3PC MODULE 098

Finite Element Methods for Structural Analysis

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Background

- This course material was developed by Dr. Naveen Rastogi while teaching graduate students at Embry-Riddle Aeronautical University, Daytona Beach, Florida, USA as an Adjunct Faculty

- This is a one-semester (3 Credit Hours) first level course in "Finite Element Methods" that can be taught to the senior-level undergraduates and graduate students in engineering

- The course is also useful for the mid-to-entry level engineering professionals who are using finite element analysis tools as part of their daily work to design, analyze and optimize various products across many industries

- Users can send their questions/comments/feedback about this course to 3pcomps@gmail.com

Self-study Course Approach

- The course material is arranged in the order of increasing complexity. Hence, it is recommended to study the Sections in the order they are arranged

- There are eleven practice problems given at the end. Users should attempt to solve these problems to gain better understanding of the subject

- It is recommended to use MS- Excel, Matlab, MathCAD or any suitable software to program the formulae and perform matrix algebra

Module Contents

- Foundations of Finite Element Method for Structural Analysis
  - Energy Principles
  - Ritz Method
  - Shape Functions
  - Jacobian and Hooke's Law
  - Element Stiffness Matrices and Force Vectors
  - Global Stiffness Matrix and Force Vector
  - Solution to System of Equations
  - Post processing - Nodal Displacements and Forces, Element Strains and Stresses, Free-Body Diagram

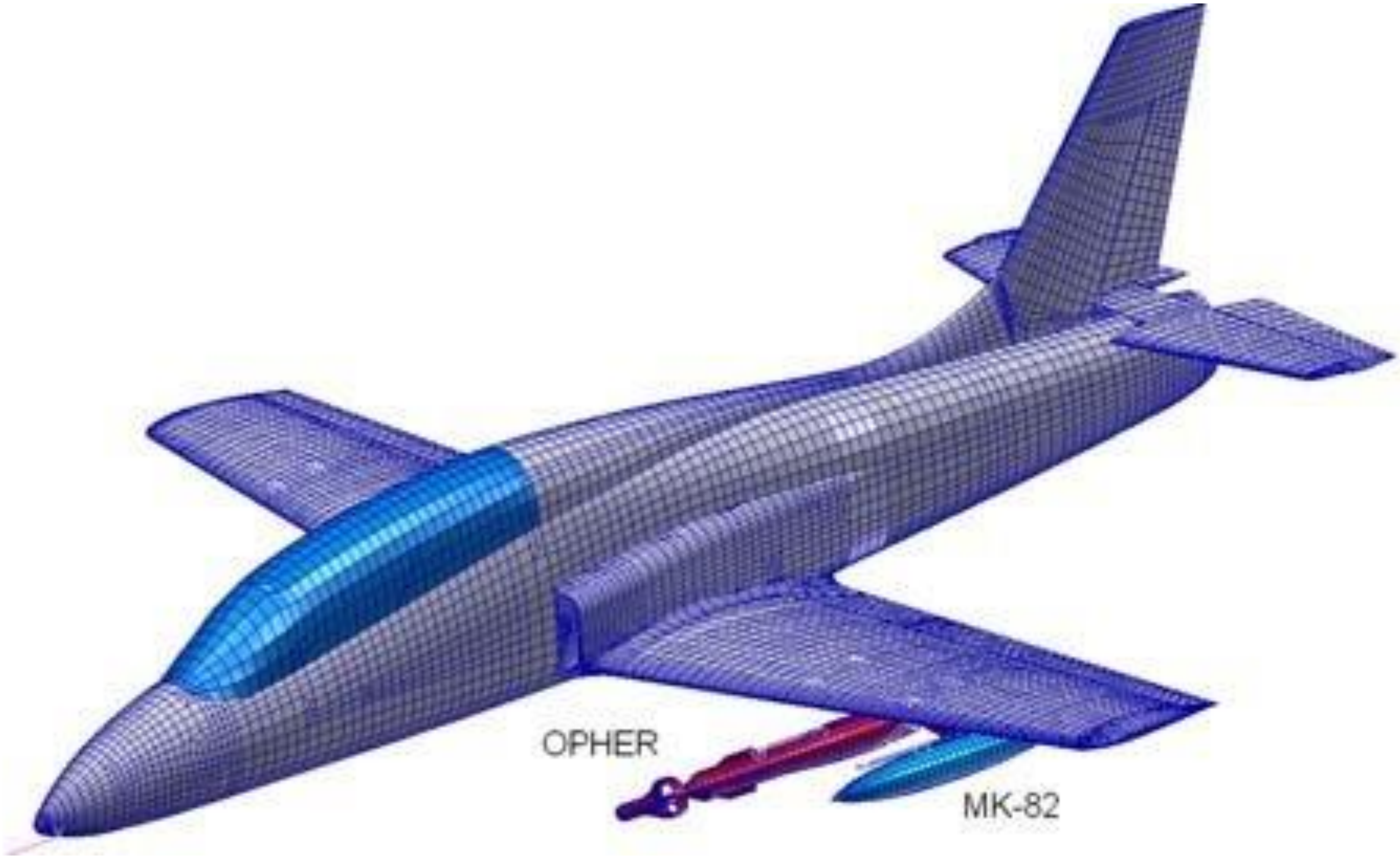
- One Dimensional Spring, Bar, Beam and Rigid Frame Elements

- Two Dimensional Constant and Linear Strain Triangular Elements, and Linear and Quadratic Quadrilateral Elements

- Static, Dynamic and Thermal analyses of Structures using Finite Element Method

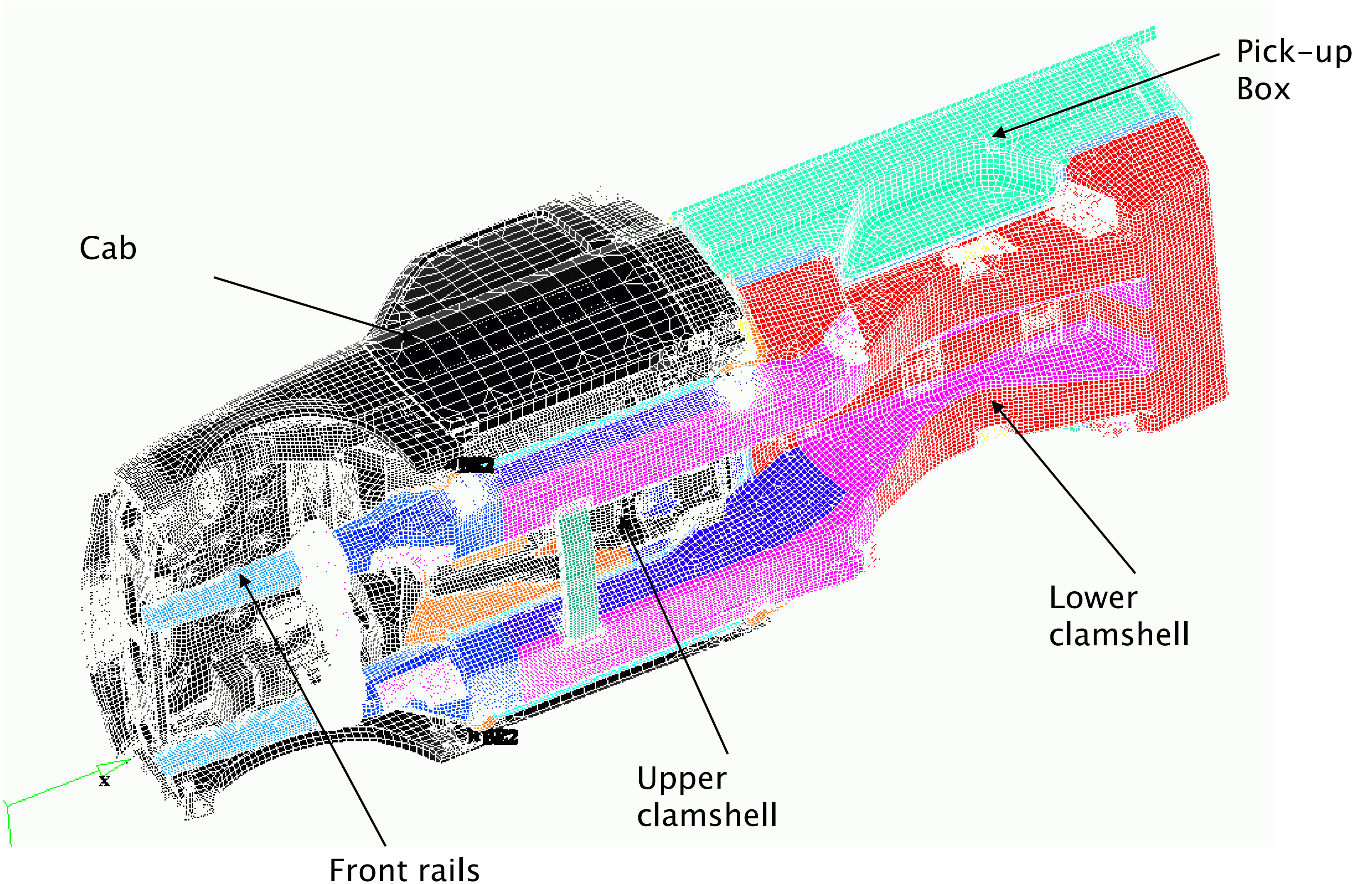
Examples

Finite Element Model - Aircraft



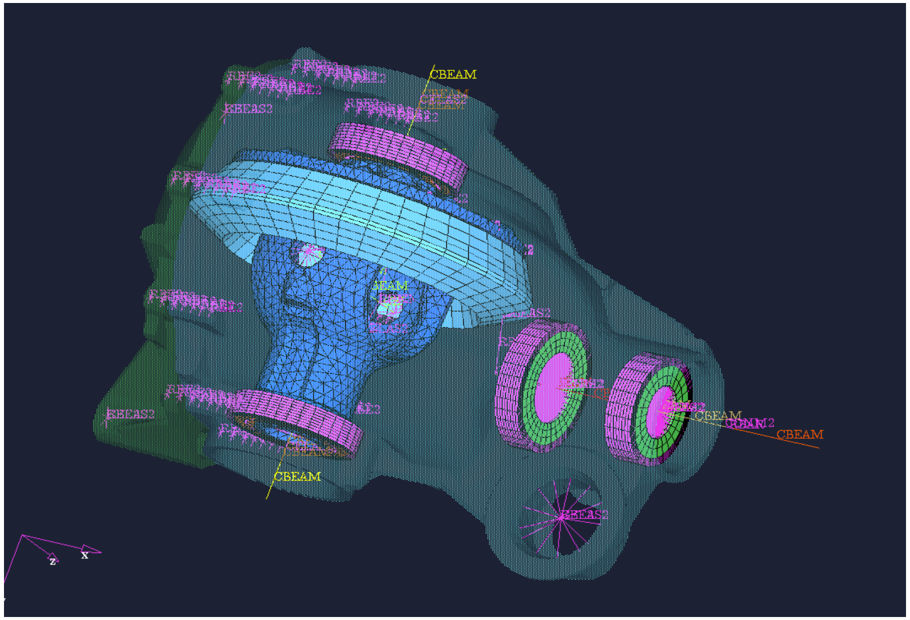
Ref: Lozici-Brînzei, Dorin & Tătaru, Simion & Bîscă, Radu. (2011). IAR-99 GVT CORRELATION FOR DYNAMICS STORES FEM. INCAS BULLETIN. 3. 10.13111/2066-8201.2011.3.1.7.

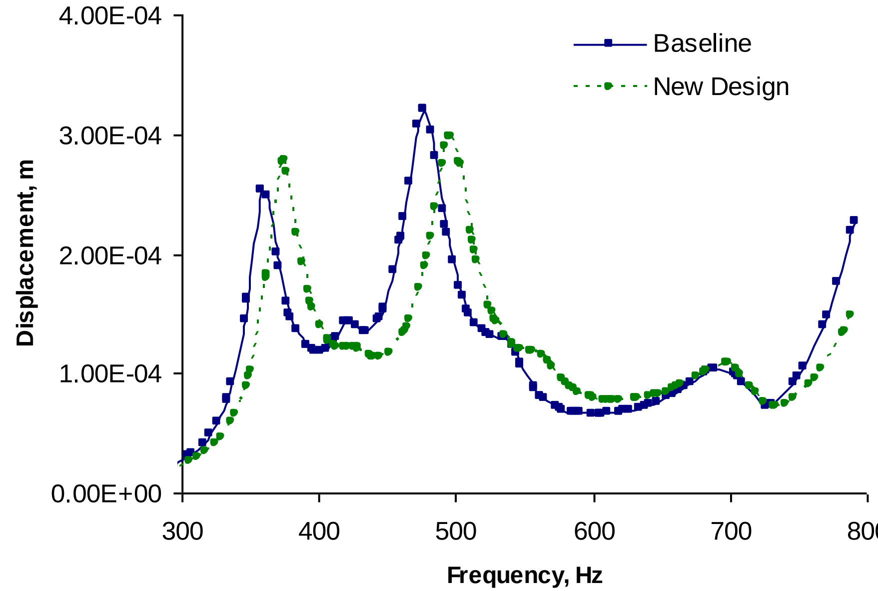
Finite Element Model – Pickup Truck



Ref: Rastogi, Naveen, "Stress Analysis and Lay-Up Optimization of an All-Composite Light Pick-Up Truck Chassis Structure", Transactions of Society of Automotive Engineers: 2004

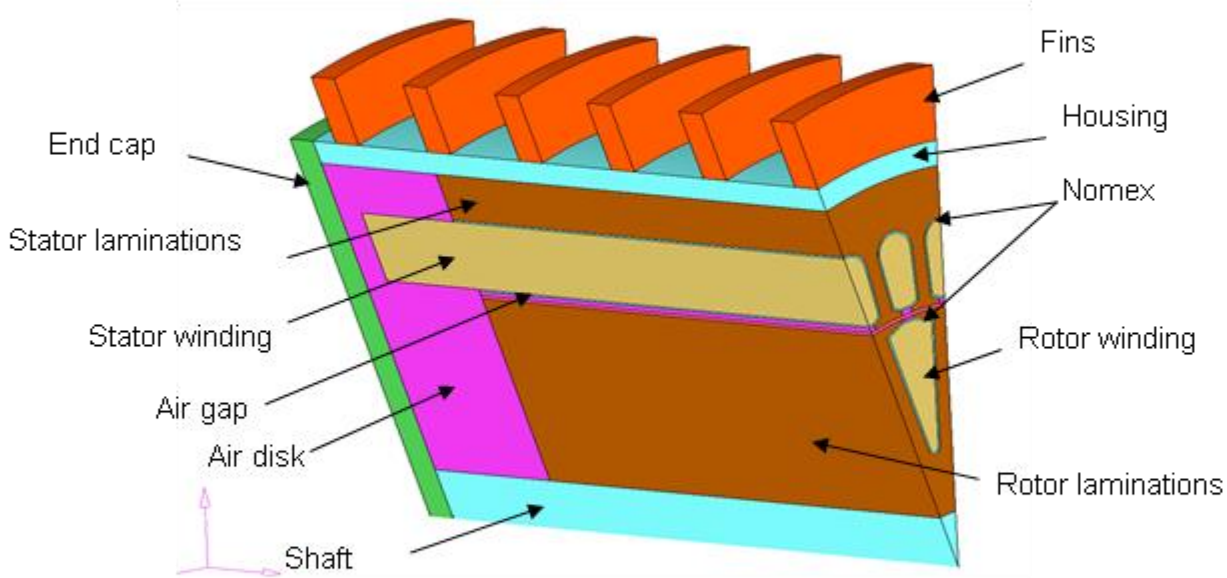
Modal Analysis



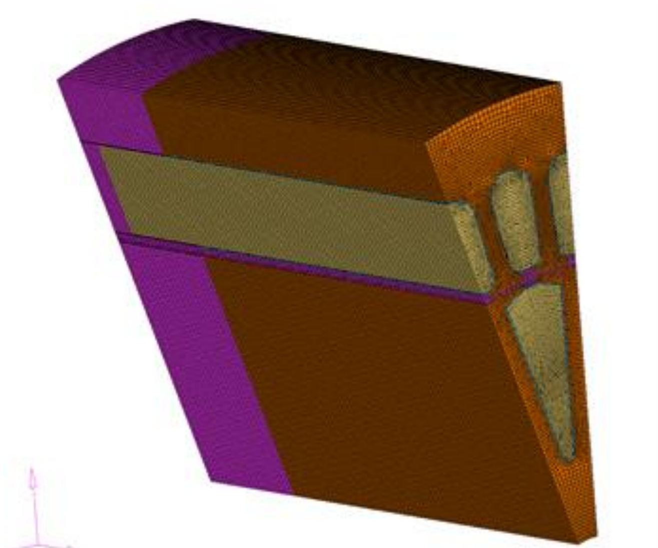


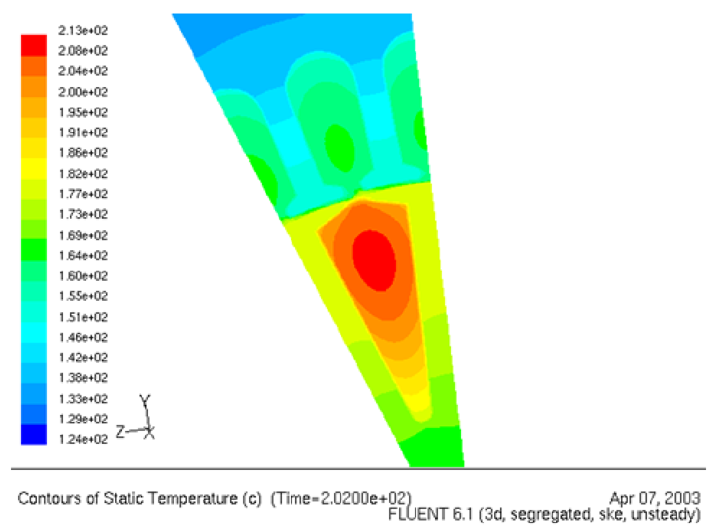
Ref: Rastogi, Naveen, "Forced Frequency Response Analysis of Multi-material Systems", 2005 SAE NVH Conference, Traverse City, MI, May 2005

