

Combinatorial Game Theory

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What is Combinatorial Game Theory?

- Different from economic game theory
- Focuses on sequential (turn-based) two-player games which do not rely on chance
- Examines games with perfect information available to both players

What is Nim?

Game played by two players who take turns removing stones from different piles. A player can take any positive integer of stones from any pile, and the player to take the last stone wins.

For our purposes, we'll denote a game of nim as $(p_1, p_2, p_3, \dots, p_n)$ denoting n piles of stones.

Definition: Nim Value

Definition: The nim value assigns a value to a nim position by considering the set of possible positions and finding the minimum value not listed. The nim value has the following properties:

- Two positions with equal nim value are considered identical
- After a move, a position with a nim value n will have a nim value strictly less than n
- Any nim value less than n is within the move set of a position with nim value n

Definition: Nim Sum

Definition: The nim sum of nim values returns the nim value of the combined game. The nim sum is equivalent to the bitwise xor function.

What is Hackenbush?

Hackenbush is a game played on a group of connected line segments, each connected. Two players take turns alternatively removing connected line segments. The game has the following rules:

- Any segment not connected to the ground plane (whether directly or indirectly) is removed
- In certain variations, players may only cut sticks of their own color
- Segments are only connected to other segments at their endpoints

What is a partisan game? (Hackenbush)

A partisan game is a game in which the moves available to each player are different.

- In hackenbush, we color some segments red and others blue to signify this
- Blue segments are cut by the “left” player
- Red segments are cut by the “right” player
- We lose the ideas of Sprague-Grundy

Partisan Hackenbush

- Blue is considered positive
- Red is considered negative
- Instead of a full set of possible moves, we use a set $\{G^L|G^R\}$

Game Simplification

- Dominated moves are considered suboptimal for the player, and so are never considered
- Reversible moves are moves in which the other player has a clear best response, and so we replace that game with the possible responses

Questions?