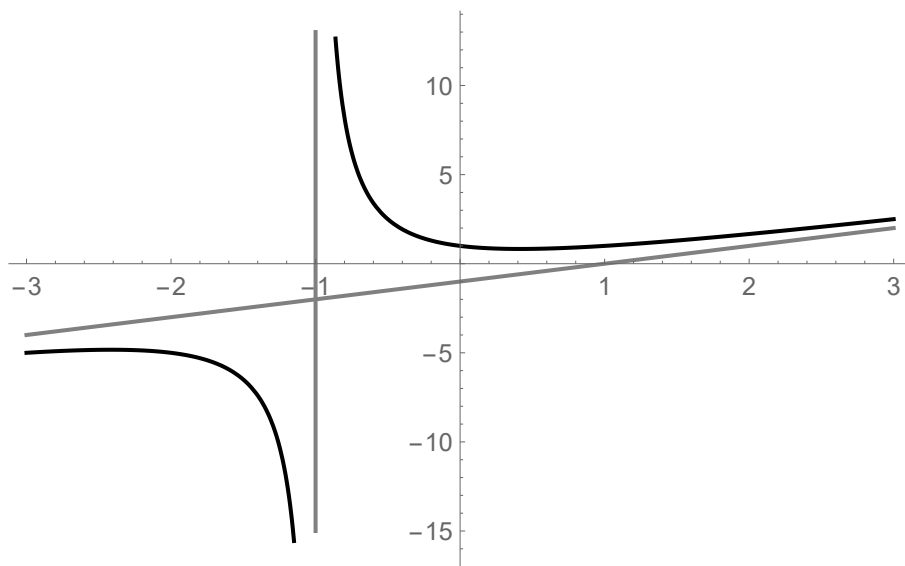
Example 2.23 Let $y = x^{-4/5}$. Calculate $\frac{dy}{dx}$.

Example 2.24 Let $y = \sqrt[3]{x^2 + 1}$. Calculate $\frac{dy}{dx}$ using the chain rule.

The aim of this section is to use the sign of the first derivative to help us to sketch the graph of a curve.

Example 2.25 *Sketch the curve $y = \frac{x^2+1}{x+1}$.*

Continued,



Exercises/Homework:

Suppose that $y(x)$ can be expressed as a product so that $y = uv$ for some functions $u(x)$ and $v(x)$. Then the **product rule** states that

$$\frac{dy}{dx} = \frac{d}{dx}(uv) = v \frac{du}{dx} + u \frac{dv}{dx} = vu'(x) + uv'(x).$$

Example 2.26 Calculate $\frac{dy}{dx}$ for $y = x^3(2x - 7)^4$ using the product rule.

Example 2.27 Calculate $\frac{dy}{dx}$ for $y = x^2 \left(x + \frac{1}{x}\right)$ using the product rule.

Example 2.28 Calculate $\frac{dy}{dx}$ for $y = (x + 1)\sqrt{x^2 - 1}$ using the product rule and the chain rule.

Let $y = \frac{u}{v} = uv^{-1} = uw$, where $w = v^{-1}$.

By the product rule, The rule

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{1}{v^2} \left(v \frac{du}{dx} - u \frac{dv}{dx} \right)$$

is called the **quotient rule**.

Example 2.29 Use the quotient rule to differentiate $y = \frac{x+2}{3x+4}$.

Example 2.30 Use the quotient rule and chain rule to differentiate $y = \frac{2x+5}{(x-6)^{1/2}}$.

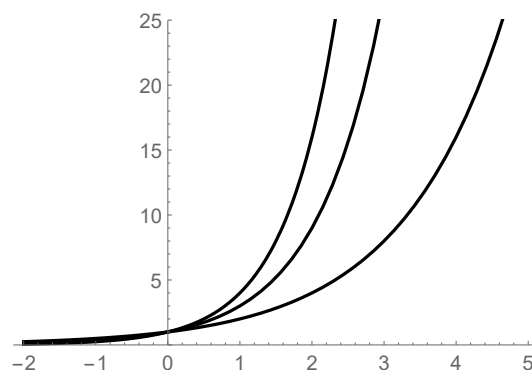
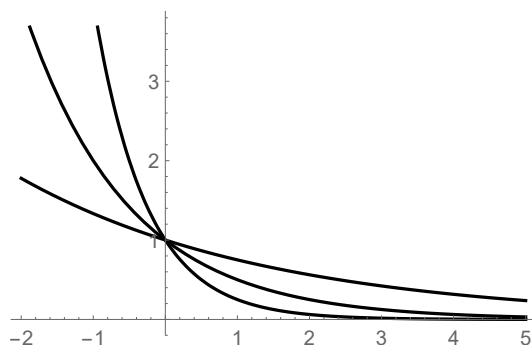
2.4.1 Sketching $y = a^x, a > 0$

If $y = a^x$, where $0 < a < 1$, then letting $b = \frac{1}{a}$ so $a = \frac{1}{b}$, we have $y = a^x = \frac{1}{b^x}$, where $b > 1$. In this case we have **exponential decay** since as $x \rightarrow \infty$, $\frac{1}{b^x} \rightarrow 0$.

When $x = 0$, $y = 1$ so we have the point $(0, 1)$.

