Chapter 8. Introduction to Trigonometry

Question-1

Simplify the following expressions: $\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta}$.

Solution:

$\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} = \frac{(\sin\theta + \cos\theta)(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta)}{\sin\theta + \cos\theta}$ =1 - sinθ cosθ.

Question-2

Prove $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 1 + \tan\theta + \cot\theta$.

Solution:

$\text{L.H.S} = \frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = \frac{-\tan^2\theta}{1-\tan\theta} + \frac{\cot\theta}{1-\tan\theta} = \frac{-\tan^2\theta + \cot\theta}{1-\tan\theta}$

Multiply tan θ/tan θ we get,

$= \frac{1-\tan^3\theta}{\tan\theta(1-\tan\theta)}$

$= \frac{(1-\tan\theta)(1+\tan\theta+\tan^2\theta)}{\tan\theta(1-\tan\theta)}$

$= \frac{(1+\tan\theta+\tan^2\theta)}{\tan\theta}$

= cotθ + 1 + tanθ = R.H.S.

Question-3

Prove $\tan^2\varphi + \cot^2\varphi + 2 = \sec^2\varphi \operatorname{cosec}^2\varphi$.

Solution:

L.H.S = $\tan^2\varphi + \cot^2\varphi + 2$

= $\sec^2\varphi - 1 + \operatorname{cosec}^2\varphi - 1 + 2$

[Using Identity $1 + \cot^2\varphi = \operatorname{cosec}^2\varphi$ and $1 + \tan^2\varphi = \sec^2\varphi$]

= $\sec^2\varphi + \operatorname{cosec}^2\varphi$ = $\frac{1}{\cos^2\phi} + \frac{1}{\sin^2\phi}$

= $\frac{\sin^2\phi + \cos^2\phi}{\cos^2\phi\sin^2\phi}$

= $\frac{1}{\cos^2\phi\sin^2\phi}$ [Using Identity $\sin^2\varphi + \cos^2\varphi = 1$]

= $\sec^2\varphi \operatorname{cosec}^2\varphi$ = R.H.S.

Question-4

Evaluate: $\left(\frac{\sin 27^\circ}{\cos 63^\circ}\right)^2 + \left(\frac{\cos 63^\circ}{\sin 27^\circ}\right)^2$.

Solution:

$$\left(\frac{\sin 27^\circ}{\cos 63^\circ}\right)^2 + \left(\frac{\cos 63^\circ}{\sin 27^\circ}\right)^2 = \left(\frac{\sin(90^\circ - 63^\circ)}{\cos 63^\circ}\right)^2 + \left(\frac{\cos(90^\circ - 27^\circ)}{\sin 27^\circ}\right)^2$$

$$= \left(\frac{\cos 63^\circ}{\cos 63^\circ}\right)^2 + \left(\frac{\sin 27^\circ}{\sin 27^\circ}\right)^2$$

$$= 1 + 1$$

$$= 2.$$

Question-5

Prove that: $\frac{\cos(90^\circ - \theta)}{\sin\theta} + \frac{\sin\theta}{\cos(90^\circ - \theta)} = 2, \theta \neq 0^\circ$.

Solution:

$$\text{L.H.S} = \frac{\cos(90^\circ - \theta)}{\sin\theta} + \frac{\sin\theta}{\cos(90^\circ - \theta)}$$

$$= \frac{\sin\theta}{\sin\theta} + \frac{\sin\theta}{\sin\theta}$$

$$= 1 + 1$$

$$= 2$$

= R.H.S.

Question-6

Prove that: $\sec^2\theta - \cot^2(90^\circ - \theta) = \cos^2(90^\circ - \theta) + \cos^2\theta$.

Solution:

$$\text{L.H.S} = \sec^2\theta - \cot^2(90 - \theta)$$

$$= \sec^2\theta - \tan^2\theta$$

$$= \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$$

$$= \frac{(1 - \sin^2\theta)}{\cos^2\theta}$$

$$= \frac{\cos^2\theta}{\cos^2\theta} = 1$$

$$\text{R.H.S} = \cos^2(90 - \theta) + \cos^2\theta$$

$$= \sin^2\theta + \cos^2\theta$$

$$= 1$$

$$\text{L.H.S} = \text{R.H.S}$$

Question-7

Prove that: $\frac{\cos(90^\circ-\theta)\cos\theta}{\tan\theta}+\cos^2(90^\circ-\theta)=1$.

