Mathematica Practical Exercise Set 2

The Mathematica Practical Exercise Sets are optional but they will help you understand Multivariate Calculus and Linear Algebra better and sooner. This exercise set covers Chapters 1 and 2 of the course notes.

Add 1+1. Press Shift and Enter to evaluate a cell.

In[ ]:=
```
1 + 1
```

A function f may be thought of as a rule
f : X → Y given by f (x) = (the rule)
such that for all x in X,
f (x) exists and is given without ambiguity.
(For each input, there is a unique output)

X is called the domain.

Y is called the codomain.

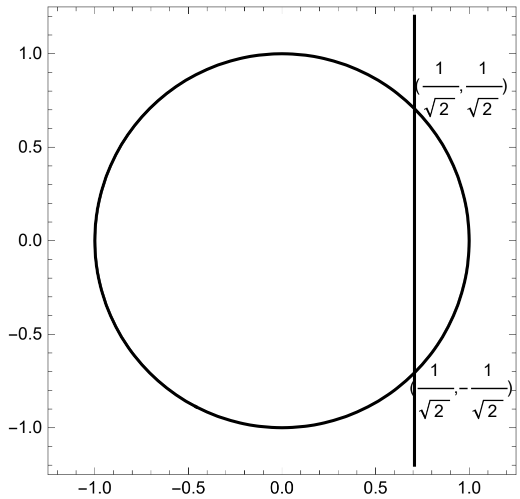
The range of f is the subset R ⊆ Y such that
for all y in R there exists x in X such that f (x) = y.

Example:
$f:\mathbb{R}\to\mathbb{R}$ by $f(x) = x^2+1$.
Plot f(x).

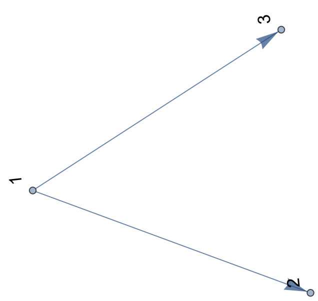
In[ ]:=
```
Plot[x^2 + 1, {x, -3, 3}]
```

Non – Example : $x^2 + y^2 = 1$.

f $\left(\frac{1}{\sqrt{2}}\right)$ is ambiguous since $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ are both points on the unit circle $x^2 + y^2 = 1$.




```
A function doesn't have:
```

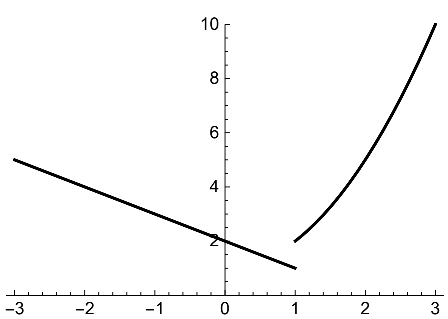


Plot the circle $x^2 + y^2 = 1$ using ContourPlot because the collection of points satisfying $x^2 + y^2 = 1$ is not a function.

```
In[∘]:= ContourPlot[x^2 + y^2 == 1, {x, -1, 1}, {y, -1, 1}]
```

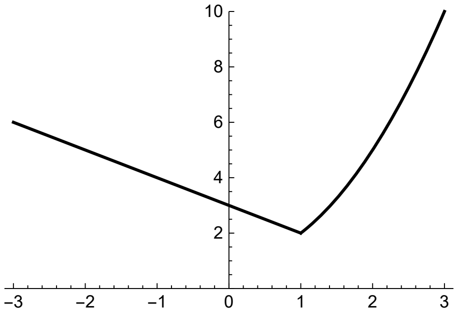
```
A piecewise function is a function
   whose rule depends on an inequality satisfied.
Example:
```

$$f : \mathbb{R} \to \mathbb{R} \quad \text{by} \quad f(x) = \begin{cases} x^2 + 1 & \text{if } x \geq 1, \\ -x + 2 & \text{if } x < 1. \end{cases}$$




```
Example:
```

$$f : \mathbb{R} \to \mathbb{R} \quad \text{by} \quad f(x) = \begin{cases} x^2 + 1 & \text{if } x \geq 1, \\ -x + 3 & \text{if } x < 1. \end{cases}$$