When $x < 0$, $y > 1$. If $x = -1$, then $y = \frac{1}{a}$ so we have the point $(-1, \frac{1}{a})$.

When $x = 1$, $y = a$, so we have the point $(1, a)$.



If $y = a^x$, where $a > 1$, then we have **exponential growth**. As $x \rightarrow -\infty$, $y \rightarrow 0$. When $x = 0$, $y = 1$ so we have the point $(0, 1)$.

Example 2.31 Sketch $y = f(x) = 0.1^x$, give the domain and range of $f(x)$, and state the asymptotes.



Example 2.32 Sketch $y = f(x) = 5^x$, give the domain and range of $f(x)$, and state the asymptotes.



Applying the translation $x \mapsto X + h$, $y \mapsto Y + k$ to the equation $Y = a^X$ gives the equation $y = a^{x-h} + k$.

Example 2.33 Sketch $y = f(x) = 3^{x+1} + 1$, give the domain and range of $f(x)$, and state the asymptotes.



Example 2.34 Sketch $y = f(x) = 2^{2x} + 3$, give the domain and range of $f(x)$, and state the asymptotes.



Exercises/Homework:

The irrational number $e = 2.71828\dots$ is an important mathematical constant defined as the limit as $n \longrightarrow \infty$ of rational numbers $\{a_n\}$, where

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

The first few terms of the sequence starting with $n = 1$ are $a_1 = 2$, $a_2 = \left(1 + \frac{1}{2}\right)^2 = \frac{9}{4} = 2.25$, $a_3 = \left(1 + \frac{1}{3}\right)^3 = \frac{64}{27} \approx 2.37$. As $n \longrightarrow \infty$, the sequence converges to the constant e . Another way to obtain e is through the infinite series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

The constants e , π , and $i = \sqrt{-1}$ are related via the formula

$$e^{\pi i} = -1.$$

To plot the curve $y = e^x$, when $x = 0$, $y = 1$ so we have the point $(0, 1)$.

When $x = -1$, $y = \frac{1}{e} \approx 0.37$ so we have the point $(-1, 0.37)$.

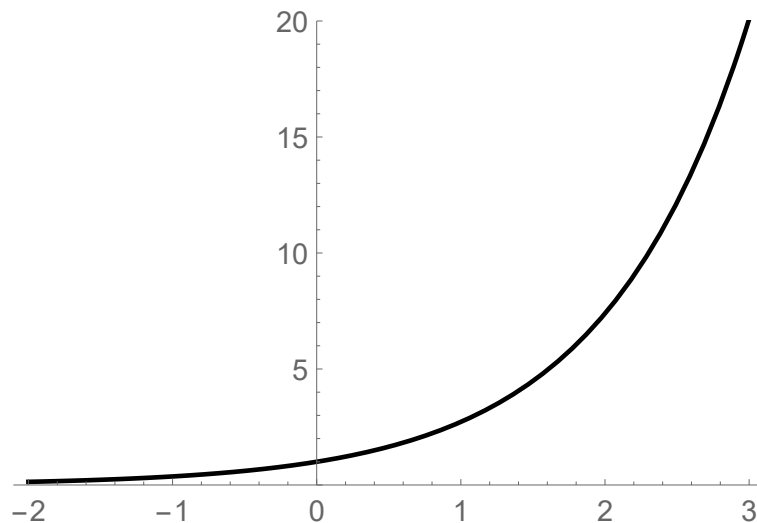
When $x = 1$, $y = e \approx 2.72$ so we have the point $(1, 2.72)$.

When $x = 2$, $y = e^2 \approx 7.39$ so we have the point $(2, 7.39)$.

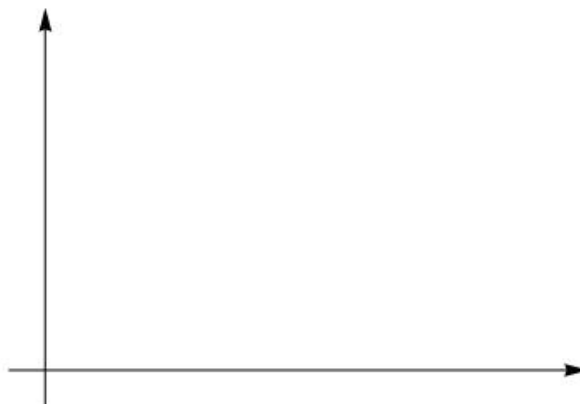
When $x = 3$, $y = e^3 \approx 20.09$ so we have the point $(3, 20.09)$.

As $x \longrightarrow \infty$, we have $y \longrightarrow \infty$,

As $x \longrightarrow -\infty$, we have $y \longrightarrow 0$.



Example 2.35 Sketch $y = f(x) = e^{x+2} + 4$, give the domain and range of $f(x)$, and state the asymptotes.



Note that since $a = e^{\log_e(a)}$, we can understand $y = a^x$ as $y = e^{x \log_e(a)}$.

Example 2.36 Sketch $y = f(x) = 3^x$ by finding a real number k such that $y = e^{kx}$.