Solution:

L.H.S = $\frac{\cos(90^\circ-\theta)\cos\theta}{\tan\theta}+\cos^2(90^\circ-\theta)$

= $\frac{\sin\theta\cos\theta}{\tan\theta}+\sin^2\theta$

= $\cos^2\theta+\sin^2\theta$ [Using Identity $\sin^2\theta + \cos^2\theta = 1$]

= 1

= R.H.S.

Question-8

Prove that: $\cos(81^\circ + \theta) = \sin(9^\circ - \theta)$.

Solution:

L.H.S = $\cos(81^\circ + \theta)$

= $\cos[90^\circ - (9^\circ - \theta)]$

= $\sin(9^\circ - \theta)$

= R.H.S.

Question-9

Prove that: $\sin\theta\cos(90^\circ-\theta) + \cos\theta\sin(90^\circ - \theta) = 1$.

Solution:

L.H.S = $\sin\theta\cos(90^\circ-\theta) + \cos\theta\sin(90^\circ - \theta)$

= $\sin\theta\sin\theta + \cos\theta\cos\theta$ = $\sin^2\theta + \cos^2\theta$ [Using Identity $\sin^2\varphi + \cos^2\varphi = 1$]

= 1

= R.H.S.

Question-10

If $\frac{2\sin\theta}{1+\cos\theta+\sin\theta}$ = y, then $\frac{1-\cos\theta+\sin\theta}{1+\sin\theta}$ is also y.

Solution:

$$\frac{1-\cos\theta+\sin\theta}{1+\sin\theta}=\frac{1-\cos\theta+\sin\theta}{1+\sin\theta}\times\frac{1+\cos\theta+\sin\theta}{1+\cos\theta+\sin\theta}$$

$$= \frac{(1+\sin\theta)^2 - \cos^2\theta}{(1+\sin\theta)(1+\cos\theta+\sin\theta)}$$

$$= \frac{(1+\sin\theta)(1+\sin\theta) - [(1+\sin\theta)(1-\sin\theta)]}{(1+\sin\theta)(1+\cos\theta+\sin\theta)}$$

$$= \frac{(1+\sin\theta)[(1+\sin\theta)-(1-\sin\theta)]}{(1+\sin\theta)(1+\cos\theta+\sin\theta)}$$

$$= \frac{2\sin\theta}{1+\cos\theta+\sin\theta}$$

$= y \quad$ (Since $\frac{2\sin\theta}{1+\cos\theta+\sin\theta} = y$).

Question-11

Prove: $\frac{1-\tan^2\phi}{\cot^2\phi - 1} = \tan^2\varphi$.

Solution:

L.H.S $= \frac{1-\tan^2\phi}{\cot^2\phi - 1}$.

$$= \frac{1 - \frac{\sin^2\phi}{\cos^2\phi}}{\frac{\cos^2\phi}{\sin^2\phi} - 1}$$

$$= \frac{\frac{\cos^2\phi - \sin^2\phi}{\cos^2\phi}}{\frac{\cos^2\phi - \sin^2\phi}{\sin^2\phi}}$$

$$= \frac{\sin^2\phi}{\cos^2\phi}$$

$= \tan^2\varphi$

= RHS.

Question-12

Prove: $(\sec\theta - \tan\theta)^2 = \frac{1-\sin\theta}{1+\sin\theta}$.

Solution:

LHS $= (\sec\theta - \tan\theta)^2$

$$= \left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)^2$$

$$= \frac{(1-\sin\theta)(1-\sin\theta)}{(1-\sin^2\theta)}$$

$$= \frac{(1-\sin\theta)(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$$

$$= \frac{1-\sin\theta}{1+\sin\theta}$$

= RHS.

Question-13

Prove that $\dfrac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\operatorname{cosec}^3 A} = \sin^2 A\cos^2 A.$

Solution:

$$\text{L.H.S} = \frac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\operatorname{cosec}^3 A}$$

$$= \frac{\left(1+\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)(\sin A-\cos A)}{\frac{1}{\cos^3 A}-\frac{1}{\sin^3 A}}$$

$$= \frac{\frac{(\cos^2 A+\sin^2 A+\cos A\sin A)(\sin A-\cos A)}{\sin A.\cos A}}{\frac{\sin^3 A-\cos^3 A}{\cos^3 A.\sin^3 A}}$$

$$= \frac{\frac{\sin^3 A-\cos^3 A}{\sin A\cos A}}{\frac{\sin^3 A-\cos^3 A}{\cos^3 A\sin^3 A}}$$

$$= \sin^2 A\cos^2 A$$

= R.H.S.

