Mathematica Practical Exercise Set 5

The Mathematica Practical Exercise Sets are optional but they will help you understand Multivariate Calculus and Linear Algebra better and sooner. This exercise set covers Chapter 5 of the course notes.

Display the surface

z = x y^2 / (x^2 + 1)

together with the region under the surface and bound by (0 ≤ x ≤ 1) and (-3 ≤ y ≤ 3).

```
In[•]:= A = Plot3D[x y^2 / (x^2 + 1), {x, -1, 2}, {y, -4, 4},
    PlotStyle → Directive[Green, Specularity[White, 20], Opacity[0.4]],
    ExclusionsStyle → {None, Red}];
B = RegionPlot3D[(0 ≤ x ≤ 1) && (-3 ≤ y ≤ 3) && (z ≤ x y^2 / (x^2 + 1)), {x, -5, 5}, {y, -5, 5},
    {z, -5, 5}, PlotPoints → 100, PlotStyle → Directive[Yellow, Opacity[0.5]], Mesh → None];
Show[
 A,
 B]
```

Display the surface

$z = x^2 \sin(y)$

together with the region under the surface and bound by $(0 \leq x \leq 1)$ and $\left(0 \leq y \leq \frac{\pi}{2}\right)$.

```
In[•]:= A = Plot3D[x^2 Sin[y], {x, -1, 2}, {y, -1, 2},
    PlotStyle → Directive[Green, Specularity[White, 20], Opacity[0.4]],
    ExclusionsStyle → {None, Red}];
B = RegionPlot3D[(0 ≤ x ≤ 1) && (0 ≤ y ≤ Pi / 2) && (z ≤ x^2 Sin[y]), {x, -1, 2}, {y, -1, 2},
    {z, 0, 5}, PlotPoints → 100, PlotStyle → Directive[Yellow, Opacity[0.5]], Mesh → None];
Show[
 A,
 B]
```

Plot the region in which points (x, y) satisfy

(-1 ≤ x ≤ 2) and (-x ≤ y ≤ x^2)

and then the region in which points (x, y) satisfy

(3 y - 5 ≤ x ≤ 2 + y^2) and (-1 ≤ y ≤ 4).

```
In[•]:= RegionPlot[(-1 ≤ x ≤ 2) && (-x ≤ y ≤ x^2), {x, -1, 3}, {y, -2, 5}, PlotPoints → 100]

RegionPlot[(3 y - 5 ≤ x ≤ 2 + y^2) && (-1 ≤ y ≤ 4), {x, -1, 8}, {y, -2, 5}, PlotPoints → 100]
```

Plot the region in which points (x, y) satisfy

(1 ≤ x ≤ 4) and (1 ≤ y ≤ - (1 / 3) x + 7 / 3).

In[∘]:=
```
A = RegionPlot[(1 ≤ x ≤ 4) && (1 ≤ y ≤ -(1/3) x + 7/3),
   {x, -1, 5}, {y, -1, 3}, PlotPoints → 100];
B = Graphics[Arrow[{{-0.5, 0}, {4.5, 0}}]];
G = Graphics[Arrow[{{0, -0.5}, {0, 2.5}}]];
Show[A, B, G]
```

Plot thesurface

z = x y

together with the region under the surface z = x y and bound by

(1 ≤ x ≤ 4) and (1 ≤ y ≤ – (1 / 3) x + 7 / 3).

In[∘]:=
```
A = Plot3D[x y, {x, 0, 4.5}, {y, 0, 2.5},
   PlotStyle → Directive[Green, Specularity[White, 20], Opacity[0.4]],
   ExclusionsStyle → {None, Red}];
B = RegionPlot3D[(1 ≤ x ≤ 4) && (1 ≤ y ≤ -(1/3) x + 7/3) && (z ≤ x y), {x, 0, 4.5}, {y, 0, 2.5},
   {z, -5, 5}, PlotPoints → 100, PlotStyle → Directive[Yellow, Opacity[0.5]], Mesh → None];
Show[
 A,
 B]
```

Plot the region of points (x, y) satisfying

(–1 ≤ x ≤ 2) and (–x ≤ y ≤ x^2)

with a pink dashed line from (0, 0) to (2, 0) separating this into two regions.