Interest in a bank account may be compounded daily, weekly, monthly, annually, or continuously, depending on the terms of the account. Let r be the interest rate per year (per annum) and compounding occurs n times per year. Let P_0 be the initial deposit (principal) and let P be the deposit after some time has elapsed. After t years,

$$P = P_0 \left(1 + \frac{r}{n}\right)^{nt}.$$

Notice that we can rewrite P as

$$P = P_0 \left(\left(1 + \frac{1}{(n/r)}\right)^{n/r} \right)^{rt}.$$

If we let $N = \frac{n}{r}$, then

$$P = P_0 \left(\left(1 + \frac{1}{N}\right)^N \right)^{rt}.$$

As $n \longrightarrow \infty$, $N \longrightarrow \infty$ so

$$\lim_{N \longrightarrow \infty} P =$$

$$=$$

$$=$$

Hence we obtain the formula for continuously compounding interest.

Example 2.37 *A bank offers an interest rate of 2% per annum. An initial deposit of \$ 100 is made.*

- (a) If compounded weekly, how many dollars are in the account after 10 year if no deposits or withdrawals are made?*
- (b) If compounded continuously, how many dollars are in the account after 10 year if no deposits or withdrawals are made?*

Recall the exponent rules for real numbers a, b, c :

$$\begin{aligned}
 a^b a^c &= a^{b+c}, & a^b / a^c &= a^{b-c}, \text{ for } a \neq 0 \\
 (a^b)^c &= a^{bc}, \\
 (ab)^c &= a^c b^c, & (a/b)^c &= a^c / b^c = a^c b^{-c}, \text{ for } b \neq 0 \\
 a^{-1} &= \frac{1}{a}, \text{ for } a \neq 0 \\
 a^0 &= 1, & 0^0 &= 1 \text{ (defined to be 1, but contraversial)} \\
 \frac{1}{a^{-b}} &= a^b, \text{ for } a \neq 0, & a^{-b} &= \frac{1}{a^b} \text{ for } a \neq 0.
 \end{aligned}$$

To solve an exponential equation like that given in the example below, we first change all of the bases into the same base. Next, once we have the same base, we equate the exponents and solve the resulting equation.

Example 2.38 Solve the exponential equation $27 * 5 - 2n = 9^{3n+4}$ for n .

Example 2.39 Solve the exponential equation $4^x = 6 - 2^x$ for x .

Recall that for real numbers a, b, c with $b, c > 0$, $a^b = c$ is equivalent to $b = \log_a(c)$. If $a = e = 2.71828\dots$, then we have $y = e^x$ is equivalent to $x = \log_e(y)$, which is sometimes written as $x = \ln(y)$. We call this the **natural logarithm**.

Recall the following log rules:

$$\begin{aligned}
 \log_a(xy) &= \log_a(x) + \log_a(y), & \text{from } a^b a^c &= a^{b+c}, \\
 \log_a(x/y) &= \log_a(x) - \log_a(y), & \text{from } a^b / a^c &= a^{b-c}, \\
 \log_a(x^n) &= n \log_a(x), & \text{from } (a^b)^c &= a^{bc}, \\
 \log_a(b) &= \frac{\log_c(b)}{\log_c(a)}, & \text{from } (c^y)^x &= c^{xy}, \\
 \log_a(1/b) &= -\log_a(b), & \text{from } \frac{1}{b} &= b^{-1}, \log_a(x^n) = n \log_a(x), \\
 b &= a^{\log_a(b)}, & \text{by letting } x &= \log_a(b), \\
 \log_a(a) &= 1, (a \neq 1) & \text{from } a^1 &= a, \\
 \log_a(1) &= 0, & \text{from } a^0 &= 1.
 \end{aligned}$$

Example 2.40 Simplify $\log_2(8) - \log_2(12)$.

Example 2.41 Solve $\log_4(2x) = 2$ for x .

Example 2.42 Solve $\log_2(3x + 2) = 6$ for x .

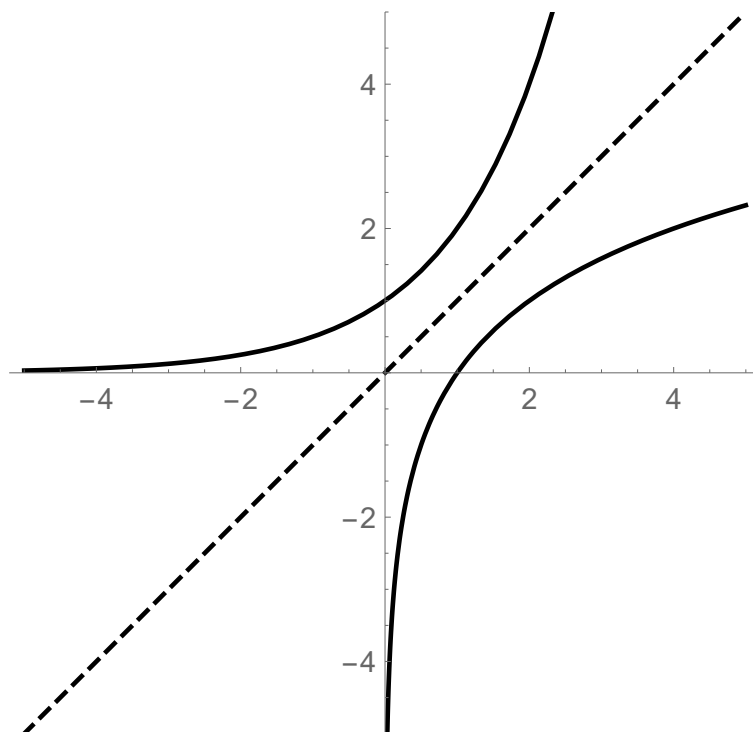
Example 2.43 Solve $\ln(4x + 4) + \ln(3x - 2) = 2\ln(x - 1)$ for x .