Heat Transfer Analysis

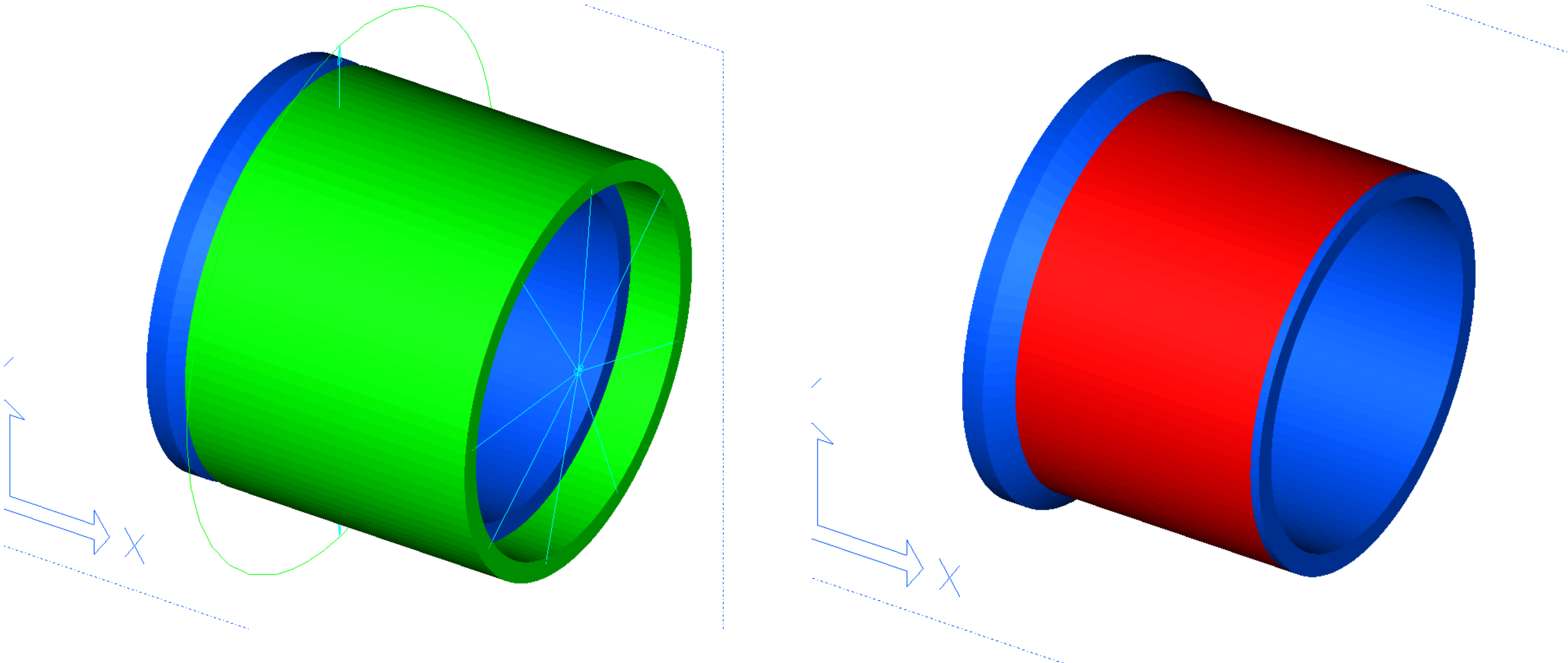


Ref: Rastogi, Naveen and Akhtar, Junaid, "Transient heat transfer analysis of electric motors ", Visteon Technical Report B600-031, Chassis Advanced Technology, Visteon Corporation, Dearborn, MI 48126, September 2003



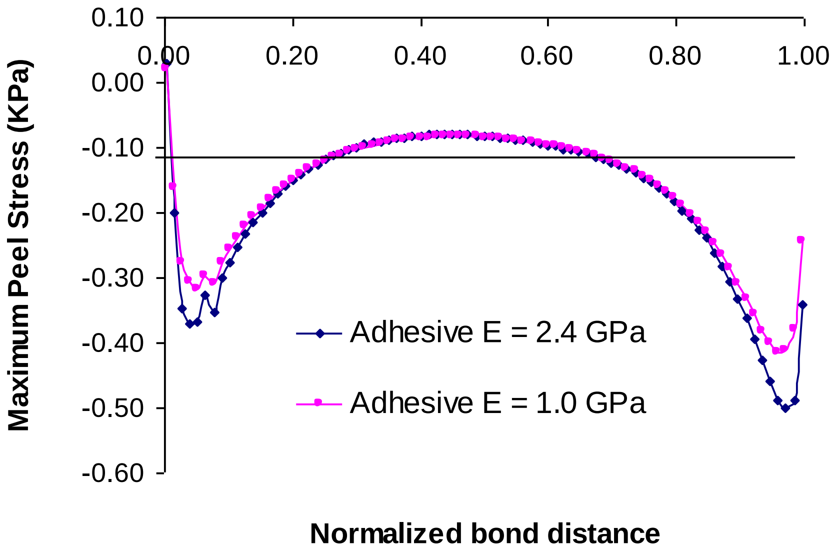


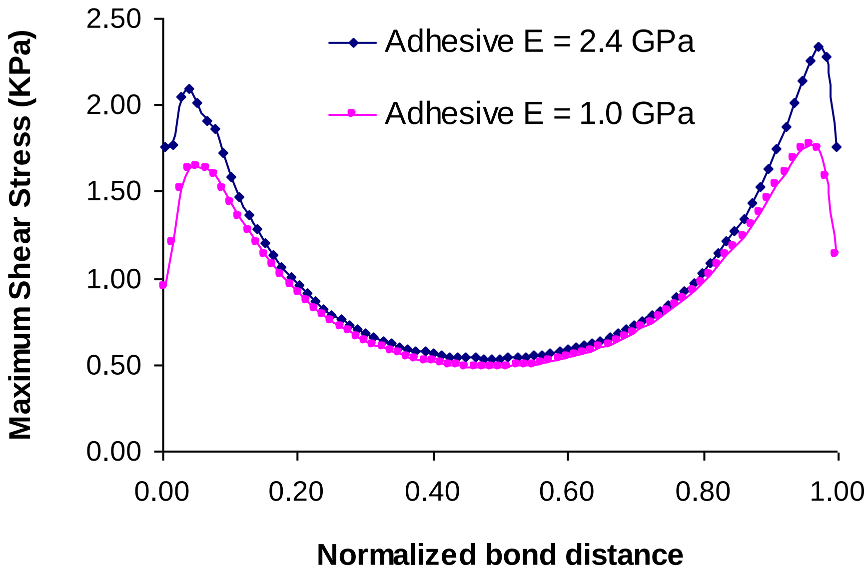
Composite Tube-Aluminum yoke joint subjected to torque



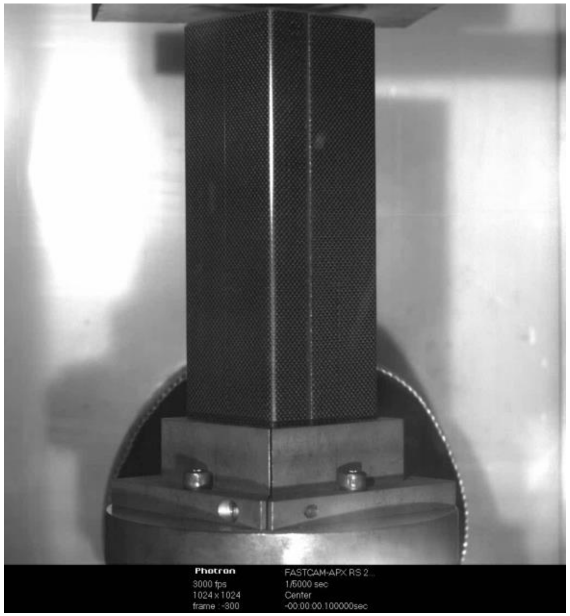
Ref: Rastogi, Naveen, "Design of Composite Driveshaft for Automotive Applications", SP-1858: Special Publication of 2004 SAE World Congress & Exhibition, SAE Paper No. 2004-01-0485, Detroit, MI, March 2004.

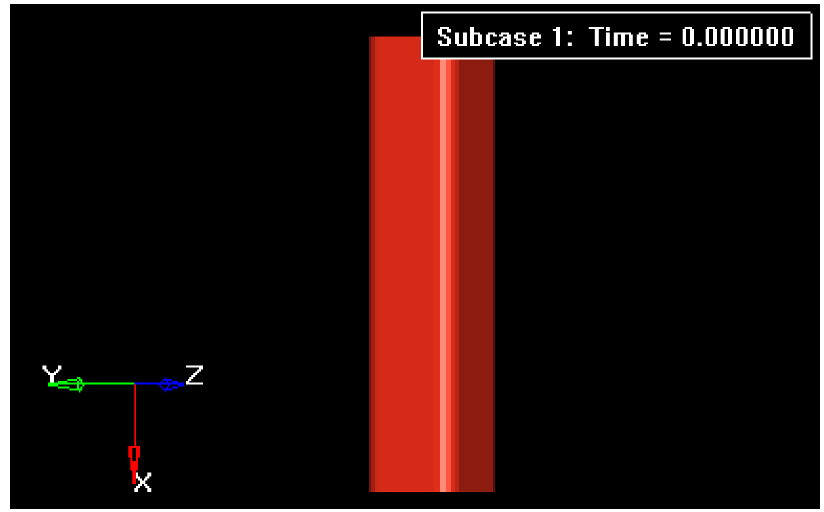
Peel and shear stresses in tubular bonded joints

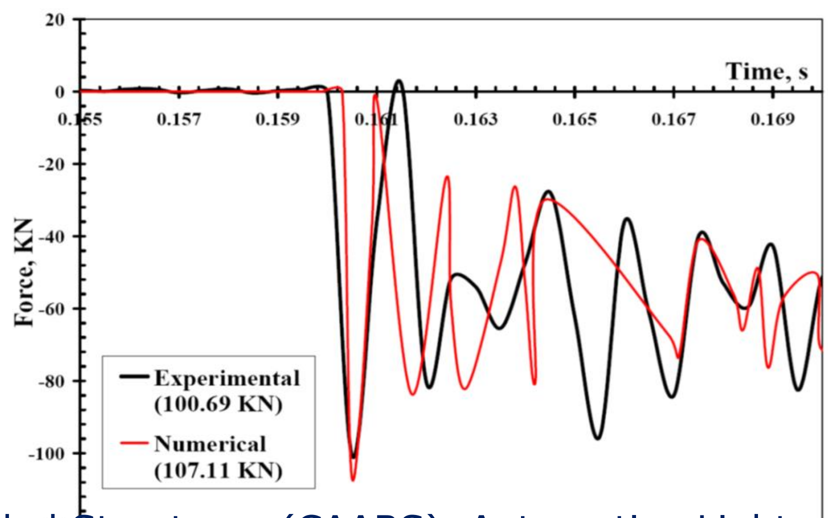




Progressive Failure Analysis







Ref: Crash Analysis of Adhesively Bonded Structures (CAABS), Automotive Lightweighting Materials, FY2004 Progress Report

Field of Mechanics

Field of Mechanics

THEORETICAL

Statics
Dynamics
Kinematics
Rigid body dynamics
Equations of Motion
Friction
Simple harmonic motion
…

APPLIED

Analytical mechanics
Computational mechanics
Contact mechanics
Continuum mechanics
Dynamics (mechanics)
Elasticity (physics)
Experimental mechanics
Fatigue (material)
Fluid mechanics
Fracture mechanics
Mechanics of materials
Mechanics of structures
Rotordynamics
Solid mechanics
Soil mechanics
Viscoelasticity
…

Computational Mechanics

- Computational mechanics uses computational methods to study physical phenomena governed by the principles of mechanics

- Computational mechanics (CM) is interdisciplinary
    - Mathematics
    - Computer Science
    - Mechanics

- Specializations within CM
    - Computational fluid dynamics (CFD)
    - Computational thermodynamics (CT)
    - Computational electromagnetics (CEM)
    - Computational solid mechanics (CSM)

    …

Computational Mechanics

- Mathematical models
  - Expressing the physical phenomenon of the engineering system in terms of partial differential equations
  - Variational Principles

- Discretization
  - Creating an approximate discrete model from the original continuous model
  - Translating system of PDEs into a system of algebraic equations as in the field of numerical analysis
  - Finite Element Method, Finite Difference Method, Boundary Element Method, Finite Volume Method

- Computer Programs
  - To solve the large system of discretized equations
  - Direct methods (which are single step methods resulting in the solution)
  - Iterative methods (which start with a trial solution and arrive at the actual solution by successive refinement)
  - Supercomputers or parallel computers, Distributed Computing

Computational Solid Mechanics

- Computational Solid Mechanics (CSM) uses computational methods to study behavior of solid matter under external actions (e.g., external forces, temperature changes, applied displacements, etc.)
    - Stress Deformation
    - Compatibility
    - Finite / Infinitesimal strain
    - Elasticity linear
    - Plasticity
    - Bending
    - Hooke's law
    - Failure theory
    - Fracture mechanics
    - Contact mechanics (Frictionless or Frictional)

Matrix Algebra

Linear System of Algebraic Equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\ldots\ldots.$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

where x_1, x_2,..., x_n are the unknowns.

In matrix form: A x = b where

$$\mathrm{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \quad \boldsymbol{x} = \{x_i\} = \begin{Bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{Bmatrix} \quad \boldsymbol{b} = \{b_i\} = \begin{Bmatrix} b_1 \\ b_2 \\ .. \\ b_n \end{Bmatrix}$$

A is called a n x n (square) matrix, and ***x*** and ***b*** are (column) vectors of dimension n

Addition, Subtraction & Multiplication

- For two matrices A and B both of the same size (m x n), the addition and subtraction are defined by

C = A + B with $c_{y} = a_{y} + b_{y}$

D = A – B with $d_{y} = a_{y} - b_{y}$

- Scalar Multiplications

$\lambda A = \lfloor \lambda a_{y} \rfloor$

- For two matrices A (of size l x m) and B (of size m x n), the product of AB is defined by C = AB with

$$c_{ij} = \sum_{k=1}^{m} a_{ik}\, b_{kj}$$ where $i = 1, 2, \ldots\ldots l$ $j = 1, 2, \ldots\ldots n$.

Transpose, Symmetric and Identity Matrices

- Transpose of a Matrix

If A = $[a_{ij}]$, then the transpose of A is $A^T = [a_{ji}]$

Note that $(AB)^T = B^T A^T$

- Symmetric Matrix : A square (n x n) matrix A is called symmetric, if

A = A^T or $a_{ij} = a_{ji}$

- Unit (Identify) Matrix

Note that AI = A, Ix = x.

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Determinant and Inverse

- The determinant of square matrix A is a scalar number denoted by A or$|A|$. For 2 x 2 and 3 x 3 matrices, the determinants are given by

det$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$, and

$$det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11}$$

- A square matrix **A** is singular if det **A** = 0
- For a square and nonsingular matrix **A** (det **A** ≠ 0), its inverse A^{-1} is given as

$$A^{-1} = \frac{1}{\det A} C^T$$

The cofactor matrix **C** of matrix **A** is defined by $C_{iy} = (-1)^{i+j} M_{ij}$ where $\mathrm{M_{ij}}$ the determined of the smaller matrix obtained by eliminating the ith row and jth column of **A**

3P COMPOSITES

Differentiation and Integration

$Let\ A(t) = \left[a_{ij}(t)\right]$

The differentiation is defined by

$$\frac{d}{dt}A(t) = \left[\frac{da_{ij}(t)}{dt}\right]$$

and the integration by

$$\int A(t)dt = \left[\int a_{ij}(t)dt\right]$$

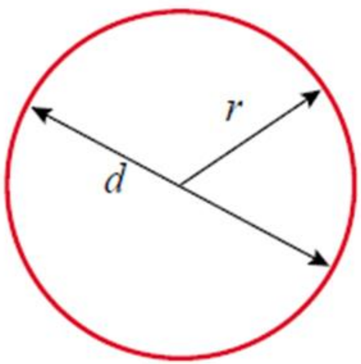
- A square (n x n) matrix **A** is said to be positive definite, if for all nonzero vector **x** of dimension n,

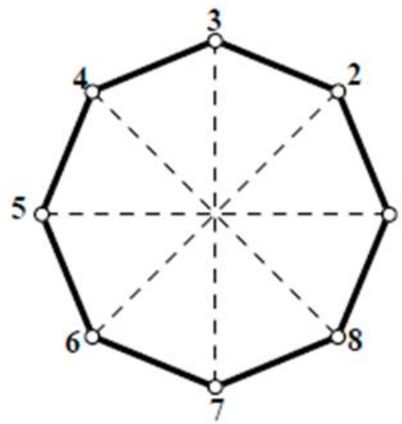
  $x^T \mathrm{Ax} > 0$

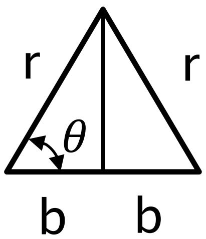
  Note that positive definite matrices are nonsingular

Classical Example

- Value of π







$$2\theta + \frac{2\pi}{n} = \pi$$

$$\theta = \pi\left(\frac{1}{2} - \frac{1}{n}\right)$$

$$\cos\theta = \frac{b}{r} = \cos\pi\left(\frac{1}{2} - \frac{1}{n}\right) = \sin\frac{\pi}{n}$$

$$\therefore 2b = 2r\ \sin\frac{\pi}{n}$$

$$2bn = 2nr\ \sin\frac{\pi}{n} = \pi d$$

$$\therefore \pi = n\ \sin\frac{\pi}{n}$$

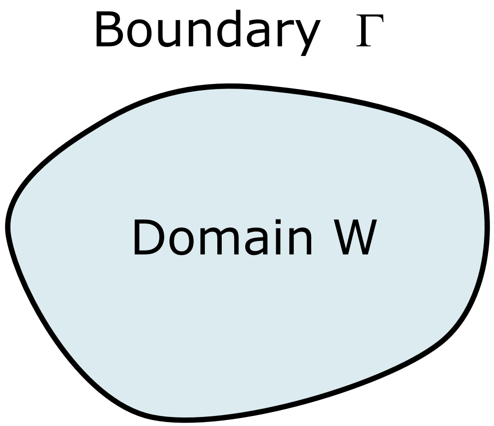
Boundary Value Problem

Boundary Value Problem

- Governing Differential Equation:

$$L\varphi = f$$

L is differential operator
f is forcing function
φ is unknown quantity



Boundary conditions on Γ that encloses domain W

- Examples:

$u_{tt} = \alpha^2 u_{xx}$ Wave Equation $u_t = \alpha u_{xx}$ Heat Equation

- Desirable to solve BVP analytically; However , approximate methods are used for complex problems

Ritz Solution

Define inner product

$$\langle \varphi, \psi \rangle = \int_\Omega \varphi\, \psi^* d\Omega$$

Operator *L* is self-adjoint

$$\langle L\varphi, \psi \rangle = \langle \phi, L\psi \rangle$$

Operator *L* is positive definite

$$\langle L\varphi, \varphi \rangle \begin{cases} > 0 & \varphi \neq 0 \\ = 0 & \varphi = 0 \end{cases}$$

Functional *F* for BVP

$$F\,(\tilde{\varphi}) = \frac{1}{2} \langle L\tilde{\varphi}, \tilde{\varphi} \rangle - \frac{1}{2} \langle \tilde{\varphi}, \mathfrak{f} \rangle - \frac{1}{2} \langle \mathfrak{f}, \tilde{\varphi} \rangle$$

$\tilde{\varphi}$ are trial functions

$$\tilde{\varphi} = \sum_{j=1}^{n} c_j v_j = \{c\}^T \{v\} = \{v\}^T \{c\}$$

Ritz Solution

$$F = \frac{1}{2}\{c\}^T \int_\Omega \{v\} L \{v\}^T d\Omega \{c\} - \{c\}^T \int_\Omega \{v\} f d\Omega$$

Functional *F*

$$\frac{\partial F}{\partial c_i} = \frac{1}{2}\int_\Omega v_i L \{v\}^T d\Omega\{c\} + \frac{1}{2}\{c\}^T \int_\Omega \{v\} L v_i d\Omega - \int_\Omega v_i f d\Omega$$

$$= \frac{1}{2}\sum_{j=1}^{N} c_j \int_\Omega (v_i L v_j + v_j L v_i) d\Omega - \int_\Omega v_i f d\Omega$$

Minimize *F*

$$= 0 \qquad \mathrm{i} = 1,2,3,\dots,\mathrm{N}$$

$$[S]\{c\} = \{b\} \qquad S_{ij} = \frac{1}{2}\int_\Omega \left(v_i L v_j + v_j L v_i\right) d\Omega \qquad b_i = \int_\Omega v_i f \, d\Omega$$

$$S_{ij} = \int_\Omega v_i L \, v_j \, d\Omega$$

Operator *L* is self-adjoint

3P COMPOSITES

Galerkin Solution

$$r = L\tilde{\varphi} - f \neq 0$$

$$R_i = \int_\Omega w_i r \, d\Omega = 0$$

$$w_i = v_i \qquad\qquad i = 1,2,3, \ldots \ldots N$$

$$R_i = \int_\Omega \left(v_i \mathcal{L} \{v\}^T \{c\} - v_i f\right) \mathrm{d}\Omega = 0 \qquad\qquad \mathrm{i} = 1,2,3, \ldots\ldots\ldots \mathrm{N}$$

$$R_i = [L\{v\}^T\{c\} - f]_{at\, point\, i} = 0$$

$$R_i = \int_{\Omega i} \left(\mathcal{L} \{v\}^T \{c\} - f\right) \mathrm{d}\Omega = 0$$

$$\mathrm{I} = \frac{1}{2} \int_\Omega r^2 \, d\,\Omega$$

$$\frac{\partial I}{\partial c_i} = \int_\Omega \mathcal{L} v_i \left(\mathcal{L} \{v\}^T \{c\} - f\right) \mathrm{d}\Omega = 0$$

Boundary Conditions

1. DIRICHLET

- Prescribed displacements (Essential, Kinematic)

u, v, w, φ

2. NEUMANN

- Prescribed derivatives (Static, Natural)

$$\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial \varphi}{\partial n}$$

Variational Principles

- Used to solve boundary value problems (inhomogeneous partial differential equations) posed as a minimization problem corresponding to the lowest energy state of the system

- Used for solving PDEs such as the heat equation, wave equation, and vibrating plate equation, etc.

