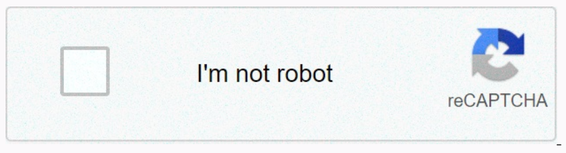
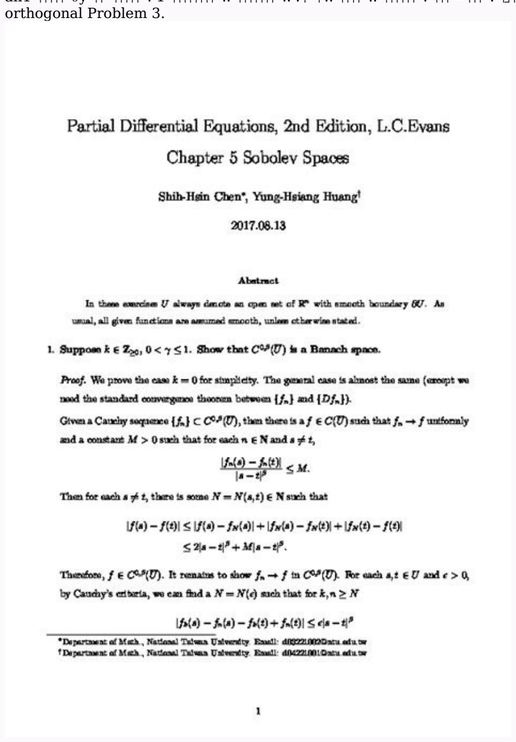


Continue

Evans pde solutions chapter 3

Homework 1 Homework 2 Homework 3 Homework 4 Embed Size (px) 344 x 292 429 x 357 514 x 422 599 x 487 Final Exam Topics The final exam for M611 will be in the usual classroom ZACH 322, Wednesday, Dec. 17,10:30 a.m.-12:30 p.m. The exam will cover the course material that was not covered on themidterm exam; in particular, Green's functions for Laplace's equation, and all of our materialon the heat equation, the wave equation, and the method of characteristics. The exam willconsist of four to six questions, with a mix of straightforward calculations and proofs. castle crashers free download pc full version I willprovide statements of the system of characteristic equations and of the d'Alembert, Poisson,and Kirchhoff solutions to the wave equation.