Example 2.44 Solve $\log_4(8^x) = 3x + 1$ for x .

Example 2.45 Solve $\log_3(9x - 8) = 2\log_3(x)$ for x .

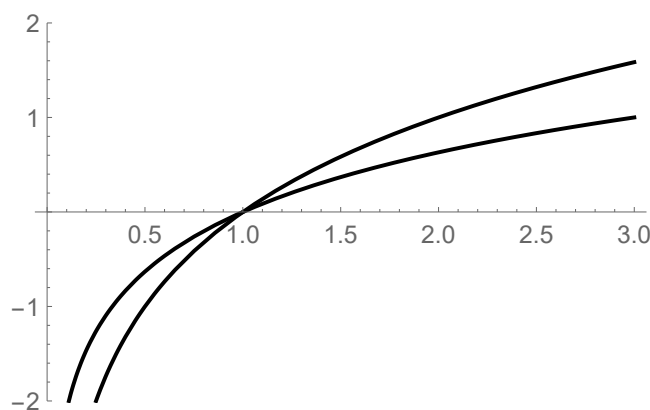
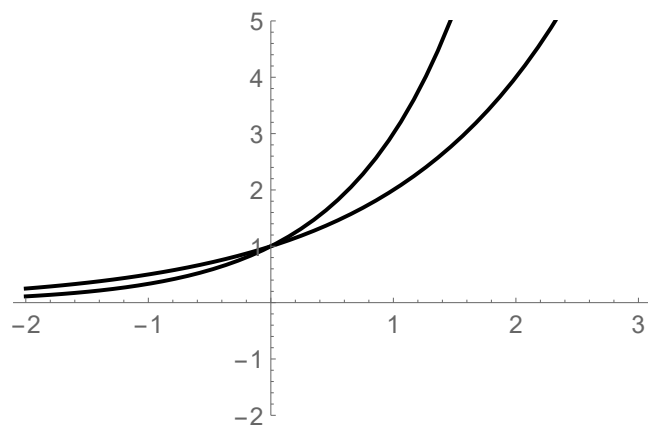
Example 2.46 Solve $4\log_x(2) = \log_2(x)$ for x .

To understand how to plot the curve $y = \log_a(x)$, we consider the equivalent equation $x = a^y$. The linear transformation $x \mapsto Y, y \mapsto X$ gives $Y = a^X$ and this transformation is a reflection about the line $y = x$.



$y = 2^x$ and $y = \log_2(x)$:

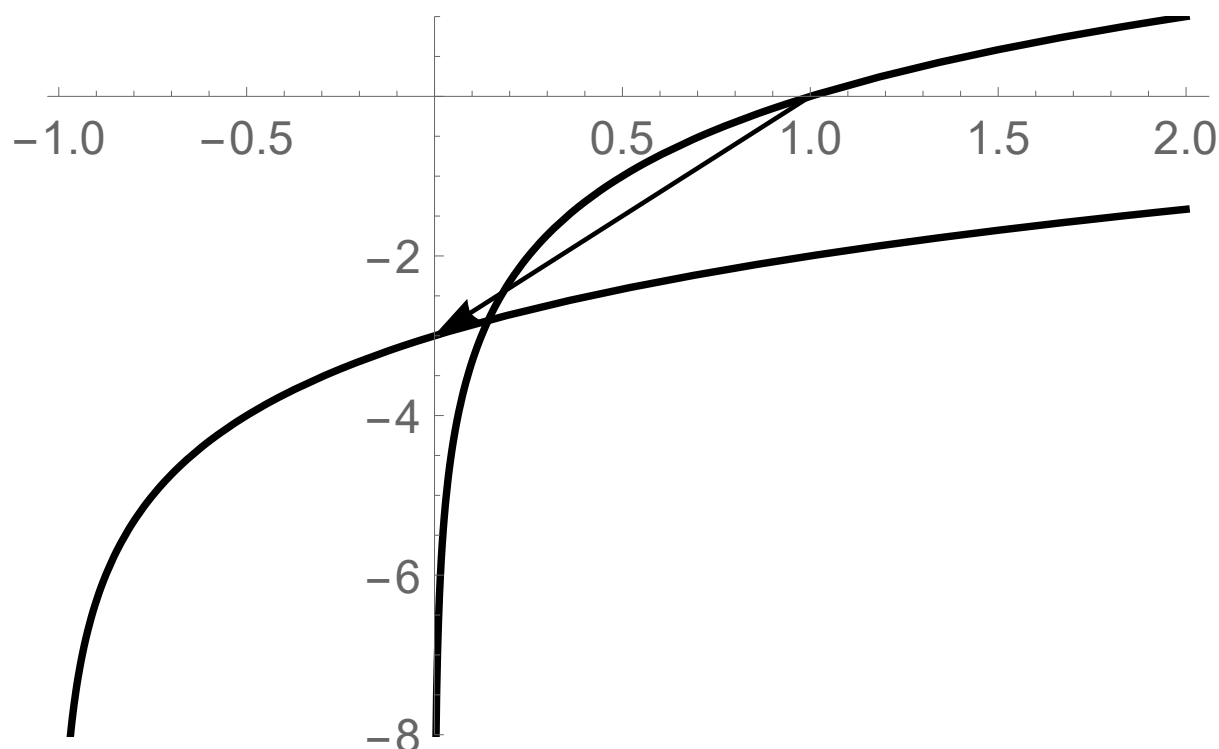
Consider the relationship between the curves $y = 2^x$ and $y = 3^x$ and the relationship between the curves $y = \log_2 x$ and $y = \log_3(x)$.