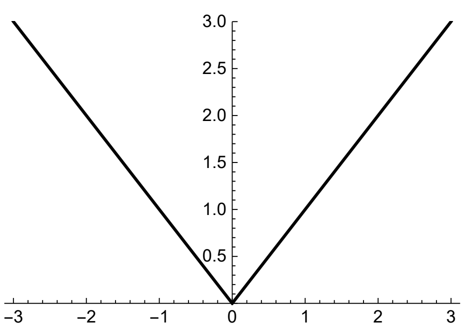


To sketch by hand,
plot $y = -x + 3$ up to when $x = 1$.
$f(0) = -0 + 3 = 3$.
We have the point $(0, 3)$.
$f(-1) = -(-1) + 3 = 1 + 3 = 4$
We have the point $(-1, 4)$.
Connect the points with a line.

Plot $y = x^2 + 1$ for $x >= 1$.

$f(2) = 2^2 + 1 = 4 + 1 = 5$.
We have the point $(2, 5)$.
$f(3) = 3^2 + 1 = 9 + 1 = 10$.
We have the point $(3, 10)$.
Connect these points with a quadratic shape.

Example :
$f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$



Range of f is $\{x : x \geq 0\}$.

Plot the piecewise function

$$f(x) = \begin{cases} x^2 + 3 & \text{if } x \le 2, \\ 3x + 7 & \text{if } x > 2. \end{cases}$$

In[∘]:=
```
Plot[If[x ≤ 2, x^2 + 3, 3 x + 7], {x, -6, 5}]
```

Let f : X → Y and g : Y → Z be functions.

Then $g \circ f$: X → Z by $g \circ f(x) = g(f(x))$.

Composition is an operation ∘ that takes two functions f and g, and produces a function $h = g \circ f$ such that $h(x) = g(f(x))$.

Example :

$f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 1$.

$g : \mathbb{R} \to \mathbb{R}$ by $g(x) = -2x + 3$.

$g \circ f : \mathbb{R} \to \mathbb{R}$ by

$g \circ f(x) = g(f(x)) = g(x^2 + 1) = -2(x^2 + 1) + 3 = -2x^2 + 1$.

$f \circ g : \mathbb{R} \to \mathbb{R}$ by

$f \circ g(x) = f(g(x)) = f(-2x + 3) = (-2x + 3)^2 + 1 = 4x^2 - 12x + 10$.

Build Mathematica functions

$f(x) = x^2 + 1$ and $g(x) = -2x + 3$, then determine $f(g(x))$ and $g(f(x))$.

In[4]:=
```
f[x_] := x^2 + 1;
g[x_] := -2 x + 3;
f[g[x]]
g[f[x]]
```

Out[6]= $1 + (3 - 2x)^2$

Out[7]= $3 - 2(1 + x^2)$

Use the Manipulate function to show the slope of a secant connecting the points (2, f(2)) and (2+h, f(2+h)) as h goes from 1 to 0 for the function $f(x) = x^4 + 5x - 1$.

In[∘]:=
```
f[x_] := x^4 + 5 x - 1;
ClearAll[h]
Manipulate[
 A = ListPlot[{{2, f[2]}, {2 + h, f[2 + h]}}, PlotStyle → {Red, PointSize[0.02]}];
 l = Line[{{2, f[2]}, {2 + h, f[2 + h]}}];
 B = Graphics[{Thick, Dashed, Red, l}];
 G = Plot[f[x], {x, 0, 3}, PlotStyle → Black];
 Show[A, B, G, PlotRange → All, AxesOrigin → True], {h, 1, 0}]
```

Take the first derivative of the function $f(x) = 9x^6 + 4x^2$.

```
D[9 x^6 + 4 x^2 + Sin[x], x]
```

Ask Wolfram alpha how to differentiate $9x^6 + 4x^2$.
Starting a cell with == calls on Wolfram alpha.

```
In[∘]:= differentiate 9 x^6+4 x^2+Sin(x)
```