Question-14

If a cosθ - b sinθ = c, show that a sinθ + b cosθ = $\pm\sqrt{a^2+b^2-c^2}$.

Solution:

$(a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2$

$= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) - 2ab\sin\theta\cos\theta + 2ab\sin\theta\cos\theta$

$= a^2 + b^2 \therefore (a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2 = a^2 + b^2$

$\therefore (a\cos\theta - b\sin\theta)^2 = a^2 + b^2 - (a\sin\theta + b\cos\theta)^2$

$= a^2 + b^2 - c^2 \therefore a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$.

Question-15

Prove that $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1 = 0$.

Solution:

$2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$

$= 2[(\sin^2\theta)^3 + (\cos^2\theta)^3] - 3[(\sin^2\theta)^2 + (\cos^2\theta)^2] + 1$

$= 2[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)] - 3[(\sin^2\theta + \cos^2\theta)^2$

$- 2\sin^2\theta\cos^2\theta] + 1$

The algebraic identity

$a^3 + b^3 = (a + b)^3 - 3ab(a + b)$ and

$a^2 + b^2 = (a + b)^2 - 2ab$

are used in the above step where

$a = \sin^2\theta$ and $b = \cos^2\theta$.

Writing $\sin^2\theta + \cos^2\theta = 1$, we have

$= 2[1 - 3\sin^2\theta\cos^2\theta] - 3[-2\sin^2\theta\cos^2\theta] + 1$

$= 2 - 6\sin^2\theta\cos^2\theta - 3 + 6\sin^2\theta\cos^2\theta + 1$

$= -3 + 3 = 0$

Question-16

Evaluate $\cos(40^o + \theta) - \sin(50^o - \theta) + \frac{\cos^2 40° + \cos^2 50°}{\sin^2 40° + \sin^2 50°}$.

Solution:

$\text{Cos}(40^o + \theta) - \sin(50^o - \theta) + \frac{\cos^2 40° + \cos^2 50°}{\sin^2 40° + \sin^2 50°}$

$= \cos[(90 - 50°) + \theta] - \sin(50° - \theta) + \frac{\cos^2(90° - 50°) + \cos^2 50°}{\sin^2(90° - 50°) + \sin^2 50°}$

$= \sin(50° - \theta) - \sin(50° - \theta) + \frac{\sin^2 50° + \cos^2 50°}{\cos^2 50° + \sin^2 50°}$

$= \frac{1}{1}$

$= 1.$

Question-17

If $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ and $x\sin\theta = y\cos\theta$, prove that $x^2 + y^2 = 1$.

Solution:

$x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$

$x\sin\theta(\sin^2\theta) + (y\cos\theta)\cos^2\theta = \sin\theta\cos\theta \Rightarrow x\sin\theta(\sin^2\theta) + (x\sin\theta)\cos^2\theta = \sin\theta\cos\theta$ [since $x\sin\theta = y\cos\theta$] $\Rightarrow x\sin\theta(\sin^2\theta + \cos^2\theta) = \sin\theta\cos\theta \Rightarrow x\sin\theta = \sin\theta\cos\theta \Rightarrow x = \cos\theta$ ... $x\sin\theta = y\cos\theta \Rightarrow y = \sin\theta$

Hence $x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$.

Question-18

Evaluate, $\sec^2 10^o - \cot^2 80^o + \frac{\sin 15°\cos 75° + \cos 15°\sin 75°}{\cos\theta\sin(90 - \theta) + \sin\theta\cos(90 - \theta)}$.