If $a > 0$ and $a \neq 1$, then the domain D of the $y = f(x) = \log_a(x)$ is the set of positive real numbers. In interval notation $D = (0, \infty) = \{x \in \mathbb{R} : x > 0\}$. The range of $f(x)$ is $\mathbb{R} = (-\infty, \infty)$.

Consider the curve $Y = \log_a(X)$ under the translation $x \mapsto X + h$, $y \mapsto Y + k$. We get $y = k + \log_a(x - h)$.

Example 2.47 Consider the relationship between $Y = \log_2(X)$ and $y = -3 + \log_2(x + 1)$.



What is the translation that relates these curves? State the domains of these curves. Find the two corresponding exponential equations and determine the translation between them.

Recall that if $y = f(x)$, then

$$\frac{dy}{dx} = f'(x) = \lim_{h \rightarrow 0} \frac{1}{h} (f(x+h) - f(x)). \quad (1)$$

Example 2.48 Let $y = e^x$. Use the definition of the derivative, Equation (1), together with the limit identity $\lim_{a \rightarrow 0} \frac{e^a - 1}{a} = 1$ to calculate $\frac{dy}{dx}$.

It follows that

$$\frac{de^x}{dx} = e^x. \quad (2)$$

Example 2.49 Recall that $a = e^{\ln(x)}$. Use this and the chain rule $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ to calculate the derivative of $y = a^x$.

Example 2.50 Calculate the derivative of $y = 3^x$.

Example 2.51 Calculate the derivative of $y = e^{-2x} + e^{3x}$.

Example 2.52 Calculate the derivative of $y = e^{x^2+4x-2}$.

Example 2.53 Calculate the derivative of $y = (x^2 + e^{x^2})^3$ and use it to calculate the tangent line to the curve $y = (x^2 + e^{x^2})^3$ at the point $(0, 1)$.

Recall that $x = e^y$ is equivalent to $y = \ln(x)$. We will learn how to differentiate $\ln(x)$ and related examples.

Example 2.54 *Let $y = \ln(x)$. Calculate the derivative of $\ln(x)$ using $\frac{de^x}{dx} = e^x$, together with the fact that if $\frac{dx}{dy} \neq 0$, then $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.*

We have

$$\frac{d}{dx} \ln(x) = \frac{1}{x}. \quad (3)$$

Example 2.55 *Calculate the derivative $\frac{d}{dx} \ln(4x + 3)$ using the chain rule.*

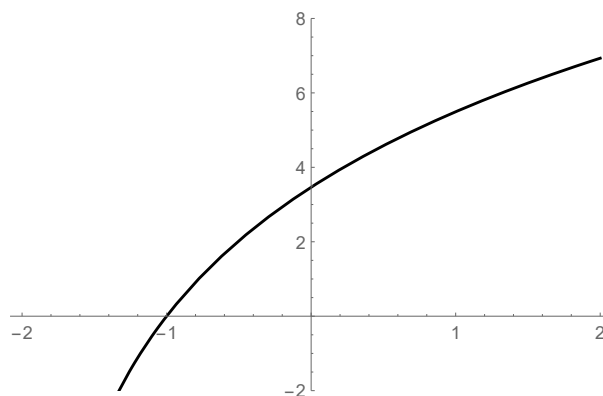
Example 2.56 Calculate the derivative $\frac{d}{dx} \ln(xe^x)$ without using the product rule.

Example 2.57 Use the chain rule to differentiate $y = (\ln(2x + x^2))^2$.

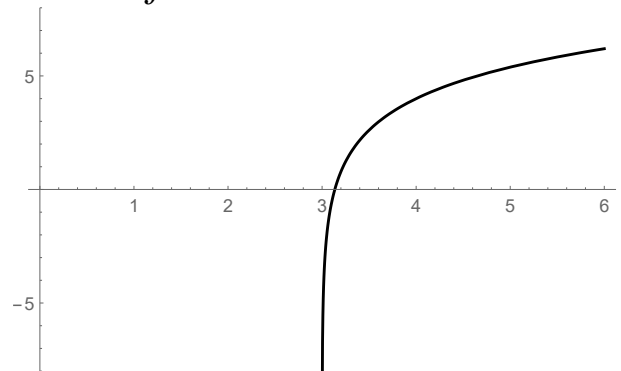
Example 2.58 Calculate the derivative of $y = \log_2(3x)$.

In this section we determine the equations of exponential curves and logarithmic curves given certain geometric information such as points on the curve and/or asymptotes.