Take the first derivative of the function $f(x) = 2e^{3x^2}\cos(x)$ and then factorize the result.

```
Factor[D[2 Exp[3 x^2] Cos[x], x]]
```

Take the second derivative of the function $f(x) = 2e^{3x^2}\cos(x)$ and then factorize the result.

```
Factor[D[2 Exp[3 x^2] Cos[x], {x, 2}]]
```

Take the first derivative of the function $f(x) = \frac{4x-5}{x^2-2x+4}$ and then factorize the result.

```
Factor[D[(4 x - 5) / (x^2 - 2 x + 4), x]]
```

Plot the inverse function $f^{-1}(x)$ of the hyperbolic sin function $f(x) = \sinh(x) = (1/2)(e^\wedge x - e^{-x})$.

```
In[∘]:= Plot[ArcSinh[x], {x, -500, 500}]
```

Build a function $f(t) = \int_0^{t^3} (1 - e^{-2x^2})\, dx$ and then differentiate it with respect to t. This is otherwise done with the fundamental theorem of calculus.

```
In[∘]:= f[t_] := Integrate[1 - Exp[-2 x^2], {x, 0, t^3}];
f[t]
Factor[D[f[t], t]]
```

Plot f(x) = sin(x), x, and the inverse function $f^{-1}(x) = \arcsin(x)$ of f(x) = sin(x) displaying that $f^{-1}(x)$ is a mirror image of f(x) about the line y = x.

```
In[∘]:= Plot[{Sin[x], x, ArcSin[x]}, {x, -Pi/2, Pi/2}, AspectRatio → {1, 1}]
```

Calculate an antiderivative (integral) $\int \frac{1}{\sqrt{1+x^2}}\, dx$ of $\frac{1}{\sqrt{1+x^2}}$. The plus c is not shown.

```
In[∘]:= Integrate[1/Sqrt[1 + x^2], x]
```

Give a numerical estimate of the base of the natural logarithm e.

```
In[∘]:= N[Exp[1.0]]
```

The Taylor series of a function f(x) about x = a is given by
$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots,$
where $f^{(n)}(a)$ is the n-th derivative of f(x) evaluated at x = a.
When |x-a| is small, say much less than 1, a truncation of the Taylor series gives a good approximation of f(x). The infinite series of f(x) about x = a converges to f(x) for all x within a region called the radius of convergence.

Calculate a truncated Taylor series of sin(x) (8 terms) about x = 0.

```
In[∘]:= Series[Sin[x], {x, 0, 8}]
```

With the Taylor polynomial f(x) of degree 7 found above, compare sin(0.5) to f(0.5).

```
x = 0.5;
x - x^3/6 + x^5/120 - x^7/5040
Sin[x]
ClearAll[x]
```

Plot the horizontal plane z = 2.

```
ContourPlot3D[z == 2, {x, -3, 3}, {y, -3, 3}, {z, -3, 3}]
```

Use the Manipulate function to plot the guitar string
f(x, t) = A sin(x) cos(t) with amplitude A = 1 over the first 15 seconds of motion.

```
A = 1;
ClearAll[x, t]
Manipulate[Plot[{A Sin[x] Cos[t], 1, -1}, {x, 0, 3.2}], {t, 0, 15}]
```

Plot the surface z = A sin(x) cos(t) with A = 1 in 3D.

```
In[•]:= Plot3D[A Sin[x] Cos[t], {x, 0, 3}, {t, 0, 15}]
```

Plot some contours of the surface z = f(x, y) = $x^2 + y^2$ and then make a 3D plot of the surface z = f(x, y) = $x^2 + y^2$.

```
In[•]:= ContourPlot[x^2 + y^2, {x, -3, 3}, {y, -3, 3}]
Plot3D[x^2 + y^2, {x, -3, 3}, {y, -3, 3}]
```

Plot some contours of the surface z = f(x, y) = $\sqrt{x^2 + y^2}$ and then make a 3D plot of the surface z = f(x, y) = $\sqrt{x^2 + y^2}$.

```
In[•]:= ContourPlot[Sqrt[x^2 + y^2], {x, -3, 3}, {y, -3, 3}]
Plot3D[Sqrt[x^2 + y^2], {x, -3, 3}, {y, -3, 3}]
```

