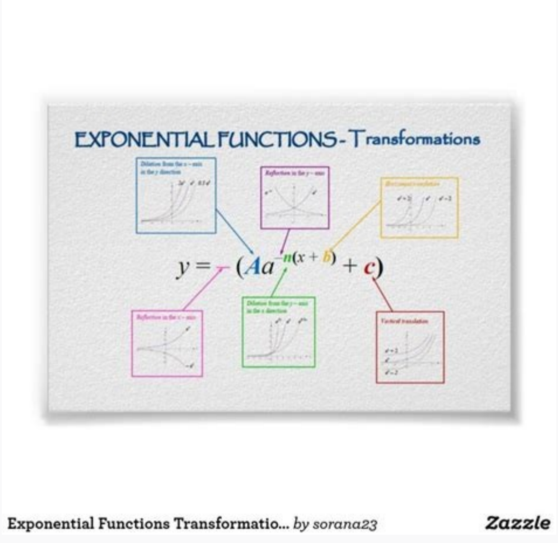
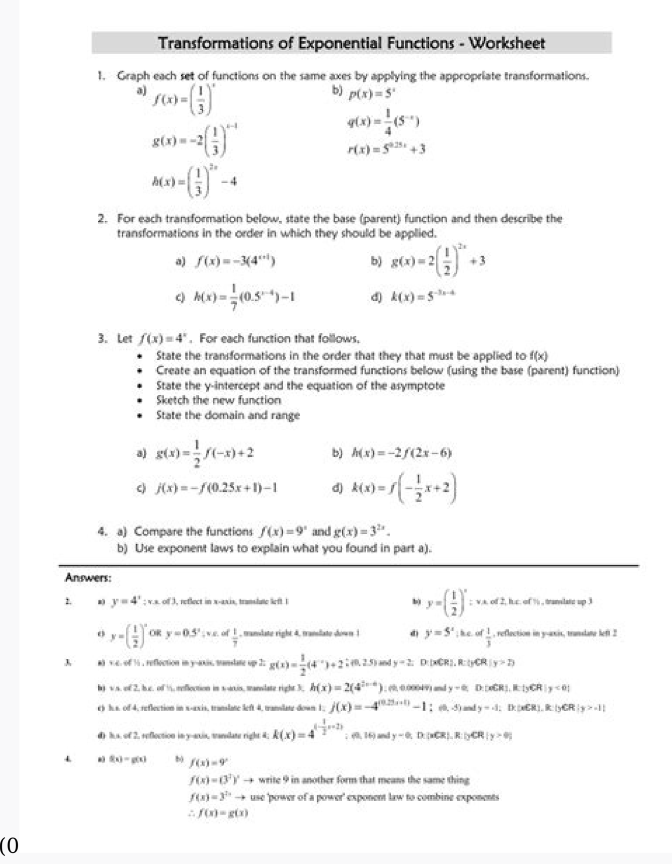
Transformations of exponential functions worksheet pdf

Learning Objectives Graph exponential functions and their transformations. Review Laws of Exponents Exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Seeing their graphs gives us another layer of insight for predicting future events. Exponential growth is modelled by functions of the form \(f(x)=b^x\) where the base is greater than one. Exponential decay occurs when the base is between zero and one.
We'll use the functions \(f(x)=2^x\) and \(g(x)={\left(\tfrac{1}{2}\right)}^x\) to get some insight into the behaviour of graphs that model exponential growth and decay. In each table of values below, observe how the output values change as the input increases by \(1\). Exponential Growth: \(f(x)=2^x\). Table \(\PageIndex{1}\): Exponential Growth \(x\) \(−3\) \(−2\) \(−1\) \(0\) \(1\) \(2\) \(3\) \(f(x)=2^x\) \(\dfrac{1}{8}\) \(\dfrac{1}{4}\) \(\dfrac{1}{2}\) \(1\) \(2\) \(4\) \(8\) Each output value is the product of the previous output and the base, \(2\). We call the base \(2\) the constant ratio. In fact, for any exponential function with the form \(f(x)=ab^x,\) \(b\) is the constant ratio of the function. This means that as the input increases by \(1\), the output value \(f(x+1)\) will be the product of the base and the previous output, \(b f(x)=b \cdot ab^x=ab^{x+1}=f(x+1)\), regardless of the value of \(a\). Notice from the table that the output values are positive for all values of \(x\); as \(x\) increases, the output values increase without bound; as \(x\) decreases, the output values grow smaller, approaching zero.

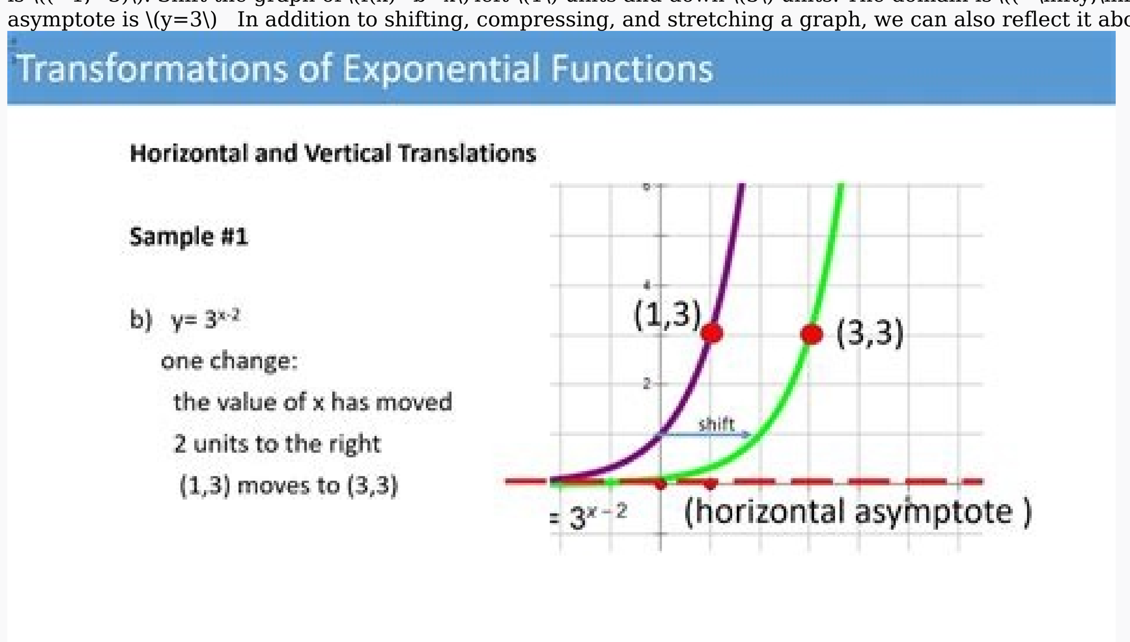


The domain of \(f(x)=2^x\) is all real numbers, the range is \((0,\infty)\), and the horizontal asymptote is \(y=0\). Figure \(\PageIndex{1}\): Graph of the exponential growth function \(f(x)=2^x\). Notice that the graph gets close to the x-axis, but never touches it. Exponential Decay: \(g(x)={\left(\frac{1}{2}\right)}^x\). Table \(\PageIndex{2}\): Exponential Decay \(x\) \(−3\) \(−2\) \(−1\) \(0\) \(1\) \(2\) \(3\) \(g(x)=\big(\dfrac{1}{2} \big) \) \(8\) \(4\) \(2\) \(1\) \(\dfrac{1}{2}\) \(\dfrac{1}{4}\) \(\dfrac{1}{8}\) Again, as the input is increases