In[∘]:=
```
A = RegionPlot[(-1 ≤ x ≤ 2) && (-x ≤ y ≤ x^2), {x, -1, 3}, {y, -2, 5}, PlotPoints → 100];
L = Line[{{0, 0}, {2, 0}}];
B = Graphics[{Thick, Dashed, Pink, L}];
Show[A, B]
```

Calculate the two relevant integrals over the above regions, and then add them.

In[1]:=
```
A = Integrate[(8/3 + 2 y^2 - ((1/3) y Sqrt[y] + y^2 Sqrt[y])), {y, 0, 4}]
B = Integrate[(8/3 + 2 y^2 - (-(1/3) y^3 - y^3)), {y, -2, 0}]
A + B
```

Plot the surface z = 1 together with the region under z = 1 bound by

(0 ≤ x^2 + y^2 ≤ 1) and (0 ≤ x ≤ 1) and (0 ≤ y ≤ 1).

In[∘]:=
```
A = Plot3D[1, {x, -1, 1}, {y, -1, 1},
   PlotStyle → Directive[Green, Specularity[White, 20], Opacity[0.4]],
   ExclusionsStyle → {None, Red}];
B = RegionPlot3D[(0 ≤ x^2 + y^2 ≤ 1) && (0 ≤ x ≤ 1) && (0 ≤ y ≤ 1) && (z ≤ 1),
   {x, -5, 5}, {y, -5, 5}, {z, -5, 5}, PlotPoints → 100,
   PlotStyle → Directive[Yellow, Opacity[0.5]], Mesh → None];
Show[
 A,
 B]
```

Plot the region bound by

(0 ≤ x^2 + y^2 ≤ 1) and (0 ≤ x ≤ 1) and (0 ≤ y ≤ 1) in the x, y plane.

In[∘]:=
```
RegionPlot[(0 ≤ x^2 + y^2 ≤ 1) && (0 ≤ x ≤ 1) && (0 ≤ y ≤ 1), {x, -0.1, 1.1}, {y, -0.1, 1.1}]
```

Plot the region bound by
$(-y-2 \le x \le 2-y)$ and $(x-2 \le y \le x)$ and $(-x \le y)$ and $(0 \le x \le 2)$ in the x, y plane.

In[∘]:=
```
RegionPlot[(-y - 2 ≤ x ≤ 2 - y) && (x - 2 ≤ y ≤ x) && (-x ≤ y) && (0 ≤ x ≤ 2),
 {x, 0, 2}, {y, -1, 1}, PlotPoints → 100]
```

Plot the surface
$z = x^2 + y^2$
together with the region under the surface $z = x^2 + y^2$ bounded by
$(-y-2 \le x \le 2-y)$ and $(x-2 \le y \le x)$ and $(-x \le y)$ and $(0 \le x \le 2)$ in the x, y plane.

In[∘]:=
```
A = Plot3D[x^2 + y^2, {x, -1, 2}, {y, -1, 1},
   PlotStyle → Directive[Green, Specularity[White, 20], Opacity[0.4]],
   ExclusionsStyle → {None, Red}];
B = RegionPlot3D[(-y - 2 ≤ x ≤ 2 - y) && (x - 2 ≤ y ≤ x) && (-x ≤ y) &&
     (0 ≤ x ≤ 2) && (z ≤ x^2 + y^2), {x, -5, 5}, {y, -5, 5}, {z, -5, 5},
   PlotPoints → 200, PlotStyle → Directive[Yellow, Opacity[0.5]], Mesh → None];
Show[
 A,
 B]
```

Plot the region bounded by
$(0 \le x \le 2)$ and $(-2 \le y \le 3)$ and $(0 \le z \le 1)$ in 3 D.

In[∘]:=
```
RegionPlot3D[(0 ≤ x ≤ 2) && (-2 ≤ y ≤ 3) && (0 ≤ z ≤ 1), {x, -1, 3}, {y, -3, 4}, {z, -1, 2},
 PlotPoints → 200, PlotStyle → Directive[Yellow, Opacity[0.5]], Mesh → None]
```

Plot the region bounded by
$(0 \le x \le 3)$ and $(0 \le y \le x)$ and $(0 \le z \le x-y)$ in 3 D.