You will need to bring your own paper. 1. Explicit calculations Examples of problems we've solved by explicit calculation include the following: • The derivation of Green's functions (by the method of images), and the application ofGreen's functions to solving Laplace's equation • Solutions of the wave equation on quarter planes • Solutions to the inhomogeneous wave equation by Duhamel's Principle • Solutions of first order linear, quasilinear and nonlinear PDE by the method of char-acteristics 2. Proofs Examples of topics from which proof-based problems may be taken include the following: • Verify properties of the heat kernel and properties of solutions to the heat equation onRn × R+ • Verify properties of the heat equation on UT • Derive properties of solutions of the wave equation, especially (though not only) in casesin which the properties can be obtained from the d'Alembert, Poisson, and Kirchhoffintegral representations • Know the properties of the straightening transformation ~Φ and its role in our develop-ment of the Method of Characteristics 1 Practice with the method of characteristics 1. (Evans 3.5.5.) Solve using characteristics: a. x1ux1 + x2ux2 = 2u; u(x1, 1) = g(x1) c. x1ux1 + 2x2ux2 + ux3 = 3u; u(x1, x2, 0) = g(x1, x2) b. uux1 + ux2 = 1; u(x1, x1) = 12x1 Solution. For (a), we begin by setting z(s) = u(x1(s), x2(s)). We computedz ds= ux1(~x(s)) dx1 ds+ ux2(~x(s)) dx2 ds, so that dx1 ds=x1; x1(0) = x10 ⇒ x1(s) = x10e s dx2 ds=x2; x2(0) = 1⇒ x2(s) = es dz ds= 2z; z(0) = g(x10)⇒ z(s) = g(x10)e 2s. Solving for x10 and s in terms of x1 and x2 we find x10 =x1 x2 es =x2, so thatu(x1, x2) = g(x1x2)x22.
For (b), we set dx1 ds=x1; x1(0) = x10 ⇒ x1(s) = x10e s dx2 ds= 2x2; x2(0) = x20 ⇒ x2(s) = x20e 2s dx3 ds= 1; x3(0) = 0⇒ x3(s) = s dz ds= 3z; z(0) = g(x10, x 20)⇒ z(s) = g(x10, x 20)e 3s. 73777763898.pdf We see that s =x3 x10 =x1e−x3 x20 =x2e−2x3 , 2 so thatu(~x) = g(x1e −x3 , x2e−2x3)e3x3 . For (c), we set dx1 ds= z(s); x1(0) = x10 ⇒ x1(s) = x10 + ˆ s 0 z(τ)dτ dx2 ds= 1; x2(0) = x10 ⇒ x2(s) = x10 + s dz ds= 1; z(0) = 1 2x10 ⇒ z(s) = 1 2x10 + s.
We see that x1(s) = x10 +s 2x10 + s2 2. Now we notice x10 = x2 − s, so that x1 = (x2 − s) +s 2(x2 − s) + s2 2= x2 − s+ s 2x2. Solving for s s =x2 − x11− 1 2x2. This in turn gives x10 =x2 − s =x2 − 1 2x22 − (x2 − x1)1− 1 2x2 =x1 − 1 2x22 1− 12x2 . We conclude u(x1, x2) =1 2x10 + s =12x1 − 1 4x22 + x2 − x1 1− 12x2 =−1 2x1 − 1 4x22 + x2 1− 12x2 . 2. Find a solution to the PDE x1ux1 + x2ux2 +u2x1 + u2x2 2=u in R× R+ u(x1, 0) =1− x21 2, x1 ∈ R. 3 (Solutions to nonlinear PDE need not be unique; in particular, you need only find onesolution.) Solution. We begin by observing F (~p, z, ~x) = x1p1 + x2p2 +1 2p21 + 1 2p22 − z, so that DpF = (x1 + p1, x2 + p2) Fz = − 1 DxF = (p1, p2). We haveux1(x1, 0) = −x1 ⇒ p10 = −x10. For p20, our equation gives the relation x10p10 + x20p 20 + (p10)2 + (p20) 2 2= 1 2− 1 2(x10) 2, which (upon noting x20 = 0) gives the equation 1 2(p20) 2 =1 2⇒ p20 = ±1. This is where the uniqueness comment comes in, and we choose to find the solution for whichp20 = +1. glidermatic grd instruction manual We obtain the system dp1 ds= 0; p1(0) = −x10 dp2 ds= 0; p2(0) = 1 dz ds= z + 1 2(p1)2 + 1 2(p2)2; z(0) = 1 2− 1 2(x10) 2 dx1 ds=x1 + p1; x1(0) = x10 dx2 ds=x2 + p2; x2(0) = 0. Solving this, we find p1(s) = − x10p2(s) = 1 z(s) = es − 1 2 ((x10) 2 + 1) x1(s) =x10x2(s) = − 1 + es. 4 We see that x10 =x1 es = 1 + x2, allowing us to write u(x1, x2) = z(s) = (1 + x2)−1 2 (x21 + 1)= −1 2x21 + x2 + 1 2. 5 (a) Try to use Jacobi identity: $$\frac{d}{ds}\det A(s)=\text{tr}\left((\text{cof}A(s))\frac{dA}{ds}(s)\right)$$ Let $A(s)=\left(X_{x_j}^{i}(s,x,t)\right)$ and $B(s)=\left(b_{x_j}^{i}(s)\right)$. We yield $$\frac{dA}{ds}(s)=B(X)A(s)$$ Plug in the identity, done. (b) Using method of characteristics is fine, $\dot{x}(t)=b$, $x(0)=y$, $\dot{z}(t)=-\text{div}bz^{-1}$, $z^{-1}(0)=g^{-1}(y)$. Note that by assumption of (b) and Jacobi identity in (a), these ODEs have unique solutions. Moreover, $$y=X(-t,x,0), X\left(s,X(-t,x,0),0\right)=X\left(X(s-t,x,0),0\right)$$ $$J(-t,x,0)=J(0,x,t)$$ $$z(t)=\frac{g(y)}{J(t,y,0)}$$ By Euler formula, $$J(t,y,0)=\exp\left(\int_{0}^{t}\text{div}b(X(s,y,0))ds\right)=\exp\left(\text{div}\left(X(s,X(-t,x,0),0)\right)ds\right)=\exp\left(\int_{0}^{t}\text{div}b(X(s-t,x,0))ds\right)=\exp\left(-\int_{-t}^{0}\text{div}b(X(\tau,x,0))d\tau\right)=J^{-1}(-t,x,0)=J^{-1}(0,x,t)$$ This suggests $$u(x,t)=g(X(0,x,t))J(0,x,t)$$ Author / Uploaded i'm not telling Authors: Joe Benson, Denis Bashkirov, Minsu Kim, Helen Li, Alex Csar Evans PDE Solutions, Chapter 2 Joe: 1, 2,11; Denis: 4, 6, 14, 18; Minsu: 2,3, 15; Helen: 5,8,13,17.
Alex:10, 16 Problem 1. Write down an explicit formula for a function u solving the initial-value problem (ut + b · Du + cu = 0 on Rn × (0, ∞) u = g on Rn × {t = 0} Here c ∈ R and b ∈ Rn are constants. Sol: Fix x and t, and consider z(s) := u(x + bs, t + s) Then z˙(s) = b · Du + ut = −cu(x + bs, t + s) = −cz(s) Therefore, z(s) = De−cs , for some constant D. We can solve for D by letting s = −t. Then, z(−t) = u(x − bt, 0) = g(x − bt) = Dect i.e. D = g(x − bt)e−ct Thus, u(x + bs, t + s) = g(x − bt)e−c(t+s) and so when s = 0, we get u(x, t) = g(x − bt)e−ct . Problem 2. Prove that Laplace's equation Δu = 0 is rotation invariant; that is, if O is an orthogonal n × n matrix and we define v(x) := u(Ox) (x ∈ R) then Δv = 0. Solution: Let y := Ox, and write O = (ai j). Thus, v(x) = u(Ox) = u(y) where y j = Pn i=1 a ji xi . This then gives that n ∂v X ∂u ∂y j = ∂xi ∂y j ∂xi j=1 1 n X ∂u = a ji ∂y j j=1 1 2 Thus, ⌈ ∂v ⌉⌈ ⌉⌈ ∂u ⌉ ⌈⌈⌈ ∂x1 ⌈⌈⌈⌈a11 . . . an1 ⌈⌈⌈⌈⌈ ∂y ⌈⌈ ⌈⌈⌈⌈ . 1 ⌈⌈⌈⌈⌈⌈⌈ .. ⌈⌈⌈⌈⌈⌈ .. . ⌈ ⌉⌈ .. ⌈⌈⌈⌈ .. ⌈⌈⌈⌈⌈ . ⌈⌈⌈ = ⌈⌈⌈ . ⌋⌈⌈⌈⌊ ⌈⌋⌊⌊⌊ ∂v ⌋ ∂u a . . . a 1n nn ∂xn ∂yn⌈ ∂u ⌉⌈⌈ ∂y1 ⌈⌈⌈⌈ ⌈⌈T⌈ = O⌈⌈⌈ ... ⌈⌈⌈⌈⌈⌊ ∂u ⌋ ∂yn D x · v = O Dy · u T Now, Δv = D x v · D x v = (OT Dy u) · (OT Dy u) = (OT Dy u)T OT Dy u = (Dy u)T (OT)T OT Dy u = (Dy u)T OOT Dy u = (Dy u)T Dy u = (Dy u) · (Dy u) = Δu(y) =0 because O is orthogonal Problem 3.


Partial Differential Equations, 2nd Edition, L.C.Evans

Chapter 5 Sobolev Spaces

Shih-Hsin Chen*, Yung-Hsiang Huang†

2017.08.13

Abstract

In these exercises U always denote an open set of $\mathbb{R}^n$ with smooth boundary ∂U. As usual, all given functions are assumed smooth, unless otherwise stated.

1. Suppose $k \in \mathbf{Z}_{\geq 0}$, $0 < \gamma \leq 1$. Show that $C^{0,\gamma}(\overline{U})$ is a Banach space.

Proof. We prove the case $k = 0$ for simplicity. The general case is almost the same (except we need the standard convergence theorem between $\{f_n\}$ and $\{Df_n\}$).

Given a Cauchy sequence $\{f_n\} \subset C^{0,\gamma}(\overline{U})$, then there is a $f \in C(\overline{U})$ such that $f_n \to f$ uniformly and a constant $M > 0$ such that for each $n \in \mathbf{N}$ and $s \neq t$,

$$\frac{|f_n(s) - f_n(t)|}{|s-t|^\gamma} \leq M.$$

Then for each $s \neq t$, there is some $N = N(s,t) \in \mathbf{N}$ such that

$$\begin{aligned} |f(s) - f(t)| &\leq |f(s) - f_N(s)| + |f_N(s) - f_N(t)| + |f_N(t) - f(t)| \\ &\leq 2|s-t|^\gamma + M|s-t|^\gamma. \end{aligned}$$