by \(1\), each output value is the product of the previous output and the base, \(\tfrac{1}{2}\), so the constant ratio of the function is \(\tfrac{1}{2}\). Notice from the table that the output values are positive for all values of \(x\); as \(x\) increases, the output values grow smaller, approaching zero; and as \(x\) decreases, the output values grow without bound. The domain of \(g\) is all real numbers, the range is \((0,\infty)\), and the horizontal asymptote is \(y=0\). Figure \(\PageIndex{2}\): Graph of the exponential decay function, \(g(x)={\left(\frac{1}{2}\right)}^x\). Notice that \(g(x)={\left(\frac{1}{2}\right)}^x = (2^{-1})^x = f(-x)\). Thus the graph of \(g\) is simply a reflection over the \(y\)-axis of the graph of \(f\). CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION \(f(x) = b^x\) An exponential function with the form \

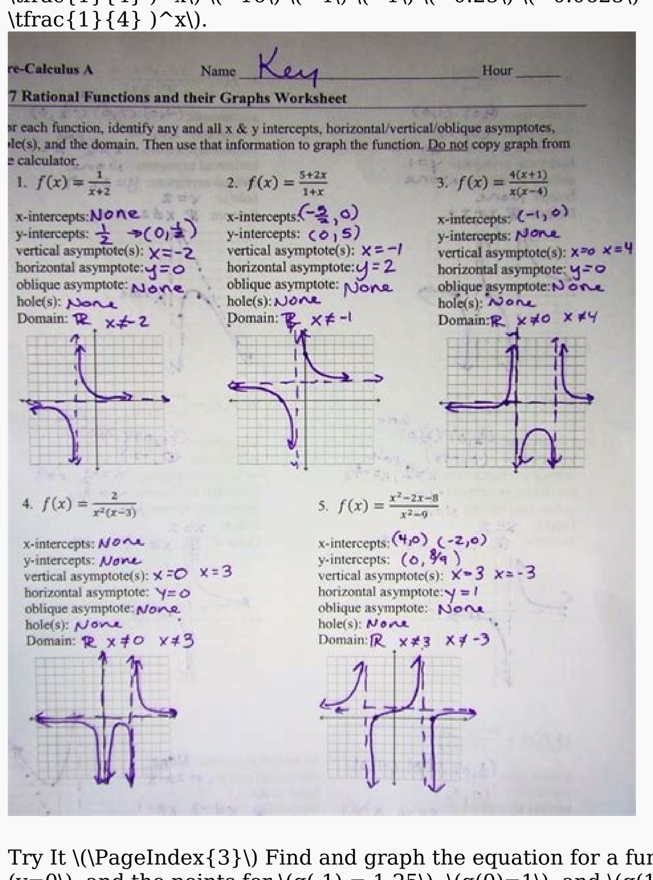


(f(x)=b^x\), \(b>0\), \(b≠1\), has these characteristics: one-to-one function horizontal asymptote: \(y=0\) \(x\)-intercept: none \(y\)-intercept: \((0,1)\) and key point \((1, b) \) domain: \((–\infty, \infty)\) range: \((0,\infty)\) increasing if \(b>1\) - "exponential growth" decreasing if \(0

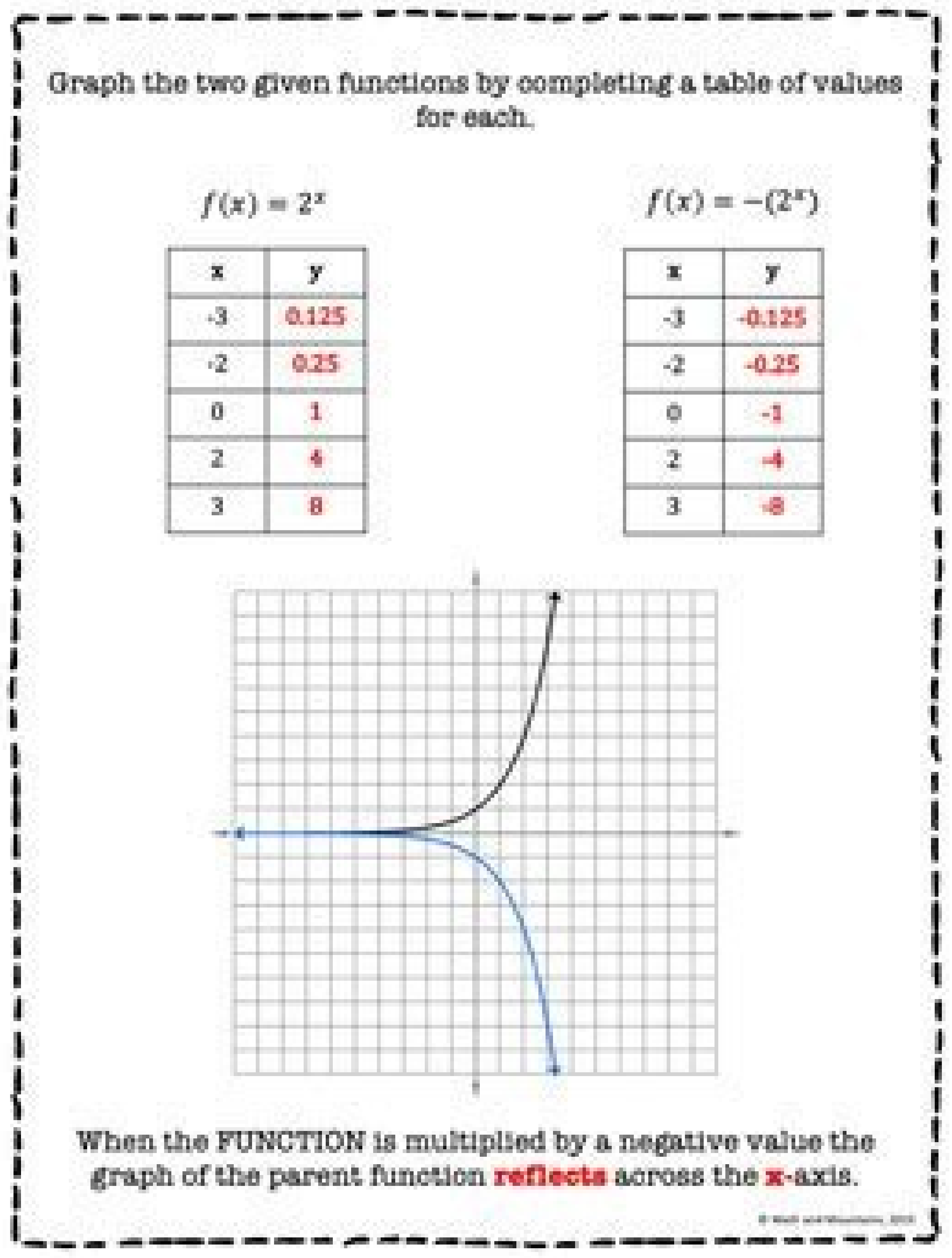
Draw a smooth curve that goes through the points and approaches the horizontal asymptote. State the domain \((−\infty,\infty)\), the range \((0,\infty)\), and the horizontal asymptote, \(y=0. \) Example \(\PageIndex{1}\): Sketch the Graph of an Exponential Function of the Form \(f(x) = b^x\) Sketch a graph of \(f(x)=0.25^x\). State the domain, range, and asymptote. Solution Before graphing, identify the behavior and create a table of points for the graph. Table \(\PageIndex{3}\) \(x\) \(−3\) \(−2\) \(−1\) \(0\) \(1\) \(2\) \(3\) \(f(x)={0.25}^x\) \(64\) \(16\) \(4\) \(1\) \(0.25\) \(0.0625\0 \(0.015625\) Since \(b=0.25\) is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote \(y=0\). Create a table of points as in Table \(\PageIndex{3}\). Plot the asymptote, and the y-intercept \((0,1)\), along with two other points. We can use \((−1,4)\) and \((1,0.25)\). Draw a smooth curve connecting the points as shown to the right. The domain is \((−\infty,\infty)\); the range is \((0,\infty)\); the horizontal asymptote is \(y=0\).
