Real Numbers

- **Euclid's Division Lemma**

For any given positive integers a and b, there exists unique integers q and r such that
$a = bq + r$ where $0 \leq r < b$

Note: If b divides a, then $r = 0$

Example 1:

For $a = 15$, $b = 3$, it can be observed that
$15 = 3 \times 5 + 0$
Here, $q = 5$ and $r = 0$
If b divides a, then $0 < r < b$

Example 2:

For $a = 20$, $b = 6$, it can be observed that $20 = 6 \times 3 + 2$
Here, $q = 6$, $r = 2$, $0 < 2 < 6$

- **Euclid's division algorithm**

Euclid's division algorithm is a series of well-defined steps based on "Euclid's division lemma", to give a procedure for calculating problems.

Steps for finding HCF of two positive integers *a* and *b* (*a* > *b*) by using Euclid's division algorithm:

Step 1: Applying Euclid's division lemma to a and b to find whole numbers q and r, such that $a = bq + r$, $0 \leq r < b$
Step 2: If $r = 0$, then HCF $(a, b) = b$
If $r \neq 0$, then again apply division lemma to b and r
Step 3: Continue the same procedure till the remainder is 0. The divisor at this stage will be the HCF of a and b.

Note: HCF (a, b) = HCF (b, r)

Example:

Find the HCF of 48 and 88.

Solution:

Take $a = 88$, $b = 48$
Applying Euclid's division lemma, we get
$88 = 48 \times 1 + 40$ (Here, $0 \leq 40 < 48$)

48 = 40 × 1 + 8 (Here, $0 \le 8 < 40$)
40 = 8 × 5 + 0 (Here, $r = 0$)
HCF (48, 88) = 8

- For any positive integer *a*, *b*, **HCF (*a*, *b*) × LCM (*a*, *b*) = *a* × *b***

Example 1:
Find the LCM of 315 and 360 by the prime factorisation method. Hence, find their HCF.

Solution:
$315 = 3 \times 3 \times 5 \times 7 = 3^2 \times 5 \times 7$
$360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5$
$\text{LCM} = 3^2 \times 5 \times 7 \times 2^3 = 2520$
$\therefore \text{HCF}(315, 360) = \frac{315 \times 360}{\text{LCM}(315, 360)} = \frac{315 \times 360}{2520} = 45$

Example 2:
Find the HCF of 300, 360 and 240 by the prime factorisation method.

Solution:
$300 = 2^2 \times 3 \times 5^2$
$360 = 2^3 \times 3^2 \times 5$
$240 = 2^4 \times 3 \times 5$
$\text{HCF}(300, 360, 240) = 2^2 \times 3 \times 5 = 60$

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Steps for finding HCF of two positive integers *a* and *b* ($a > b$) by using Euclid's division algorithm:

Step 1: Applying Euclid's division lemma to a and b to find whole numbers q and r, such that $a = bq + r$, $0 \leq r < b$

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- **Using Euclid's division lemma to prove mathematical relationships**

Result 1:

Every positive even integer is of the form $2q$, while every positive odd integer is of the form $2q + 1$, where q is some integer.

Proof:

Let a be any given positive integer.

Take $b = 2$

By applying Euclid's division lemma, we have

$a = 2q + r$ where $0 \leq r < 2$

As $0 \leq r < 2$, either $r = 0$ or $r = 1$

If $r = 0$, then $a = 2q$, which tells us that a is an even integer.

If $r = 1$, then $a = 2q + 1$

It is known that every positive integer is either even or odd.

Therefore, a positive odd integer is of the form $2q + 1$.

Result 2:

Any positive integer is of the form $3q$, $3q + 1$ or $3q + 2$, where q is an integer.

Proof:

Let a be any positive integer.

Take $b = 3$

Applying Euclid's division lemma, we have

$a = 3q + r$, where $0 \le r < 3$ and q is an integer
Now, $0 \le r < 3$ Þ $r = 0, 1$, or 2
$\therefore a = 3q + r$
$\Rightarrow a = 3q + 0, a = 3q + 1, a = 3q + 2$
Thus, $a = 3q$ or $a = 3q + 1$ or $a = 3q + 2$, where q is an integer.

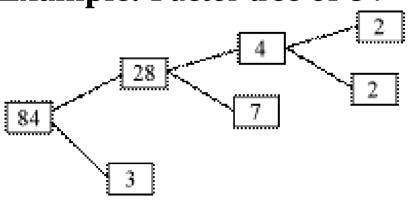
- Fundamental theorem of arithmetic states that very composite number can be uniquely expressed (factorised) as a product of primes apart from the order in which the prime factors occur.

Example: 1260 can be uniquely factorised as

2	1260
2	630
3	315
3	105
5	35
	7

$1260 = 2 \times 2 \times 3 \times 3 \times 5 \times 7$

Example: Factor tree of 84



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Find the HCF of 300, 360 and 240 by the prime factorisation method.