Therefore, $f \in C^{0,\gamma}(\overline{U})$. It remains to show $f_n \to f$ in $C^{0,\gamma}(\overline{U})$. For each $s,t \in U$ and $\epsilon > 0$, by Cauchy's criteria, we can find a $N = N(\epsilon)$ such that for $k, n \geq N$

$$|f_k(s) - f_n(s) - f_k(t) + f_n(t)| \leq \epsilon|s-t|^\gamma$$

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Modify the proof of the mean value formulas to show for n ≥ 3 that Z Z 1 1 1 1 gdS + − f dx, u(0) = nα(n)rn−1 ∂B(0,r) n(n − 2)α(n) B(0,r) |x|n−2 rn−2 provided ⌈ ⌋ ⌊ ⌉ ⌊−Δu = f ⌋ ⌊ ⌉ ⌊ u=g in B0 (0, r) on ∂B(0, r). Solution: Set 1 φ(t) = nα(n)tn−1 Z u(y)dS (y), 0 ≤ t < r, ∂B(0,t) and 1 φ(r) = nα(n)rn−1 Z 1 u(y)dS (y) = nα(n)rn−1 ∂B(0,r) Z gdS . ∂B(0,r) Then, t 1 φ0(t) = n α(n)tn (See the proof of Thm2) Z Z t 1 Z −1 Δu(y)dy = − f dy = f dy. n α(n)tn B(0,t) α(n)tn−1 B(0,t) B(0,t) 3 Let > 0 be given. φ() = φ(r) − (1) r Z 1 φ0(t)dt = nα(n)rn−1 Z Z gdS − ∂B(0,r) r φ0(t)dt. Using integration by parts, we compute Z r Z r Z 1 − φ0(t)dt = f dydt n−1 nα(n)t B(0,t) Z r Z 1 1 f dydt = nα(n) tn−1 B(0,t) Z r Z r 1 1 Z 1 1 f dy − f dS dt = n−2 nα(n) 2 − n tn−2 B(0,t) 2−nt ∂B(0,t) Z Z Z Z r 1 1 1 = f dS dt − f dy + f dy n(n − 2)α(n) tn−2 ∂B(0,t) rn−2 B(0,r) n−2 B(0,) Z 1 1 =: I − n−2 f dy + J . n(n − 2)α(n) r B(0,r) Observe that J: and Z 1 n−2 Z f dy ≤ C · 2 , for some constant C > 0 B(0,) B(0,) 1 f (x)dx = |x|n−2 Z r Z dt 0 ∂B(0,t) 1 tn−2 f dS . R As → 0, I + J → B(0,) |x|1n−2 f (x)dx. Thus, Z Z r Z 1 1 1 f dy lim − φ0(t)dt = f (x)dx − →0 n(n − 2)α(n) B(0,r) |x|n−2 rn−2 B(0,r) Z 1 1 1 = − f dx. n(n − 2)α(n) B(0,r) |x|n−2 rn−2 Therefore, letting → 0, we have from (1) Z Z 1 1 1 1 u(0) = φ(0) = gdS + − f dx. nα(n)rn−1 ∂B(0,r) n(n − 2)α(n) B(0,r) |x|n−2 rn−2 ¯ is subharmonic if Problem 4. We say v ∈ C 2 (U) −Δv ≤ 0 (a) Prove for subharmonic v that in U. ? v(x) ≤ v dy for all B(x, r) ⊂ U. B(x,r) (b) Prove that therefore maxU¯ v = max∂U v. (c) Let φ : R → R be smooth and convex. Assume u is harmonic and v := φ(u). Prove v is subharmonic. 4 (d) Prove v := |Du|2 is subharmonic, whenever u is harmonic. Solution. > (a) As in the proof of Theorem 2, set φ(r) := ∂B(x,r) v dS (y) and obtain ? r 0 φ (r) = Δv(y)dy ≥ 0.

.1

MATH 6662: Differential Equations I **Fall 2018**

Lecture 4.1: Separation of Variables

Disclaimer: *These notes are based on the textbook* Partial Differential Equations *by Lawrence C. Evans. Do not distribute without permission.*

4.1 General Overview

When using this method called "separation of variables" the goal is to construct a solution u to a given partial differential equation as a combination of functions of fewer variables. Or in other words we try to guess that that u can be written as a sum or product of undetermined constituent functions then to plug this guess into the PDE and choose the simpler functions to ensure that u is really a solution.

Example 4.1 *This is the first Example of the lecture.*
Let $U \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary. Consider the initial/boundary-value problem for the heat equation

$$(1) \begin{cases} u_t - \Delta u = 0 & \text{in } U \times (0, \infty) \\ u = 0 & \text{on } \partial U \times [0, \infty) \\ u = g & \text{on} U \times \{t = 0\} \end{cases}$$

where $g : U \to \mathbb{R}$ is given. We conjecture there exists a solution having the multiplicative form

$$u(x,t) = v(t)w(x) \ (x \in U, t \geq 0)$$

Proof: That is, we look for a solution with the variables $x = (x_1, \ldots, x_n) \in U$ to be "separated" from the variable $t \in [0, T]$. The question then is will this work? to find out we compute

$$(2) \ \ u_t(x,t) = v'(t)w(x), \ \Delta u(x,t) = v(t)\Delta w(x)$$

Hence

$$0 = u_t(x,t) - \Delta u(x,t) = v'(t)w(x) - v(t)\Delta w(x)$$

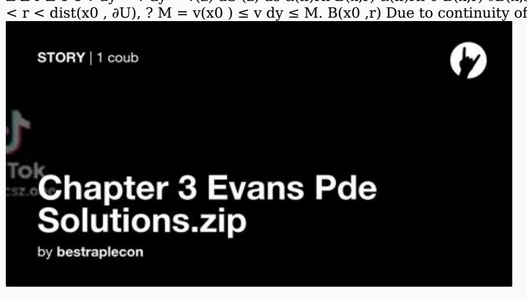
if and only if

$$(3) \ \ \frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)}$$

for all $x \in U$ and $t > 0$ such that $w(x), v(t) \neq 0$. Now we can observe that the LHS of (3) depends only on t and the RHS depends on x. This is only possible if each is constant as below

$$\frac{v'(t)}{v(t)} = \mu = \frac{\Delta w(x)}{w(x)} \ (t \geq 0, x \in U)$$

n B(x,r) For 0 < < r, r Z φ0 (s)ds = φ(r) − φ() ≥ 0. Hence, φ(r) ≥ lim φ() = v(x). Therefore, ¬0 ! ?
Z Z r Z 1 1 v dy = v dy = v(z) dS (z) ds α(n)rn B(x,r) α(n)rn 0 B(x,r) ∂B(x,s) Z r Z 1 1 r n−1 n−1 = nα(n)s φ(s) ds ≥ n ns v(x) ds = v(x) α(n)rn 0 r 0 (b) We assume that U ⊂ Rn is open and bounded. For a moment, we assume also that U is connected. Suppose that x0 ∈ U is such a point that v(x0) = M := maxU¯ v. bmw_e90_owners_manual.pdf Then for 0 < r < dist(x0 , ∂U), ? M = v(x0) ≤ v dy ≤ M. B(x0 ,r) Due to continuity of v, an equality holds only if v ≡ M within B(x0 , r). Therefore, the set u−1 ({M}) ∩ U = {x ∈ U|u(x) = M} is both open and relatively closed in U. By the connectedness of U, v is constant within the set U. Hence, it is constant within U¯ and we conclude that maxU¯ v = max∂U v.