- A trial function which depends on the variational parameters is used to minimize the function using these parameters. The accuracy of the solution dependents on the number of variational parameters and the type of trial function

- Ritz and Galerkin methods are based on variational principles where functional is minimized to obtain an approximate solution to boundary value problems subjected to boundary conditions

General Formulation

Finite Element Analysis

- Linear
- Nonlinear
  - Geometric
  - Material
  - Contact

- Static
- Modal
- Dynamic
- Fatigue

- Numerical analysis
- Approximate solution
- Ritz method (Minimize Functional / Energy)

Finite Element Analysis

- **Displacement based (widely applicable)**
- Stress based
- Mixed (approximated by two different variables such as displacements and stresses)
- Hybrid (Uses multifield variational principle , yet displacements are the only unknowns; use of Lagrange multiplier)

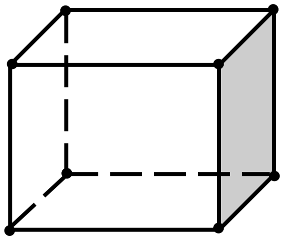
- Sub-parameteric
  - Geometric interpolation lower than displacement

- **Iso-parameteric**
  - **Geometric interpolation same as displacement**

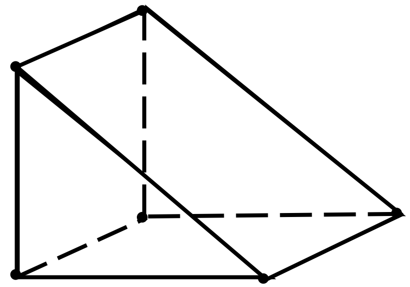
- Super-parameteric
  - Geometric interpolation higher than displacement

Types of Elements

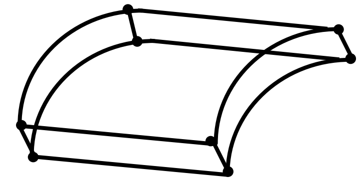
3-D



Rectangular Brick



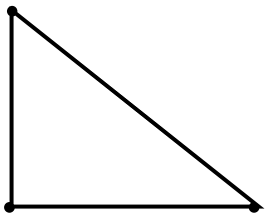
Wedge



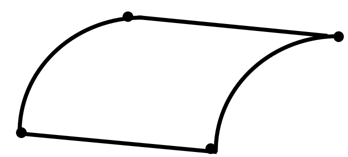
Cylindrical Brick

2-D

Quadrilateral



Triangular



Shell/Plate

[Plane Stress, Plane Strain or Axisymmetric]

1-D

Spring

Bar, Beam, Rod

MPCs

Steps in Finite Element Analysis

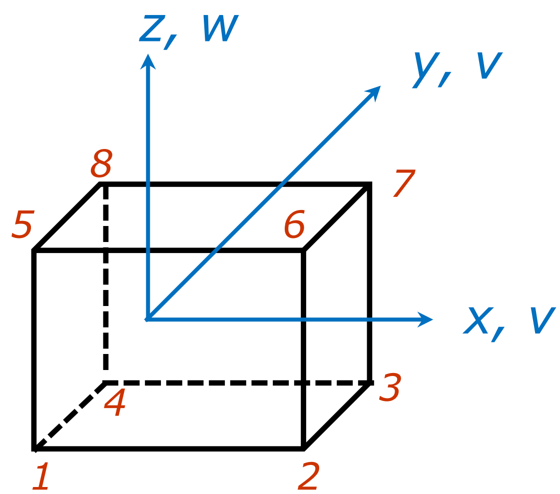
1. Select Element type
2. Select Displacement (Shape) functions
3. Define Strain-Displacement relationship
4. Define Hooke's law
5. Derive Element Stiffness Matrix
6. Assemble Global Stiffness Matrix [K]
7. Apply Boundary Conditions, i.e. known displacements/rotations
8. Assemble Global Displacement Vector {q} and Force Vector {f}
9. Solve [K]{q} = {f}
10. Post process and Interpret results

General Formulation

- Uses Energy Principle and Variational Formulation (Ritz Solution)

$$\boxed{q = \{u_1, v_1, w_1, u_2, v_2, w_2, \ldots\}}$$ 3 N X 1 Vector

$$u = \sum_{i=1}^{N} N_i\, u_i; \quad v = \sum_{i=1}^{N} N_i\, v_i; \quad w = \sum_{i=1}^{N} N_i\, w_i$$



$$\boxed{\{\boldsymbol{U}\} = [\mathrm{N}]\{q\}}$$ 3 X 1 Vector

$$[N] = \begin{bmatrix} N_1 & 0 & 0 & N_2 & \ldots \\ 0 & N_1 & 0 & 0 & \ldots \\ 0 & 0 & N_1 & 0 & \ldots \end{bmatrix} \qquad \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \{\boldsymbol{U}\}$$

3 X 3 N Matrix

General Formulation

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$\{\varepsilon\} = [B]\{q\}$ 6 X 1 Vector $[B] = [B_1, B_2, \ldots]$ 6 X 3 N Matrix

$$[B_i] = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix}$$

6 X 3 Matrix

General Formulation

$$\{\sigma\} = [C]\{\varepsilon\}$$

6 X 1 Vector

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & SYM & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}$$

6 X 6 Matrix

$$\{\sigma\} = \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz}\}$$

$$\{\varepsilon\} = \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}\}$$

General Formulation

$$\Pi = U - W$$

$$U = \frac{1}{2}\int_V \{\epsilon\}^T\{\sigma\}dV \qquad W = \int_V \{U\}^T\{F\}dV \qquad \{F\} = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix}$$

$$\Pi = \frac{1}{2}\int_V \{\epsilon\}^T[C]\{\varepsilon\}dV - \int_V \{q\}^T[N]^T\{F\}dV$$

$$\Pi = \frac{1}{2}\int_V \{q\}^T[B]^T[C][B]\{q\}dV - \int_V \{q\}^T[N]^T\{F\}dV$$

$$\frac{\partial \Pi}{\partial q} = 0 \rightarrow \quad \int_V [B]^T[C][B]\{q\}dV - \int_V [N]^T\{F\}dV = 0$$

$$\boxed{[K]\{q\} = \{f\}} \qquad \boxed{[K] = \int_V [B]^T[C][B]\, dV} \qquad \boxed{\{f\} = \int_V [N]^T\{F\}dV}$$

Displacement Shape Functions

- Shape functions or Basis functions or Interpolation functions
  - Defines deformation shape (are assumed or approximated)

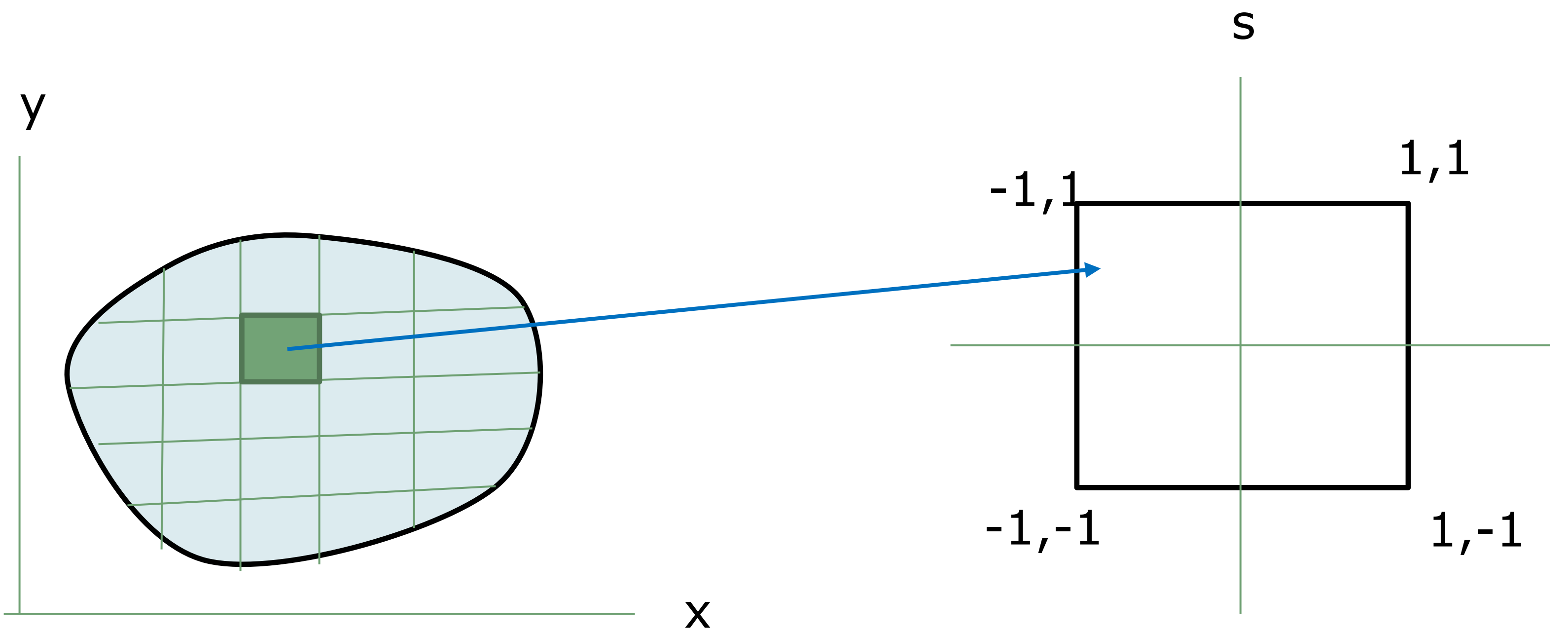
$$u(x) = \sum_{i=1}^{3} N_i\, u_i \; or \;\; u(x) = N_1(x)u_1 + N_2(x)u_2 + N_3(x)u_3$$

$$u(x) = u_1 + (x)u_2 + (x^2)u_3$$

$$N_1(x) = 1$$
$$N_2(x) = x$$
$$N_3(x) = x^2$$

Natural Coordinates

❖ Introduce a two-dimensional Natural Coordinate System *r-s*



Geometric Shape Functions

- Shape functions also define global – to – local (element) coordinate transformations

$$x = \sum_{i=1}^{N} N_i \, x_i$$

$$x = \sum_{i=1}^{3} N_i \, x_i \ or \ \ x = N_1(r)x_1 + N_2(r)x_2 + N_3(r)x_3$$

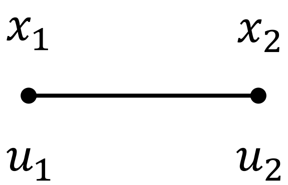
$$y = \sum_{i=1}^{N} N_i \, y_i$$

.. And so on

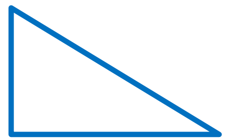
$$z = \sum_{i=1}^{N} N_i \, z_i$$

3P COMPOSITES

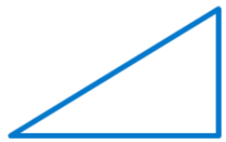
Linear Shape Functions



$$N_1(r) = \frac{1}{2}(1 - r)$$



$$N_2(r) = \frac{1}{2}(1 + r)$$



$$u(r) = \sum_{i=1}^{2} N_i\, u_i \; or \;\; u(r) = N_1(r)u_1 + N_2(r)u_2$$

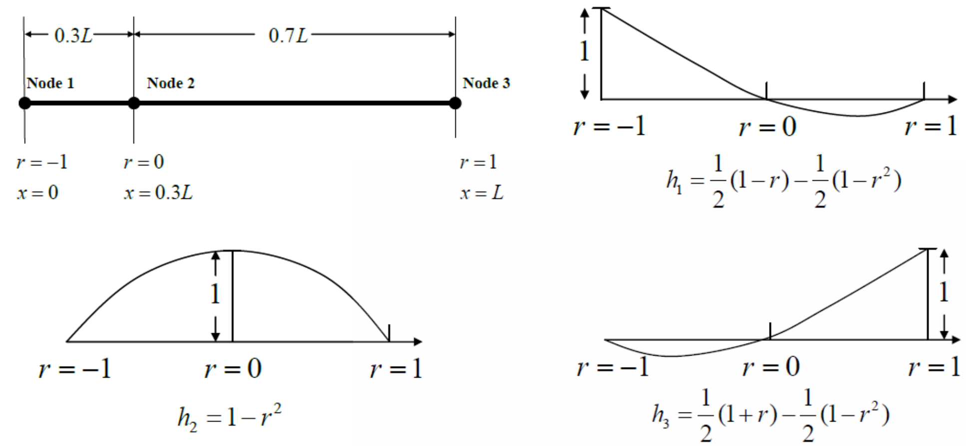
$$x(r) = \sum_{i=1}^{2} N_i\, x_i \; or \;\; x = N_1(r)x_1 + N_2(r)x_2$$

Quadratic Shape Functions

$$N_1(r) = \frac{1}{2}(1-r) - \frac{1}{2}(1-r^2)$$

$$N_2(r) = (1-r^2)$$

$$N_3(r) = \frac{1}{2}(1+r) - \frac{1}{2}(1-r^2)$$



$$u(r) = \sum_{i=1}^{3} N_i\, u_i \ or\ \ u(r) = N_1(r)u_1 + N_2(r)u_2 + N_3(r)u_3$$

$$x(r) = \sum_{i=1}^{3} N_i\, x_i \ or\ \ x = N_1(r)x_1 + N_2(r)x_2 + N_3(r)x_3$$

Differentiation - Chain rule

$$\frac{\partial \varphi}{\partial r} = \frac{\partial \varphi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \varphi}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial \varphi}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial \varphi}{\partial s} = \frac{\partial \varphi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \varphi}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial \varphi}{\partial z}\frac{\partial z}{\partial s} \Rightarrow \begin{Bmatrix} \frac{\partial \varphi}{\partial r} \\ \frac{\partial \varphi}{\partial s} \\ \frac{\partial \varphi}{\partial t} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{bmatrix} \begin{Bmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \\ \frac{\partial \varphi}{\partial z} \end{Bmatrix} \qquad \frac{\partial}{\partial \boldsymbol{r}} = \mathbf{J}\frac{\partial}{\partial \boldsymbol{x}} \Rightarrow \frac{\partial}{\partial \boldsymbol{x}} = \mathbf{J}^{-1}\frac{\partial}{\partial \boldsymbol{r}}$$

$$\frac{\partial \varphi}{\partial t} = \frac{\partial \varphi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \varphi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \varphi}{\partial z}\frac{\partial z}{\partial t}$$

$\mathbf{J}$ = Jacobian Matrix

$$\varphi = \{\boldsymbol{U}\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \qquad K = \int_V B^T C B \, dV$$

$$dV = dx\,dy\,dz = |J|\,\mathrm{dr\,ds\,dt}$$

$$K = \int_V B^T C B |J| drdsdt$$

Jacobian

- Jacobian transforms the physical coordinates to the natural coordinates. This transformation matrix is known as the Jacobian matrix and has terms that are functions of natural coordinates *r, s* and *t*

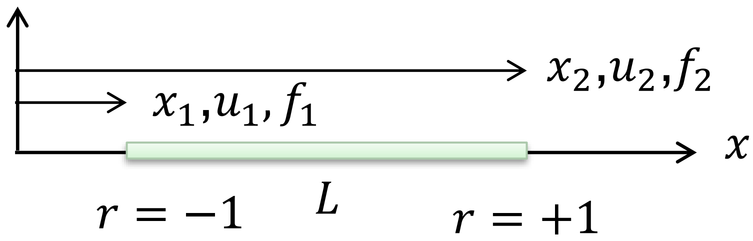
- Jacobian can be regarded as a scale factor / ratio between the length of physical coordinates and the length of natural coordinates

- Since the natural coordinates *r, s* and *t* has between +1 and -1 irrespective of the physical coordinates, numerical integration techniques can be used with ease to evaluate the Jacobian

- Jacobian Matrix is a square matrix which have dimension of 1x1 for 1D elements, 2x2 2D elements and 3x3 for 3D elements.