In[∘]:=
```
RegionPlot3D[(0 ≤ x ≤ 3) && (0 ≤ y ≤ x) && (0 ≤ z ≤ x - y), {x, -1, 4}, {y, -3, 5}, {z, -1, 4},
 PlotPoints → 200, PlotStyle → Directive[Yellow, Opacity[0.5]], Mesh → None]
```

Plot the vector field $F = (x^2 + y^2,\ 4y)$ together with the curve
$C : \{(2t+4,\ -t^2+2)\ :\ 0 \le t \le 2\}$ and the vector $r(t)$ in red,
the vector $r'(t)$ in black, and the vector $n(t)$ in green.

```
Manipulate[A = VectorPlot[{x^2 + y^2, 4 y}, {x, 2, 8.5}, {y, -3, 3}];
 B = ParametricPlot[{2 s + 4, -s^2 + 2}, {s, 0, 2}, PlotStyle → Black];
 H = Graphics[{Red, Arrow[{{0, 0}, {2 * t + 4, -t^2 + 2}}]}];
 L = Graphics[{Black, Arrow[{{2 * t + 4, -t^2 + 2}, {2 * t + 4, -t^2 + 2} + {2, -2 t}}]}];
 M = Graphics[
    {Green, Arrow[{{2 * t + 4, -t^2 + 2}, {2 * t + 4, -t^2 + 2} + (1 / Sqrt[t^2 + 1]) * {t, 1}}]}];
 G = ListPlot[{{4, 2}, {8, -2}}, PlotStyle → {Black, PointSize[0.02]}];
 Show[A, B, G, H, L, M], {t, 0, 2}]
```

Plot the vector field $F = (x + y,\ -x^2 - y^2)$ together with the curve
$C : \{(1-t,\ t)\ :\ 0 \le t \le 1\}$.

In[ ]:=
```
A = VectorPlot[{x + y, -x^2 - y^2}, {x, 0, 1.1}, {y, 0, 1.1}];
B = ParametricPlot[{1 - t, t}, {t, 0, 1}, PlotStyle → Black];
G = ListPlot[{{0, 1}, {1, 0}}, PlotStyle → {Black, PointSize[0.02]}];
Show[A, B, G]
```

Plot the vector field F = (3 x, 4 y, 5 z) together with the sphere of radius 2 centred at the point (0, 0, 0).

```
A = VectorPlot3D[{3 x, 4 y, 5 z}, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}];
R = Sphere[{0, 0, 0}, 2];
B = Graphics3D[{{Opacity[0.5], LightBlue, R}}];
Show[A, B]
```