Now let {Ui |i ∈ I} be the connected components of U. Pick any x ∈ U and find j ∈ I such that x ∈ U j . We obtain v(x) ≤ max v = max v ≤ max v U¯ j ∂U j ∂U and conclude that maxU¯ v = max∂U v. (c) For x = (x1 , ..., xn) ∈ U and 1 ≤ i, j ≤ n, ∂2 v ∂2 ∂u ∂u ∂2 u (x) = φ(u(x)) = φ00 (u(x)) · (x) · (x) + φ0 (u(x)) · (x). ∂xi ∂x j ∂xi ∂x j ∂xi ∂x j ∂xi ∂x j Since φ is convex, then φ00 (x) ≥ 0 for any x ∈ R. Recall that u is harmonic and obtain !2 !2 n n X X ∂u ∂u 00 00 Δv = φ (u) · + Δu = φ (u) · ≥ 0. ∂xi ∂xi i=1 i=1 n P ∂u 2 . For x = (x1 , ..., xn) ∈ U and 1 ≤ i, j ≤ n, (d) We set v := |Du|2 = ∂xk k=1 # n " X ∂2 v ∂2 u ∂2 u ∂u ∂u ∂3 u (x) = 2 (x) · (x) + (x) · (x) . ∂xi ∂x j ∂x ∂x ∂x ∂x ∂x ∂x ∂x i k i j k i j k k=1 5 Therefore, ⌈ ! n X ⌈⌈⌈ ∂2 u 2 ∂u ∂ ∂2 v ⌈⌈⌊ =2 + · ∂xi 2 ∂xi ∂xk ∂xk ∂xk k=1 !2 X n X ∂u ∂2 u + · Δv = 2 ∂x ∂x ∂x i k k k=1 1≤i,k≤n ∂2 u ∂xi 2 !⌉ ⌈⌈⌈ ⌈⌋ , !2 X ∂ ∂2 u ≥ 0. Δu = 2 ∂xk ∂x ∂x i k 1≤i,k≤n Problem 5: Prove that there exists a constant C, depending only on n, such that ! max |u| ≤ C max |g| + max | f | ∂B(0,1) B(0,1) whenever u is a smooth solution of ⌈ ⌉ ⌊ ⌉ ⌊− 4 u = f ⌉ ⌉ ⌊u = g B(0,1) in B0 (0, 1) on ∂B(0, 1). Proof: Let M := maxB(0,1) | f |, then we define v(x) = u(x) + first consider v(x) . Note that M |x|2 2n and w(x) = −u(x) + M |x|2 . 2n We − 4 v = − 4 u − M = f − M ≤ 0. So, v(x) is a subharmonic funcion. From Problem 4 (b), we have max v(x) = max v(x) ≤ max |g| + B(0,1) ∂B(0,1) ∂B(0,1) M . 2n That is max u(x) ≤ max v(x) ≤ max |g| + B(0,1) B(0,1) ∂B(0,1) 1 max | f |.