Solution:

L.H.S. $= \sec^2 10^\circ - \cot^2 80^\circ + \dfrac{\sin 15^\circ \cos 75^\circ + \cos 15^\circ \sin 75^\circ}{\cos\theta \sin(90^\circ - \theta) + \sin\theta \cos(90^\circ - \theta)}$

$= \sec^2(90^\circ - 80^\circ) - \cot^2 80^\circ + \dfrac{\sin(90^\circ - 75^\circ)\cos 75^\circ + \cos(90^\circ - 75^\circ)\sin 75^\circ}{\cos\theta \sin(90^\circ - \theta) + \sin\theta \cos(90^\circ - \theta)}$

$= \mathrm{cosec}^2 80^\circ - \cot^2 80^\circ + \dfrac{\cos 75^\circ \times \cos 75^\circ + \sin 75^\circ \times \sin 75^\circ}{\cos\theta \times \cos\theta + \sin\theta \times \sin\theta}$

$= \dfrac{1}{\sin^2 80^\circ} - \dfrac{\cos^2 80^\circ}{\sin^2 80^\circ} + \dfrac{1}{1}$

$= \dfrac{1 - \cos^2 80^\circ}{\sin^2 80^\circ} + 1$

$= \dfrac{\sin^2 80^\circ}{\sin^2 80^\circ} + 1$

$= 1 + 1 = 2 =$ R.H.S.

Question-19

Prove the following identity: $\dfrac{\cot^2 A(\sec A - 1)}{(1 + \sin A)} + \dfrac{\sec^2 A(\sin A - 1)}{(1 + \sec A)} = 0.$

Solution:

LHS $= \dfrac{\cot^2 A(\sec A - 1)}{(1 + \sin A)} + \dfrac{\sec^2 A(\sin A - 1)}{(1 + \sec A)}$

$= \dfrac{\cot^2 A(\sec^2 A - 1) + \sec^2 A(\sin^2 A - 1)}{(1 + \sin A)(1 + \sec A)}$

$= \dfrac{\cot^2 A \tan^2 A - \sec^2 A \cos^2 A}{(1 + \sin A)(1 + \sec A)}$

$= \dfrac{1 - 1}{(1 + \sin A)(1 + \sec A)} = 0.$

Question-20

Prove: $\dfrac{\tan^2\theta}{\tan^2\theta - 1} + \dfrac{\mathrm{cosec}^2\theta}{\sec^2\theta - \mathrm{cosec}^2\theta} = \dfrac{1}{\sin^2\theta - \cos^2\theta}.$

Solution:

L.H.S. $= \dfrac{\tan^2\theta}{\tan^2\theta - 1} + \dfrac{\mathrm{cosec}^2\theta}{\sec^2\theta - \mathrm{cosec}^2\theta}$

$= \dfrac{\dfrac{\sin^2\theta}{\cos^2\theta}}{\dfrac{\sin^2\theta}{\cos^2\theta} - 1} + \dfrac{\dfrac{1}{\sin^2\theta}}{\dfrac{1}{\cos^2\theta} - \dfrac{1}{\sin^2\theta}}$

$= \dfrac{\dfrac{\sin^2\theta}{\cos^2\theta}}{\dfrac{\sin^2\theta - \cos^2\theta}{\cos^2\theta}} + \dfrac{\dfrac{1}{\sin^2\theta}}{\dfrac{\sin^2\theta - \cos^2\theta}{\cos^2\theta \sin^2\theta}}$

$$= \frac{\sin^2\theta}{\cos^2\theta} \times \frac{\cos^2\theta}{\sin^2\theta - \cos^2\theta} + \frac{1}{\sin^2\theta} \times \frac{\cos^2\theta\sin^2\theta}{\sin^2\theta - \cos^2\theta}$$

$$= \frac{\sin^2\theta}{\sin^2\theta - \cos^2\theta} + \frac{\cos^2\theta}{\sin^2\theta - \cos^2\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta - \cos^2\theta}$$

$$= \frac{1}{\sin^2\theta - \cos^2\theta}$$

$$= \text{R.H.S.} \;\therefore\; \frac{\tan^2\theta}{\tan^2\theta - 1} + \frac{\text{cosec}^2\theta}{\sec^2\theta - \text{cosec}^2\theta} = \frac{1}{\sin^2\theta - \cos^2\theta}.$$

Question-21

Solve the following equation for $0° < \theta \leq 90°$: $3\tan\theta + \cot\theta = 5\,\text{cosec}\,\theta$.