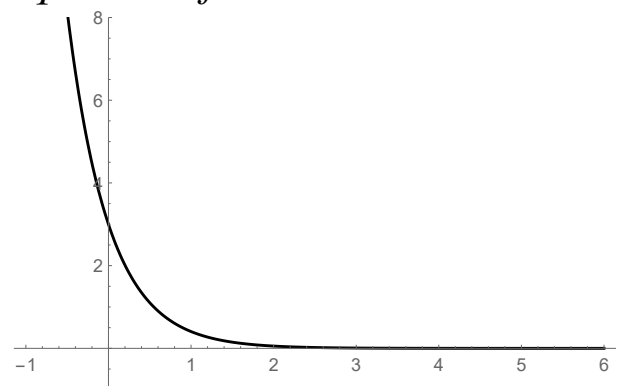
Example 2.59 *The curve shown below is of the form $y = k \ln(x + c)$ and has points $(-1, 0)$ and $(0, \ln(32))$. Find c and k and hence determine the equation of the curve.*



Example 2.60 The curve shown below is of the form $y = k \ln(x + c) + d$ and has the points $(4, 4)$ and $\left(3 + \frac{1}{\sqrt{e}}, 0\right)$, and asymptote $x = 3$. Find c, d and k and hence determine the equation of the curve.



Example 2.61 The curve shown below is of the form $y = ce^{kx}$ and has the point $(0, 3)$ and the tangent line at the point $(0, 3)$ is $y = -6x + 3$. Find c and k and hence determine the equation of the curve.



2.4.9 Differentiating Trigonometric Functions $\sin(x)$, $\cos(x)$, and $\tan(x)$ and the Chain, Product, and Quotient Rules

We derive the derivatives of $\sin(x)$, $\cos(x)$, and $\tan(x)$ calculate the derivatives of other trigonometric functions.

Example 2.62 *Using the definition of the derivative*

$$f'(x) = \lim_{h \rightarrow 0} \frac{1}{h} (f(x+h) - f(x))$$

together with the angle addition formulas

$$\begin{aligned}\sin(a+b) &= \cos(a)\sin(b) + \sin(a)\cos(b), \\ \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b),\end{aligned}$$

and the limits:

$$\lim_{a \rightarrow 0} \frac{\sin(a)}{a} = 1, \quad \lim_{a \rightarrow 0} \frac{\cos(a) - 1}{a} = 0,$$

calculate the derivatives of $\sin(x)$ and $\cos(x)$.

Continued...

We have

$$\frac{d}{dx}\sin(x) = \cos(x), \quad (4)$$

$$\frac{d}{dx}\cos(x) = -\sin(x). \quad (5)$$

Example 2.63 Using the quotient rule $\frac{d(u/v)}{dx} = \frac{1}{v^2} \left(v \frac{du}{dx} - u \frac{dv}{dx} \right)$ and $\tan(x) = \frac{\sin(x)}{\cos(x)}$, calculate the derivative of $\tan(x)$

Example 2.64 Calculate the derivative of $\cos(3x^2 - 2)$ using the chain rule.

Example 2.65 Calculate the derivative of $\sin^2(2x + 2)$ using the chain rule.

Example 2.66 Calculate the derivative of $\cos(x) \sin(x)$ using the product rule $\frac{d(uv)}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$.

Example 2.67 Calculate the derivative of $\frac{\cos(x)}{e^x}$ using the quotient rule $\frac{d(u/v)}{dx} = \frac{1}{v^2} \left(v \frac{du}{dx} - u \frac{dv}{dx} \right)$.

Example 2.68 Calculate the derivative of $\frac{\sin(x^2)}{\cos(e^x)}$ using the quotient rule $\frac{d(u/v)}{dx} = \frac{1}{v^2} \left(v \frac{du}{dx} - u \frac{dv}{dx} \right)$ and the chain rule.

The **second derivative** of a function $y = f(x)$ is the derivative of the first derivative $f'(x)$ or $\frac{dy}{dx}$. The second derivative is written $f''(x)$ or $f^{(2)}(x)$ or $\frac{d^2y}{dx^2}$, which is $\frac{d}{dx} \left(\frac{dy}{dx} \right)$.

Example 2.69 Calculate the second derivative of

$$y = f(x) = \sin(x) + \sqrt{x}.$$

Let $s(t)$ be a scalar function of time t that represents the position of a particle moving in a straight line. Then $v(t) = s'(t)$ is the speed of the particle at time t and $a(t) = v'(t) = s''(t)$ is the acceleration of the particle at time t .

Example 2.70 A particle moving in a straight line has position given by the point $(s(t), 0)$, where $s(t) = \cos(t) + e^{2t}$. Calculate the velocity of the particle, the initial velocity of the particle, and the acceleration of the particle at time t .

Recall that the point $P = (x_0, y_0)$ of the curve $y = f(x)$ is a stationary point if $f'(x_0) = 0$.

Example 2.71 Find all stationary points of the curve

$$y = f(x) = \frac{1}{2}e^{2x} - 5e^x + 4x.$$

Example 2.72 The curve $y = f(x) = x^4 + ax + b$ has a stationary point $(-1, 0)$. Find a and b . Are there any other stationary points with real coordinates?

Recall that a stationary point $P_0 = (x_0, y_0)$ of a continuous curve $y = f(x)$ is a point with $f'(x_0) = 0$. If the sign of the first derivative changes at P_0 , then the point P_0 is a maximum or minimum, which can be classified using the first derivative test. The second derivative test is sometimes easier to use than the first derivative test.

