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End of Semester Examination,

Multivariate Calculus

Time: 90 Minutes for working

No perusal time before examination begins

Total marks available: 90

Full working must be shown on the pages provided.

Permitted materials: A pocket calculator or graphics calculator.

Mobile phones and laptops are not permitted. Please switch phones off.

Name:

1. Let

$$f(x, y) = y\sin(x) + x\cos(y).$$

   (a) Calculate the partial derivatives: $f_x(x, y) = \frac{\partial f}{\partial x}$ and $f_y(x, y) = \frac{\partial f}{\partial y}$. **(4 Marks)**

   (b) Calculate $\nabla f(0, 0)$, the gradient vector of $f(x, y)$ at the point $(0, 0)$.
   . **(2 Marks)**

   (c) Calculate the slope of the function $f(x, y)$ in the direction of the vector $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\mathbf{i} + \mathbf{j}$ at the point $(0, 0)$, the directional derivative $f_{\mathbf{v}}(0, 0)$.
   . **(4 Marks)**

2. The volume under the plane $z = x - y$ and above the plane $z = 0$ between $x = 0$ and $x = 2$, and between $y = 0$ and $y = 2$ is given by the double integral $V = \int_0^2 \int_0^x (x - y)\, dy\, dx = \frac{4}{3}$. See Figure 1 below.

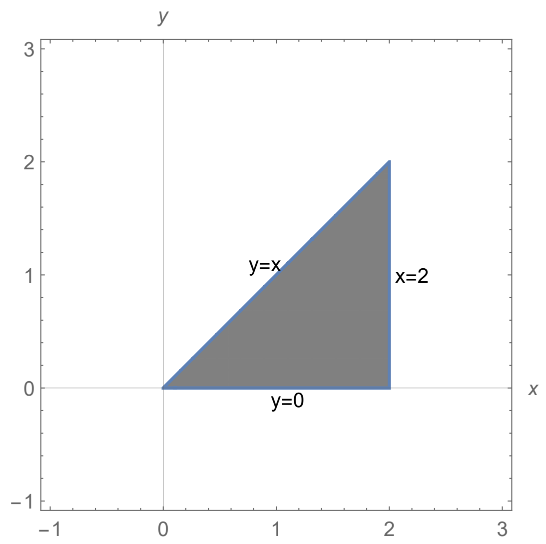

Figure 1: The region we are integrating over.

(a) Show that $V = \int_0^2 \int_0^x (x - y)\, dy\, dx = \frac{4}{3}$. **(10 Marks)**

(b) Change the order of integration and show that $V = \frac{4}{3}$ again with this new order of integration. Hint: $V = \int_0^2 \int_{f(y)}^2 (x - y)\, dx\, dy$, where $f(y)$ is a particular function of y that you must determine by consideration of a point (x, y) that is inside the shaded region of Figure 1 satisfying $0 \leq y \leq 2$ and $f(y) \leq x \leq 2$ for some particular $f(x)$. **(4 Marks)**

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3. The velocity of water particles in a river is described by the vector field $\mathbf{F} = (x + 2y, -x, 0)$ shown in Figure 2 below.

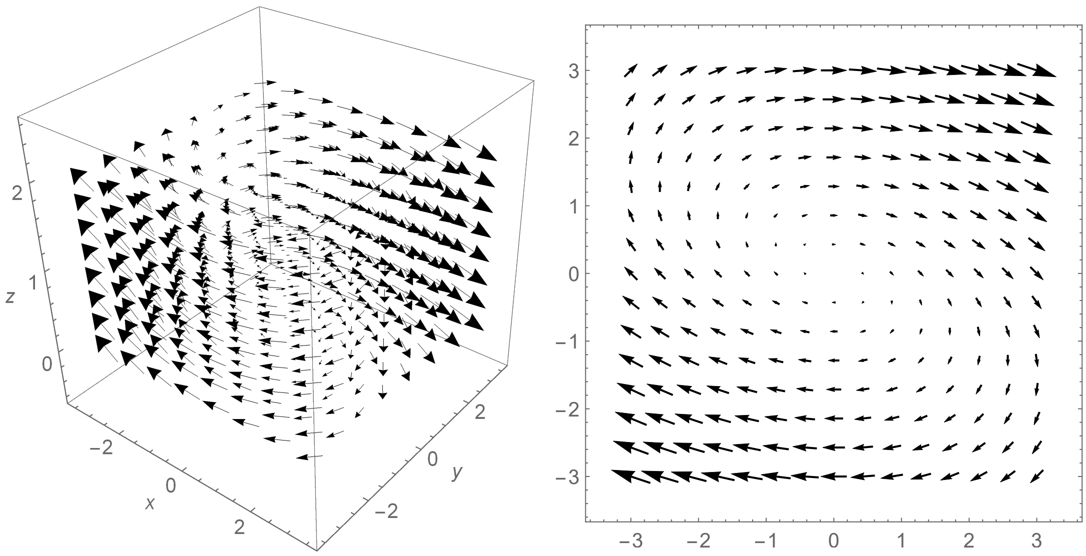

Figure 2: Left: The vector field $\mathbf{F} = (x + 2y, -x, 0)$. Right: The vector field from above.