Plot z = f(x, y) = $x^2 + y^2$ together with the particular contours z = 1, 2, 3.

```
A = ContourPlot3D[z == x^2 + y^2, {x, -3, 3}, {y, -3, 3}, {z, -3, 3}];
B = ContourPlot3D[{z == 1, z == 2, z == 3}, {x, -3, 3}, {y, -3, 3}, {z, -3, 3}];
Show[A, B]
```

Plot the surface z = f(x, y) = $x^2 + 4y^2$-2 x+1 together with the particular contours z = 1, 2, 3.

```
In[11]:= A = ContourPlot3D[z == 1 - 2 x + x^2 + 4 y^2, {x, -3, 3}, {y, -3, 3}, {z, -3, 4}];
B = ContourPlot3D[{z == 1, z == 2, z == 3}, {x, -3, 3}, {y, -3, 3}, {z, -3, 3}];
Show[A, B]
```

Plot the surface
z = $1 - x^2 - y^2$ together with cross sections x = 1, and y = -1.

```
A = ContourPlot3D[z == 1 - x^2 - y^2, {x, -3, 3}, {y, -3, 3}, {z, -3, 3}];
B = ContourPlot3D[{y == -1, x == 1}, {x, -3, 3}, {y, -3, 3}, {z, -3, 3}];
G = ListPointPlot3D[{{1, -1, -1}}, PlotStyle → {Black, PointSize[0.02]}];
Show[A, B, G]
```

Plot the surface

$z = e^{-x^2-y^2}$ together with contours z = 1 and z = 2.

```
A = Plot3D[Exp[-x^2 - y^2], {x, -2, 2},
   {y, -2, 2}, PlotRange → {-.3, 1.0}, BoxRatios -> {1, 1, 1}];
B = Plot3D[2, {x, -3, 3}, {y, -3, 3}, PlotStyle -> Opacity[0.5]];
G = Plot3D[1, {x, -3, 3}, {y, -3, 3}, PlotStyle -> Opacity[0.5]];
F = Show[A]
```

Give a numerical estimate of $\frac{2 \text{Sqrt}[2]}{3}$ to 14 decimal places.

```
In[∘]:= N[2 Sqrt[2] / 3, 14]
```

Find three points of the plane z = 6 x+4 y-22 that are on the axes and plot them with ListPointPlot3D, call the result A. Then use ContourPlot3D to plot the equation z = 6 x+4 y-22 and call the result B. Finally, show A and B together using Show.

```
px = Solve[((6 x + 4 y - z - 22) /. {y → 0, z → 0}) == 0]
Px = {px[[1, 1, 2]], 0, 0}
py = Solve[((6 x + 4 y - z - 22) /. {x → 0, z → 0}) == 0]
Py = {0, py[[1, 1, 2]], 0}
pz = Solve[((6 x + 4 y - z - 22) /. {x → 0, y → 0}) == 0]
Pz = {0, 0, pz[[1, 1, 2]]}
A = ListPointPlot3D[{Px, Py, Pz}];
B = ContourPlot3D[6 x + 4 y - z - 22 == 0, {x, -25, 25},
   {y, -25, 25}, {z, -25, 25}, ContourStyle → Opacity[0.5]];
Show[A, B, PlotRange → All]
```

Plot the surface $z = 1-x^2-y^2$ together with the intersection of the plane y=-1 and the surface. It shows, with y fixed, that the slope of a vertical slice depends on x;
$z_x = -2x$.

```
In[∘]:= A = Plot3D[1 - x^2 - y^2, {x, -2, 2}, {y, -2, 2}, BoxRatios → {1, 1, 1}];
B =
  ParametricPlot3D[{t, -1, -t^2}, {t, -2.5, 2.0}, PlotRange → All, PlotStyle → Tube[0.1]];
G = ParametricPlot3D[{t, -1, -t}, {t, -2.5, 2.0}, PlotRange → All, PlotStyle → Tube[0.1]];
F = Show[A, B, G]
```