1-D Elements

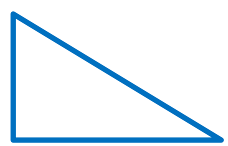
Bar Element



$$\varepsilon = \frac{du}{dx} = \frac{du}{dr}\frac{dr}{dx}$$

$$u(r) = \sum_{i=1}^{2} N_i\, u_i \;\; or \;\; u(r) = N_1(r)u_1 + N_2(r)u_2 \qquad x(r) = N_1(r)x_1 + N_2(r)x_2$$

$$N_1(r) = \frac{1}{2}(1-r)$$



$$N_2(r) = \frac{1}{2}(1+r)$$

$$\frac{du}{dr} = \frac{1}{2}(u_2 - u_1) \qquad \frac{dx}{dr} = \frac{1}{2}(x_2 - x_1) = \frac{L}{2} = J$$

$$\frac{du}{dx} = \frac{1}{L}(u_2 - u_1)$$

Bar Element

$$\varepsilon = [B]\{u\}$$

$$[B] = \frac{1}{L}[-1 \quad 1]$$

$$\{u\} = \{u_1, u_2\}$$

$$\sigma = E\varepsilon$$

$$K = \int_V B^T C B dV$$

$$K = \frac{1}{L^2}\int_{-1}^{1}[-1 \quad 1]^T E[-1 \quad 1]A\boldsymbol{J}dr$$

$$\boxed{[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

$$[K]\{u\} = \{f\}; \{f\} = \{f_1, f_2\}$$

Spring Element

L

x_1, u_1, f_1 $\qquad$ x_2, u_2, f_2

$$k = \frac{AE}{L}$$

$$\boxed{[K] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}$$

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Spring Element (Alternate)

$$\delta = u(L) - u(0) = u_2 - u_1$$

$$T = k\delta$$

$$T = k(u_2 - u_1)$$

$$f_{1x} = -T \qquad f_{2x} = T$$

$$T = -f_{1x} = k(u_2 - u_1)$$

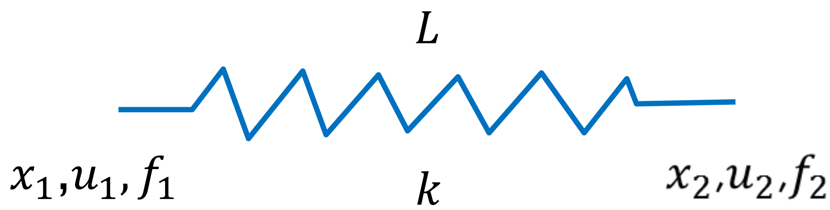
$$T = f_{2x} = k(u_2 - u_1)$$

$$f_{1x} = k(u_1 - u_2)$$

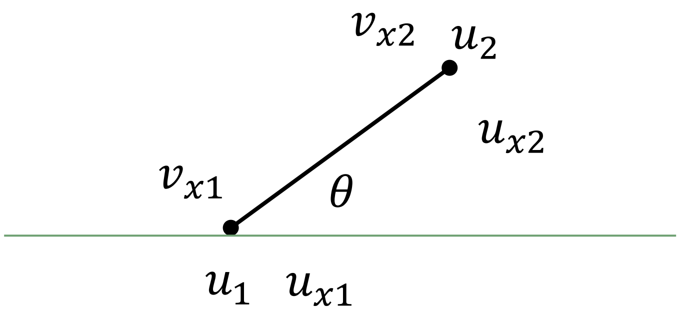
$$f_{2x} = k(u_2 - u_1)$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$[k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$



Bar Element (Rotated)



$$\{f'\} = [K']\{u'\}$$

$$u_2 - u_1 = \delta = (u_{x2} - u_{x1})\cos\theta + (v_{y2} - v_{y1})\sin\theta$$

$$F = (f_{x2} - f_{x1})\cos\theta + (f_{y2} - f_{y1})\sin\theta$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{Bmatrix} u_{x1} \\ v_{y1} \\ u_{x2} \\ v_{y2} \end{Bmatrix} \qquad \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{Bmatrix}$$

$$\{u\} = [T']\{u'\} \qquad \{f\} = [T']\{f'\}$$

Bar Element (Rotated)

$$[K]\{u\} = \{f\}$$

$$[K][T']\{u'\} = [T']\{f'\} \qquad \{f\} = [T']\{f'\} \qquad \{u\} = [T']\{u'\}$$

Can not invert $[T']$, so {u}, {f} and [K] need to expanded to 4x4 order

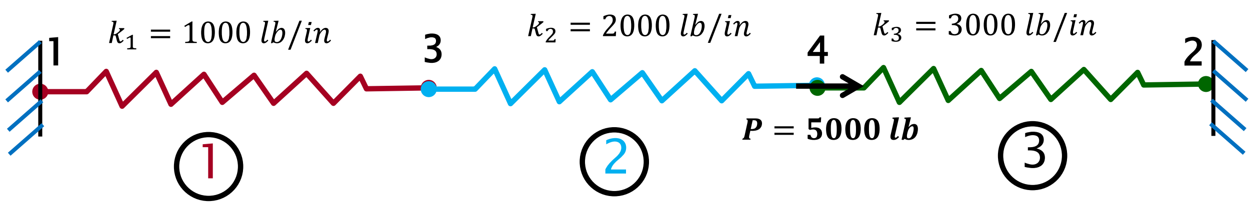
$$[T] = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \qquad [K] = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ v_{y1} \\ u_{x2} \\ v_{y2} \end{Bmatrix}}$$

$$[K'] = [T^T][K][T]$$

$$\{f'\} = [K']\{u'\}$$

Example 1 – Spring Elements



Element Matrices:

$$\begin{array}{cc} & \begin{array}{cc} 1 & 3 \end{array} \\ [k^{(1)}] = & \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{array}{c} 1 \\ 3 \end{array} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} 3 & 4 \end{array} \\ [k^{(2)}] = & \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix} \begin{array}{c} 3 \\ 4 \end{array} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} 4 & 2 \end{array} \\ [k^{(3)}] = & \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix} \begin{array}{c} 4 \\ 2 \end{array} \end{array}$$

Example 1 – Spring Elements

Global Stiffness Matrix:

$$[K] = \left[k^{(1)}\right] + \left[k^{(2)}\right] + \left[k^{(3)}\right]$$

$$[K] = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 1000+2000 & -2000 \\ 0 & -3000 & -2000 & 2000+3000 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}$$

Global System of Equations:

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

After Boundary Conditions:

$$\begin{Bmatrix} 0 \\ 5000 \end{Bmatrix} = \begin{bmatrix} 3000 & -2000 \\ -2000 & 5000 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

Solution:

$$u_3 = \frac{10}{11} \text{ in.} \qquad u_4 = \frac{15}{11} \text{ in.}$$

Example 1 – Spring Elements

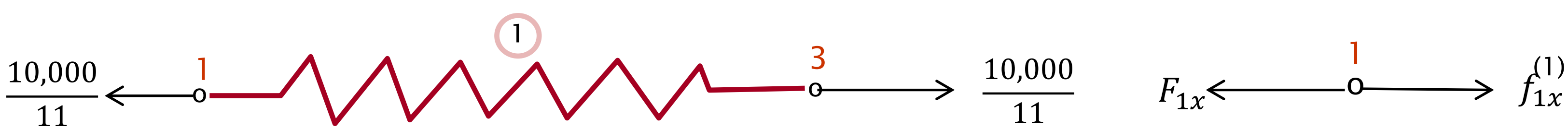
<u>Nodal Forces:</u>

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \frac{10}{11} \\ \frac{15}{11} \end{Bmatrix}$$

$$F_{1x} = \frac{-10{,}000}{11}\,\mathrm{lb} \qquad F_{2x} = \frac{-45{,}000}{11}\,\mathrm{lb} \qquad F_{3x} = 0 \qquad F_{4x} = \frac{55{,}000}{11}\,\mathrm{lb} = 5000\,\mathrm{lb}$$

<u>Element 1:</u>

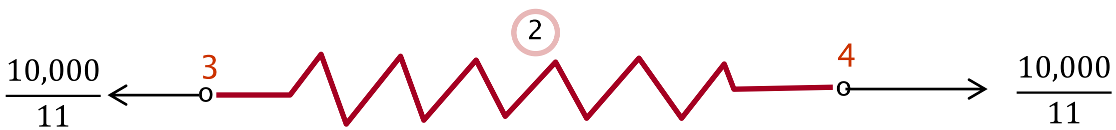
$$\begin{Bmatrix} f_{1x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ \frac{10}{11} \end{Bmatrix} \qquad f_{1x} = \frac{-10{,}000}{11}\,lb \qquad f_{3x} = \frac{10{,}000}{11}\,lb$$



Example 1 – Spring Elements

Element 2:

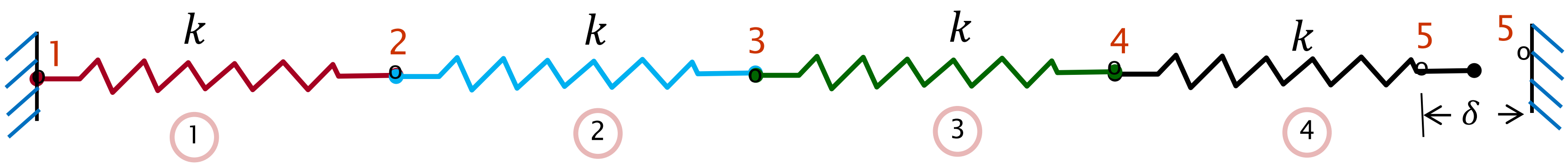
$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix} \begin{Bmatrix} \frac{10}{11} \\ \frac{15}{11} \end{Bmatrix} \qquad f_{3x} = \frac{-10{,}000}{11} lb \qquad f_{4x} = \frac{10{,}000}{11} lb$$



Element 3:

$$\begin{Bmatrix} f_{4x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix} \begin{Bmatrix} \frac{15}{11} \\ 0 \end{Bmatrix} \qquad f_{4x} = \frac{45{,}000}{11} lb \qquad f_{2x} = \frac{-45{,}000}{11} lb$$

Example 2 – Spring Elements



Element Matrices:

$$[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{4}] = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix}$$

Global Stiffness Matrix:

$$[K] = \begin{bmatrix} 200 & -200 & 0 & 0 & 0 \\ -200 & 400 & -200 & 0 & 0 \\ 0 & -200 & 400 & -200 & 0 \\ 0 & 0 & -200 & 400 & -200 \\ 0 & 0 & 0 & -200 & 200 \end{bmatrix} \frac{k\text{N}}{m}$$

Example 2 – Spring Elements

Global System of Equations:

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \\ F_{5x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 & 0 & 0 & 0 \\ -200 & 400 & -200 & 0 & 0 \\ 0 & -200 & 400 & -200 & 0 \\ 0 & 0 & -200 & 400 & -200 \\ 0 & 0 & 0 & -200 & 200 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$$

After Boundary Conditions:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} -200 & 400 & -200 & 0 & 0 \\ 0 & -200 & 400 & -200 & 0 \\ 0 & 0 & -200 & 400 & -200 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ u_4 \\ 0.02\ \mathrm{m} \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 4\ k\mathrm{N} \end{Bmatrix} = \begin{bmatrix} 400 & -200 & 0 \\ -200 & 400 & -200 \\ 0 & -200 & 400 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Example 2 – Spring Elements

Solution:

$u_2 = 0.005\text{m}$ $\quad$ $u_3 = 0.01\text{m}$ $\quad$ $u_4 = 0.015\text{m}$

Nodal Forces:

$$F_{1X} = (-200)(0.005) = -1.0k\text{N}$$

$$F_{2x} = (400)(0.005) - (200)(0.01) = 0$$

$$F_{3x} = (-200)(0.005) + (400)(0.01) - (200)(0.015) = 0$$

$$F_{4x} = (-200)(0.01) + (400)(0.015) - (200)(0.02) = 0$$

$$F_{5x} = (-200)(0.015) + (200)(0.02) = 1.0k\text{N}$$

Element 1

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.005 \end{Bmatrix} \qquad f_{1x}^{(1)} = -1.0\ k\text{N} \qquad f_{2x}^{(1)} = 1.0\ k\text{N}$$

Example 2 – Spring Elements

Element 2

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.005 \\ 0.01 \end{Bmatrix}$$

$f_{2x}^{(2)} = -1\ k\mathrm{N}$ $\quad f_{3x}^{(2)} = 1\ k\mathrm{N}$

Element 3

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.01 \\ 0.015 \end{Bmatrix}$$

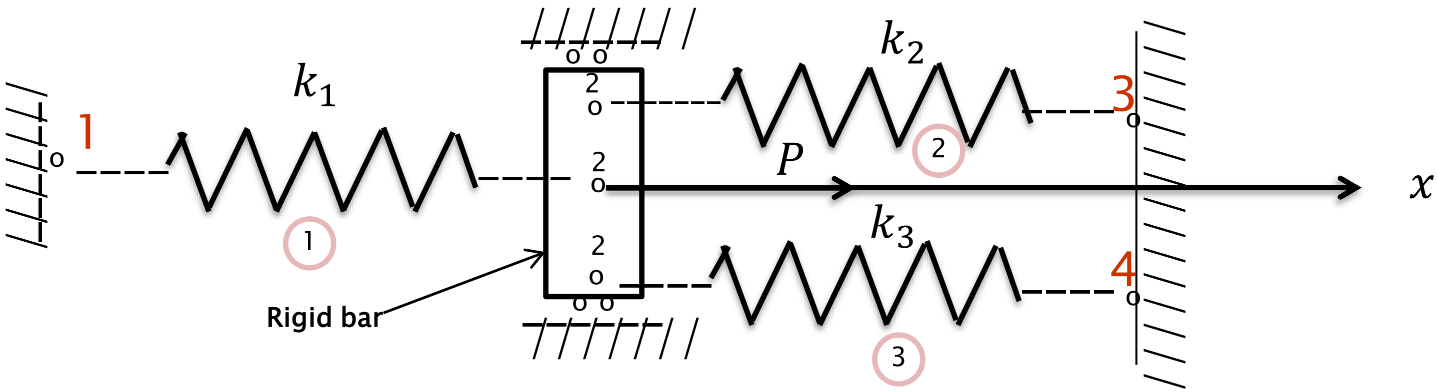
$f_{3x}^{(3)} = -1\ k\mathrm{N}$ $\quad f_{4x}^{(3)} = 1\ k\mathrm{N}$

Element 4

$$\begin{Bmatrix} f_{4x} \\ f_{5x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.015 \\ 0.02 \end{Bmatrix}$$

$f_{4x}^{(4)} = -1\ k\mathrm{N}$ $\quad f_{5x}^{(4)} = 1\ k\mathrm{N}$

Example 3 – Spring Elements



Boundary Conditions:

$u_1 = 0 \qquad u_3 = 0 \qquad u_4 = 0$

Compatibility Condition @ Node 2:

$$u_2^{(1)} = u_2^{(2)} = u_2^{(3)} = u_2$$

Nodal Equilibrium Conditions:

$$F_{1x} = f_{1x}^{(1)} \qquad P = f_{2x}^{(1)} + f_{2x}^{(2)} + f_{2x}^{(3)} \qquad F_{3x} = f_{3x}^{(2)} \qquad F_{4x} = f_{4x}^{(3)}$$

Example 3 – Spring Elements

Element Matrices:

$$\begin{array}{c} \\ [k^{(1)}] = \end{array}\begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ [k^{(2)}] = \end{array}\begin{array}{c} \begin{array}{cc} u_2 & u_3 \end{array} \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ [k^{(3)}] = \end{array}\begin{array}{c} \begin{array}{cc} u_2 & u_4 \end{array} \\ \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \end{array}$$

Global Stiffness Matrix:

$$\begin{array}{c} \\ [K] = \end{array}\begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \end{array}$$

Example 3 – Spring Elements

Global System of Equations:

$$\begin{Bmatrix} F_{1x} \\ P \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Solution:

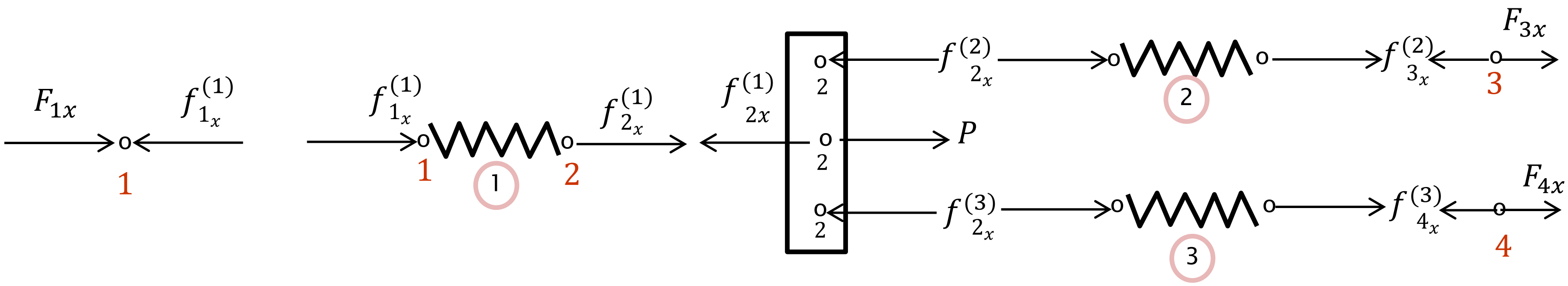
$$u_2 = \frac{P}{k_1 + k_2 + k_3}$$ After BCs

Nodal Forces:

$F_{1x} = -k_1 u_2$ $F_{3x} = -k_2 u_2$ $F_{4x} = -k_3 u_2$

Example 3 – Spring Elements

Free body diagram:



Global Equilibrium Equations:

$$F_{1x} = k_1u_1 - k_1u_2$$

$$P = -k_1u_1 + k_1u_2 + k_2u_2 - k_2u_3 + k_3u_3 - k_3u_4$$

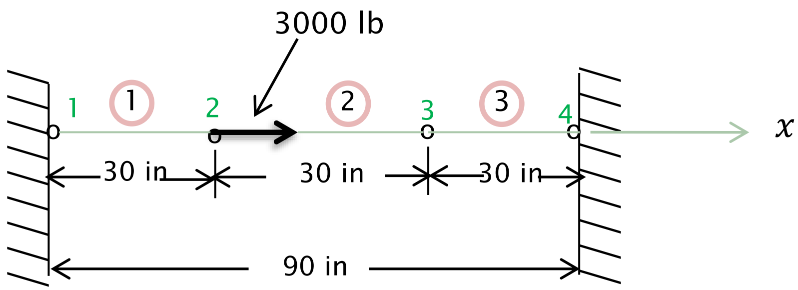
$$F_{3x} = -k_2u_2 + k_2u_3$$

$$F_{4x} = -k_3u_2 + k_3u_4$$

Same as earlier !

Example 4 – Bar Elements

Three bar assemblage



Element Matrices:

$$[k^{(1)}] = [k^{(2)}] = \frac{(1)(30\text{ x}10^6)}{30}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{array}{c} \begin{matrix} 1 & 2^{(1)} \\ 2 & 3^{(2)} \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}}$$

$$[k^{(3)}] = \frac{(2)(15\text{ x}10^6)}{30}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{array}{c} \begin{matrix} 3 & 4 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}}$$

Example 4 – Bar Elements

Global Stiffness Matrix:

$$[K] = 10^6 \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+1 & -1 & 0 \\ 0 & -1 & 1+1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}}$$

Global system of equations:

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

After boundary conditions:

$$\begin{Bmatrix} 3000 \\ 0 \end{Bmatrix} = 10^6 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \qquad u_1 = 0 \qquad u_4 = 0$$

Solution:

$u_2 = 0.002$ in. $\qquad$ $u_3 = 0.001$ in.