Theorem 14 (Second Derivative Test) *Let $y = f(x)$, where $f(x)$ is continuous and $f'(x)$ and $f''(x)$ exist at the point $P_0 = (x_0, y_0)$.*

- *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then the point P_0 is a local minimum.*
- *If $f'(x_0) = 0$ and $f''(x_0) < 0$, then the point P_0 is a local maximum.*
- *If $f'(x_0) = 0$ and $f''(x_0) = 0$, then the test is inconclusive; P_0 could be an inflection point or an undulation point.*

Example 2.73 *Find and classify all stationary points of the curve $y = x^2$ using the second derivative test.*

An **inflection point** $P_0 = (x_0, y_0)$ of a continuous curve $y = f(x)$ is a point with a change in the curvature of the curve at the point P_0 , that is a change in the sign of the second derivative so that $f''(x_0 + \varepsilon)$ and $f''(x_0 - \varepsilon)$ have opposite signs for all real $\varepsilon > 0$ and sufficiently small.

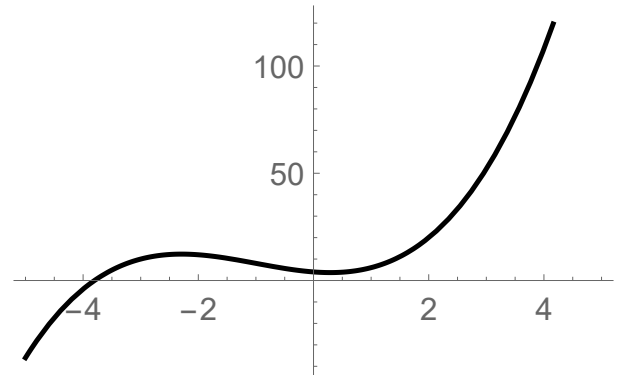
An **undulation point** $P_0 = (x_0, y_0)$ of a continuous curve $y = f(x)$ is a point with $f''(x_0) = 0$ such that there is no change of sign of the second derivative at P_0 .

Theorem 15 (Inflection implies f'' is zero) *If $P_0 = (x_0, y_0)$ is an inflection point of the curve $y = f(x)$, then $f''(x_0) = 0$.*

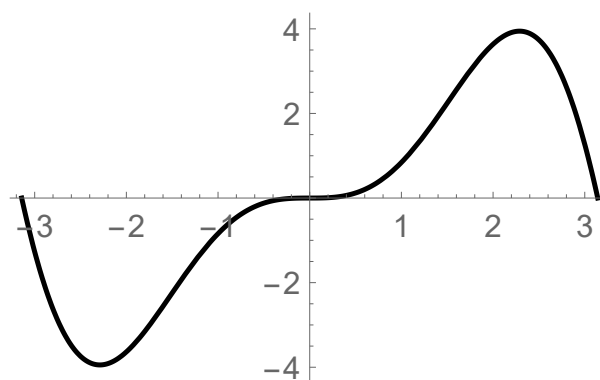
Example 2.74 *Find and classify all stationary points of the curve $y = x^3$. Identify all inflection points of the curve.*

Example 2.75 *Find and classify all stationary points of the curve $y = x^4$. Is the point $(0, 0)$ an inflection point?*

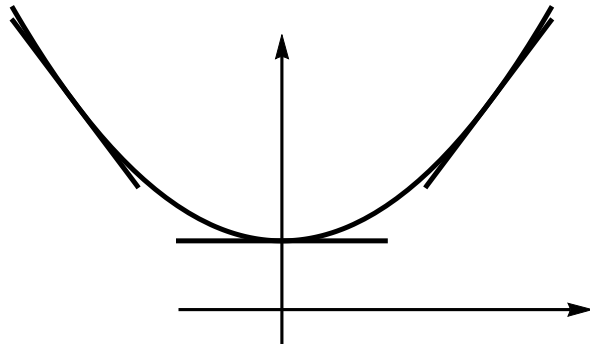
Example 2.76 Find and classify all stationary points of the curve $y = f(x) = x^3 + 3x^2 - 2x + 4$. Identify all inflection points of the curve.



Example 2.77 Find and classify all stationary points of the curve $y = f(x) = x^2 \sin(x)$ defined on the domain $D = [-\pi, \pi]$. Identify all inflection points of the curve in D .



The curve $y = f(x)$ is **concave up** if $f''(x) > 0$. This means that $f'(x)$ is increasing as x increases.