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$240 = 2^4 \times 3 \times 5$
HCF (300, 360, 240) = $2^2 \times 3 \times 5 = 60$

- According to fundamental theorem of arithmetic, a number can be represented as the product of primes having a unique factorisation.

  Example:
  Check whether 15^n in divisible by 10 or not for any natural number *n*. Justify your answer.

  Solution:
  A number is divisible by 10 if it is divisible by both 2 and 5.
  $15^n = (3 \,.\, 5)^n$
  3 and 5 are the only primes that occur in the factorisation of 15^n
  By uniqueness of fundamental theorem of arithmetic, there is no other prime except 3 and 5 in the factorisation of 15^n.
  2 does not occur in the factorisation of 15^n.
  Hence, 15^n is not divisible by 10.

- Every number of the form $\sqrt{p}$, where *p* is a prime number is called an irrational number. For example, $\sqrt{3}, \sqrt{11}, \sqrt{12}$ etc.

  Theorem: If a prime number *p* divides a^2, then *p* divides *a*, where *a* is a positive integer.

  Example:
  Prove that $\sqrt{7}$ is an irrational number.

  Solution:
  If possible, suppose $\sqrt{7}$ is a rational number.
  Then, $\sqrt{7} = \frac{p}{q}$, where *p*, *q* are integers, $q \neq 0$.
  If HCF $(p, q) \neq 1$, then by dividing *p* and *q* by HCF (p, q), $\sqrt{7}$ can be reduced as
  $\sqrt{7} = \frac{a}{b}$ where HCF $(a, b) = 1$ … (1)
  $\Rightarrow \sqrt{7}b = a$
  $\Rightarrow 7b^2 = a^2$
  $\Rightarrow a^2$ is divisible by 7
  $\Rightarrow a$ is divisible by 7 … (2)
  $\Rightarrow a = 7c$, where *c* is an integer
  $\therefore \sqrt{7}c = b$
  $\Rightarrow 7b^2 = 49c^2$
  $\Rightarrow b^2 = 7c^2$

⇒ b^2 is divisible by 7

⇒ b is divisible by 7 … (3)

From (2) and (3), 7 is a common factor of a and b. which contradicts (1)

∴ $\sqrt{7}$ is an irrational number.

Example:

Show that $\sqrt{12} - 6$ is an irrational number.

Solution:

If possible, suppose $\sqrt{12} - 6$ is a rational number.

Then $\sqrt{12} - 6 = \frac{p}{q}$ for some integers p, q (q ¹ 0)

Now,

$$\sqrt{12} - 6 = \frac{p}{q}$$

$$\Rightarrow 2\sqrt{3} = \frac{p}{q} + 6$$

$$\Rightarrow \sqrt{3} = \frac{1}{2}\left(\frac{p}{q} + 6\right)$$

As p, q, 6 and 2 are integers, $\frac{1}{2}\left(\frac{p}{q} + 6\right)$ is rational number, so is $\sqrt{3}$.

This conclusion contradicts the fact that $\sqrt{3}$ is irrational.

Thus, $\sqrt{12} - 6$ is an irrational number.

- Decimal expansion of a rational number can be of two types:

(i) Terminating

(ii) Non-terminating and repetitive

In order to find decimal expansion of rational numbers we use long division method.

For example, to find the decimal expansion of $\frac{1237}{25}$.

We perform the long division of 1237 by 25.

```
       49.48
25 ) 1237.00
     100
     ----
      237
      225
     ----
       120
       100
     ----
        200
        200
     ----
          0
     ----
```

Hence, the decimal expansion of $\frac{1237}{25}$ is 49.48. Since the remainder is obtained as zero, the decimal number is terminating.

- If x is a rational number with terminating decimal expansion then it can be expressed in the $\frac{p}{q}$ form, where p and q are co-prime (the HCF of p and q is 1) and the prime factorisation of q is of the form 2^n5^m, where n and m are non-negative integers.

- Let $x = \frac{p}{q}$ be any rational number.

i. If the prime factorization of q is of the form 2^m5^n, where m and n are non-negative integers, then x has a terminating decimal expansion.

ii. If the prime factorisation of q is not of the form 2^m5^n, where m and n are non-negative integers, then x has a non-terminating and repetitive decimal expansion.

For example, $\frac{17}{1600} = \frac{17}{2^6 \times 5^2}$ has the denominator in the form 2^n5^m, where n = 6and m = 2 are non-negative integers. So, it has a terminating decimal expansion.

$\frac{723}{392} = \frac{3 \times 241}{2^3 \times 7^2}$ has the denominator not in the form 2^n5^m, where n and m are non-negative integers. So, it has a non-terminating decimal expansion.