Try It \(\PageIndex{1}\) Sketch the graph of \(f(x)=4^x\). State the domain, range, and asymptote Answer The domain is \((−\infty,\infty)\); the range is \((0,\infty)\); the horizontal asymptote is \(y=0\). Transformations of exponential graphs behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the parent function \(f(x)=b^x\) without loss of general shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied. The first transformation occurs when we add a constant \(d\) to the parent function \(f(x)=b^x\), giving us a vertical shift \(d\) units in the same direction as the sign. For example, if we begin by graphing a parent function, \(f(x)=2^x\), we can then graph two vertical shifts alongside it, using \(d=3\): the upward shift, \(g(x)=2^x+3\) and the downward shift, \(h(x)=2^x−3\). Both shifts are shown in the figure to the right. Observe the results of shifting \(f(x)=2^x\) vertically: The domain, \((−\infty,\infty)\) remains unchanged. When the function is shifted up \(3\) units to \(g(x)=2^x+3\): The \(y\)-intercept shifts up \(3\) units to \((0,4)\). The asymptote shifts up \(3\) units to \(y=3\). The range becomes \((3,\infty)\). When the function is shifted down \(3\) units to \(h(x)=2^x−3\): The \(y\)-intercept shifts down \(3\) units to \((0,−2)\).
The asymptote also shifts down \(3\) units to \(y=−3\). The range becomes \((−3,\infty)\). The next transformation occurs when we add a constant \(c\) to the input of the parent function \(f(x)=b^x\), giving us a horizontal shift \(c\) units in the opposite direction of the sign. For example, if we begin by graphing the parent function \(f(x)=2^x\), we can then graph two horizontal shifts alongside it, using \(c=3\): the shift left, \(g(x)=2^{x+3}\), and the shift right, \(h(x)=2^{x−3}\). Both horizontal shifts are shown in the figure to the right. Observe the results of shifting \(f(x)=2^x\) horizontally: The domain, \((−\infty,\infty)\), remains unchanged. The asymptote, \(y=0\), remains unchanged. The \(y\)-intercept shifts such that: When the function is shifted left \(3\) units to \(g(x)=2^{x+3}\), the \(y\)-intercept becomes \((0,8)\). This is because \(2^{x+3}=(8)2^x\), so the initial value of the function is \(8\). When the function is shifted right \(3\) units to \(h(x)=2^{x−3}\), the \(y\)-intercept becomes \((0,\frac{1}{8})\). Again, see that \(2^{x−3}=(\frac{1}{8})2^x\), so the initial value of the function is \(\frac{1}{8}\). SHIFTS OF THE PARENT FUNCTION \(y = b^x\) For any constants \(c\) and \(d\), the function \(f(x)=b^{x+c}+d\) shifts the parent function \(y=b^x\) Vertically \(d\) units, in the same direction of the sign of \(d\). If \(d>0\) the parent function is shifted up \(d\) units If \(d<0\) the parent function is shifted down \(d\) units The new \(y\)-coordinates are equal to \(y+d\) The horizontal asymptote becomes \(y=d\), and occurs when the exponential part of the function, \(b^{x+c}\) approaches zero. The range becomes \((d,\infty)\). Horizontally \(c\) units, in the opposite direction of the sign of \(c\). If \(c>0\) the parent function is shifted left \(c\) units If \(c<0\) the parent function is shifted right \(c\) units The new \(x\)-coordinates are equal to \(x-c\).; The new y-intercept, at \((0,1)\) in the parent function, occuring at \(f(0)\), becomes \((0,b^c+d)\). The domain, \((−\infty,\infty)\), remains unchanged. How to: Graph an exponential function of the form \(f(x)=b^{x+c}+d\) Draw the horizontal asymptote \(y=d\). Identify the shift as \((−c,d)\). Shift the graph of \(f(x)=b^x\) left \(c\) units if \(c\) is positive, and right \(c\) units if \(c\) is negative. Shift the graph of \(f(x)=b^x\) up \(d\) units if \(d\) is positive, and down \(d\) units if \(d\) is negative. State the domain, \((−\infty,\infty)\), the range, \((d,\infty)\), and the horizontal asymptote \(y=d\).
Example \(\PageIndex{2}\): Graphing a Shift of an Exponential Function Graph \(f(x)=2^{x+1}−3\). State the domain, range, and asymptote. Solution We have an exponential equation of the form \(f(x)=b^{x+c}+d\), with \(b=2\), \(c=1\), and \(d=−3\). Draw the horizontal asymptote \(y=d\), so draw \(y=−3\). Identify the shift as \((−c,d)\), so the shift is \((−1,−3)\). Shift the graph of \(f(x)=b^x\) left \(1\) units and down \(3\) units. The domain is \((−\infty,\infty)\); the range is \((−3,\infty)\); the horizontal asymptote is \(y=−3\). Try It \(\PageIndex{2}\) Graph \(f(x)=2^{x−1}+3\). State domain, range, and asymptote. Answer The domain is \((−\infty,\infty)\); the range is \((3,\infty)\); the horizontal asymptote is \(y=3\) In addition to shifting, compressing, and stretching a graph, we can also reflect it about the \(x\)-axis or the \(y\)-axis. When we multiply the parent function \(f(x)=b^x\) by \(−1\), we get a reflection about the \(x\)-axis.



When we multiply the input by \(−1\), we get a reflection about the \(y\)-axis. For example, if we begin by graphing the parent function \(f(x)=2^x\), we can then graph the two reflections alongside it. The reflection about the \(x\)-axis, \(g(x)=−2^x\), is illustrated below in the graph on the left, and the reflection about the \(y\)-axis \(h(x)=2^{−x}\), is shown in the graph on the right. REFLECTIONS OF THE PARENT FUNCTION \(f(x) = b^x\) The function \(f(x)=−b^x\) reflects the parent function \(f(x)=b^x\) about the \(x\)-axis. has a \(y\)-intercept of \((0,−1)\). has a range of \((−\infty,0)\) has a horizontal asymptote at \(y=0\) and domain of \((−\infty,\infty)\), which are unchanged from the parent function. The function \(f(x)=−b^x + d\) has both a vertical shift and reflection about the \(x\)-axis. In this situation, always do the vertical shift LAST. The function \(f(x)=b^{−x}\) reflects the parent function \(f(x)=b^x\) about the \(y\)-axis. has a \(y\)-intercept of \((0,1)\), a horizontal asymptote at \(y=0\), a range of \((0,\infty)\), and a domain of \((−\infty,\infty)\), which are unchanged from the parent function. The function \(f(x)=b^{-x+c}\) has both a horizontal shift and reflection about the \(y\)-axis. In this situation, always do the horizontal shift FIRST. Example \(\PageIndex{3}\): Construct an Equation for a Reflected Exponential Function Find and graph the equation for a function, \(g(x)\), that reflects \(f(x)=(\tfrac{1}{4})^x\) about the \(x\)-axis. State its domain, range, and asymptote. Solution Since we want to reflect the parent function \(f(x)=(\tfrac{1}{4})^x\) about the \(x\)-axis, we multiply \(f(x)\) by \(−1\) to get, \(g(x)=−f(x)=−(\tfrac{1}{4})^x\). Next we create a table of points below. \(x\) \(−2\) \(−1\) \(0\) \(1\) \(2\) \(g(x)=−(\tfrac{1}{4})^x\) \(−16\) \(−4\) \(−1\) \(−0.25\) \(−0.0625\) Plot the \(y\)-intercept, \((0,−1)\), along with two other points. We can use \((−1,−4)\) and \((1,−0.25)\). Draw a smooth curve connecting the points. The domain is \((−\infty,\infty)\); the range is \((−\infty,0)\); the horizontal asymptote is \(y=0\). Figure \(\PageIndex{11}\). Graph of \(g(x)=−(\tfrac{1}{4})^x\).