Solution:

$3\tan\theta + \cot\theta = 5\,\text{cosec}\,\theta \Rightarrow 3\tan\theta + \frac{1}{\tan\theta} = 5\,\text{cosec}\,\theta \Rightarrow 3\tan^2\theta + 1 = 5$ $\text{cosec}\,\theta\tan\theta \Rightarrow 3\tan^2\theta + 1 = 5\sec\theta \Rightarrow 3(\sec^2\theta - 1) + 1 = 5\sec\theta \Rightarrow 3\sec^2\theta$ $- 5\sec\theta - 2 = 0 \Rightarrow \sec\theta = \frac{5 \pm \sqrt{25+24}}{6} = \frac{5 \pm 7}{6}$

$\sec\theta = 2$ or $\frac{-1}{3}$

$\sec\theta \neq \frac{-1}{3}$ (as $-1 < \cos\theta < 1$) $\therefore \theta = 60°$ for $0° < \theta \leq 90°$.

Question-22

Solve for θ : $\frac{\cos^2\theta - 3\cos\theta + 2}{\sin^2\theta} = 1, (\theta \neq 0)$.

Solution:

$$\frac{\cos^2\theta - 3\cos\theta + 2}{\sin^2\theta} = 1$$

$$\frac{1 - \sin^2\theta - 3\cos\theta + 2}{\sin^2\theta} = 1$$

$$\frac{-\sin^2\theta - 3\cos\theta + 3}{\sin^2\theta} = 1$$

$-\sin^2\theta - 3\cos\theta + 3 = \sin^2\theta\; -2\sin^2\theta - 3\cos\theta = -3$

$-2(\cos^2\theta - 1) - 3\cos\theta = -3$

$-2\cos^2\theta + 2 - 3\cos\theta = -3$

$-2\cos^2\theta + 5 - 3\cos\theta = 0$

$2\cos^2\theta + 3\cos\theta - 5 = 0$

$\cos\theta = \frac{-3 \pm \sqrt{9+40}}{4} = \frac{-3 \pm \sqrt{49}}{4} = \frac{-3 \pm 7}{4} = -\frac{5}{2}$ or 1 if $\cos\theta = 1$, $\therefore \theta = 0°$.

Question-23

Using the formula $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$, obtain the value of $\tan 15°$.

Solution:

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

Putting $\theta = 15°$,

$\tan 30° = \frac{2\tan 15°}{1-\tan^2 15°}$

$\frac{1}{\sqrt{3}} - \frac{2\tan 15°}{1-\tan^2 15°}$

$(1 - \tan^2 15°) = \sqrt{3}(2\tan 15°)$

$\tan^2 15° + 2\sqrt{3}\tan 15° - 1 = 0$

$\tan 15° = \frac{-2\sqrt{3} \pm \sqrt{12+4}}{2} = \frac{-2\sqrt{3} \pm 4}{2} = -\sqrt{3} \pm 2$.

Question-24

If $a\cos\theta - b\sin\theta = x$ and $a\sin\theta + b\cos\theta = y$, prove that $a^2 + b^2 = x^2 + y^2$.

Solution:

$a\cos\theta - b\sin\theta = x$ and $a\sin\theta + b\cos\theta = y$

R.H.S. $= x^2 + y^2$

$= (a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2$

$= a^2\cos^2\theta - 2ab\cos\theta\sin\theta + b^2\sin^2\theta + a^2\sin^2\theta + 2ab\sin\theta\cos\theta + b^2\cos^2\theta$ $= (a^2 + b^2)\cos^2\theta + (b^2 + a^2)\sin^2\theta$ $= (a^2 + b^2)\cos^2\theta + (a^2 + b^2)\sin^2\theta$ $= (a^2 + b^2)(\cos^2\theta + \sin^2\theta)$

$= (a^2 + b^2)$ $[\because \cos^2\theta + \sin^2\theta = 1]$

$=$ L.H.S. $\therefore a^2 + b^2 = x^2 + y^2$.

Question-25

If $x = p\sec\theta + q\tan\theta$ and $y = p\tan\theta + q\sec\theta$, prove that $x^2 - y^2 = p^2 - q^2$.

Solution:

$x = p\sec\theta + q\tan\theta$ and $y = p\tan\theta + q\sec\theta$

L.H.S. $= x^2 - y^2$

$= (p\sec\theta + q\tan\theta)^2 - (p\tan\theta + q\sec\theta)^2$

$= p^2\sec^2\theta + 2pq\sec\theta\tan\theta + q^2\tan^2\theta - (p^2\tan^2\theta + 2pq\tan\theta\sec\theta + q^2\sec^2\theta)$

$= p^2\sec^2\theta + 2pq\sec\theta\tan\theta + q^2\tan^2\theta - p^2\tan^2\theta - 2pq\tan\theta\sec\theta - q^2\sec^2\theta$ $= (p^2 - q^2)\sec^2\theta + (q^2 - p^2)\tan^2\theta$

$= (p^2 - q^2)\sec^2\theta - (p^2 - q^2)\tan^2\theta$ $= (p^2 - q^2)(\sec^2\theta - \tan^2\theta)$

$= (p^2 - q^2)$ [Since $1 + \tan^2\theta = \sec^2\theta$]

$=$ R.H.S. $\therefore x^2 - y^2 = p^2 - q^2$.