Example 4 – Bar Elements

Nodal Forces:

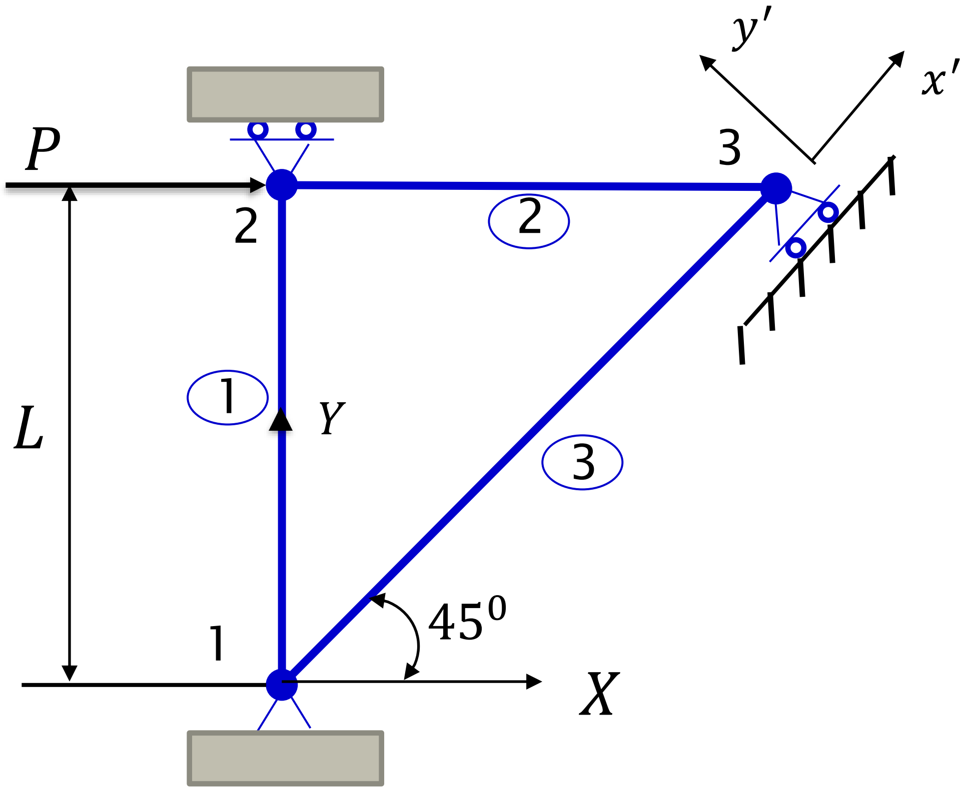
$$F_{1x} = 10^6(\,u_1 - u_2) = 10^6(0 - 0.002) = -2000\ \text{lb}$$

$$F_{2x} = 10^6(-u_1 + 2u_2 - u_3) = 10^6[0 + 2(0.002) - 0.001] = 3000\ \text{lb}$$

$$F_{3x} = 10^6(-u_2 + 2u_3 - u_4) = 10^6[-0.002 + 2(0.001) - 0] = 0$$

$$F_{4x} = 10^6(-u_3 + u_4) = 10^6(-0.001 + 0) = -1000\ \text{lb}$$

Example 5 – Bar Elements



Properties for plane truss:

$$P = 1000kN$$

$$L = 1m$$

$$E = 210GPa$$

$$A_1 = A_2 = 6.0 \text{ x } 10^{-4} m^2$$

$$A_3 = 6\sqrt{2} \text{ x } 10^{-4} m^2$$

An example of multipoint constraints !

Example 5 – Bar Elements

<u>Stiffness Matrix Element 1:</u>

$\theta = 90^0, \qquad c = 0, \qquad s = 1$

$$k_1 = \frac{(210 \text{ x } 10^9)(6.0 \text{ x } 10^{-4})}{1} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} (\text{N/m})$$

<u>Stiffness Matrix Element 2:</u>

$\theta = 0^0, \qquad c = 1, \qquad s = 0$

$$k_2 = \frac{(210 \text{ x } 10^9)(6.0 \text{ x } 10^{-4})}{1} \begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} (\text{N/m})$$

Example 5 – Bar Elements

Stiffness Matrix Element 3:

$$\theta = 45^0, \qquad c = \frac{1}{\sqrt{2}}, \qquad s = \frac{1}{\sqrt{2}}$$

$$\mathrm{k}_3 = \frac{(210 \text{ x } 10^9)(6\sqrt{2} \text{ x } 10^{-4})}{\sqrt{2}} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \end{array} (\mathrm{N/m})$$

Global Equations:

$$1260 \text{ x } 10^5 \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ & 1.5 & 0 & -1 & -0.5 & -0.5 \\ & & 1 & 0 & -1 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1.5 & 0.5 \\ sym. & & & & & 0.5 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{1X} \\ F_{1Y} \\ F_{2X} \\ F_{2Y} \\ F_{3X} \\ F_{3Y} \end{Bmatrix}$$

Example 5 – Bar Elements

Boundary conditions:

$u_1 = v_1 = v_2 = 0$ and $\grave{v}_3 = 0$ $\quad F_{2x} = P, \quad F_{3\grave{x}} = 0$

Transformed Boundary conditions:

$$\grave{v}_3 = [-s \;\; c]\begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \frac{1}{\sqrt{2}}(-u_3 + v_3) = 0 \qquad \theta = 45^0, \qquad c = \frac{1}{\sqrt{2}}, \qquad s = \frac{1}{\sqrt{2}}$$

$u_3 - v_3 = 0; MPC$

$$F_{3\grave{x}} = [c \;\; s]\begin{Bmatrix} F_{3X} \\ F_{3Y} \end{Bmatrix} = \frac{1}{\sqrt{2}}(F_{3X} + F_{3Y}) = 0$$

$F_{3X} + F_{3Y} = 0$; Force Relation

Example 5 – Bar Elements

Global Equations after BCs:

$$1260 \text{ x } 10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} P \\ F_{3X} \\ F_{3Y} \end{Bmatrix}$$

Global Equations after MPCs:

$$1260 \text{ x } 10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P \\ F_{3X} \\ -F_{3X} \end{Bmatrix} \equiv \quad 1260 \text{ x } 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P \\ F_{3X} \\ -F_{3X} \end{Bmatrix}$$

$$F_{3X} = -1260 \text{ x } 10^5 u_3$$

Reduced Global Equation:

$$1260 \text{ x } 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

Example 5 – Bar Elements

Solution:

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2520 \text{ x } 10^5} \begin{Bmatrix} 3P \\ P \end{Bmatrix} = \begin{Bmatrix} 0.01191 \\ 0.003968 \end{Bmatrix} \text{ (m)}$$

$$v_3 = u_3 = 0.003968\, m$$

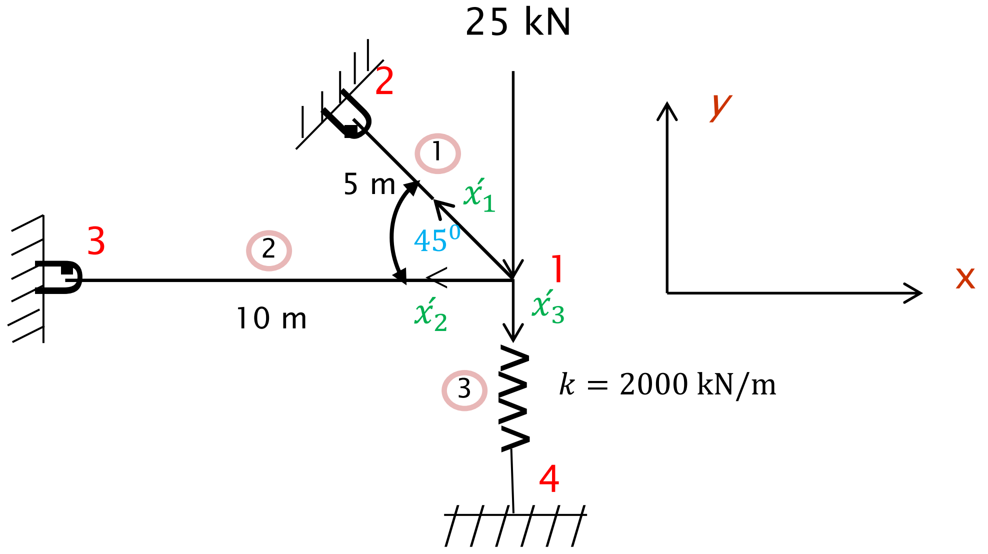
Nodal Forces:

$$\begin{Bmatrix} F_{1X} \\ F_{1Y} \\ F_{2Y} \\ F_{3X} \\ F_{3Y} \end{Bmatrix} = 1260 \text{ x } 10^5 \begin{bmatrix} 0 & -0.5 & -0.5 \\ 0 & -0.5 & -0.5 \\ 0 & 0 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} -500 \\ -500 \\ 0.0 \\ -500 \\ 500 \end{Bmatrix} \text{(kN)}$$

$$F_{2X} = \text{P}$$

Example 6 – Bar & Spring Elements

Two-bar truss with spring support



Example 6 – Bar & Spring Elements

Stiffness Matrix Element 1:

$$\theta^{(1)} = 135^0, \qquad \cos\theta^{(1)} = -\sqrt{2}/2, \qquad \sin\theta^{(1)} = \sqrt{2}/2$$

$$[k^{(1)}] = \frac{(5.0 \text{ x } 10^{-4}\text{m}^2)(210 \text{ x}10^6 k\text{ N/m}^2)}{5\text{m}} \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

$$[k^{(1)}] = 105 \text{ x } 10^2 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 6 – Bar & Spring Elements

Stiffness Matrix Element 2:

$$\theta^{(2)} = 180^0, \qquad \cos\theta^{(2)} = -1.0, \qquad \sin\theta^{(2)} = 0$$

$$\left[k^{(2)}\right] = \frac{(5 \text{ x } 10^{-4}\text{m}^2)(210 \text{ x} 10^6 k\text{ N/m}^2)}{10\text{m}}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[k^{(2)}\right] = 105 \text{ x } 10^2 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 6 – Bar & Spring Elements

Stiffness Matrix Element 3:

$$\theta^{(3)} = 270^0, \qquad \cos\theta^{(3)} = 0, \qquad \sin\theta^{(3)} = -1.0$$

$$\left[k^{(3)}\right] = 20 \text{ x } 10^2 \begin{array}{c} \begin{matrix} u_1 & v_1 & u_4 & v_4 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Boundary conditions:

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

Global system of equations after BCs:

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{1y} = -25kN \end{Bmatrix} = 10^2 \begin{bmatrix} 210 & -105 \\ -105 & 125 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

Solution:

$$u_1 = -1.724 \text{ x } 10^{-3}\text{m} \qquad v_1 = -3.448 \text{ x } 10^{-3}\text{m}$$

Example 6 – Bar & Spring Elements

Stresses in Bar Elements:

$$\sigma^{(1)} = \frac{210 \text{ x } 10^3 \text{ MN/m}^2}{5\text{m}} \begin{bmatrix} 0.707 & -0.707 & -0.707 & 0.707 \end{bmatrix} \begin{Bmatrix} -1.724 \text{ x } 10^{-3} \\ -3.448 \text{ x } 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

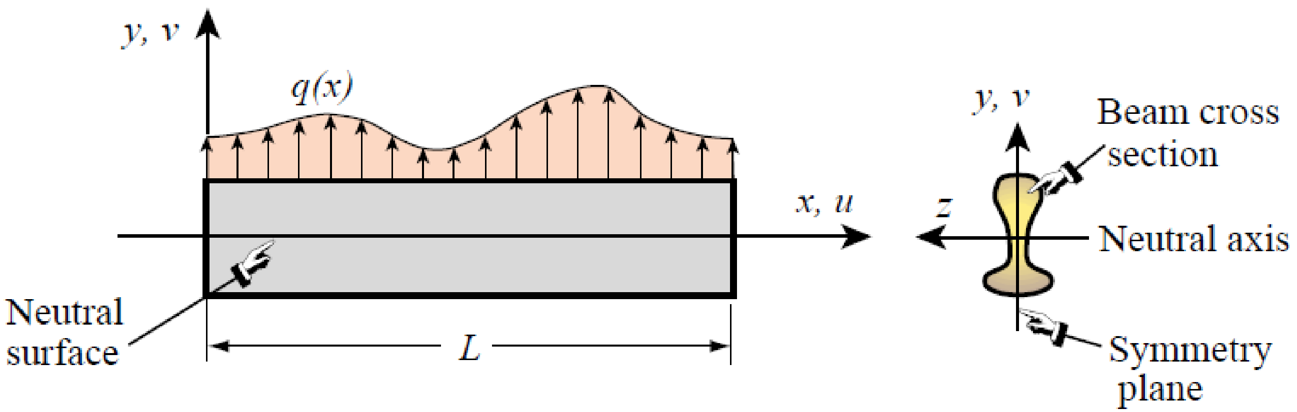
$$\sigma^{(1)} = 51.2 \text{ MP}a(T)$$

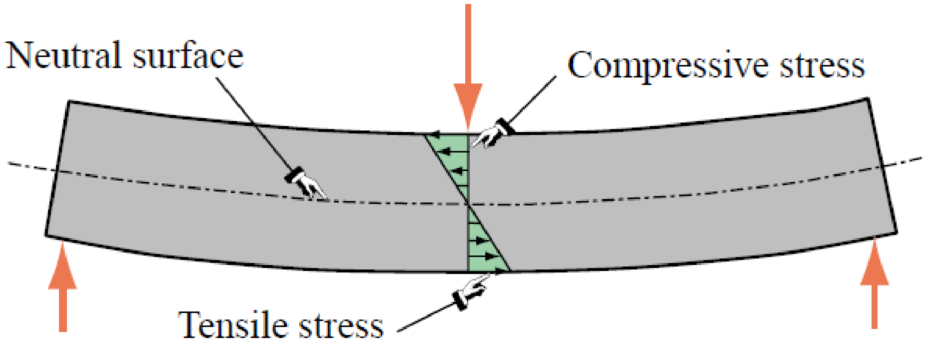
$$\sigma^{(2)} = \frac{210 \text{ x } 10^3 \text{ MN/m}^2}{10\text{m}} \begin{bmatrix} 1.0 & 0 & -1.0 & 0 \end{bmatrix} \begin{Bmatrix} -1.724 \text{ x } 10^{-3} \\ -3.448 \text{ x } 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma^{(2)} = -36.2 \text{ MP}a(\text{C})$$

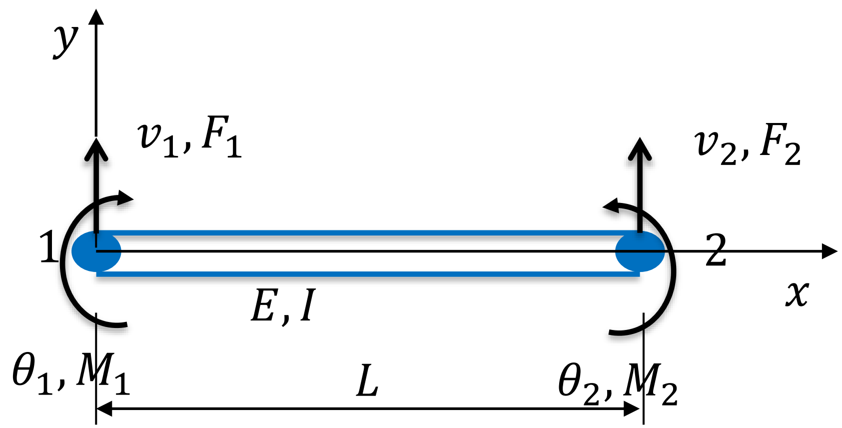
Beam Element

BEAM ELEMENT





Beam Element



L	*length of the beam*
I	*moment of inertia of the cross-sectional area*
E	*elastic modulus*
$v = v(x)$	*deflection (lateral displacement) of the neutral axis*
$\theta = \theta(x)$	*slope of the neutral axis*
$\kappa = \kappa(x)$	*curvature of the neutral axis*

Beam Element (Direct Method)

Displacement Approximations:

$$v(x) = a_1x^3 + a_2x^2 + a_3x + a_4$$

Why Cubic?

- Four dofs
- Continuity of displacement & slope
- *Continuity of moment and non-zero shear force* $(V = \frac{dM}{dx})$

Hermite Cubic Shape Functions:

$$v(0) = v_1 = a_4$$

$$\frac{dv(0)}{dx} = \theta_1 = a_3 \qquad where\ \theta = \frac{dv}{dx}$$

$$v(L) = v_2 = a_1L^3 + a_2L^2 + a_3L + a_4$$

$$\frac{dv(L)}{dx} = \theta_2 = 3a_1L^2 + 2a_2L = a_3$$

Beam Element (Direct Method)

Hermite Cubic Shape Functions:

$$v = \left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\theta_1 + \theta_2)\right] x^3 + \left[-\frac{3}{L^2}(v_1 - v_2) - \frac{1}{L^2}(2\theta_1 + \theta_2)\right] x^2 + \theta_1 x + v_1$$

$v = [N]\{d\}$ where

$$[N] = [N_1\ N_2\ N_3\ N_4] \qquad \{d\} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$N_1 = \frac{1}{L^3}(2x^3 - 3x^2L + L^3)$$

$$N_2 = \frac{1}{L^3}(x^3L - 2x^2L^2 + xL^3)$$

$$N_3 = \frac{1}{L^3}(-2x^3 + 3x^2L)$$

$$N_4 = \frac{1}{L^3}(x^3L - x^2L^2)$$

Hermite Cubic Interpolation Functions !