Try It \(\PageIndex{3}\) Find and graph the equation for a function, \(g(x)\), that reflects \(f(x)={1.25}^x\) about the \(y\)-axis. State its domain, range, and asymptote. Answer \(g(x) = f(-x) = 1.25^{-x}\) The domain is \((−\infty,\infty)\); the range is \((0,\infty)\); the horizontal asymptote is \(y=0\). Included in the graph is the horizontal asymptote \(y=0\), and the points for \(g(-1) = 1.25\), \(g(0)=1\), and \(g(1) = .8\). While horizontal and vertical shifts involve adding constants to the input or to the function itself, a stretch or compression occurs when we multiply the parent function \(f(x)=b^x\) by a constant \(|a|>0\). For example, if we begin by graphing the parent function \(f(x)=2^x\), we can then graph the stretch, using \(a=3\), to get \(g(x)=3{(2)}^x\) as shown in Figure (a), and the compression, using \(a=\dfrac{1}{3}\), to get \(h(x)=\dfrac{1}{3}{(2)}^x\) as shown on the right in Figure (b). VERTICAL STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION \(y = b^x\) For any factor \(a e 0\), the function \(f(x)=a{(b)}^x\) is stretched vertically by a factor of \(a\) if \(|a|>1\). is compressed vertically by a factor of \(a\) if \(0 < |a|<1\). The new \(y\)-coordinates are equal to \(ay\). This would include vertical reflection if present. has a \(y\)-intercept of \((0,a)\). has a horizontal asymptote at \(y=0\), a range of \((0,\infty)\), and a domain of \((−\infty,\infty)\), which are unchanged from the parent function. If a vertically stretched, compressed and/or reflected function also has a vertical shift, like \(g(x)=a{(b)}^x + d, \) then the vertical shift, (\(d\) units up or down), must be done AFTER performing the vertical stretching, compression, and/or reflection.



Example \(\PageIndex{4}\): Graphing the Vertical Stretch of an Exponential Function Sketch a graph of \(f(x)=4{\Big(\dfrac{1}{2}\Big)}^x\). State the domain, range, and asymptote. Solution Before graphing, identify the behavior and key points on the graph. Since \(b=\dfrac{1}{2}\) is between zero and one, the left tail of the graph will increase without bound as \(x\) decreases, and the right tail will approach the \(x\) -axis as \(x\) increases. Since \(a=4\), the graph of the parent function \(y={\Big(\dfrac{1}{2}\Big)}^x\) will be stretched vertically by a factor of \(4\). Create a table of points of the parent function as shown in Table \(\PageIndex{4}\). Then multiply all the \(y\) coordinates by 4. Table \(\PageIndex{4}\) \(x\) \(−3\) \(−2\) \(−1\) \(0\) \(1\) \(2\) \(3\) \(y={\Big(\dfrac{1}{2}\Big)}^x\) \(8\) \(4\) \(2\) \(1\) \(\dfrac{1}{2}\) \(\dfrac{1}{4}\) \(\dfrac{1}{8}\) \(f(x)=4{\Big(\dfrac{1}{2}\Big)}^x\) \(32\) \(16\) \(8\) \(4\) \(2\) \(1\) \(0.5\)   Plot the \(y\)-intercept, \((0,4)\), along with two other points. We can use \((−1,8)\) and \((1,2)\). Draw a smooth curve connecting the points, as shown in the figure on the right.
The domain is \((−\infty,\infty)\); the range is \((0,\infty)\); the horizontal asymptote is \(y=0\). Try It \(\PageIndex{4}\) Sketch the graph of \(f(x)=\dfrac{1}{2}{(4)}^x\). State the domain, range, and asymptote. Answer The domain is \((−\infty,\infty)\); the range is \((0,\infty)\); the horizontal asymptote is \(y=0\). Now that we have worked with each type of translation for the exponential function, we can summarize them. How to: GRAPH TRANSFORMATIONS OF EXPONENTIAL FUNCTIONS A transformation of an exponential function has the form \(f(x)=ab^{mx+c}+d\) The transformations to the parent function, \(y=b^x\), \(b>1\), needed to obtain \(f\) are given below. The order these are done is important. FIRST. Shift horizontally \(c\) units.
\(\quad   x \rightarrow (x - c) \).
If \(c > 0\) shift left \(c\) units.   If \(c < 0\) shift right \(c\) units..   Horizontal reflection, compression and/or stretching.   \(\quad \) \(x \rightarrow \frac{1}{m} x \).