(a) In which direction does curl($\mathbf{F}$) point? Explain. **(2 Marks)**

(b) Is $\mathbf{F}$ irrotational? Explain. **(2 Marks)**

(c) Calculate div($\mathbf{F}$). What does the sign of div($\mathbf{F}$) tell you? **(2 Marks)**

(d) By calculating curl($\mathbf{F}$), show that $\mathbf{F}$ is not conservative. **(6 Marks)**

(e) Show that the parametrization of the linear path from the point $A = (1, 1, 1)$ to the point $B = (0, 0, 0)$ is given by $C: \ \mathbf{r}(t) = (1 - t, 1 - t, 1 - t)$, $0 \le t \le 1$.
. **(2 Marks)**

(f) Calculate the work done in moving a particle through the current from the point $A = (1, 1, 1)$ to the point $B = (0, 0, 0)$ along a linear path. Recall that $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(r(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt$, where $\mathbf{r}(t) = (1-t, 1-t, 1-t)$. **(6 Marks)**

(g) Now let $\mathbf{F} = (x + 2y, -x)$. Use Green's theorem in the plane,

$$W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (Q_x - P_y)\, dx\, dy, \text{ where } \mathbf{F} = (P, Q), P_y = \frac{\partial P}{\partial y}, Q_x = \frac{\partial Q}{\partial x}$$

to calculate the work done in moving a particle from the point $(1, 0)$ to $(1, 0)$ counterclockwise around the path $C : \mathbf{r}(t) = (\cos(t), \sin(t))$, $0 \le t \le 2\pi$. Note that the region D enclosed by C is the unit disc $D = \{(x, y) \ : \ x^2 + y^2 \le 1\}$, the disc whose boundary is the a circle of radius 1 and centre the origin. You may assume that $\iint_D 1\, dx\, dy = \iint_D r\, dr\, d\theta = \int_0^1 \int_0^{2\pi} r\, dr\, d\theta$.
. **(5 Marks)**

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4. Consider the vector field $\mathbf{F}(x, y) = (2xy + y^2, 2xy + x^2)$.

   (a) Show that $\mathbf{F}$ is conservative by showing that $\operatorname{curl}(2xy + y^2, 2xy + x^2, 0)$ is the zero vector $(0, 0, 0)$. **(5 Marks)**

   (b) Show that $f(x, y) = x^2y + xy^2$ is a potential function for $\mathbf{F}$. i.e. $\mathbf{F} = \nabla f$ by solving for $f(x, y)$ such that $f_x = 2xy + y^2$ and $f_y = 2xy + x^2$. Note that if you only differentiate $f(x, y)$, then you will not get full marks for this part but part marks. **(5 Marks)**

   (c) Calculate the work done $W = \int_{t=0}^{1} \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt$ in moving a particle from the point $(0, 0)$ to the point $(1, 1)$ along the path C, where C is the line connecting $(0, 0)$ and $(1, 1)$ by parametrizing C. Since $\mathbf{F}$ is conservative and hence W is path independent, you may choose a linear path C. **(5 Marks)**

   (d) Calculate the work done in moving a particle from the point $A = (0, 0)$ to the point $B = (1, 1)$ using the fundamental theorem for line integrals, $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$. **(6 Marks)**

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5. Use the fact that $\iiint_V 1\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2\sin(\phi)\,dr\,d\phi\,d\theta$, where V is the interior of a sphere of radius 1 centred at the origin, to show that the volume of a sphere of radius 1 is $\frac{4}{3}\pi$. **(10 Marks)**

6. Find the flux of the vector field $\mathbf{F}(x, y, z) = (z, y, x)$ over the unit sphere $x^2 + y^2 + z^2 = 1$. You may assume that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$, i.e. $\iiint_V 1\,dV = \frac{4}{3}\pi r^3$, where V is the interior of a sphere of radius r. Note that the divergence theorem gives Flux $= \iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_V \operatorname{div}(\mathbf{F})\,dV$.
. **(10 Marks)**