Question-26

Show that (1 + cot θ - cosec θ)(1 + tan θ + sec θ) = 2.

Solution:

L.H.S = (1 + cot θ - cosec θ)(1 + tan θ + sec θ)

$= (1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta})(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta})$

$= (\frac{\sin\theta + \cos\theta - 1}{\sin\theta})(\frac{\sin\theta + \cos\theta + 1}{\cos\theta})$

$= \frac{(\sin\theta + \cos\theta)^2 - 1}{\sin\theta\cos\theta}$

$= \frac{2\sin\theta\cos\theta + \sin^2\theta + \cos^2\theta - 1}{\sin\theta\cos\theta}$

$= \frac{1 + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta}$

$= \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$

= 2

= R.H.S.

Question-27

If cosec θ - sin θ = a, sec θ - cos θ = b, prove that $a^2b^2(a^2 + b^2 + 3) = 1$.

Solution:

L.H.S = $a^2b^2(a^2 + b^2 + 3)$

$= (\text{cosec}\,\theta - \sin\theta)^2(\sec\theta - \cos\theta)^2[(\text{cosec}\,\theta - \sin\theta)^2 + (\sec\theta - \cos\theta)^2 + 3]$

$= (\text{cosec}\,\theta - \sin\theta)^2(\sec\theta - \cos\theta)^2[\text{cosec}^2\theta + \sin^2\theta - 2\,\text{cosec}\,\theta\sin\theta + \sec^2\theta + \cos^2\theta - 2\sec\theta\cos\theta + 3]$

$= (\text{cosec}\,\theta - \sin\theta)^2(\sec\theta - \cos\theta)^2[\text{cosec}^2\theta + 1 - 2 + \sec^2\theta - 2 + 3]$

$= (\text{cosec}\,\theta - \sin\theta)^2(\sec\theta - \cos\theta)^2(\text{cosec}^2\theta + \sec^2\theta)$

$= \left(\frac{1}{\sin\theta} - \sin\theta\right)^2\left(\frac{1}{\cos\theta} - \cos\theta\right)^2\left(\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}\right)$

$= \left(\frac{1 - \sin^2\theta}{\sin\theta}\right)^2\left(\frac{1 - \cos^2\theta}{\cos\theta}\right)^2\left(\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta}\right)$

$= \left(\frac{\cos^2\theta}{\sin\theta}\right)^2\left(\frac{\sin^2\theta}{\cos\theta}\right)^2\left(\frac{1}{\sin^2\theta\cos^2\theta}\right)$

$= \frac{\cos^4\theta}{\sin^2\theta} \times \frac{\sin^4\theta}{\cos^2\theta} \times \frac{1}{\sin^2\theta\cos^2\theta}$

= 1

= R.H.S

Question-28

Prove that $(\sin\theta + \sec\theta)^2 + (\cos\theta + \text{cosec}\,\theta)^2 = (1 + \sec\theta\,\text{cosec}\,\theta)^2$.

Solution:

L.H.S = $(\sin\theta + \sec\theta)^2 + (\cos\theta + \text{cosec}\,\theta)^2$

$= (\sin\theta + \frac{1}{\cos\theta})^2 + (\cos\theta + \frac{1}{\sin\theta})^2$

$= \sin^2\theta + \frac{1}{\cos^2\theta} + 2\sin\theta\frac{1}{\cos\theta} + \cos^2\theta + \frac{1}{\sin^2\theta} + 2\cos\theta\frac{1}{\sin\theta}$

$= 1 + \left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right) + 2\left(\sin\theta \frac{1}{\cos\theta} + \cos\theta \frac{1}{\sin\theta}\right)$

$= 1 + \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta \sin^2\theta} + 2 \frac{\sin^2\theta + \cos^2\theta}{\cos\theta \sin\theta}$

$= 1 + \frac{1}{\cos^2\theta \sin^2\theta} + 2 \frac{1}{\cos\theta \sin\theta}$

$= 1 + \sec^2\theta \operatorname{cosec}^2\theta + 2\sec\theta \operatorname{cosec}\theta$

$= (1 + \sec\theta \operatorname{cosec}\theta)^2$

= R.H.S.