If \(m \) is negative, reflect over the \(y\)-axis.   \(\quad \) \((x \rightarrow -x).\) If \(|m| > 1 \) shrink the graph horizontally by a factor of \(\frac{1}{|m|}. \)   If \(0<|m|<1 \) stretch the graph horizontally by a factor of \(\frac{1}{|m|}. \)   Vertical reflection, compression and/or stretching.   \(\quad \) \(y \rightarrow ay \). If \(a\) is negative, reflect over the \(x\)-axis. \(\quad \) \((y \rightarrow -y) \). If \(|a|>0\) stretch the graph vertically by a factor of \(|a| \).   If \(0<|a|<1\)compress the graph vertically by a factor of \(|a| \).   LAST. Shift vertically \(d\) units. \(\quad \) \(y \rightarrow y + d \). If \(d > 0\) shift up \(d\) units. If \(d < 0\) shift down \(d\) units. If the exponential function is written in the form \(f(x)=ab^{m(x + c)}+d\) then reverse the order of steps 1 and 2 -- do reflecting and stretching first, then do the horizontal shift.   Example \(\PageIndex{5}\) Sketch the graphs of \(g(x) = -2 \cdot 3^{x-5}+6 \) and \(f(x) = 3 \cdot 2^{-x+1}-4 \) and the corresponding original parent function. State the domain, range, and horizontal asymptote of the transformation. Solution The basic parent function for \(g(x) = -2 \cdot 3^{x-5}+6 \) is \(y = 3^x\).   The transformations needed to obtain the graph of \(g(x)\) from the graph of \(y\) are: Shift right 5 units (\(x \rightarrow x+5 \)) Reflect over the \(x\)-axis (\(y \rightarrow -y \)) Vertically stretch by a factor of 2 (\(y \rightarrow 2y \)) Shift up \(6\) units (\(y \rightarrow y-4 \)) The graph of \(g(x)\) and its parent function is on the right. The domain is \((−\infty,\infty)\); the range is \((-\infty, 6)\); the horizontal asymptote is \(y=6\). If tables are used to graph the function, coordinate points for the parent function appear in the table below \(\begin{array}{|r|c|c|c|c|c|} \hline \text{Parent function: }x & \text{HA} & -1 & 0 & 1 & 2 \\ \hline y = 3^x & 0 & \frac{1}{3} & 1 & 3 & 9 \\[2pt] \hline \end{array} \) Corresponding coordinate points for the transformation \(g(x) = -2 \cdot 3^{x-5}+6 \) would be \(\begin{array}{|r|c|c|c|c|c|} \hline \text{New }x \text{ is: } \:\;\quad x+5 & \text{HA} & 4 & 5 & 6 & 7 \\ \hline \text{New } y \text{ is: } -2y+6 & 6 & 5\frac{1}{3} & 4 & 0 & -12 \\[2pt] \hline \end{array} \) The basic parent function for \(f(x) = 3 \cdot 2^{-x+1}-4 \) is \(y = 2^x\).   The transformations needed to obtain the graph of \(f(x)\) from the graph of \(y\) are: Shift left 1 unit (\(x \rightarrow x-1 \)) Reflect over the \(y\)-axis (\(x \rightarrow -x \)) Vertically stretch by a factor of 3 (\(y \rightarrow 3y \)) Shift down \(4\) units (\(y \rightarrow y-4 \)) If instead we re-wrote the function in the form: \(f(x) = 3 \cdot 2^{-(x-1)}-4 \), the transformations would be Reflect over the \(y\)-axis (\(x \rightarrow -x \)) Shift right 1 unit (\(x \rightarrow x+1 \)) Vertically stretch by a factor of 3 (\(y \rightarrow 3y \)) Shift down \(4\) units (\(y \rightarrow y-4 \)) The graph of \(f(x)\) and its parent function is on the right. The domain is \((−\infty,\infty)\); the range is \((\infty, -4)\); the horizontal asymptote is \(y=-4\). If tables are used to graph the function, coordinate points for the parent function appear in the table below \(\begin{array}{|r|c|c|c|c| c |} \hline \text{Parent function: }x & \text{HA} & -1 & 0 & 1 & 2 \\ \hline y = 2^x & 0 & \frac{1}{2} & 1 & 2 & 4 \\[2pt] \hline \end{array} \) Corresponding coordinate points for the transformation \(f(x) = 3 \cdot 2^{-x+1}-4 \) would be \(\begin{array}{|r|c|c|c|c| c |} \hline \text{New } x \text{ is: } -x+1 & \text{HA} & 2 & 1 & 0 & -1 \\ \hline \text{New } y \text{ is: } 3y-4 & -4 & -2\frac{1}{2} & -1 & 2 & 8 \\[2pt] \hline \end{array} \) Example \(\PageIndex{6}\) Sketch the graph of \(f(x) = 2 ^ {\frac{1}{2} x + 4} - 3 \). State the domain, range, and horizontal asymptote of the transformation. Solution The basic parent function is \(y = 2^x\).   The transformations needed to obtain the graph of \(f(x)\) from the graph of \(y\) are: Shift left \(4\) units (\(x \rightarrow x-4 \)) Horizontally stretch by a factor of 2 (\(x \rightarrow 2x \)) Shift down \(3\) units (\(y \rightarrow y-3 \)) The graph of \(f(x)\) and its parent function is on the right. The domain is \((−\infty,\infty)\); the range is \((-3, \infty)\); the horizontal asymptote is \(y=-3\). If tables are used to graph the function, coordinate points for the parent function appear in the table below \(\begin{array}{|r|c|c|c|c| c |} \hline \text{Parent function: }x & \text{HA} & -1 & 0 & 1 & 2 \\ \hline y = 2^x & 0 & \frac{1}{2} & 1 & 2 & 4 \\[2pt] \hline \end{array} \) Corresponding coordinate points for the transformation would be \(\begin{array}{|r|c|c|c|c| c |} \hline \text{New }x \text{ is: } 2(x-4) & \text{HA} & -10 & -8 & -6 & -4 \\ \hline \text{New } y \text{ is: } y-3 & -3 & -2.5 & -2 & -1 & 1 \\[2pt] \hline \end{array} \) Note that if the function were rewritten in the form: \(f(x) = 2 ^ {\frac{1}{2} (x + 8)} - 3 \) then the transformations indicated by this version of the function would be Horizontally stretch by a factor of 2 (\(x \rightarrow 2x \)) Shift left \(8\) units (\(x \rightarrow x-8 \)) Shift down \(3\) units (\(y \rightarrow y-3 \)) Try It \(\PageIndex{7}\) Sketch the graph of \(f(x) = 2 ^ {-2 x - 4} + 3 \). State the domain, range, and horizontal asymptote of the transformation.   Answer The domain is \((−\infty,\infty)\); the range is \((3, \infty)\); the horizontal asymptote is \(y=3\). As written the transformations done on the parent function \(y = 2^x\) would be \(\quad \) right 4, reflect over y axis, \(x \rightarrow \frac{1}{2} x \), up 3. However, if rewritten as \(f(x) = 2 ^ {-2 (x +2)} + 3 \), the transformations would be \(\quad \) reflect over y axis, \(x \rightarrow \frac{1}{2} x \), left 2, up 3. Example \(\PageIndex{8}\): Write a Function from a Description Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range. \(f(x)=e^x\) is vertically stretched by a factor of \(2\) , reflected across the y-axis, and then shifted up \(4\) units. Solution We want to find an equation of the general form \(f(x)=ab^{x+c}+d\). We use the description provided to find \(a\), \(b\), \(c\), and \(d\). We are given the parent function \(f(x)=e^x\), so \(b=e\). The function is vertically stretched by a factor of \(2\), so \(a=2\). The function is reflected about the \(y\)-axis.
We replace \(x\) with \(−x\) to get: \(e^{−x}\). The graph is shifted vertically \(4\) units, so \(d=4\). Substituting in the general form we get, \(f(x)=ab^{x+c}+d\) \(=2e^{−x+0}+4\) \(=2e^{−x}+4\) The domain is \((−\infty,\infty)\); the range is \((4,\infty)\); the horizontal asymptote is \(y=4\). Try It \(\PageIndex{8}\) Write the equation for function described below. Give the horizontal asymptote, the domain, and the range. \(f(x)=e^x\) is compressed vertically by a factor of \(\dfrac{1}{3}\), reflected across the \(x\)-axis and then shifted down \(2\) units. Answer \(f(x)=−\dfrac{1}{3}e^{x}−2\); the domain is \((−\infty,\infty)\); the range is \((−\infty,2)\); the horizontal asymptote is \(y=2\). When learning about exponents, we are typically given expressions that have a whole number exponent and a base that is a variable, like \(x^2\). These type of expressions are technically called power functions. In contrast, the base can be a constant and the exponent can have a variable in it, like \(2^x\). Expressions in this form are called exponential functions. When simplifying expressions with the variable in the exponent, we often use the Laws of Exponents "backwards". A review of properties of exponents from a different perspective is briefly discussed here. Competency with exponents is a very useful skill to have. The table below summarizes these properties. Rules of Exponents For nonzero real numbers a and b and integers m and n   Product Rule Quotient Rule Power Rule Like Bases Rule \(a^m·a^n=a^{m+n}\) \(\dfrac{a^m}{a^n}=a^{m−n}\) \((a^m)^n=a^{m·n}\) Reverse of Like Bases Rule \(a^{m+n}=a^m·a^n \) \(a^{m−n}= \dfrac{a^m}{a^n}\) \(a^{m·n}=(a^m)^n\) Like Exponents Rule \(a^n·b^n = (a·b)^n \) \(\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n\) \(\dfrac{1}{b^n} = \left(\dfrac{1}{b}\right)^n\) Example \(\PageIndex{9}\) Rewrite the following exponential expressions such that the result uses only one exponent, \(x\).