Beam Element (Direct Method)

Beam Theory:

$$M(x) = EI\frac{d^2v}{dx^2}; \qquad F(x) = EI\frac{d^3v}{dx^3}$$

Element Forces and Moments:

$$F_1 = V(0) = EI\frac{d^3v(0)}{dx^3} = \frac{EI}{L^3}(12v_1 + 6L\theta_1 - 12v_2 + 6L\theta_2)$$

$$M_1 = -M(0) = -EI\frac{d^2v(0)}{dx^2} = \frac{EI}{L^3}(6Lv_1 + 4L^2\theta_1 - 6Lv_2 + 2L^2\theta_2)$$

$$F_2 = -V(L) = -EI\frac{d^3v(L)}{dx^3} = \frac{EI}{L^3}(-12v_1 - 6L\theta_1 + 12v_2 - 6L\theta_2)$$

$$M_2 = M(L) = -EI\frac{d^2v(L)}{dx^2} = \frac{EI}{L^3}(6Lv_1 + 2L^2\theta_1 - 6Lv_2 + 4L^2\theta_2)$$

Beam Element (Direct Method)

Element system of equations :

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

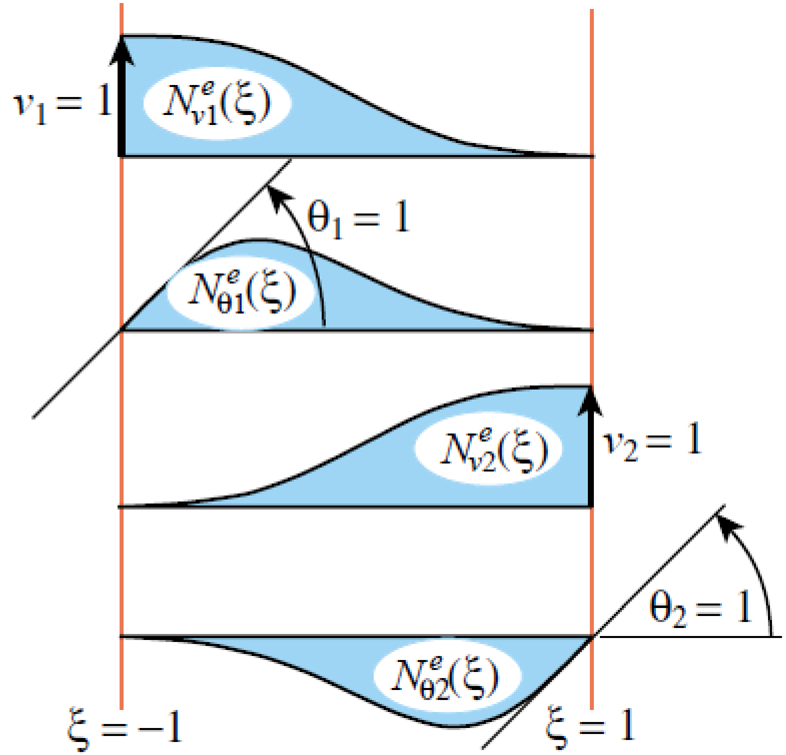
Element Stiffness :

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Beam Element (Alternate Method)

Shape Functions:

$$v = \left[N_{v1} N_{\theta 1} N_{v2} N_{\theta 2}\right] \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$



$$N_{v1}(r) = \frac{1}{4}(1-r)^2(2+r)$$

$$N_{v2}(r) = \frac{1}{4}(1+r)^2(2-r)$$

$$N_{\theta 1}(r) = \frac{1}{8}L(1-r)^2(1+r)$$

$$N_{\theta 2}(r) = -\frac{1}{8}L(1+r)^2(1-r)$$

$$r = \frac{2x}{L} - 1$$

Beam Element (Alternate Method)

Strain– Displacement & Hooke's Law:

$$U(x,y) = -y\frac{dv}{dx}$$

$$V(x,y) = v(x)$$

$$\varepsilon_x(x,y) = \frac{dU}{dx} = -y\frac{d^2v}{dx^2} = \kappa y$$

$$\sigma_x(x,y) = E\varepsilon = -Ey\frac{d^2v}{dx^2}$$

$$\varepsilon = y[B]\{v\}$$

$$\kappa = -\left[\ \frac{d^2N_{v1}}{dx^2}\ \frac{d^2N_{\theta 1}}{dx^2}\ \frac{d^2N_{v2}}{dx^2}\ \frac{d^2N_{\theta 2}}{dx^2}\ \right]\begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

$$[B] = -\left[\ \frac{d^2N_{v1}}{dx^2}\ \frac{d^2N_{\theta 1}}{dx^2}\ \frac{d^2N_{v2}}{dx^2}\ \frac{d^2N_{\theta 2}}{dx^2}\ \right]$$

Beam Element (Alternate Method)

<u>Chain Rule:</u>

$$\frac{df(x)}{dx} = \frac{df(r)}{dr}\frac{dr}{dx} = \frac{2}{L}\frac{df(r)}{dr} \qquad r = \frac{2(x - x_1)}{L} - 1$$

$$\frac{d^2f(x)}{dx^2} = \cancelto{0}{\frac{d(2/L)}{dx}}\frac{df(r)}{dr} + \frac{2}{L}\frac{d}{dx}\left(\frac{df(r)}{dr}\right) = \frac{4}{L^2}\frac{d^2f(r)}{dr}$$

$$[B] = -\frac{4}{L^2}\left[\ \frac{d^2N_{v1}}{dr^2}\ \frac{d^2N_{\theta 1}}{dr^2}\ \frac{d^2N_{v2}}{dr^2}\ \frac{d^2N_{\theta 2}}{dr^2}\ \right]$$

$$\boxed{[B] = -\frac{1}{L}\left[6\frac{r}{L} \quad 3r - 1 \quad -6\frac{r}{L} \quad 3r + 1\right]}$$

Beam Element (Alternate Method)

Element Stiffness Matrix:

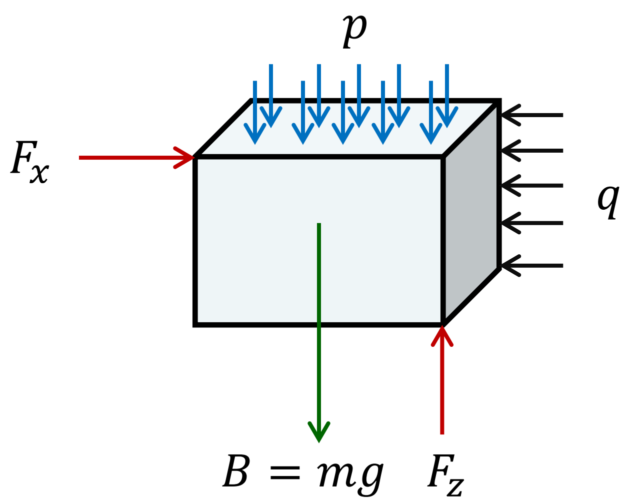
$$[K] = \int_V \mathbf{B}^T E\, \mathbf{B}\ y^2 dV = \int_{-1}^{1} EI\, \mathbf{B}^T \mathbf{B}\ \frac{L}{2}\, dr$$

$$I = \int_A y^2 dA \qquad dx = \frac{L}{2}\, dr$$

$$[K] = \frac{EI}{2L^3} \int_{-1}^{1} \begin{bmatrix} 36r^2 & 6r(3r-1) & -36r^2 & 6r(3r+1)L \\ & (3r-1)^2 L^2 & -6r(3r-1)L & (9r^2-1)L^2 \\ & & 36r^2 & -6r(3r+1)L \\ symm & & & (3r+1)^2 L^2 \end{bmatrix} dr$$

$$[K] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Beam Element – Force Vector



$$\{f\} = \int_V [N]^T \{F\} dV$$

$$\boxed{\{f\} = \int_L [N]^T \{q\} dx + \int_A [N]^T \{p\} dA + \int_V [N]^T \{B\} dV + \sum N_i F_i}$$

Beam Element – Force Vector

$$\{f\} = \int_0^L \mathbf{N}^T q \, dx = \int_{-1}^{1} \mathbf{N}^T \, q \, \frac{L}{2} \, dr$$

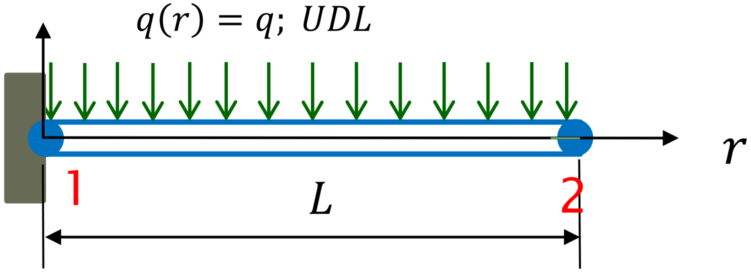
$$N_{v1}(r) = \frac{1}{4}(1-r)^2(2+r)$$

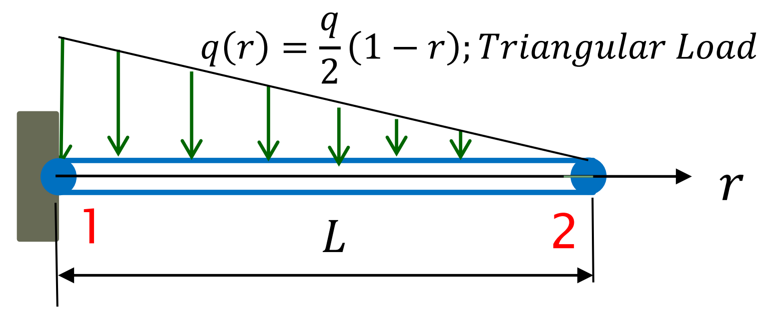
$$N_{v2}(r) = \frac{1}{4}\,(1+r)^2(2-r)$$

$$N_{\theta 1}(r) = \frac{1}{8}\,L(1-r)^2(1+r)$$

$$N_{\theta 2}(r) = -\frac{1}{8}L\,(1+r)^2(1-r)$$

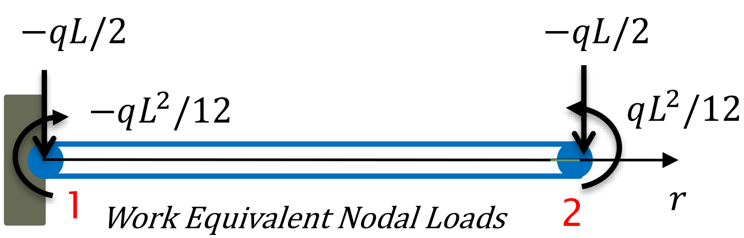
Beam Element – Force Vector

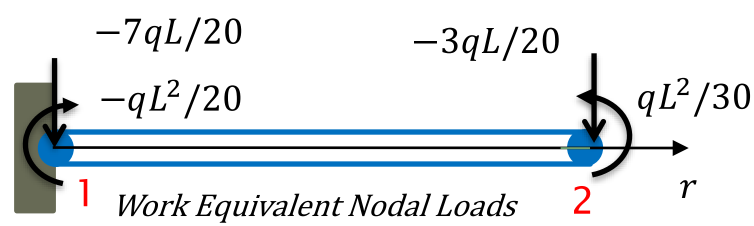




$$\{f\} = \frac{qL}{2}\begin{Bmatrix} -1 \\ -\frac{L}{6} \\ -1 \\ \frac{L}{6} \end{Bmatrix}$$

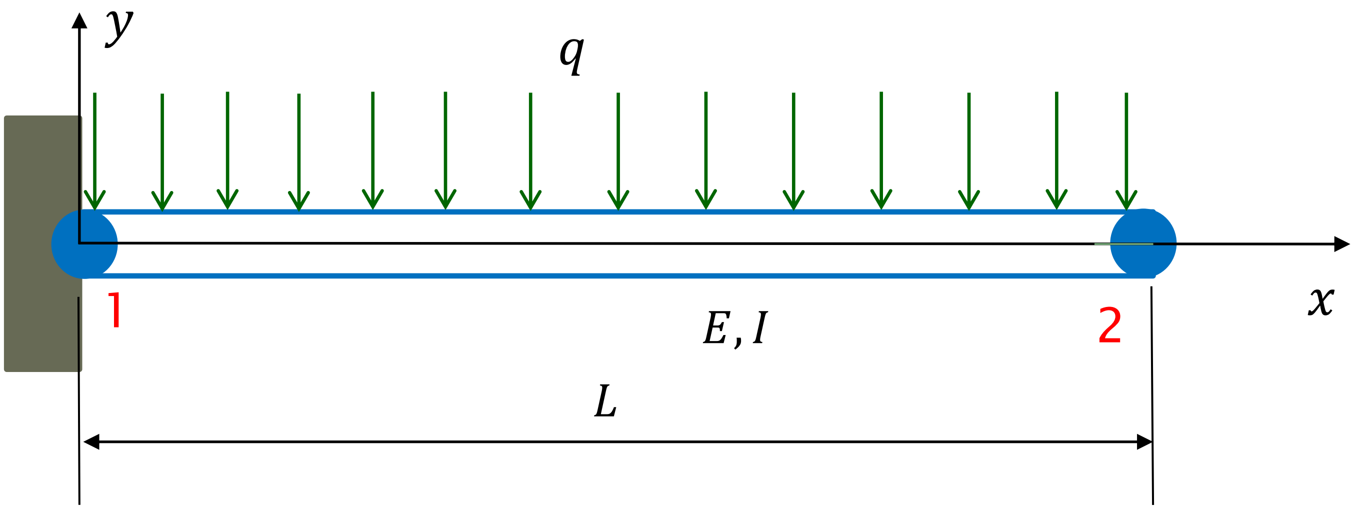
$$\{f\} = \frac{qL}{2}\begin{Bmatrix} -\frac{7}{10} \\ -\frac{L}{10} \\ -\frac{3}{10} \\ \frac{L}{15} \end{Bmatrix}$$





Example 7 - Beam Element

A cantilever beam with distributed lateral load q



Global [K] :

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = -qL\begin{Bmatrix} \frac{1}{2} \\ \frac{L}{12} \\ \frac{1}{2} \\ -\frac{L}{12} \end{Bmatrix}$$

Example 7 - Beam Element

BCs and Loads:

$$v_1 = \theta_1 = 0$$

Reduced Equations:

$$\frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = qL\begin{Bmatrix} -1/2 \\ L/12 \end{Bmatrix}$$

Solution:

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -qL^4/8EI \\ -qL^3/6EI \end{Bmatrix}$$

Downward displacement
Clockwise rotation
Exact at nodes !

Example 7 - Beam Element

<u>Nodal Forces:</u>

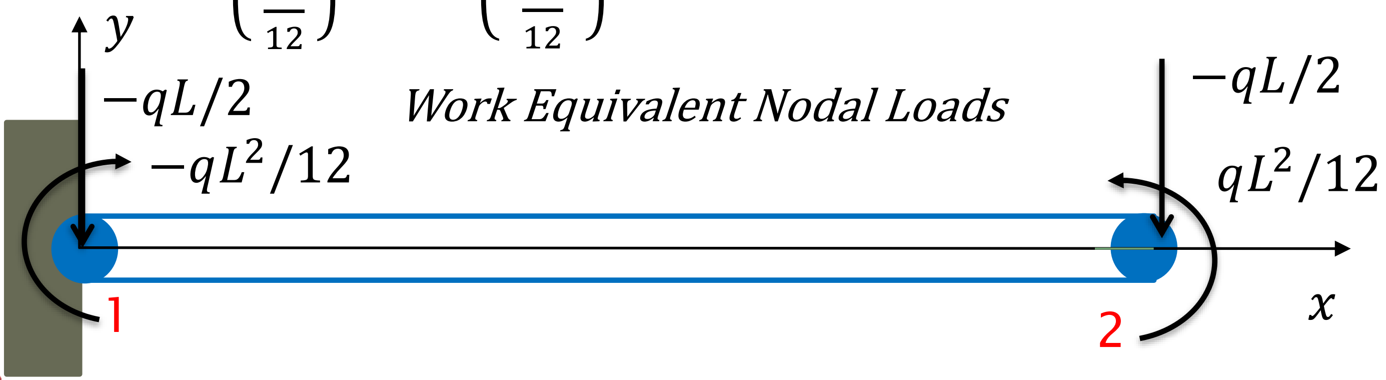
$$\begin{Bmatrix} F_1 \\ M_1 \end{Bmatrix} = \frac{L^3}{EI}\begin{bmatrix} -12 & 6L \\ -6L & 2L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} q\,L/2 \\ 5qL^2/12 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ M_2 \end{Bmatrix} = \frac{L^3}{EI}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -q\,L/2 \\ qL^2/12 \end{Bmatrix}$$

Effective Global Nodal Forces

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = qL\begin{Bmatrix} \frac{1}{2} \\ \frac{5L}{12} \\ -\frac{1}{2} \\ \frac{L}{12} \end{Bmatrix} - qL\begin{Bmatrix} -\frac{1}{2} \\ -\frac{L}{12} \\ -\frac{1}{2} \\ \frac{L}{12} \end{Bmatrix} = qL\begin{Bmatrix} 1 \\ \frac{L}{2} \\ 0 \\ 0 \end{Bmatrix}$$

Eq. 4.4.11: Subtract applied forces
Correct Nodal Forces !



Example 7 - Beam Element

Exact Beam Solution:

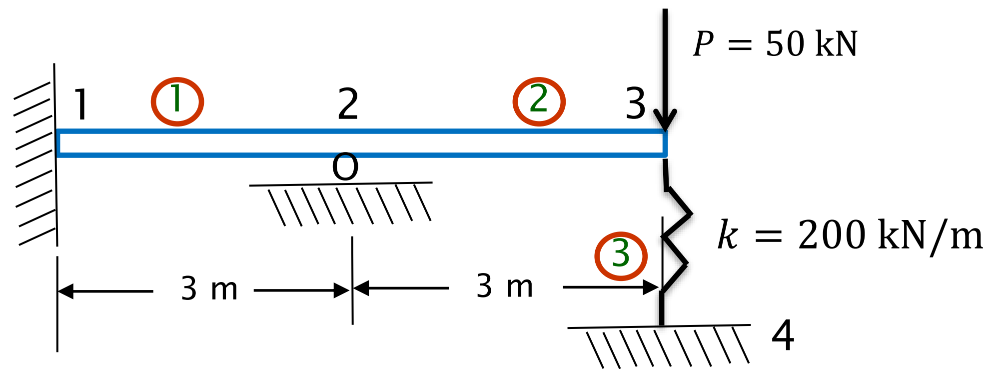
$$v(x) = \frac{1}{EI}\left(\frac{-qx^4}{24} + \frac{qLx^3}{6} - \frac{qL^2x^2}{4}\right)$$

$$\theta(x) = \frac{1}{EI}\left(\frac{-qx^3}{6} + \frac{qLx^2}{2} - \frac{qL^2x}{2}\right)$$

$$\begin{Bmatrix} v(L) \\ \theta(L) \end{Bmatrix} = \begin{Bmatrix} -qL^4/8EI \\ -qL^3/6EI \end{Bmatrix}$$

- Beam theory predicts a quartic (4th order) polynomial for *v(x)*
- FEA assumes a cubic polynomial for *v(x)*
- FEA solution is exact at nodes
- FEA solution predicts lower displacement for $0 < x < L$ (Prove?)