2n B(0,1) Then, for w(x), we have − 4 w = 4u − M = − f − M ≤ 0.
Again, we can get max w(x) = max w(x) ≤ max |g| + B(0,1) ∂B(0,1) ∂B(0,1) M . nutoxadukarazegit.pdf
2n i.e. 1 max | f |. B(0,1) B(0,1) ∂B(0,1) 2n B(0,1) Combining these two together, we finally proved the problem. max −u(x) ≤ max w(x) ≤ max |g| + Problem 6. Use Poisson's formula for the ball to prove r − |x| r + |x| rn−2 u(0) ≤ u(x) ≤ rn−2 u(0) n−1 (r + |x|) (r − |x|)n−1 whenever u is positive and harmonic in B0 (0, r). This is an explicit form of Harnack's inequality. 6 Solution. Since y ∈ ∂B(0, r), then |x − y| ≤ |x| + r. Therefore, Z r2 − |x|2 g(y) u(x) = dS (y) nα(n)r ∂B(0,r) |x − y|n Z Z g(y) 1 r2 − |x|2 r − |x| n−2 dS (y) = r · g(y)dS (y) ≥ nα(n)r ∂B(0,r) (r + |x|)n (r + |x|)n−1 nα(n)rn−1 ∂B(0,r) ? r − |x| r − |x| n−2 =r g(y)dS (y) = rn−2 u(0) n−1 (r + |x|) (r + |x|)n−1 ∂B(0,r) r+|x| The inequality u(x) ≤ rn−2 (r−|x|) n−1 u(0) can be proven in a similar way. Problem 7. Prove Poisson's formula for a ball: Assume g ∈ C(∂B(0, r)) and let Z r 2 − x2 g(y) u(x) = dS (y) for x ∈ B0 (0, r). nα(n)r ∂B(0,r) |x − y|n Show that Proof. Problem 8. Let u be the solution of ⎧ ⎪ ⎨4u = 0 ⎪ ⎩u = g in Rn+ on ∂Rn+ given by Poisson's formula for the half-space. Assume g is bounded and g(x) = |x| for x ∈ ∂Rn+ , |x| le1. Show Du is not bounded near x = 0. (Hint: Estimate u(λen)−u(0) .) Proof: From formula (33) on page 37, we have 2xn u(x) = nα(n) Z ∂Rn+ g(y) dy, |x − y|n and u(0) = g(0) = 0. Thus, using hint, we get Z u(λen) − u(0) 2 g(y) dy = n λ nα(n) ∂R+ |λen − y|n Z Z 2 g(y) 2 g(y) dy + dy = n T T nα(n) |y|≤1 ∂Rn+ |λen − y| nα(n) |y|>1 ∂Rn+ |λen − y|n Taking absolute value on both sides, we have Z u(λen) − u(0) 2 Z g(y) 2 |g(y)| ≥ dy − dy λ nα(n) |y|≤1 T ∂Rn+ |λen − y|n nα(n) |y|>1 T ∂Rn+ |λen − y|n =I1 − I2 . johnson and johnson swot analysis pdf 7 Since g is bounded, so it is obvious that I2 is bounded and independent of λ.
For I1 , in this case, g(y) = |y|, so Z 2 |y| I1 = dy nα(n) |y|≤1 T ∂Rn+ |λen − y|n Z 2 |y| ≥ dy T nα(n) |y|≤1 ∂Rn+ (λ + |y|)n Note that for fixed y, theorem, we have |y| (λ+|y|)n is increasing when λ is decreasing to 0, so by Monotone Convergence Z 2 |y| dy lim λ→0 nα(n) |y|≤1 T ∂Rn+ (λ + |y|)n + Z |y| = dy n T |y|≤1 ∂Rn+ |y| Z |y| = dy n Bn−1 (0,1) |y| Z 1 Z 1 Z 1 n−2 1 dS (y) = C r dr = ∞. = dr n−1 n−1 0 r 0 ∂Bn−1 (0,r) |y| So, Du is unbounded near x = 0. Problem 10. Suppose u is smootha nd solves ut − Δu = 0 in Rn × (0, ∞). (i) Show uλ (x, t) := u(λx, λ2 t) also solves the heat equation for each λ ∈ R. (ii) Use (i) to show v(x, t) := x · Du(x, t) + 2tut (x, t) solves the heat equation as well. (i) uλt (x, t) = λ2 ut (λx, λ2 t) and uλxi (x, t) = λu(λx, λ2 t) for each i. Then uλxi xi (x, t) = λ2 u xi (λx, λ2 t). Consequently, Δuλ = λ2 Δu and uλt − Δuλ = λ2 (ut − Δu), so uλ solves the heat equation for all λ ∈ R. 18930332195.pdf (ii) We differentiate u(λx, λ2 t) = u(λx1 , . . . , λxk , λ2 t) + 2λtut (λx1 , . . . , λxn , λ2 t) = x · D(λx, λ2 t) + 2tut (λx, λ2 t). k Taking λ = 1, we then have that v(x, t) = x · Du(x, t) + 2tut (x, t). u is smooth, so the second derivatives of u(λx, λ2 t) are continuous, meaning the mixed partials are equal. Therefore, ∂ ∂ ∂ ∂ u(λx, λ2 t) − Δ ∂λ u(λx, λ2 t) = ∂λ∂t u(λx, λ2 t) − ∂λ Δu(λx, λ2 t) = ∂λ (uλt − Δuλ) = 0, vt − Δv = ∂t∂λ since uλ satisfies the heat equation for all λ.
Thus v does as well. 2 Problem 11: Assume n = 1 and u(x, t) = v(xt). a) Show ut = u xx if and only if (2) 4zv″(z) + (2 + z)v0 (z) = 0 (z > 0) 8 b) Show that the general solution of (1) is v(z) = c z Z e−s/4 s−1/2 ds + d 0 2 c) Differentiate v(xt) with respect to x and select the constant c properly, so as to obtain the fundamental solution Φ for n = 1.
Solution: a) Assume that ut = u xx .
Then x2 x2 ut = − 2 v0 t t ! and ! ! 2 x2 2 00 x u xx = 2v + 4x v t t 0 So ut = u xx implies that ! ! ! 2 2 x2 0 x2 0 x 2 00 x − 2v = 2v + 4x v t t t t or ! ! ! 2 x2 0 x2 4x2 00 x2 + + 2 v =0 v t2 t t t t If we let z = x2 , t we get ! 2 z 0 4z 00 v (z) + + v (z) = 0 t t t Multiplying this equation by t gives the desired equality. For the other direction, reverse the steps, and hence our proof is done. b) 4zv00 + (2 + z)v0 = 0 ⇐⇒ ⇐⇒ (by integrating) v00 11 1 = − − v0 2z 4 log(v0) = − log √ z− z +c 4 ⇐⇒ v0 = Cz−1/2 e−z/4 ⇐⇒ v=C z Z e−s/4 s−1/2 ds + d 0 9 as is desired. c) v(z) = c Z z e−s/4 s−1/2 ds + d 0 ⇐⇒ ! Z xt2 x2 v e−s/4 s−1/2 ds + d =c t 0 ⇐⇒ ! !−1/2 x2 2x − x4t2 x2 v =c e t t t or ! 