Simplify. 1. \(2^{x+3}\) 2.
\(3^{x-2}\) 3. \(5^{3+x}\) 4. \(2^x \cdot 11^x\) 5. \(5^{3+2x} \) 6. \(\dfrac{4^x}{12^x} \) 7. \(\dfrac{6^{3x}}{9^{2x}} \) 8. \(\dfrac{8^{x-1}}{2^{2x-5}} \) Solution 1. \(2^{x+3} = 2^x 2^3 = 2^x \cdot 8 = 8(2^x)\) 2. \(3^{x-2} = 3^x3^{-2} = \dfrac{3^x}{3^2} = \dfrac{3^x}{9} \) 3. \(5^{3-x} = 5^3 5^{-x} = \dfrac{5^3}{5^x} = \dfrac{125}{5^x} = 125 \left(\dfrac{1}{5} \right)^x \) 4. \(2^x \cdot 11^x = (2 \cdot 11)^x = 22^x \) 5. \(5^{3+2x} = 5^3 5^{2x} = 125 (5^2)^x = 125(25^x) \) 6. \(\dfrac{4^x}{12^x} = \left(\dfrac{4}{12} \right)^x = \left(\dfrac{1}{3} \right)^x \) 7. \(\dfrac{6^{3x}}{9^{2x}} = \dfrac{(6^3)^x}{(9^2)^x} = \left(\dfrac{6^3}{9^2} \right)^x = \left(\dfrac{8}{3} \right)^x \) 8. \(\dfrac{8^{x-1}}{2^{2x-5}} = \dfrac{8^x 8^{-1}}{2^{2x} 2^{-5}} = \dfrac{8^x}{2^{2x}} \cdot \dfrac{2^5}{8} = \dfrac{2^5}{8} \cdot \dfrac{8^x}{4^x}= \dfrac{32}{8} \cdot \left(\dfrac{8}{4} \right)^x \\ \) \(\quad = 4 (2^x) \) \(\PageIndex{9}\) Rewrite the following exponential expressions such that the result uses only one exponent, \(x\). Simplify. \(1.\quad 3^{x+1}\cdot 4^{-2x} \) \(2. \quad \dfrac{2^{3x+4}}{7^{2x+1}} \) Answer \(1.\quad 3 \left(\dfrac{3}{16} \right) ^x \qquad 2. \quad\dfrac{16}{7} \left(\dfrac{8}{49} \right) ^x   \) Try It \(\PageIndex{9b}\) Use exponent properties to describe the transformations needed to graph the function \(f(x) = 5 \cdot 3^{-4x+2}-6\) from a parent function. Answer \(f(x) = 5 \cdot 3^{-4x+2}-6\) = \(f(x) = 5 \cdot 3^{-4x} \cdot 3^{2}-6\) = \(f(x) = 5 \cdot 3^{2} \cdot (3^4)^{-x} -6\) = \(f(x) =45 \cdot 81^{-x} -6\) Parent function is now \(y = 81^x\) Reflect over the \(y\)-axis (\(x \rightarrow -x \)) Vertically stretch by a factor of 45 (\(y \rightarrow 45y \)) Shift down \(6\) units (\(y \rightarrow y-6 \)) Example \(\PageIndex{A}\) 1) What role does the horizontal asymptote of an exponential function play in telling us about the end behavior of the graph? 2) What is the advantage of knowing how to recognize transformations of the graph of a parent function algebraically? 3) Explore and discuss the graphs of \(F(x)=(b)^x\) and \(G(x)=\left (\frac{1}{b} \right)^x\). Then make a conjecture about the relationship between the graphs of the functions \(b^x\) and \(\left (\frac{1}{b} \right)^x\) for any real number \(b>0\). 4) Prove the conjecture made in the previous exercise. 5) Explore and discuss the graphs of \(f(x) = 4^x\), \(g(x)=4^{x-2}\), and \(h(x)=\left (\frac{1}{16} \right)4^x\). Then make a conjecture about the relationship between the graphs of the functions \(b^x\) and \(\left (\frac{1}{b^n} \right)b^x\) for any real number \(n\) and real number \(b>0\). 6) Prove the conjecture made in the previous exercise. For the following exercises, match each function with one of the graphs in Figure at the right. 7) \(f(x)=2(0.69)^x \\[3pt] \) 8) \(f(x)=2(1.28)^x \\[3pt] \) 9) \(f(x)=2(0.81)^x \\[3pt] \) 10) \(f(x)=4(1.28)^x \\[3pt] \) 11) \(f(x)=2(1.59)^x \\[3pt] \) 12) \(f(x)=4(0.69)^x\) For the following exercises, use the graphs shown in Figure at the right. All have the form \(f(x)=ab^x\). 13) Which graph has the largest value for \(b? \\[3pt] \) 14) Which graph has the smallest value for \(b? \\[3pt] \) 15) Which graph has the largest value for \(a? \\[3pt] \) 16) Which graph has the smallest value for \(a\)?\ Answers to odd exercises: 1. An asymptote is a line that the graph of a function approaches, as \(x\) either increases or decreases without bound. The horizontal asymptote of an exponential function tells us the limit of the function’s values as the independent variable gets either extremely large or extremely small. 3. The graph of \(G(x)=\left (\frac{1}{b} \right)^x\) is the refelction about the \(y\)-axis of the graph of \(F(x)=(b)^x\); For any real number \(b>0\) and function \(f(x)=(b)^x\) , the graph of \(\left (\frac{1}{b} \right)^x\) is the the reflection about the \(y\)-axis, \(F(-x)\).
5. \(g(x)\) and \(h(x)\) are the same graph and are a horizontal shift to the right of the graph of \(f(x)\); For any real number \(n\), real number \(b>0\), and function \(f(x)=b^x\), the graph of \(\left (\frac{1}{b^n} \right)b^x\) is the horizontal shift \(f(x-n)\). 7. B, 9. A, 11. E, 13.
D, 15. C B: Match Graphs With Equations Exercise \(\PageIndex{B}\) \(\bigstar \) In the following exercises, match the graphs to one of the following functions: a. \(2^{x}\) b. \(2^{x+1}\) c. \(2^{x-1}\) d. \(2^{x}+2\) e. \(2^{x}-2\) f. \(3^{x}\) 17.   18. 19. 20.
21.
22. \(\bigstar \) For the following exercises, match the exponential equation to the correct graph. a. \(y=4^{−x}\) b. \(y=3^{x−1}\) c. \(y=2^{x+1}\) d. \(y=(\frac{1}{2})^x+2\) e. \(y=−3^{−x}\) f. \(y=1−5^x\) 23) 24) 25) 26) 27) 28) Answers to odd exercises: 17. f,   19. a,   21. e;   23. d,   25.
b,   27. e C: Graph Basic Exponential Functions Exercise \(\PageIndex{C}\) \(\bigstar \) In the following exercises, graph each exponential function. 29.
