Stochastic Processes

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Review: Random Variables

Expected Value:

$E[X] = \Sigma x \cdot p_x$ for discrete X

p_x denotes the probability of x occurring.

Variance:

$Var[X] = E[(X - E[X])^2]$

$Var[X] = E[X^2] - E[X]^2$

Stochastic Processes

Definition:

Sequence of random variables indexed by a set (usually time)

Often denoted $\{X_N\}$ for a random variable X and indices N in a set T.

Examples:

- repeated coin tosses
- brownian motion
- random walks

Examples of Stochastic Process problems

- Expected number of coin flips until you get two heads in a row?
- Repeatedly roll and add 2 standard die. Probability you roll a 12 before you roll two 7’s in a row?
- Random walk on the number line. Start at 1, and stop if you hit 0 or 5. Probability you stop at 5?

Markov Chain

Markov Property:

Future events occur independent of previous states

Aka, “Memorylessness”

Definition:

A Markov chain is a stochastic process which has the Markov Property

Modeling with Markov Chains

Assumptions for simplicity:

We have perfect information

There are a finite number of states

States in a Markov Chain:

Each state is a "position"

Markov states transition between each other with some probabilities

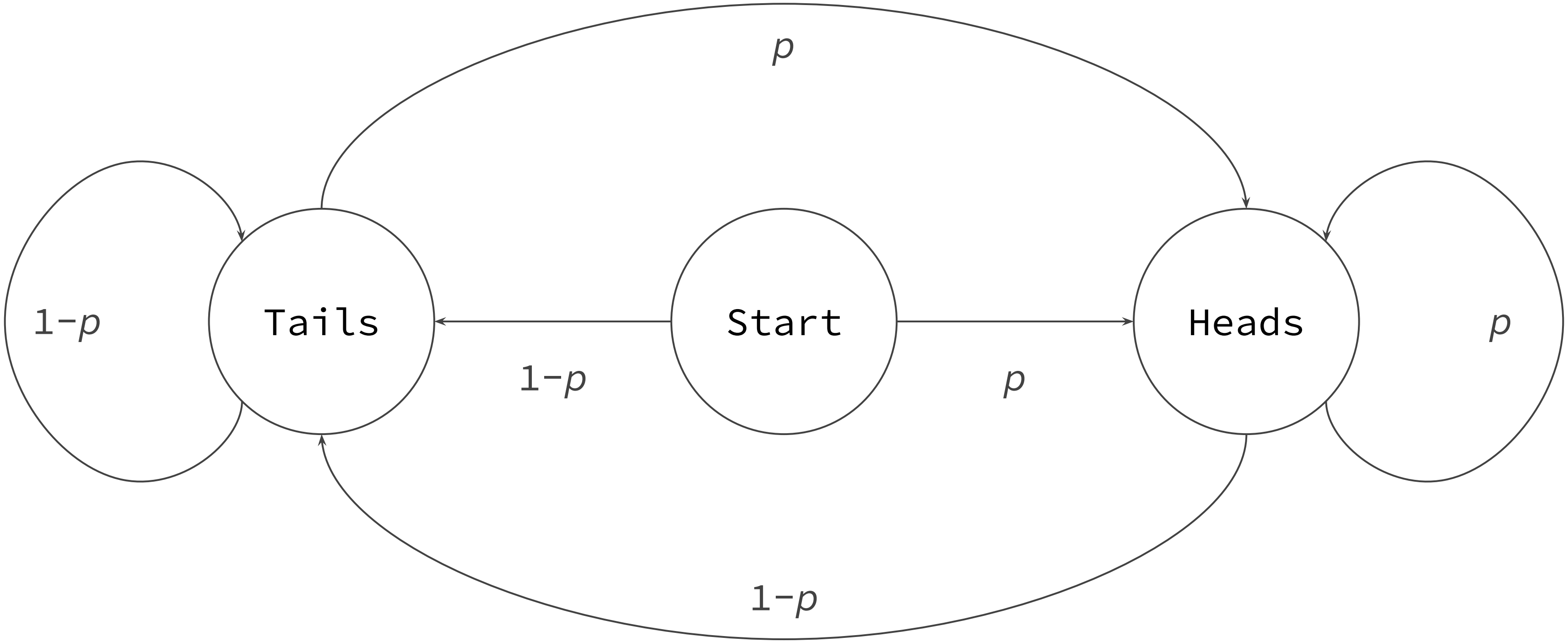
Modeling coin flips

How do we model a sequence of coin flips?

Assume the coin has probability p of landing heads and $(1 - p)$ of landing tails.

States:

- Starting
- Previous flip H
- Previous flip T



Markov Chain representation of a coin flip sequence

Markov Chains as equations

Notation:

Let S_K denote the state K.