Example 8 - Beam Element



Global [K] :

$$
[K] = \frac{EI}{L^3}
\begin{array}{c}
\begin{array}{ccccccc} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 & v_4 \end{array} \\
\begin{bmatrix}
12 & 6L & -12 & 6L & 0 & 0 & 0 \\
 & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\
 & & 24 & 0 & -12 & 6L & 0 \\
 & & & 8L^2 & -6L & 2L^2 & 0 \\
 & & & & 12 + \dfrac{kL^3}{EI} & -6L & -\dfrac{kL^3}{EI} \\
 & & & & & 4L^2 & 0 \\
symm & & & & & & \dfrac{kL^3}{EI}
\end{bmatrix}
\end{array}
$$

Example 8 - Beam Element

Global system of equations :

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \\ F_4 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ & & 24 & 0 & -12 & 6L & 0 \\ & & & 8L^2 & -6L & 2L^2 & 0 \\ & & & & 12+k' & -6L & -k' \\ & & & & & 4L^2 & 0 \\ symm & & & & & & k' \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \end{Bmatrix}$$

$$k' = \frac{kL^3}{EI}$$

BCs:

$v_1 = 0$ $\theta_1 = 0$ $v_2 = 0$ $v_4 = 0$

Reduced equations :

$$\begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 8L^2 & -6L & -2L^2 \\ -6L & 12+k' & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

Example 8 - Beam Element

Solution:

$$v_3 = -\frac{7PL^3}{EI}\left(\frac{1}{12+7k'}\right) \qquad \theta_2 = -\frac{3PL^2}{EI}\left(\frac{1}{12+7k'}\right) \qquad \theta_3 = -\frac{9PL^2}{EI}\left(\frac{1}{12+7k'}\right)$$

Numerical Solution:

$$v_3 = \frac{-7(50\text{kN})(3\text{m})^3}{(210 \text{ x } 10^6 \text{ kN/m}^2)(2 \text{ x} 10^{-4}\text{m}^4)}\left(\frac{1}{12+7(0.129)}\right) = -0.0174 \text{ m}$$

$\theta_2 = -0.00249 \text{ rad}$ $\qquad$ $\theta_3 = -0.00747 \text{ rad}$

$F_1 = -69.9\text{kN}$ $\qquad$ $M_1 = -69.7\text{kN}\cdot\text{m}$

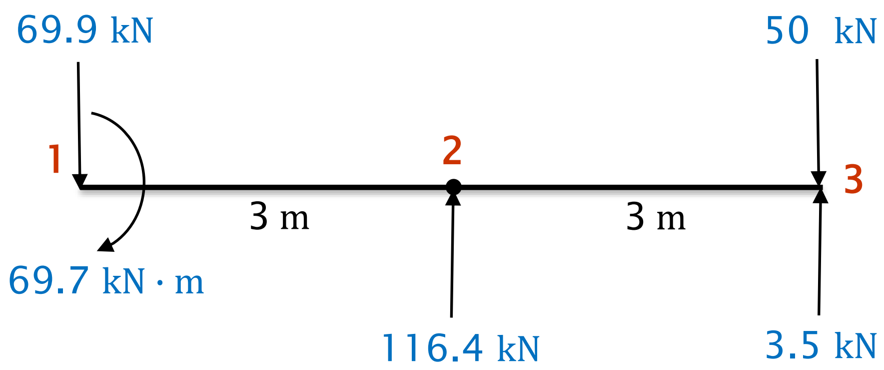
$F_2 = 116.4\text{kN}$ $\qquad$ $M_2 = 0.0\text{kN}\cdot\text{m}$

$F_3 = -50.0\text{kN}$ $\qquad$ $M_3 = 0.0\text{kN}\cdot\text{m}$

$F_4 = -v_3\text{k} = (0.0174)200 = 3.5 \text{ kN}$

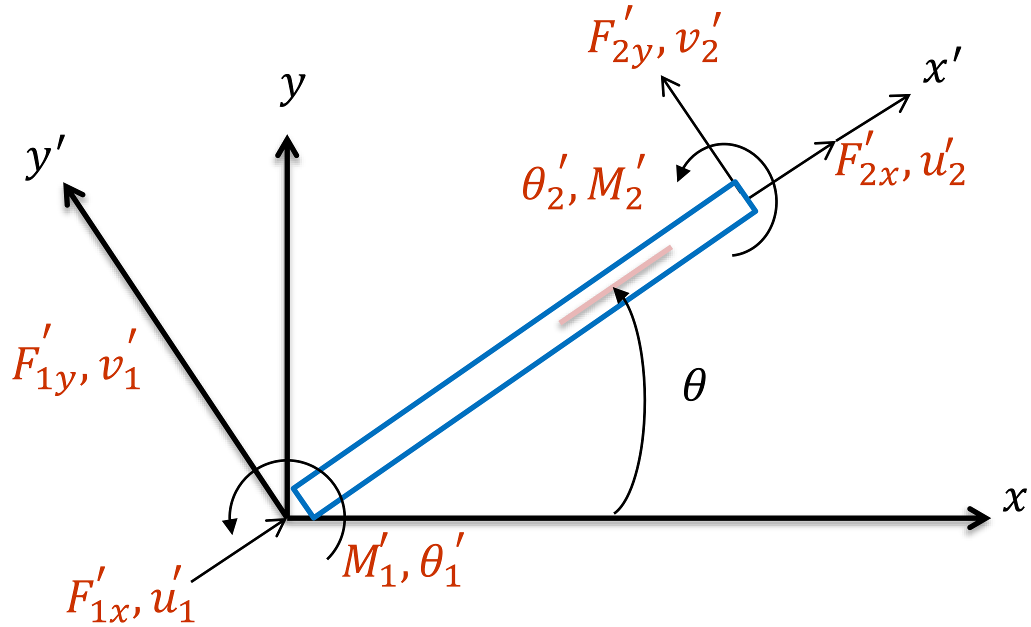
Example 8 - Beam Element

Free Body Diagram



Rigid Frame Element

Rigid Frame Element



A rigid frame is defined as series of beam elements rigidly connected to each other:

- Moments transmitted from one element to another
- Moment continuity at rigid joints
- Elements and loads lie in common x-y plane
- Both axial and transverse loads (beam columns)

Rigid Frame Element

Local (x' – y') Stiffness Matrix :

$$[K']\{u'\} = \{f'\}$$

$$[K'] = \frac{E}{L}\begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ & \dfrac{12I}{L^2} & \dfrac{6I}{L} & 0 & -\dfrac{12I}{L^2} & \dfrac{6I}{L} \\ & & 4I & 0 & -\dfrac{6I}{L} & 2I \\ & & & A & 0 & 0 \\ & & & & \dfrac{12I}{L^2} & -\dfrac{6I}{L} \\ symm & & & & & 4I \end{bmatrix}$$

$$\{f'\} = \left\{F'_{1x}, F'_{1y}, M'_1, F'_{2x}, F'_{2y}, M'_2\right\}^T \qquad \{u'\} = \left\{u'_1, v'_1, \theta'_1, u'_2, v'_2, \theta'_2\right\}^T$$

Rigid Frame Element

Transformation Matrix:

$$[T] = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad C = Cos\theta \qquad S = Sin\theta \qquad [T]^{-1} = [T]^T$$

Transformations:

$$\{u'\} = [T]\,\{u\}$$

$$\{f'\} = [T]\,\{f\}$$

$$[K] = [T]^T [K']\ [T]$$

Rigid Frame Element

Global ($x - y$) Transformed Stiffness Matrix :

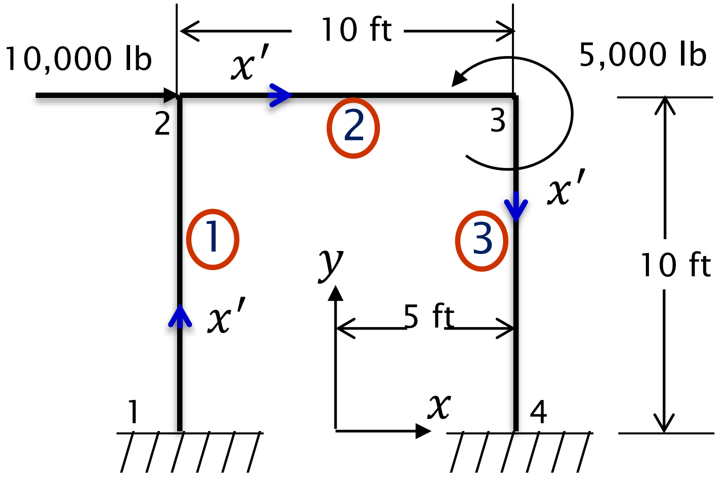
$$[K]\{u\} = \{f\}$$

$$[K] = \frac{E}{L}\begin{bmatrix} AC^2 + \frac{12I}{L^2}S^2 & \left(A - \frac{12I}{L^2}\right)CS & -\frac{6I}{L}S & -\left(AC^2 + \frac{12I}{L^2}S^2\right) & -\left(A - \frac{12I}{L^2}\right)CS & -\frac{6I}{L}S \\ & AS^2 + \frac{12I}{L^2}C^2 & \frac{6I}{L}C & -\left(A - \frac{12I}{L^2}\right)CS & -\left(AS^2 + \frac{12I}{L^2}C^2\right) & \frac{6I}{L}C \\ & & 4I & \frac{6I}{L}S & -\frac{6I}{L}C & 2I \\ & & & AC^2 + \frac{12I}{L^2}S^2 & \left(A - \frac{12I}{L^2}\right)CS & \frac{6I}{L}S \\ & & & & AS^2 + \frac{12I}{L^2}C^2 & -\frac{6I}{L}C \\ & symm & & & & 4I \end{bmatrix}$$

$$\{f\} = \{F_{1x}, F_{1y}, M_1, F_{2x}, F_{2y}, M_2\}^T \qquad \{u\} = \{u_1, v_1, \theta_1, u_2, v_2, \theta_2\}^T$$

- Axially and Transversely Loaded Beam
- Can recover Bar/Spring/Rotated Bars/Rotated Springs and Beam [K]

Example 9 - Rigid Frame Element



$E = 30 \text{ x } 10^6$psi
A $= 10$ in.^{2}
$\mathrm{I} = 200$ in.^{4} for 1 & 3
$\mathrm{I} = 100$ in.^{4} for 2

Example 9 - Rigid Frame Element

Element 1

$$C = \cos 90^0 = \frac{x_2 - x_1}{L^{(1)}} = \frac{-60-(-60)}{120} = 0 \qquad S = \sin 90^0 = \frac{y_2 - y_1}{L^{(1)}} = \frac{120-0}{120} = 1$$

$$\frac{12I}{L^2} = \frac{12(200)}{(10 \text{ x } 12)^2} = 0.167 \text{ in.}^2 \qquad \frac{6I}{L} = \frac{6\,(200)}{10 \text{ x } 12} = 10.0 \text{ in.}^3$$

$$\frac{E}{L} = \frac{30 \text{ x } 10^6}{10 \text{ x } 12} = 250{,}000 \text{ lb/ in.}^3$$

$$[k^{(1)}] = 250{,}000 \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 0.167 & 0 & -10 & -0.167 & 0 & -10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ -10 & 0 & 800 & 10 & 0 & 400 \\ -0.167 & 0 & 10 & 0.167 & 0 & 10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ -10 & 0 & 400 & 10 & 0 & 800 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}}$$

Example 9 - Rigid Frame Element

Element 2

$$C = 1 \quad S = 0$$

$$\frac{12I}{L^2} = \frac{12(200)}{120^2} = 0.0835 \text{ in.}^2 \qquad \frac{6I}{L} = \frac{6(100)}{120} = 5 \text{ in.}^3 \qquad \frac{E}{L} = 250{,}000 \text{ lb/ in.}^3$$

$$[k^{(2)}] = 250{,}000 \begin{array}{c} \begin{array}{cccccc} u_2 & v_2 & \theta_2 & u_3 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.0835 & 5 & 0 & -0.0835 & 5 \\ 0 & 5 & 400 & 0 & -5 & 200 \\ -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & -0.0835 & -5 & 0 & 0.0835 & -5 \\ 0 & 5 & 200 & 0 & -5 & 400 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}}$$

Example 9 - Rigid Frame Element

Element 3

$C = 0 \quad S = -1$

$$[k^{(3)}] = 250{,}000 \begin{array}{c} \begin{array}{cccccc} u_3 & v_3 & \theta_3 & u_4 & v_4 & \theta_4 \end{array} \\ \begin{bmatrix} 0.167 & 0 & 10 & -0.167 & 0 & 10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ 10 & 0 & 800 & -10 & 0 & 400 \\ -0.167 & 0 & -10 & 0.167 & 0 & -10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ 10 & 0 & 400 & -10 & 0 & 800 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}}$$

Global Reduced Equations:

$$\begin{Bmatrix} 10{,}000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5000 \end{Bmatrix} = 250{,}000 \begin{bmatrix} 10.167 & 0 & 10 & -10 & 0 & 0 \\ 0 & 10.0835 & 5 & 0 & -0.0835 & 5 \\ 10 & 5 & 1200 & 0 & -5 & 200 \\ -10 & 0 & 0 & 10.167 & 0 & 10 \\ 0 & -0.0835 & -5 & 0 & 10.0835 & -5 \\ 0 & 5 & 200 & 10 & -5 & 1200 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

Example 9 - Rigid Frame Element

Solution:

$$\begin{Bmatrix} u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0.211 \text{ in.} \\ 0.00148 \text{ in.} \\ -0.00153 \text{ rad} \\ 0.209 \text{ in.} \\ -0.00148 \text{ in.} \\ -0.00149 \text{ rad} \end{Bmatrix}$$

Top frames move to the right; small vertical Displacement and small rotations

Element 1 local displacements:

$$[T]\{d\} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ \theta_1 = 0 \\ u_2 = 0.211 \\ v_2 = 0.00148 \\ \theta_2 = -0.00153 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00148 \\ -0.211 \\ -0.00153 \end{Bmatrix}$$

Example 9 - Rigid Frame Element

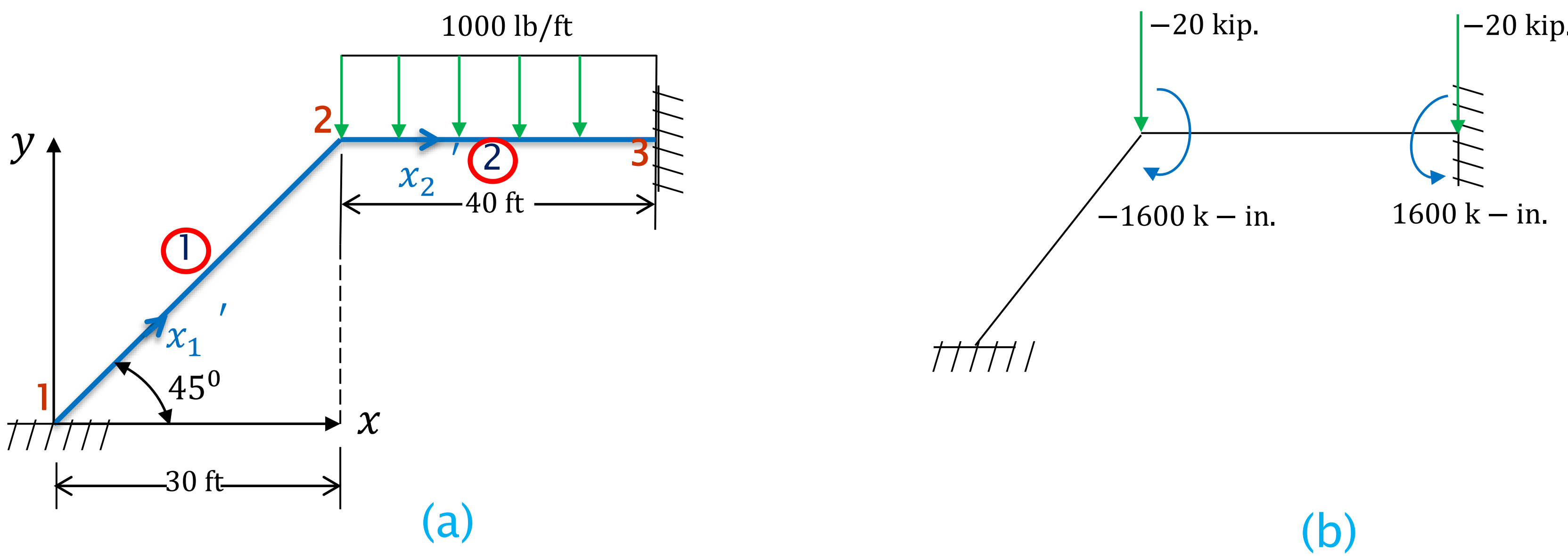
$\underline{Element\ 1\ local\ forces}$:

$$\{f'\} = [k'][T]\{d\} = 250{,}000 \begin{bmatrix} 10 & 0 & 1 & -10 & 0 & 0 \\ 0 & 0.167 & 10 & 0 & -0.167 & 10 \\ 0 & 10 & 800 & 0 & -10 & 400 \\ -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & -0.167 & -10 & 0 & 0.167 & -10 \\ 0 & 10 & 400 & 0 & -10 & 800 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00148 \\ -0.211 \\ -0.00153 \end{Bmatrix}$$

Eq. 5.1.6

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m'_1 \\ f'_{2x} \\ f'_{2y} \\ m'_2 \end{Bmatrix} = \begin{Bmatrix} -3700\ \mathbf{lb} \\ 4990\ \mathbf{lb} \\ 376{,}000\ \mathbf{lb\text{-}in.} \\ 3700\ \mathbf{lb} \\ -4990\ \mathbf{lb} \\ 223{,}000\ \mathbf{lb\text{-}in.} \end{Bmatrix}$$

Example 10 - Rigid Frame Element



(a) Plane frame for analysis and (b) equivalent nodal forces on frame

- To illustrate the procedure for solving frames subjected to distributed loads, solve the rigid plane frame shown in Figure. The frame is fixed at nodes 1 and 3 and subjected to a uniformly distributed load of 1000 lb/ft applied downward over element 2. The global coordinate axes have been established at node 1. The element lengths are shown. Let $E =$ 30 x 10^6 psi, $A = 100$ in.^{2}, and $I = 1000\text{in}^4$ for both elements

Example 10 - Rigid Frame Element

Solution:

Replacing the distributed load acting on element 2 by nodal forces and moments acting at nodes 2 and 3.

$$f_{2y} = -\frac{wL}{2} - \frac{(1000)40}{2} = -20{,}000 \text{ lb} = -20 \text{ kip}$$

$$m_2 = -\frac{wL^2}{12} - \frac{(1000)40^2}{12} = -133{,}333 \text{ lb}-\text{ft} = -1600 \text{ k}-\text{in.}$$

$$f_{3y} = -\frac{wL}{2} - \frac{(1000)40}{2} = -20{,}000 \text{ lb} = -20 \text{ kip}$$

$$m_3 = \frac{wL^2}{12} - \frac{(1000)40^2}{12} = 133{,}333 \text{ lb}-\text{ft} = 1600 \text{ k}-\text{in.}$$

Example 10 - Rigid Frame Element

Element 1:

$\theta^{(1)} = 45^0 \quad C = 0.707 \quad S = 0.707 \quad L^{(1)} = 42.4 \text{ ft} = 509.0 \text{ in.}$

$$\frac{E}{L} = \frac{30 \text{ x } 10^3}{509} = 58.93$$

$$[k^{(1)}] = 58.93 \begin{bmatrix} 50.02 & 49.98 & 8.33 \\ 49.98 & 50.02 & -8.33 \\ 8.33 & -8.33 & 4000 \end{bmatrix} \frac{\text{kip}}{\text{in.}}$$

$$[k^{(1)}] = \begin{array}{c} \begin{array}{ccc} u_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 2948 & 2945 & 491 \\ 2945 & 2948 & -491 \\ 491 & -491 & 235{,}700 \end{bmatrix} \end{array} \frac{\text{kip}}{\text{in.}}$$

Element 2:

$\theta^{(2)} = 0^0 \quad C = 1 \quad S = 0 \quad L^{(2)} = 40 \text{ ft} = 480 \text{ in.}$

$$\frac{E}{L} = \frac{30 \text{ x } 10^3}{480} = 62.50$$

$$[k^{(2)}] = 62.50 \begin{bmatrix} 100 & 0 & 0 \\ 0 & 0.052 & 12.5 \\ 0 & 12.5 & 4000 \end{bmatrix} \frac{\text{kip}}{\text{in.}}$$

$$[k^{(2)}] = \begin{array}{c} \begin{array}{ccc} u_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 6250 & 0 & 0 \\ 0 & 3.25 & 781.25 \\ 0 & 781.25 & 250{,}000 \end{bmatrix} \end{array} \frac{\text{kip}}{\text{in.}}$$

Example 10 - Rigid Frame Element

$$\begin{Bmatrix} F_{2x} = 0 \\ F_{2y} = -20 \\ M_2 = -1600 \end{Bmatrix} = \begin{bmatrix} 9198 & 2945 & 491 \\ 2945 & 2951 & 290 \\ 491 & 290 & 485{,}700 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} \qquad \begin{Bmatrix} u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} = z \begin{Bmatrix} 0.0033 \text{ in.} \\ -0.0097\text{in.} \\ -0.0033\text{rad} \end{Bmatrix}$$

The results indicate that node 2 moves to the right ($u_2 = 0.0033$in.) and down ($v_2 = -0.0097$in.) and the rotation of the joint is clockwise ($\theta_2 = -0.0033$ rad).