2 2c x2 0 x v = √ e− 4t t t Now we want to integrate over R and set the integral equal to 1. Thus we get Z 2c ∞ − x4t2 1= √ e dx t ∞ 0 Letting y = we get dy = (4t)−1/2 dx and substituting, we get Z 2c ∞ √ −y2 1= √ 4te dy t ∞ or Z ∞ 2 1 = 4c e−y dy ∞ R∞ 2 √ −y Employing the identity ∞ e dy = π and solving for c, we get √x , 4t c= 1 √ 4 π Thus, ! x2 Φ(x, t) : = v t 2c x2 = √ e− 4t t 1 −x4t2 = √ e 2 πt 0 is easily shown to solve the equation Φt = Φ xx Problem 12. haynes pdf repair manuals free Write down an explicit formula for a solution of ⎧ ⎪ ⎨ ⎪ ⎩ in Rn x(0, ∞) ⎧ut − Δu + cu = f ⎪ ⎨ ⎪ ⎩ u = g on Rn x{t = 0}, where c∈ R. 10 Solution: Set v(x, t) = u(x, t)eCt . Then, vt = ut eCt + CeCt u and v xi xi = u xi xi eCt . aldi annual report 2020 pdf ⇒ vt − Δv = ut eCt + CeCt u − eCt Δu = eCt (ut − Δu + Cu) = eCt f. So, v is a solution of ⎧ ⎪ ⎨ ⎪ ⎩vt − Δv = eCt f ⎪ ⎨ ⎪ ⎩ v=g in Rn x(0, ∞) on Rn x{t = 0}, By (17) (p.51), v(x, t) = Z Φ(x − y, t)g(y)dy + Rn Z tZ Φ(x − y, t − s)eCs f (y, s)dyds Rn 0 where Φ is the fundamental solution of the hear equation. Since v(x, t) = u(x, t)eCt , we have Z tZ Z Ct u(x, t) = e Φ(x − y, t)g(y)dy + Φ(x − y, t − s)eCs f (y, s)dyds . Rn 0 Rn Problem 13: Given g : [0, ∞] → R, with g(0) = 0, derive the formula Z t −x2 1 x 4(t−s) g(s)ds, x > 0 e u(x, t) = √ 4π 0 (t − s)3/2 for a solution of the initial/boundary-value problem ⎧ ⎪ ⎨ ⎪ ⎩ ut − u xx = 0 inR+ × (0, ∞) ⎧ ⎪ ⎨ ⎪ ⎩ u = 0 onR+ × {t = 0}, ⎧ ⎪ ⎨ ⎪ ⎩ u = g on{x = 0} × [0, ∞). Proof. We define ⎧ ⎪ ⎨ ⎪ ⎩u(x, t) − g(t) v(x, t) = ⎧ ⎪ ⎨ ⎪ ⎩−u(−x, t) + g(t) x > 0, x ≤ 0. So, we have ⎧ ⎪ ⎨ ⎪ ⎩ut (x, t) − g0 (t) vt (x, t) = ⎧ ⎪ ⎨ ⎪ ⎩−ut (−x, t) + g0 (t) x > 0, x ≤ 0, and ⎧ ⎪ ⎨ ⎪ ⎩u xx (x, t) v xx (x, t) = ⎧ ⎪ ⎨ ⎪ ⎩−u xx (−x, t) x > 0, x ≤ 0. 11 Hence, ⎧ ⎪ ⎪ ⎪ ⎪ ⎪ ⎪ ⎪ ⎪ ⎨ ⎪ ⎪ ⎪ ⎪ ⎪ ⎪ ⎪ ⎪ ⎩ ⎧ ⎪ ⎪ ⎪ ⎪ ⎪ ⎪ ⎪ ⎪ ⎨ ⎪ ⎪ ⎪ ⎪ ⎪ ⎪ ⎪ ⎪ ⎩ ⎧ ⎪ ⎨ ⎪ ⎩−g0 (t) vt (x, t) − v xx (x, t) = ⎧ ⎪ ⎨ ⎪ ⎩g0 (t) v(x, 0) = 0, v(0, t) = 0. x > 0, x ≤ 0.
By formula (13) on page 49, we get) (Z 0 Z ∞ Z t −(y−x)2 −(y−x)2 1 0 0 e 4(t−s) g (s)dyds e 4(t−s) g (s)dyds − v(x, t) = √ 4π(t − s) −∞ 0 0 Note that(page 46 Lemma) Z ∞ −(y−x)2 1 e 4(t−s) dy = 1, √ 4π(t − s) −∞ so when x > 0, we let y − x = −z and obtain u(x, t) = v(x, t) + g(t) Z t Z 0 = v(x, t) + g (s)ds 0 =2 Z = Z t Z ∞ −∞ 0 −(y−x)2 1 e 4(t−s) dy √ 4π(t − s) −(y−x)2 1 1 e 4(t−s) dy g0 (s)ds √ (t − s)− 2 −∞ 0 4π Z ∞ 2 t −z 1 1 e 4(t−s) dz dg(s) √ (t − s)− 2 π 0 x Integrating by parts, we get Z ∞ 2 −z 1 −1/2 u(x, t) = √ (t − s) e 4(t−s) dz g(s)| s=t s=0 π x Z ∞ 2 Z t −z 1 1 − e 4(t−s) dz g(s) √ (t − s)−3/2 ds π2 x 0 Z t Z ∞ 2 −z 1 −z2 − g(s) √ (t − s)−1/2 ds e 4(t−s) dz 4(t − s)2 π 0 x Z t Z ∞ 2 −z 1 1 −3/2 e 4(t−s) dz = I1 − g(s) √ (t − s) ds π2 x 0 Z t Z ∞ −z2 1 −z + g(s) √ (t − s)−1/2 ds de 4(t−s) π 0 x 2(t − s) Z t Z ∞ 2 −z 1 1 −3/2 = I1 − g(s) √ (t − s) ds e 4(t−s) dz π2 0 x Z t −z2 1 + g(s) √ (t − s)−3/2 ds (−z) e 4(t−s) |z=∞ z=x 0 4π Z t Z ∞ 2 −z 1 1 + g(s) √ (t − s)−3/2 ds e 4(t−s) dz π2 0 x Z t 2 −x 1 x = I1 + √ e 4(t−s) g(s)ds. 3/2 4π 0 (t − s) 12
Now, we focus on I1 and define w2 to be z2 , 4 Z 1 −1/2 ∞ −z42 I1 = lim+ √ e dz g(t −) →0 π x 2 /4 Thus, we proved x u(x, t) = √ 4π t Z 0 −x2 1 4(t−s) g(s)ds, x > 0. e (t − s)3/2 Next, we need to show that lim u(x, t) = g(t). x→0+ Note that for any fixed δ > 0. Z t −x2 1 x 4(t−s) g(s)ds lim+ u(x, t) = lim+ √ e x→0 x→0 4π t−δ (t − s)3/2 Z t−δ −x2 x 1 + lim+ √ e 4(t−s) g(s)ds 3/2 x→0 4π 0 (t − s) Z t −x2 1 x 4(t−s) ds = g(t) lim+ √ e x→0 4π t−δ (t − s)3/2 Z δ 1 −x4s2 x = g(t) lim+ √ e ds x→0 4π s3/2 For fixed x, we let s = x2 /w2 and get lim+ u(x, t) = g(t) lim+ x→0 x √ Z x2 /δ w3 −w4 2 −2x2 e dw x3 w3 2 π ∞ Z ∞ 1 −w2 = g(t) √ e 4 dw = g(t). π 0 x→0 Hence, we are done. Problem 14.
We say v ∈ C12 (UT) is a subsolution of the heat equation if vt − Δv ≤ 0 in UT . (a) Prove for a subsolution v that 1 v(x, t) ≤ n 4r Z Z v(y, s) E(x,t;r) for all E(x, t; r) ⊂ UT . (b) Prove that maxU¯ T v = maxΓT v |x − y|2 dyds (t − s)2 13 Solution. (a) We may well assume upon translating the space and time coordinates that x = 0 and t = 0. As in the proof of Theorem 3, set Z Z 1 |y|2 φ(r) := n v(y, s) 2 dyds, r s E(r) n |y|2 ψ(y, s) := − log(−4πs) + + n log r 2 4s and derive Z Z n 1 2n X 0 vy yi dyds φ (r) ≥ n+1 −4nΔvψ − r s i=1 i E(r) Z Z n X 1 2n = 4nvyi ψyi − vyi yi dyds = 0. n+1 r s E(r) i=1 For 0 < < r, Zr φ0 (z)dz = φ(r) − φ() ≥ 0. R R |y|2 Hence, φ(r) ≥ lim φ() = v(0, 0) · lim 1n dyds = 4v(0, 0), and the statement follows. E() s2 →0 →0 (b) Suppose there exists a point (x0 , t0) ∈ UT with u(x0 , t0) = M := maxU¯ T u. Then for all sufficiently small r > 0, E(x0 , t0 ; r) ⊂ UT . Using the result proved above, we deduce Z Z 1 |x − y|2 M = v(x0 , t0) ≤ n v(y, s) dyds ≤ M, 4r (t − s)2 E(x0 ,t0 ;r) since 1 1= n 4r Z Z E(x0 ,t0 ;r) |x0 − y|2 dyds. (t0 − s)2 Conclude that u|E(x0 ,t0 ;r) = M. The argument used in the proof of Theorem 4 will finish the proof. Problem 15. (a) Show the general solution of the PDE u xy = 0 is u(x, y) = F(x) + G(y) for arbitrary functions F,G. (b) Using the change of variables ξ = x + t, η = x − t, show utt − u xx = 0 if and only if uξη = 0. (c) Use (a),(b) to rederive d'Alembert's formula. Solution: (a) R u xy = 0 ⇒ u x = f (x) ⇒ u(x, y) = R f (x)dx + G(y) uyx = 0 ⇒ uy = g(y) ⇒ u(x, y) = g(y)dy + F(x) 14 This implies u(x, y) = F(x) + G(y). (b) x = ξ+η , y = ξ−η 2 2 Define u˜ := u ξ+η , ξ−η 2 2 1 1 u˜ ξ = u x + ut and 2 2 Hence, u˜ ξη = 0 ⇔ utt − u xx = 0. 1 1 1 1 1 u˜ ξη = u xx − u xt + utx − utt = (u xx − utt) 4 4 4 4 4 (c) By (b), utt − u xx = 0 ⇔ uξη = 0, and u(ξ, η) = F(ξ) + G(η) by (a) ,i.e, u(x, y) = F(x + t) + G(x − t). Since u(x, 0) = g, ut (x, 0) = h, (3) u(x, 0) = F(x) + G(x) = g(x), ut (x, 0) = F 0 (x) − G0 (x) = h(x) Integration ⇒ (4) F(x) − G(x) = Z x h(y)dy + C, C:constant. 0 (2) + (3); (2) − (3); Z x 1 F(x) = g(x) + h(y)dy + C 2 Z0 x 1 G(x) = g(x) − h(y)dy − C 2 0