\(f(x)=2^{x}\) 30. \(g(x)=3^{x}\) 31. \(f(x)=6^{x}\) 32. \(g(x)=7^{x}\) 33. \(f(x)=(1.5)^{x}\) 34. \(g(x)=(2.5)^{x}\) 35. \(f(x)=\left(\frac{1}{2}\right)^{x}\) 36. \(g(x)=\left(\frac{1}{3}\right)^{x}\) 37. \(f(x)=\left(\frac{1}{6}\right)^{x}\) 38. \(g(x)=\left(\frac{1}{7}\right)^{x}\) 39. \(f(x)=(0.4)^{x}\) 40. \(g(x)=(0.6)^{x}\) 41. \(f(x)=4^{x}\) 42. \(g(x)=5^{x}\) 43. \(f(x)=\left(\frac{1}{4}\right)^{x}\) 44. \(h(x)=\left(\frac{1}{5}\right)^{x}\) Answers to odd exercises: 29. \(f(x)=2^{x}\) 31.
\(f(x)=6^{x}\) 33. \(f(x)=(1.5)^{x}\) 35. \(f(x)=\left(\frac{1}{2}\right)^{x}\) 37. \(f(x)=\left(\frac{1}{6}\right)^{x}\) 39.
\(f(x)=(0.4)^{x}\) 41. \(f(x)=4^{x}\) 43. \(f(x)=\left(\frac{1}{4}\right)^{x}\)   D: Graph Shifts of Exponential Functions Exercise \(\PageIndex{D}\) \(\bigstar \) In the following exercises, use transformations to graph each exponential function. State the transformations that must be done to the parent function in order to obtain the graph. 45. \(g(x)=2^{x}+1\) 46.\(g(x)=2^{x}-1\) 47.
\(g(x)=2^{x-2}\) 48. \(g(x)=2^{x+2}\) 49. \(g(x)=3^{x}+2\) 50.
\(f(x)=3^{x}-6\) 51. \(g(x)=4^{x-1}\) 52. \(f(x)=4^{x+2}\) 53. \(f(x)=2^{x}+3\) 54. \(f(x)=2^{x}-3\) 55. \(f(x)=3^{x+2}\) 56. \(f(x)=3^{x+1}\) 57. \(h(x)=2^{x-3}-2\) 58. \(g(x)=3^{x-1}+3\) 59.
\(f(x)=3^{x+1}-4\) 60. \(h(x)=3^{x-2}+4\) 61. \(f(x)=4^{x}+2\) 62. \(f(x)=4^x−1\) 63. \(f(x)=5^{x+1}+2\) 64. \(f(x)=10^{x-4}+2\) Answers to odd exercises: 45. Up 1 \(g(x)=2^{x}+1\) 47. Right 2 \(g(x)=2^{x-2}\) 49. Up 2 \(g(x)=3^{x}+2\) 51.
Right 1 \(g(x)=4^{x-1}\) 53. Up 3 \(f(x)=2^{x}+3\) 55. Left 2 \(f(x)=3^{x+2}\) 57. Right 3, Down 2; Domain: \((-\infty, \infty)\); Range: \((-2, \infty)\) \(h(x)=2^{x-3}-2\) 59.
Left 1, Down 4; Domain: \((-\infty, \infty)\); Range: \((-4, \infty)\) \(f(x)=3^{x+1}-4\) 61. up 2; Domain: \((-\infty, \infty)\); Range: \((2, \infty)\) \(f(x)=4^{x}+2\) 63. Left 1, up 2 63.
\(f(x)=5^{x+1}+2\)   \(\bigstar \) In the following exercises, use transformations to graph each exponential function. State the transformations that must be done to the parent function in order to obtain the graph. 65. \(f(x)=\left(\frac{1}{2}\right)^{x-4}\) 66. \(f(x)=\left(\frac{1}{2}\right)^{x}-3\) 67. \(f(x)=\left(\frac{1}{4}\right)^{x}-2\) 68. \(h(x)=\left(\frac{1}{3}\right)^{x}+2\) 69. \(f(x)=e^{x}-3\) 70. \(f(x)=e^{x}+2\) 71.
\(f(x)=e^{x}+1\) 72. \(f(x)=e^{x-2}\) 73. \(f(x)=e^{x+1}\) 74. \(f(x)=e^{x-3}\) 75. \(f(x)=e^{x-2}+1\) 76. \(f(x)=e^{x+2}-1\) Answers to odd exercises: 65. Right 4 \(f(x)=\left(\frac{1}{2}\right)^{x-4}\) 67. Down 2; Domain: \((-\infty, \infty)\); Range: \((-2, \infty)\) \(f(x)=\left(\frac{1}{4}\right)^{x}-2\) 69. Down 3. Domain: \((-\infty, \infty)\); Range: \((-3, \infty)\) \(f(x)=e^{x}-3\) 71. Up 1 \(f(x)=e^{x}+1\) 73. Left 1; Domain: \((-\infty, \infty)\); Range: \((0, \infty)\) \(f(x)=e^{x+1}\) 75. Right 2, up 1; Domain: \((-\infty, \infty)\); Range: \((1, \infty)\) \(f(x)=e^{x-2}+1\) E: Graph Reflections and Stretches of Exponential Functions Exercise \(\PageIndex{E}\) \(\bigstar \) In the following exercises, graph each exponential function. State the transformations that must be done to the parent function in order to obtain the graph and state its domain, range, and horizontal asymptote. 77.
\(f(x)=-2^{x}\) 78. \(f(x)=2^{-x-1}-1\) 79. \(g(x)=2^{-x}-3\) 80. \(g(x)=3^{-x}+1\) 81. \(f(x)=6-10^{-x}\) 82. \(g(x)=5-4^{-x}\) 83. \(f(x)=5-2^{x}\) 84. \(f(x)=3-3^{x}\) 84.1 \(f(x) = 3 \cdot 2^{x-4}-5 \) 84.2 \(y = 5-\frac{1}{2} \cdot 4^{x-3} \) 84.3 \(f(x) = 3^{x/2+3}-4\) 84.4 \(f(x) = 5-2^{-x/3+1}\) 85) \(f(x)=1−2^{−x}\) 86) \(f(x)=2^{-x}\) 87. \(g(x)=-e^{x}\) 88. \(g(x)=e^{-x}\) 89. \(h(x)=-e^{x+1}\) 90. \(h(x)=-e^{x}+3\) 91) \(f(x)=e^{−x}−1\) \(\bigstar \) For the following exercises, graph each set of functions on the same axes. 93) \(f(x)=3\left (\frac{1}{4} \right)^x, g(x)=3(2)^x, h(x)=3(4)^x\) 94) \(f(x)=\frac{1}{4}(3)^x, g(x)=2(3)^x, h(x)=4(3)^x\) \(\bigstar \) For the following exercises, describe the end behavior of the graphs of the functions and the transformations needed to do on the parent graph to obtain the graph of the function. 95) \(f(x)=-5(4)^x-1\) 96) \(f(x)=3\left (\frac{1}{2} \right)^x-2\) 97) \(f(x)=3(4)^{-x}+2\) Answers to odd exercises: 77. Reflect over x-axis HA: y = 0 Domain: \((-\infty, \infty)\); Range: \((-\infty, 0)\) \(f(x)=-2^{x}\) 79. Reflect over y-axis, down 3; HA: y = -3 Domain: \((-\infty, \infty)\); Range: \((-3, \infty)\) \(g(x)=2^{-x}-3\) 81. Reflect over y-axis, reflect over x-axis, up 6; HA: y = 6; Domain: \((-\infty, \infty)\); Range: \((-\infty, 6)\) \(f(x)=6-10^{-x}\) 83. Reflect over x-axis, up 5; HA: y = 5; Domain: \((-\infty, \infty)\); Range: \((-\infty, 5)\) \(f(x)=5-2^{x}\) 85. Reflect over y-axis, repfect over x-axis up 1; HA: y=1; Domain: all real numbers, range: \((−∞,1) \) \(f(x)=1−2^{−x}\) 87. Reflect over x-axis; HA: y = 0; Domain: \((-\infty, \infty)\); Range: \((-\infty, 0)\) \(g(x)=-e^{x}\) 89. Left 1, Reflect over x-axis; HA: y=0; Domain: \((-\infty, \infty)\); Range: \((-\infty, 0)\) \(h(x)=-e^{x+1}\) 91. Reflect over y-axis, down 1; HA: y = -1; Domain: all real numbers, range: \((−1,∞)\) \(f(x)=e^{−x}−1\) 93.   