Plot the surface $z = 1-x^2-y^2$ together with the intersection of the plane x = 1 and the surface. It shows, with x fixed, that the slope of a vertical slice depends on y;
$z_y = -2y$.

```
A = Plot3D[1 - x^2 - y^2, {x, -2, 2}, {y, -2, 2}, BoxRatios → {1, 1, 1}];
B =
  ParametricPlot3D[{1, t, -t^2}, {t, -2.5, 2.0}, PlotRange → All, PlotStyle → Tube[0.1]];
G =
 Show[
  A,
  B]
```

Let f = x Sin[y] + y Cos[x]. Then

$f_x = \frac{\partial f}{\partial x}$,

$= \frac{\partial}{\partial x}(x \, \mathrm{Sin}[y] + y \, \mathrm{Cos}[x]),$

$= \frac{\partial}{\partial x}(x \, \mathrm{Sin}[y]) + \frac{\partial}{\partial x}(y \, \mathrm{Cos}[x]),$

$= \mathrm{Sin}[y] \frac{\partial}{\partial x}(x) + y \frac{\partial}{\partial x}(\mathrm{Cos}[x]),$

$= \mathrm{Sin}[y] (1) + y(-\mathrm{Sin}[x]),$

$= \mathrm{Sin}[y] - y \, \mathrm{Sin}[x].$

$f_y = \frac{\partial f}{\partial y},$

$= \frac{\partial}{\partial y}(x \, \mathrm{Sin}[y] + y \, \mathrm{Cos}[x]),$

$= \frac{\partial}{\partial y}(x \, \mathrm{Sin}[y]) + \frac{\partial}{\partial y}(y \, \mathrm{Cos}[x]),$

$= x \frac{\partial}{\partial y}(\mathrm{Sin}[y]) + \mathrm{Cos}[x] \frac{\partial}{\partial y}(y),$

$= x (\mathrm{Cos}[y]) + \mathrm{Cos}[x] (1),$

$= x \, \mathrm{Cos}[y] + \mathrm{Cos}[x].$

We have

$f_x (1, 1) = \mathrm{Sin}[1] - \mathrm{Sin}[1] = 0,$

$f_y (1, 1) = 2 \, \mathrm{Cos}[1] \approx 1.080604611736279.$

Use Mathematica to verify these calculations.

```
In[26]:= f = x Sin[y] + y Cos[x];
fx = D[f, x]
fy = D[f, y]
fx /. {x → 1.0, y → 1.0}
fy /. {x → 1.0, y → 1.0}
```

Let $f = x \, \mathrm{Sin}[y] + y \, \mathrm{Cos}[x]$. Then

$f_{xy} = \frac{\partial}{\partial y} \frac{\partial f}{\partial x},$

$= \frac{\partial}{\partial y}(\mathrm{Sin}[y] - y \, \mathrm{Sin}[x]),$

$= \frac{\partial}{\partial y}(\mathrm{Sin}[y]) - \frac{\partial}{\partial y}(y \, \mathrm{Sin}[x]),$

$= \mathrm{Cos}[y] - \mathrm{Sin}[x].$

Use Mathematica to verify this.

```
D[D[x Sin[y] + y Cos[x], x], y]
D[D[x Sin[y] + y Cos[x], y], x]
```

If z = f(x,y) is a function of x and y, then the plane that meets the surface
z = f(x,y) at the point (x_0, y_0, z_0) is given by the formula
$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

Plot the surface z =100-x^2-y^2 , verify that the tangent plane at the point (x_0, y_0, z_0) = (5, 0,75) is z = 125-10 y, and plot the plane together with the surface z =100-x^2-y^2.

```
Expand[100 - x0^2 - y0^2 - 2 x0 (x - x0) - 2 y0 (y - y0)] /. {x0 → 5, y0 → 0}
A = Plot3D[100 - x^2 - y^2, {x, -4, 4}, {y, -2, 8}, BoxRatios → {1, 1, 1}];
B = Plot3D[125 - 10 y, {x, -4, 4},
    {y, -2, 8}, BoxRatios → {1, 1, 1}, PlotStyle -> Opacity[0.5]];
G =
 Show[
  A,
  B]
```