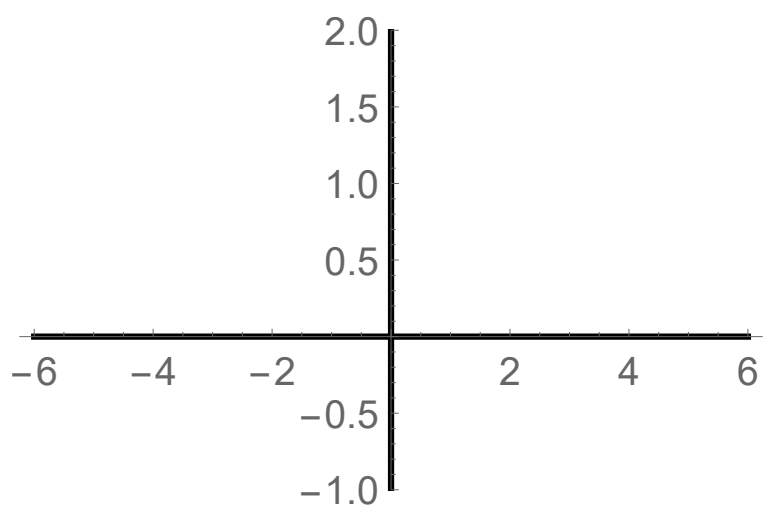
The curve $y = f(x)$ is **concave down** if $f''(x) < 0$. This means that $f'(x)$ is decreasing as x increases.

When sketching the curve $y = f(x)$ defined on the domain D , where $f(x)$ is a function of x , we first calculate the following details:

- The y -intercept. Note that there is only one since f is a function.
- The x -intercept(s).
- Local maxima and local minima points.
- Global maxima and minima points on the domain D .
- Any inflection points.
- Any asymptotes.
- Regions of D in which the function $f(x)$ is concave up.
- Regions of D in which the function $f(x)$ is concave down.

Example 2.78 *Sketch the curve $y = f(x) = x^4 - 32x - 12$ on the domain $D = [-3, 3]$.*

Example 2.79 Sketch the curve $y = f(x) = -\frac{1}{720}x^4 + \frac{1}{24}x^2 + \frac{1}{x^2} - \frac{1}{2}$



Exercises/Homework:

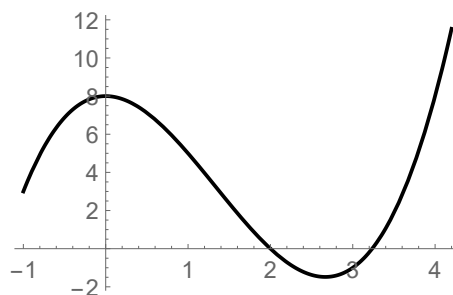
Recall the definition of global maxima and minima from Section 2.3.9. In this section we will present the concept again.

Let $y = f(x)$ be a continuous function on the domain $D = [a, b]$. If for all x in the domain D , $f(x) \leq v = f(u)$, then the point (u, v) is a **global maximum** point.

To find a global maximum point, we find and classify all of the local maximum points and compare their y coordinates to that of the boundaries of the domain, $f(a)$ and $f(b)$. Among those, a point with the greatest y -value corresponds to a global maximum.

Let $y = f(x)$ be a continuous function on the domain $D = [a, b]$. If for all x in the domain D , $f(x) \geq v = f(u)$, then the point (u, v) is a **global minimum** point.

To find a global minimum point, we find and classify all of the local minimum points and compare their y coordinates to that of the boundaries of the domain, $f(a)$ and $f(b)$. Among those, a point with the least y -value corresponds to a global minimum.



Example 2.80 Find the global maximum and global minimum of $y = x^4 - 64x^2 + 800$ on the domain $D = [-3, 8]$.

Example 2.81 Find the global maximum and global minimum of $y = x^6 - 80x^3 + 1800$ on the domain $D = [-3, 5]$.

Example 2.82 A 10000 m^3 enclosure is to be constructed for penguins at a zoo made from expensive glass and inexpensive rock. One vertical side of the enclosure and the horizontal bottom is to be made from rock. Three vertical sides are to be made from rectangular sheets of glass and the top of the enclosure is to be open. Since the glass is expensive, the zoo aims to minimise the surface area of glass used. The side opposite the vertical rock face of the enclosure must have width equal to 20 metres. Find the dimensions of the enclosure that give the least glass costs.

In an **optimisation problem**, we seek to find a maximum or a minimum value of a function of (usually multiple variables) subject to some constraint. The constraint might be a budget, or a volume, or some similar relationship between the variables that restricts the possible values of the variables. When we determine the maxima and minima, we classify these often using the second derivative test and we must consider the domain of definition of the relevant functions since we require global maxima or global minima to answer such questions.

The process we will use in our optimisation problems is to use the constraint to eliminate variables and then find maxima or minima of the resulting function of one variable.

Example 2.83 *The product of two positive numbers x and y is 35. Find x and y such that $f(x, y) = 2x + 3y$ is least.*

Example 2.84 *A farmer growing strawberries in a field wishes to find the maximum crop yield per dollar spent on plants. A high plant density reduces the strawberry yield since they do not get enough light and nutrients. If x strawberry plants are planted, then y strawberries are expected per square metre, where $y = 2500 - 10000x^{-1/2} - \frac{1}{10}x^2$. The cost of each strawberry plant is \$ 3. If the field is 1000 m^2 , what is the maximum number of strawberries that can be produced, and what is the production cost of a strawberry at the maximum yield? If the farm is only viable if at least 20000 strawberries are produced, find the x such that the production cost of a strawberry is least by finding the minimum value of $C = \frac{3x}{y}$.*

Continued...

Exercises/Homework:

References

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- [2] Michael Evans, Kay Lipson, Douglas Wallace, David Greenwood, Mathematical Methods Units 1 & 2, Cambridge Senior Mathematics for Queensland, 2nd Ed., 2024.
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