We have the following states:

S_S → Start
S_H → Heads
S_T → Tails

State Transitions:

$S_S = 0$

$S_H = p \cdot S_H + p \cdot S_T$

$S_T = (1-p)S_H + (1-p)S_T$

Stopping Criteria

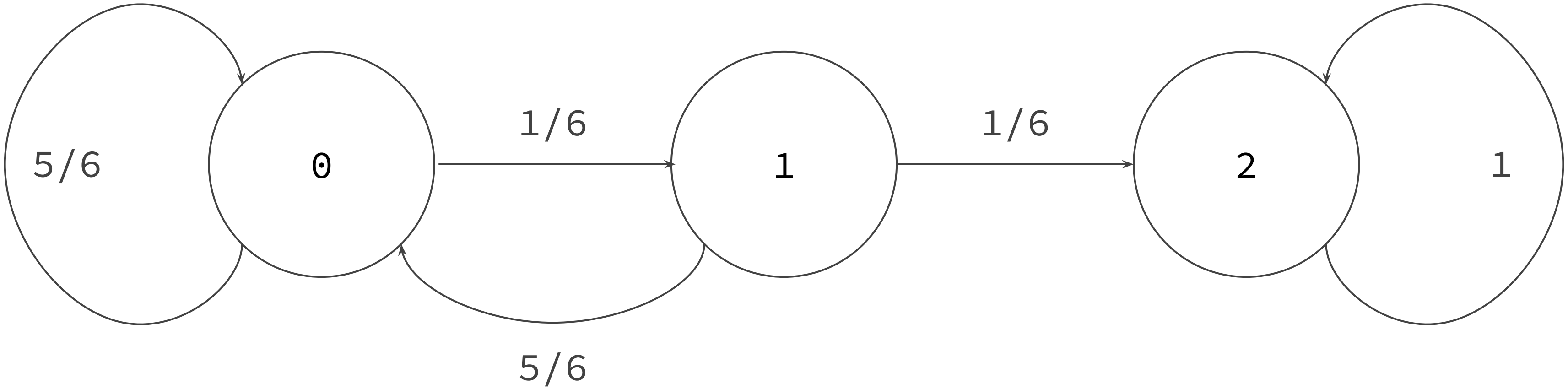
In the previous example, we have no stopping criteria, so the chain loops endlessly.

How do we represent stopping criteria?

Problem:

Roll a die until you roll two 6’s in a row.

What is the expected number of rolls?



Markov Chain Representation

Absorption States

- The state of two 6’s in a row is an “absorption state”
- We stop the process once we reach this state.
- Usually represented by a loop edge with probability 1.

Markov Chains for Expected Value

Notation:

Let E_T be the expected number of moves from state T to stopping.

We have the states E_0, E_1, E_2.

Equations:

$E_0 = 1 + \frac{5}{6}E_0 + \frac{1}{6}E_1$

$E_1 = 1 + \frac{5}{6}E_0 + \frac{1}{6}E_2$

$E_2 = 0$

Recall E_2 is our stopping criteria.

Random Walks

Particle on a grid which randomly moves one unit in a given direction.

A wide variety of problems arise.

Example:

Consider the random walk with given conditions:

- Start at: 1
- Stop at 0, 10

What is the probability the particle stops at 10? Expected number of steps?

Markov Chain Approach

General Overview:

Create positional states:
0, 1, 2, …, 10

Write out the system of equations for probability and for expected number of steps.

Probabilistic Equations:

$S_0 = 0,\ S_{10} = 1$

$S_i = ½S_{i-1} + ½S_{i+1}$

Expected Value Equations:

$S_0 = 0,\ S_{10} = 0$

$S_i = ½S_{i-1} + ½S_{i+1} + 1$

Martingales

Definition:

A stochastic process which satisfies the conditional expected value

$E[X_{t+1}|X_1,\dots,X_t] = X_t$.

Relation to Random Walks:

Random Walks are martingales.

$X_{t+1} = X_t \pm 1$.

Martingale Trickery

Why is this useful?

Let N be the last step, so $X_N = 0$ or 10.

What is $E[X_N]$?

X_N is a martingale process, so $E[X_N] = X_0 = 1$.

Let p_{10} be the probability X_N stops at 10. Then,

$E[X_N] = 10p_{10} + 0(1 - p_{10}) = 1$

Solving, $p_{10} = 1/10 = 0.1$

Martingale Trickery, Expected Value

Variance of Random Walk:

Consider $Var[X_t]$ now.

$Var[X_t] = E[X_t^2] - E[X_t]^2$.

Let $X_t = \Sigma e_i$, where
$e_i = Bern(-1, 1)$.

$E[X_t^2] = \Sigma e_i^2 = t$

$E[X_t^2] = t$, so

$E[X_t^2 - t] = 0$.

$E[X_{t-1}^2 - (t - 1)] = 0$,
so the process

$V_t = X_t^2 - t$ is a
Martingale.

More trickery

End States:

Let N be the last state.

$E[X_N^2 - N] = E[X_N^2] - N$

By martingale properties,

$E[X_N^2] - N = E[X_0^2] - 0$, so

$N = E[X_N^2] - E[X_0^2]$

Solving:

Let p_{10} be the probability of stopping at 10. We know $p_{10} = 10$

$E[X_N^2] = 10^2 p_{10} + 0^2(1 - p_{10})$

$N = 100p_{10} - 1.$

$N = 9$

Summary

- Random variables: expectation, variance
- Stochastic processes are an indexed sequence of random variables
- A Markov process is a “memoryless” process
- Martingales properties simplify random walks

Questions?

Future continuations

- Bertrand's Ballot Theorem
- Catalan Numbers