Thus, Z x+t Z x−t 1 1 u(x, y) = F(x + t) + G(x − t) = g(x + t) + h(y)dy + C + g(x − t) − h(y)dy − C 2 2 0 0 Z x+t Z 0 1 = g(x + t) + h(y)dy + C + g(x − t) + h(y)dy − C 2 0 x−t Z x+t 1 1 = g(x + t) + g(x − t) + h(y)dy (x ∈ R, t ≥ 0). 2 2 x−t Problem 16. Assume E = (E 1 , E 2 , E 3) and B = (B1 , B2 , B3) solve Maxwell's equations: Et = curl B Bt = − curl E div B = div E = 0 Show that utt − Δu = 0 where u = Bi or E i for i = 1, 2, 3. Solution.
15 curl(curl E) = curl(−Bt) ! ∂2 B3 ∂2 B2 ∂2 B3 ∂2 B1 ∂2 B2 ∂B1 + ,− + ,− + = − ∂y∂t ∂z∂t ∂x∂t ∂z∂t ∂x∂t ∂y∂t ∂ = − curl B ∂t ∂ = − Et ∂t ∂2 E =− 2 ∂t However, we also know that curl(curl E) = ∇(div E) − ∇2 E = −∇2 E. sevel gelato case manual version download Then E i satisfies utt − Δu = 0 for i = 1, 2, 3.
Similarly, curl(curl B) = curl Et = − ∂∂tB2 , and curl(curl B) = ∇(div B) − ∇2 B = −∇2 B, so Bi satisfies utt − Δu = 0 for i = 1, 2, 3. 2 Problem 17.(Equipartition of energy) Let u ∈ C 2 (R × [0, ∞)) solve the initial value problem for the wave equation in one dimension: ⎧ ⎪ ⎨ in R × (0, ∞) ⎧utt − u xx = 0 ⎪ ⎩u = g; ut = h on R × {t = 0}. R∞ Suppose g, h have compact support. The kinetic energy is k(t) := 12 −∞ u2t (x, t)dx and the potential R∞ energy is p(t) := 12 −∞ u2x (x, t)dx.
Prove (i) k(t) + p(t) is constant in t. (ii) k(t) = p(t) for all large enough times t. R∞ Proof. (i.) We define e(t) = k(t) + p(t) = 12 −∞ u2t + u2x dx. digogu.pdf Since g, h have compact support, so we have Z d e(t) 1 ∞ = 2ut utt + 2u x ut xt dx dt 2 −∞ Z ∞ Z ∞ ut utt dx − u xx ut dx −∞ −∞ Z ∞ = ut (utt − u xx) dx = 0. −∞ Hence, e(t) ≡ e(0). (ii.)By d'Alembert's formula on page 68, we have 1 1 u(x, t) = g(x + t) + g(x − t) + 2 2 Z x+t h(y)dy. x−t So, ut = 1 1 0 g (x + t) − g0 (x − t) + [h(x + t) + h(x − t)] , 2 2 ux = 1 1 0 g (x + t) + g0 (x − t) + [h(x + t) − h(x − t)] . 2 2 and 16 We assume that there exists a positive constant M so that [−M, M] ⊇ supp(g0) and [−M, M] ⊇ supp(h). Note that for a fixed t > M, −M ≤ x − t ≤ M ⇔ 0 < t − M ≤ x ≤ t + M and −M ≤ x + t ≤ M ⇔ −t − M ≤ x ≤ −t + M < 0. Thus, when t > M : (a) 0 < t − M ≤ x ≤ t + M. Then we have h(x + t) = g(x + t) = 0. So, 1 1 1 u2t = g0 (x − t)2 + h(x − t)2 − g0 (x − t)h(x − t) = u2x . 4 4 2 (b) −t − M ≤ x ≤ −t + M < 0. Then, 1 1 1 u2t = g0 (x + t)2 + h(x + t)2 + g0 (x + t)h(x + t) = u2x . 4 4 2 (c) Otherwise g0 (x + t) = g0 (x − t) = h(x + t) = h(x − t) = 0. So, combining all the cases, it is obvious that when t > M, k(t) = p(t). Problem 18. Let u solve (utt − Δu = 0 in R3 × (0, ∞) u = g, ut = h on R3 × {t = 0}, where g, h are smooth and have compact support. Show there exists a constant C such that |u(x, t)| ≤ C/t (x ∈ R3 , t > 0).
Solution. From the conditions it follows that there exist R, M > 0 such that spt g, spt h ⊂ B(0, R) and g(y) ≤ M, |Dg(y)| ≤ M, h(y) ≤ M for any y ∈ R3 . Kirchhoff's formula gives the solution of the initial-value problem: ? u(x, t) = th(y) + g(y) + Dg(y) · (y − x) dS (y).
∂B(x,t) Denote by Σ the intersection ∂B(x, t) ∩ B(0, R). Observe that the area of Σ is not greater than the area of the sphere ∂B(0, R). Then, for t > 0, we obtain ? Z 1 th(y) + Dg(y) · (y − x) dS (y) = th(y) + Dg(y) · (y − x) dS (y) 2 4πt ∂B(x,t)∩B(0,R) ∂B(x,t) Z 1 t · |h(y)| + |Dg(y)| · |y − x| dS (y) ≤ 4πt2 ∂B(x,t)∩B(0,R) 1 2R2 M 2 ≤ · 4πR · (tM + tM) = . 4πt2 t 17 For t > 1, using the same argument, we get ? Z 1 1 R2 M R2 M 2 = ≤ g(y) dS (y) g(y) dS (y) · 4πR · M = ≤ . 4πt2 ∂B(x,t)∩B(0,R) 4πt2 t2 t ∂B(x,t) Notice now that the area Σ is not greater than the area of the sphere ∂B(x, t). Then for 0 < t ≤ 1, ? Z 1 1 M g(y) dS (y) = g(y) dS (y) · 4πt2 · M ≤ . texasexiganux.pdf
≤ 2 2 4πt 4πt t ∂B(x,t) ∂B(x,t)∩B(0,R) Without loss of generality, we can take R > 1. Then, combining the estimates obtained above, we 2 conclude |u(x, t)| ≤ 3Rt M . Evans PDE Solutions, Chapter 5 Alex: 4, Helen: 5, Rob H.: 1 Problem 1. ¯ is a Banach space. Suppose k ∈ {0, 1, . .