Question-29

Find the value of $\frac{4}{3}\tan^2 30^o + \sin^2 60^o - 3\cos^2 60^o + \frac{3}{4}\tan^2 60^o - 2\tan^2 45^o$.

Solution:

$\frac{4}{3}\tan^2 30^o + \sin^2 60^o - 3\cos^2 60^o + \frac{3}{4}\tan^2 60^o - 2\tan^2 45^o$

$= \frac{4}{3} \times \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 - 3\left(\frac{1}{2}\right)^2 + \frac{3}{4}(\sqrt{3})^2 - 2$

$= \frac{4}{3} \times \frac{1}{3} + \frac{3}{4} - \frac{3}{4} + \frac{3}{4} \times 3 - 2$

$= \frac{4}{9} + \frac{3}{4} \times 3 - 2$

$= \frac{4}{9} + \frac{9}{4} - 2$

$= \frac{16 + 81 - 72}{36}$

$= \frac{25}{36}$.

Question-30

Find the value of $4(\sin^4 30^o + \cos^4 60^o) - 3(\cos^2 45^o + \sin^2 90^o)$.

Solution:

$4(\sin^4 30^o + \cos^4 60^o) - 3(\cos^2 45^o + \sin^2 90^o)$

$= 4\left[\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4\right] - 3\left[\left(\frac{1}{\sqrt{2}}\right)^2 + 1^2\right]$

$= 4\left[2 \times \frac{1}{16}\right] - 3\left[\frac{1}{2} + 1\right]$

$= \frac{1}{2} - 3\left[\frac{3}{2}\right]$

$= \frac{1}{2} - \frac{9}{2}$

$= -\frac{8}{2}$

$= -4$.

Question-31

Prove that $\left(\frac{1}{\sec^2\theta - \cos^2\theta} + \frac{1}{\operatorname{cosec}^2\theta - \sin^2\theta}\right)\sin^2\theta \cos^2\theta = \frac{1 - \sin^2\theta \cos^2\theta}{2 + \sin^2\theta \cos^2\theta}$.

Solution:

$$\left(\frac{1}{\sec^2\theta - \cos^2\theta} + \frac{1}{\operatorname{cosec}^2\theta - \sin^2\theta}\right)\sin^2\theta\cos^2\theta = \left(\frac{\cos^2\theta}{1-\cos^4\theta} + \frac{\sin^2\theta}{1-\sin^4\theta}\right)\sin^2\theta\cos^2\theta$$

$$= \left(\frac{\cos^2\theta - \cos^2\theta\sin^4\theta + \sin^2\theta - \sin^2\theta\cos^4\theta}{(1+\cos^2\theta)\sin^2\theta(1+\sin^2\theta)\cos^2\theta}\right)\sin^2\theta\cos^2\theta$$

$$= \frac{1-\cos^2\theta\sin^2\theta(\cos^2\theta+\sin^2\theta)}{1+\cos^2\theta+\sin^2\theta+\cos^2\theta\sin^2\theta}$$

$$= \frac{1-\sin^2\theta\cos^2\theta}{2+\sin^2\theta\cos^2\theta}.$$

Question-32

Find the value of $\frac{4}{3}\tan^2 30^o + \sin^2 60^o - 3\cos^2 60^o + \frac{3}{4}\tan^2 60^o - 2\tan^2 45^o$.

Solution:

$$\frac{4}{3}\tan^2 30^\circ + \sin^2 60^\circ - 3\cos^2 60^\circ + \frac{3}{4}\tan^2 60^\circ - 2\tan^2 45^\circ$$

$$= \frac{4}{3}\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 - 3\left(\frac{1}{2}\right)^2 + \frac{3}{4}(\sqrt{3})^2 - (2\times 1)$$

$$= \left(\frac{4}{3}\times\frac{1}{3}\right) + \left(\frac{3}{4}\right) - \left(\frac{3}{4}\right) + \left(\frac{3}{4}\times 3\right) - 2$$

$$= \frac{4}{9} + \frac{9}{4} - 2$$

$$= \frac{16+81-72}{36}$$

$$= \frac{25}{36}$$