Display how spherical coordinates work in 3 D.

```
x = r Sin[ϕ] Cos[θ];
y = r Sin[ϕ] Sin[θ];
z = r Cos[ϕ];

G = ListPointPlot3D[{{2, 2, 2}, {-2, -2, -2}, {2, -2, -2},
    {2, -2, 2}, {-2, -2, 2}, {2, 2, -2}, {-2, 2, -2}, {-2, 2, 2}},
   PlotStyle → {Black, PointSize[0.02]}, PlotRange → All, AspectRatio → 1];
H = Plot3D[0, {x, -3, 2}, {y, -2, 2}, PlotStyle → Opacity[0.4]];
M = ContourPlot3D[x == 0, {x, -3, 3}, {y, -3, 3}, {z, -4, 4}, ContourStyle → Opacity[0.4]];
Manipulate[R = Sphere[{0, 0, 0}, r];
 B = Graphics3D[{{Opacity[0.5], LightBlue, R}}];
 A = ListPointPlot3D[{{0, 0, 0}, {r Sin[ϕ] Cos[θ], r Sin[ϕ] Sin[θ], r Cos[ϕ]},
    {r Sin[ϕ] Cos[θ], r Sin[ϕ] Sin[θ], 0}}, PlotStyle → {Black, PointSize[0.02]}];
 W = ListPointPlot3D[{{2, 2, 2}, {-2, -2, -2}, {2, -2, -2}, {2, -2, 2}, {-2, -2, 2},
    {2, 2, -2}, {-2, 2, -2}, {-2, 2, 2}}, PlotStyle → {Gray, PointSize[0.01]}];
 M = ContourPlot3D[x == 0, {x, -3, 3}, {y, -3, 3}, {z, -4, 4}, ContourStyle → Opacity[0.4]];
 L = Line[{{0, 0, 0}, {r Sin[ϕ] Cos[θ], r Sin[ϕ] Sin[θ], 0}}];
 LL = Line[{{0, 0, 0}, {r Sin[ϕ] Cos[θ], r Sin[ϕ] Sin[θ], r Cos[ϕ]}}];
 Z = Graphics3D[{Thick, Black, L}];
 ZZ = Graphics3D[{Thick, Red, LL}];
 Show[W, A, B, G, H, Z, ZZ, AspectRatio → 1], {θ, 0, 2 Pi}, {ϕ, 0, Pi}, {r, 0, 2}]
```

```
In[•]:= ℛ = Sphere[{0, 0, 0}, 4];
B = Graphics3D[{{Opacity[0.1], LightBlue, ℛ}}];
X = Graphics3D[Arrow[{{0, 0, 0}, {2, 0, 0}}]];
Y = Graphics3D[Arrow[{{0, 0, 0}, {0, 2, 0}}]];
Z = Graphics3D[Arrow[{{0, 0, 0}, {0, 0, 2}}]];
t1 = Graphics3D[Text["x", {2.1, 0, 0}]];
t2 = Graphics3D[Text["y", {0, 2.1, 0}]];
t3 = Graphics3D[Text["z", {0, 0, 2.1}]];
P =
  ListPointPlot3D[{{0, 0, 0}, {1, 2, 0}, {1, 2, 3}}, PlotStyle → {Black, PointSize[0.03]}];
L = Line[{{0, 0, 0}, {1, 2, 3}}];
L2 = Line[{{1, 2, 0}, {1, 2, 3}}];
L3 = Line[{{0, 0, 0}, {1, 2, 0}}];
R = Graphics3D[{Thick, Black, L}];
Q = Graphics3D[{Thick, Dashed, Pink, L2}];
S = Graphics3D[{Thick, Dashed, Pink, L3}];
t4 = Graphics3D[Text["θ", {0.5, 0.5, 0}]];
Show[B, X, Y, Z, t1, t2, t3, t4, P, Q, R, S]
```

Calculate the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}$$

and its determinant for spherical coordinates.

```
In[11]:= x = r Sin[ϕ] Cos[θ];
y = r Sin[ϕ] Sin[θ];
z = r Cos[ϕ];

J = {{D[x, r], D[x, ϕ], D[x, θ]}, {D[y, r], D[y, ϕ], D[y, θ]}, {D[z, r], D[z, ϕ], D[z, θ]}};
MatrixForm[J]
Factor[Det[J]]
```

Display a cylinder of radius 1 and height 1 using parametric coordinated.

```
In[•]:= ParametricPlot3D[ {Cos[u], Sin[u], v}, {u, 0, 2 Pi}, {v, 0, 1},
 MaxRecursion → 4, PlotStyle → {Orange, Opacity[0.3]}, Mesh → None]
```

Display how dS works.

```
In[•]:= A = ParametricPlot[{t^2 + 2 Cos[t], 3 - t Sin[t]},
    {t, 0, 3}, PlotRange → {0, 3}, PlotStyle → Black];
B = Graphics[Arrow[{{0, 0}, {1.5^2 + 2 Cos[1.5], 3 - 1.5 Sin[1.5]}}]];
G = Graphics[Arrow[{{0, 0}, {1.7^2 + 2 Cos[1.7], 3 - 1.7 Sin[1.7]}}]];
L = Line[{{1.5^2 + 2 Cos[1.5], 3 - 1.5 Sin[1.5]}, {1.7^2 + 2 Cos[1.7], 3 - 1.7 Sin[1.7]}}];
H = Graphics[{Thick, Red, L}];
T = Graphics[Text["r(t)", 0.5 * {1.5^2 + 2 Cos[1.5], 0.4 + 3 - 1.5 Sin[1.5]}]];
U = Graphics[Text["r(t+Δt)", 0.5 * {1.7^2 + 2 Cos[1.7], -0.4 + 3 - 1.7 Sin[1.7]}]];
V = Graphics[Text["Δs", {-0.1 + 1.7^2 + 2 Cos[1.7], 0.2 + 3 - 1.7 Sin[1.7]}]];
Show[A, B, G, H, T, U, V]
```