.}, 0 < γ < 1. Prove C k,γ (U) Solution: 1. First we show that || · ||C k,γ (U) ¯ is a norm, where we recall that ||u||C k,γ (U) ¯ = X ||Dα u||C(U) ¯ + X [Dα u]C 0,γ (U) ¯ , |α|=k |α|≤k and ([u]C 0,γ (U) ¯ = sup x,y∈U) |u(x) − u(y)| . |x − y|γ For the sake of opaqueness we now omit subscripts on all norms unless it is unclear from context. 2. For any λ ∈ R we have first [λu] = sup x,y∈U and certainly |λu(x) − λu(y)| |u(x) − u(y)| = |λ| sup = |λ| [u] , γ |x − y| |x − y|γ x,y∈U α α ||Dα (λu)||C(U) ¯ = ||λD u|| = |λ| · ||D u|| . So ||λu|| = X ||Dα (λu)|| + |α|≤k = |λ| X X [Dα (λu)] |α|=k ||Dα u|| + |λ| |α|≤k X [Dα u] |α|=k = |λ| · ||u|| . 3. If u = 0 it is obvious that ||u|| = 0. On the other hand, ||u|| = 0 implies that ||Dα u||C(U) ¯ = 0 for every |α| ≤ k.
In particular this is true for α = 0 so that the supremum of D0 u = u on U is 0, i.e. u ≡ 0. 4. Finally we must prove the triangle inequality. We know the triangle inequality is true for the sup norm || · ||C(U) ¯ . We can also see that for any α which makes sense [Dα (u + v)] = [Dα u + Dα v] ≤ [Dα u] + [Dα v] . Therefore we can easily conclude ||u + v|| = X ||Dα (u + v)|| + ≤ [Dα (u + v)] |α|=k |α|≤k X X (||Dα u|| + ||Dα v||) + |α|≤k X ([Dα u] + [Dα v]) |α|=k = ||u|| + ||v||. 5. We need only show that C k,γ (U) is complete. So let {um } be a Cauchy sequence. Then {um (x){ is a Cauchy sequence for every x, so define u to be the pointwise limit of the um . Now if V is any bounded subset of U, then V¯ is compact, so that um ⇒ u uniformly on any V. Since the um are uniformly continuous on V¯ by assumption, this implies that u is uniformly continuous on V¯ as well ¯ (and so, a fortiori u ∈ C(U)). Therefore u ∈ C(U). ¯ But similar arguments show that u has What we would really like would be to have u ∈ C k (U). α derivatives D u for all |α| ≤ k on U by restricting first to bounded subsets of U to find the derivatives and then using uniform convergence on these subsets to show the derivatives must also be uniformly continuous on bounded subsets since the Dα um were. This leaves us with only showing that the norm of u is finite, so that in fact u ∈ C k,γ (U). But for every n we have X X |Dα un (x) − Dα un (y) − Dα u(x) + Dα u(y)| ||un − u|| = sup |Dα un (x) − Dα u(x)| + sup |x − y|γ |α|≤k x∈U |α|=k x,y∈U ⎛ ⎞ α α α α X ||| X |||u (x) − D u (y) − D u (x) + D u (y)| |D n n m m |||= lim ||| sup |Dα un (x) − Dα um (x)| + sup ⎠γ m⇒∞ |x − y| x∈U x,y∈U |α|≤k |α|=k = lim ||un − um ||. m⇒∞ In particular, since {um } is Cauchy there is some N so that n, m ≥ N implies ||un − um || ≤ 1. Letting m approach ∞, this implies that ||uN − u|| < 1. Now the triangle inequality applies to give ||u|| ≤ ||uN − u|| + ||uN || < 1 + ||uN || < ∞. Problem 4. SN Vi . Show there exist C ∞ functions ζi (i = 1, . . . , N) such that Assume U is bounded and U ⊂⊂ i=1 ⎛ ⎞ i=1 i The functions {ζi }1N for a partition of unity. 19 SN Solution. 4840613263.pdf Assume U is bounded and U ⊂⊂ i=1 Vi . Without loss of generality, we may assume that the Vi are open, for if they are not, we can replace Vi by its interior. We note that, since U is bounded, U is compact. Each x ∈ U has a compact neighbourhood N x contained in Vi for some i. Then {N x◦ } is an open cover of U, which then has a finite subcover N x◦1 , .
. . , N x◦n .
We now let Fi be the union of the N xk contained in Vi . Fi is the compact since it is the finite union of compact sets. The C ∞ version of Urysohn's Lemma (Folland, p.245) allows us to find smooth Pn functions ξ1 , . . . , ξN such that ξi = 1 on Fi and supp(ξi) ⊂ Vi . Since the Fi cover U, U ⊂ {x : 1 ξi (x) > 0} and we can P use Urysohn again to find ζ ∈ C ∞ with ζ = 1 on U and supp(ζ) ⊂ {x : n1 ξi (x) > 0}. Now, we let PN+1 ξN1 = 1 − ζ, so 1 ξi > 0 everywhere. magic quadrant for unified threat management 2018 pdf We then take ξi ζi = PN+1 ξj 1 as our partition of unity. Problem 5 (Helen) Prove that if n = 1 and u ∈ W 1,p (0, 1) for some 1 ≤ p < ∞, then u is equal a.e. to an absolutely continuous function, and u0 which exists a.e. belongs to L p (0, 1). Proof. Since u ∈ W 1,p (0, 1), so by definition on page 242 and 244, we have some function v ∈ L p (0, 1) such that Z Z u Dφdx = − vφdx, ∀φ ∈ Cc∞ ((0, 1)) . (0,1) (0,1) Note that v ∈ L p (0, 1), so by H¨older's inequality, we have kvkL1 ≤ kvkL p k1kLq < ∞, which means v ∈ L1 (0, 1). Thus, we can define function f (x) on (0, 1) by the following formula Z x 1 f (x) = u( ) + v(t)dt, ∀x ∈ (0, 1). 1 2 2 According to the Fundamental Theorem of Calcalus, f is absolutely continuous. what is learner centred approach to language teaching Now we will prove u = f a.e. By the definition of f , we have f 0 = v a.e. So for any φ ∈ Cc∞ ((0, 1)) we get Z Z Z 0 f Dφdx = − f φdx = − vφdx. (0,1) (0,1) (0,1) Therefore, Z (f − u) Dφdx = 0 ∀φ ∈ Cc∞ ((0, 1)) , (0,1) which means u = f + const. And note that u(21) = f (12), hence u = f a.e. So u0 exists a.e. and satisfy u0 = v a.e., so u0 ∈ L p (0, 1).
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