95. As \(x\rightarrow \infty , f(x)\rightarrow -\infty\); As \(x\rightarrow -\infty , f(x)\rightarrow -1\); Reflect over x-axis, y's stretched by a factor of 5, down 1 97. As \(x\rightarrow \infty , f(x)\rightarrow 2\); As \(x\rightarrow -\infty , f(x)\rightarrow \infty\); Reflect over y-axis, y's get stretched by a factor of 3, up 2. F: Construct an Equation Given a Description Exercise \(\PageIndex{F}\) \(\bigstar \) For the following exercises, start with the graph of \(f(x)=4^x\). Then write a function that results from the given transformation. 98) Shift \(f(x)\) \(4\) units upward 99) Shift \(f(x)\) \(3\) units downward 100) Shift \(f(x)\) \(2\) units left 101) Shift \(f(x)\) \(5\) units right 102) Reflect \(f(x)\) about the \(x\)-axis 103) Reflect \(f(x)\) about the \(y\)-axis \(\bigstar \) For the following exercises, (a) graph the function, (b) graph its reflection about the \(y\)-axis and construct the function for the reflected graph, and (c) give the \(y\)-intercept. 104) \(f(x)=3\left (\frac{1}{2} \right)^x\) 105) \(g(x)=-2(0.25)^x\) 106) \(h(x)=6(1.75)^{-x}\) \(\bigstar \) For the following exercises, (a) graph the function, (b) graph its reflection about the \(x\)-axis and construct the function for the reflected graph. 107) \(f(x)=\frac{1}{2}(4)^x\) 108) \(f(x)=3(0.75)^x-1\) 109) \(f(x)=-4(2)^x+2\) \(\bigstar \) For the following exercises, (a) construct the equation of the exponential function given a description. (b) State its \(y\)-intercept, domain, and range.
111) The graph of \(f(x)=10^x\) is reflected about the \(x\)-axis and shifted upward \(7\) units. What is the equation of the new function, \(g(x)\)? State its \(y\)-intercept, domain, and range. 112) The graph of \(f(x)=\left (\frac{1}{2} \right)^{-x}\) is reflected about the \(y\)-axis and compressed vertically by a factor of \(\left (\frac{1}{5} \right)\)
What is the equation of the new function, \(g(x)\) State its \(y\)-intercept, domain, and range. 113) The graph of \(f(x) = 3^x\) is reflected about the \(y\)-axis and stretched vertically by a factor of \(4\).
What is the equation of the new function, \(g(x)\). State its \(y\)-intercept, domain, and range. 114) The graph of \(f(x)=(1.68)^x\) is shifted right \(3\) units, stretched vertically by a factor of \(2\), reflected about the \(x\)-axis, and then shifted downward \(3\) units. What is the equation of the new function, \(g(x)\)? State its \(y\)-intercept (to the nearest thousandth), domain, and range. 115) The graph of \(f(x)=2\left (\frac{1}{4} \right)^{x-1}\) is shifted left \(2\) units, stretched vertically by a factor of \(4\), reflected about the \(x\)-axis, and then shifted downward \(4\) units.
What is the equation of the new function, \(g(x)\)? State the y-intercept, domain, and range of \(g(x)\). Answers to odd exercises: 99. \(f(x)=4^x-3\) \(\qquad\) 101. \(f(x)=4^{x-5}\) \(\qquad\) 103.
\(f(x)=4^{-x}\) 105. \(y\)-intercept: \((0,-2)\) 107 109 111. \(g(x)=-10^x + 7\); \(y\)-intercept: \((0,6)\); Domain: all real numbers; Range: all real numbers less than \(7\). 113. \(g(x)=4(3)^{-x}\) \((0,4)\) Domain: all real numbers; Range: all real numbers greater than \(0\). 115.
\(g(x)=-8 \left (\frac{1}{4} \right)^{x+1} -4\); \(y\)-intercept: \((0,-6)\);   Domain: \((-\infty, \infty)\); Range: \((-\infty, -4)\) G: Construct an Equation Given a Graph Exercise \(\PageIndex{G}\) \(\bigstar \) Write an equation for each of the functions graphed below. The graphs #117-123 are transformations of the parent function \(f(x)=2^x\). The parent functions for #124-127 must be determined. 117. 118. 119. 120. 121. 122. 123. 124. 125. 126. 127. (Hint: this graph has been stretched) Answer to odd exercises: 117. \(y=-2^x+3\) 119. \(y=2^x+3\) 121. \(y=-2^{-x+1}+3\) 123. \(y=2^{-x+1}-4\) 125. \(y=4^x-1\) 127. \(y=-2(3)^x+7\) H: Practice with Exponent Properties Exercise \(\PageIndex{H}\) \(\bigstar \) Write the following with only one exponent, \(x\).
Simplify. \(2^{x+3} \) \(7^{2+x} \) \(4^{x-3} \) \(9^{x-2} \) \(4^{4-x} \) \(6^{2-x} \) \(5^{2x} \) \(4^ {2x} \) \(5^x \cdot 7^x \) \(9^x \cdot 4^x \) \(2^x \cdot 3^{2x} \) \(x^2 \cdot y^2 \) \(2^{3x+2} \) \(3^{2x+3} \) \(4^{3x-2} \) \(2^{5x-3} \) \(6^{x+1} \cdot 4^{x-2} \) \(8^{2x+1} \cdot 7^{x-1} \) \(\dfrac{ 3^x}{5^x} \\[3pt] \) \(\dfrac{ 4^x}{w^x} \\[3pt] \) \(\dfrac{ 2^{4x}}{8^x} \) \(\dfrac{ 9^x}{6^{2x}} \\[3pt] \) \(\dfrac{ 3^{x+1}}{2^{ 2x-3 }} \\[3pt] \) \(\dfrac{ 8^{x-1}}{4^{3x+2}} \) \(\dfrac{ 6^{x+2}}{4^{2x+3}} \\[3pt] \) \(\dfrac{ 5^{2x-1}}{3^{3x-2}} \) Answers to odd exercises: 131. \(8(2^x) \quad \) 133. \(\dfrac{4^x}{64} \quad \) 135. \(\dfrac{256}{4^x} = 256 \left(\dfrac{1}{4} \right)^x \quad \) 137. \(25^x \quad \) 139. \(35^x \) \(\quad\) 141. \(18^x \quad \) 143. \(4(8^x) \) 145. \(\dfrac{64^x}{16} \quad \) 147. \(\dfrac{3}{8}24^x \quad \) 149. \(\left(\dfrac{3}{5} \right)^x\) \(\quad\) 151. \(2^x \quad \) 153. \(24 \left(\dfrac{3}{4} \right)^x \quad \) 155.   \(\dfrac{9}{16} \left(\dfrac{3}{8} \right)^x \)