The local forces in each element can now be determined. The procedure for elements that are subjected to a distributed load must be applied to element 2. Recall that the local forces are given by $\{f'\} = [k'][T]\{u\}$.

For element 1, we have

$$[T]\{u\} = \begin{bmatrix} 0.707 & 0.707 & 1 & 0 & 0 & 0 \\ -0.707 & 0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.707 & 0.707 & 0 \\ 0 & 0 & 0 & -0.707 & 0.707 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.0033 \\ -0.0097 \\ -0.0033 \end{Bmatrix} \qquad \{u'\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00452 \\ -0.0092 \\ -0.0033 \end{Bmatrix}$$

Example 10 - Rigid Frame Element

$$\{f'\} = [K']\{u'\}$$

$$\begin{Bmatrix} f_{1x}^{'} \\ f_{1y}^{'} \\ m_1^{'} \\ f_{2x}^{'} \\ f_{2y}^{'} \\ m_2^{'} \end{Bmatrix} = \begin{bmatrix} 5893 & 0 & 0 & -5893 & 0 & 0 \\ & 2.730 & 694.8 & 0 & -2.730 & 694.8 \\ & & 117{,}900 & 0 & -694.8 & 117{,}900 \\ & & & 5893 & 0 & 0 \\ & & & & 2.730 & -694.8 \\ symm & & & & & 235{,}800 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00452 \\ -0.0092 \\ -0.0033 \end{Bmatrix}$$

$f_{1x}' = 26.64$ kip $\quad f_{1y}' = -2.268$ kip $\quad m_{1x}' = -389.1$ k–in.

$f_{2x}' = -26.64$ kip $\quad f_{2y}' = 2.268$ kip $\quad m_{2x}' = -778.2$ k–in.

For element 2, we have

$$[T]\{u\} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.0033 \\ -0.0097 \\ -0.0033 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \{u'\} = \begin{Bmatrix} 0.0033 \\ -0.0097 \\ -0.0033 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Example 10 - Rigid Frame Element

$$[K']\{u'\} = [f'] = \begin{bmatrix} 6250 & 0 & 0 & -6250 & 0 & 0 \\ & 3.25 & 781.1 & 0 & -3.25 & 781.1 \\ & & 250{,}000 & 0 & -781.1 & 125{,}000 \\ & & & 6250 & 0 & 0 \\ & & & & 3.25 & -781.1 \\ symm & & & & & 250{,}000 \end{bmatrix} \begin{Bmatrix} 0.0033 \\ -0.0097 \\ -0.0033 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

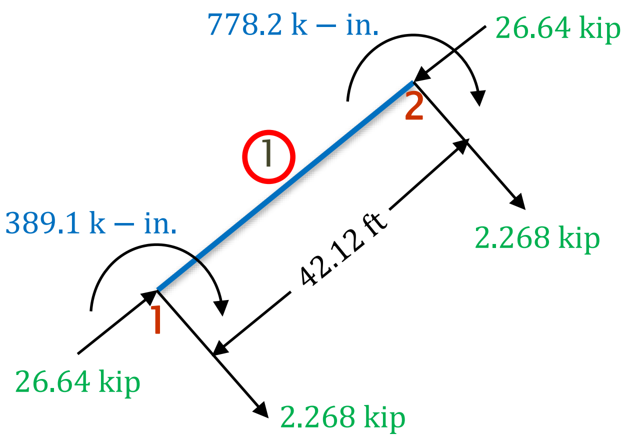
$$\{f'\} = \begin{Bmatrix} 20.63 \\ -2.58 \\ -832.57 \\ -20.63 \\ 2.58 \\ -412.50 \end{Bmatrix} \qquad \begin{Bmatrix} f_{2x}' \\ f_{2y}' \\ m_2' \\ f_{3x}' \\ f_{3y}' \\ m_3' \end{Bmatrix} = \begin{Bmatrix} 20.63 \\ -2.58 \\ -832.57 \\ -20.63 \\ 2.58 \\ 412.50 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -20 \\ -1600 \\ 0 \\ -20 \\ 1600 \end{Bmatrix}$$

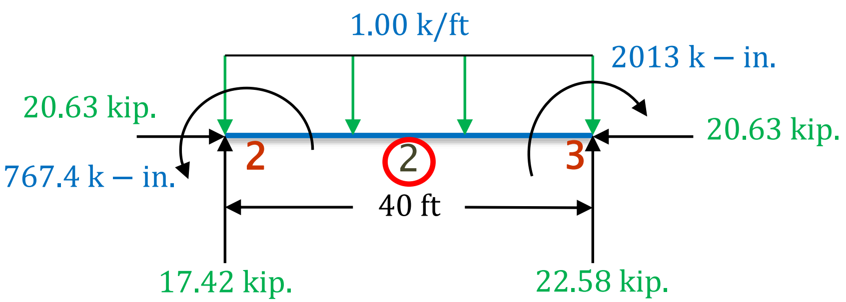
$f_{2x}' = 20.63$ kip $\qquad f_{2y}' = 17.42$kip $\qquad m_2' = 767.4$k−in.

$f_{3x}' = -20.63$ kip $\qquad f_{3y}' = 22.58$ kip $\qquad m_3' = -2013$ k−in.

Example 10 - Rigid Frame Element

Free body diagrams of elements 1 and 2





Two-Dimensional Elements

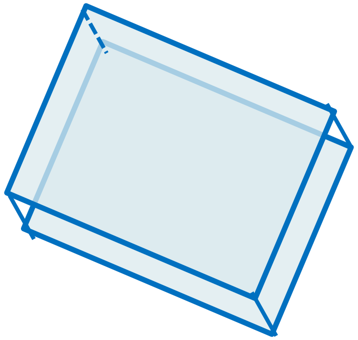
Triangular Element

1. Plane Stress

- State of stress in which normal and shear stresses perpendicular to the (x-y) plane are assumed zero

$\sigma_z = 0;\ \tau_{xz} = 0;\ \tau_{yz} = 0$

- Thin structures having small z- dimension as compared x-y plane
- Loads act only in x-y plane

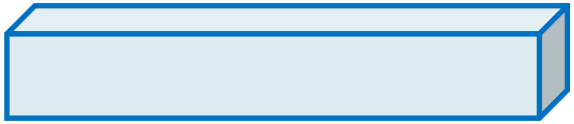


2. Plane Strain

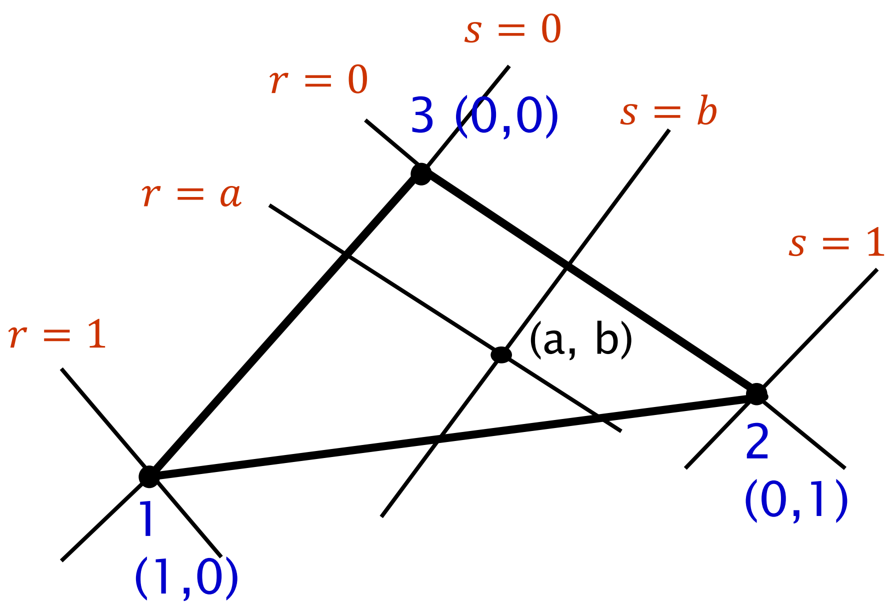
- State of strain in which normal and shear strains perpendicular to the (x-y) plane are assumed zero

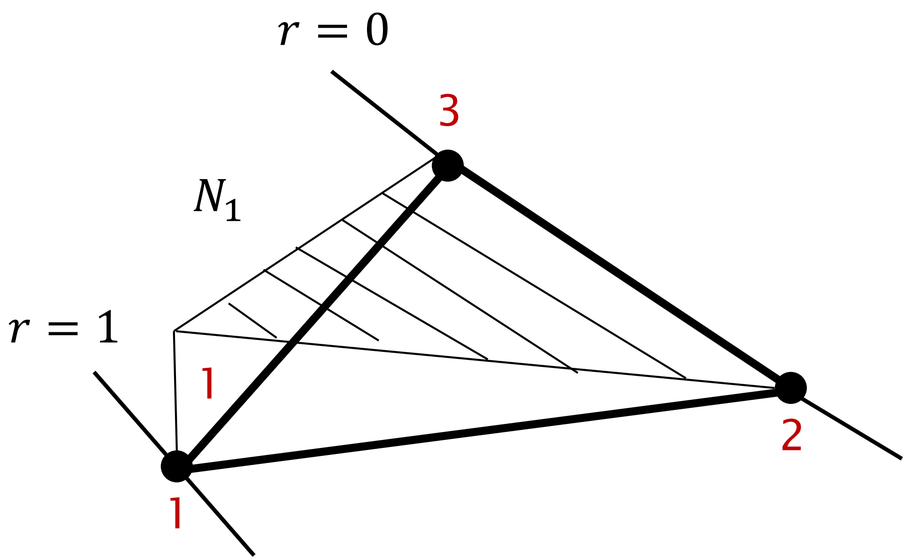
$\epsilon_z = 0;\ \gamma_{xz} = 0;\ \gamma_{yz} = 0$

- Long structures in z- dimension with constant cross section in x-y plane
- Loads act only in x-y plane and not vary along z-axis



Linear Triangular Element





$N_1 = r, \quad N_2 = s, \quad N_3 = 1 - r - s \qquad N_1 + N_2 + N_3 = 1$

$$N_i = \begin{cases} 1, & \text{at node i;} \\ 0, & \text{at the other nodes} \end{cases}$$

- Linear triangular element is also known as Constant Strain Triangular (CST) Element

Constant Strain Triangular Element

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 \qquad y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

or

$$x = x_{13} r + x_{23} s + x_3 \qquad y = y_{13} r + y_{23} s + y_3$$

$$x_{ij} = x_i - x_j \text{ and } y_{ij} = y_i - y_j \ (i, j = 1,2,3)$$

$$u(r,s) = N_1 u_1 + N_2 u_2 + N_3 u_3 \qquad v(r,s) = N_1 v_1 + N_2 v_2 + N_3 v_3$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\{\mathbf{u}\} = [\mathrm{N}]\{u\} \qquad \{u\} = \{u_1, v_1, u_2, v_2, u_3, v_3\}$$

Constant Strain Triangular Element

$$\begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial x}{\partial y} \end{Bmatrix}$$

$\mathbf{J} = Jacobian\ matrix$

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} \qquad \mathbf{J}^{-1} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}$$

Jacobian is independent of r and s !

$$|\mathbf{J}| = x_{13}y_{23} - x_{23}y_{13} = 2A \qquad 2A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \end{Bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} u_1 - u_3 \\ u_2 - u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} v_1 - v_3 \\ v_2 - v_3 \end{Bmatrix}$$

Constant Strain Triangular Element

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{1}{2A}\left[y_{23}(u_1 - u_3) - y_{13}(u_2 - u_3)\right] = \frac{1}{2A}\left[y_{23}u_1 - y_{13}u_2 + (y_{13} - y_{23})u_3\right]$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{1}{2A}\left[y_{23}u_1 + y_{31}u_2 + y_{12}u_3\right] = \frac{1}{2A}\left[\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3\right]$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{1}{2A}\left[\gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3\right]$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{2A}\left[\gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3\right]$$

$$\{\varepsilon\} = [\mathrm{B}]\{\mathrm{u}\}$$

Strains are constant in the element and are independent of r and s !

$$[\mathrm{B}] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

Constant Strain Triangular Element

$\{\sigma\} = [C]\{\epsilon\}$

$$\{\sigma\} = \{\sigma_{xx}, \sigma_{yy}, \tau_{xy}\}$$

$$[C] = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix}$$

Stresses are constant in the element and are independent of r and s !

$$C_{11} = C_{22} = \begin{cases} \dfrac{E}{1-v^2} & \text{plane stress} \\ \dfrac{E(1-v)}{(1+v)(1-2v)} & \text{plane strain} \end{cases}$$

$$C_{12} = \begin{cases} \dfrac{Ev}{1-v^2} & \text{plane stress} \\ \dfrac{Ev}{(1+v)(1-2v)} & \text{plane strain} \end{cases}$$

$$C_{66} = \frac{E}{2(1+v)} \qquad \text{plane stress and plane strain}$$

Constant Strain Triangular Element

$$[K] = \int_V [B]^T[C][B]\, dV$$

[B] is independent of r and s !

$$[K] = tA[B]^T[C][B];\ V = t\iint dxdy = tA$$

$$[\mathrm{K}] = \frac{t}{4A}\begin{bmatrix} \beta_1^2C_{11}+\gamma_1^2C_{66} & \beta_1\gamma_1C_{12}+\beta_1\gamma_1C_{66} & \beta_1\beta_2C_{11}+\gamma_1\gamma_2C_{66} & \beta_1\gamma_2C_{12}+\beta_2\gamma_1C_{66} & \beta_1\beta_3C_{11}+\gamma_1\gamma_3C_{66} & \beta_1\gamma_3C_{12}+\beta_3\gamma_1C_{66} \\ & \gamma_1^2C_{22}+\beta_1^2C_{66} & \beta_2\gamma_1C_{12}+\beta_1\gamma_2C_{66} & \gamma_1\gamma_2C_{22}+\beta_1\beta_2C_{66} & \beta_3\gamma_1C_{12}+\beta_1\gamma_3C_{66} & \gamma_1\gamma_3C_{22}+\beta_1\beta_3C_{66} \\ & & \beta_2^2C_{11}+\gamma_2^2C_{66} & \beta_2\gamma_2C_{12}+\beta_2\gamma_2C_{66} & \beta_2\beta_3C_{11}+\gamma_2\gamma_3C_{66} & \beta_2\gamma_3C_{12}+\beta_3\gamma_2C_{66} \\ & & & \gamma_2^2C_{22}+\beta_2^2C_{66} & \beta_3\gamma_2C_{12}+\beta_2\gamma_3C_{66} & \gamma_2\gamma_3C_{22}+\beta_2\beta_3C_{66} \\ & & & & \beta_3^2C_{11}+\gamma_3^2C_{66} & \beta_3\gamma_3C_{12}+\beta_3\gamma_3C_{66} \\ & SYMM & & & & \gamma_3^2C_{22}+\beta_3^2C_{66} \end{bmatrix}$$

$$\beta_1 = y_2 - y_3 \qquad \gamma_1 = x_3 - x_2$$
$$\beta_2 = y_3 - y_1 \qquad \gamma_2 = x_1 - x_3$$
$$\beta_3 = y_1 - y_2 \qquad \gamma_3 = x_2 - x_1$$

$$2A = \beta_1x_1 + \beta_2x_2 + \beta_3x_3$$

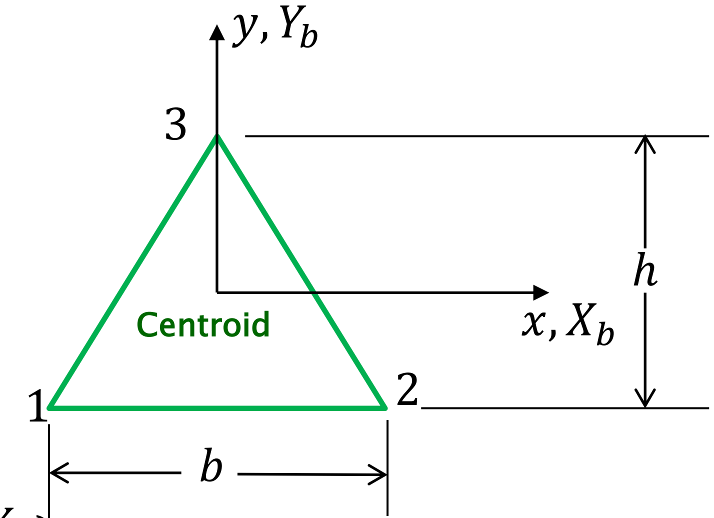
CST Element – Body Force Vector

$$\{f\} = \int_L [N]^T \{q\} dx + \int_A [N]^T \{p\} dA + \int_V [N]^T \{X\} dV + \sum N_i F_i$$

$$\{f_b\} = \iiint_V [N]^T \{X\} dV = t \iint_A [N]^T \{X\} |J| dr ds$$

$$\{X\} = \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix}$$

Weight densities/volume



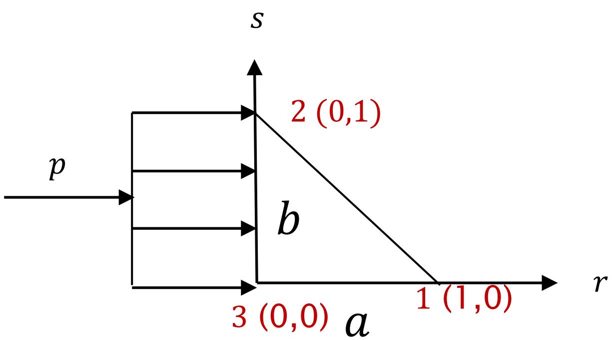
$$\iint_A N_1 |J| dr ds$$

$$= \int_0^1 \int_0^{1-r} r\, 2A\, dr\, ds; s = 1 - r$$

$$= 2A \int_0^1 r(1-r) dr = \frac{A}{3}$$

$$\{f_b\} = \begin{Bmatrix} f_{b1x} \\ f_{b1y} \\ f_{b2x} \\ f_{b2y} \\ f_{b3x} \\ f_{b3y} \end{Bmatrix} = \begin{Bmatrix} X_b \\ Y_b \\ X_b \\ Y_b \\ X_b \\ Y_b \end{Bmatrix} \frac{tA}{3}$$

CST Element – Surface Force Vector



$$\{f_p\} = \int_A [N]^T \{p\} dA$$

$$\{f_p\} = t \int_s [N_{Along\ r=0}]^T \{p\} ds$$

$$N_1 = r, \qquad N_2 = s, \qquad N_3 = 1 - r - s$$

$$ALong\ r = 0: \ N_1 = 0,\ N_2 = s,\ N_3 = 1 - s$$

$$[N_{Along\ r=0}]^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ s & 0 \\ 0 & s \\ 1-s & 0 \\ 0 & 1-s \end{bmatrix}$$

$$\{p\} = \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = \begin{Bmatrix} p \\ 0 \end{Bmatrix}$$

CST Element – Surface Force Vector

$$\{f_p\} = tb\int_0^1 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ s & 0 \\ 0 & s \\ 1-s & 0 \\ 0 & 1-s \end{bmatrix} \begin{Bmatrix} p \\ 0 \end{Bmatrix} ds$$

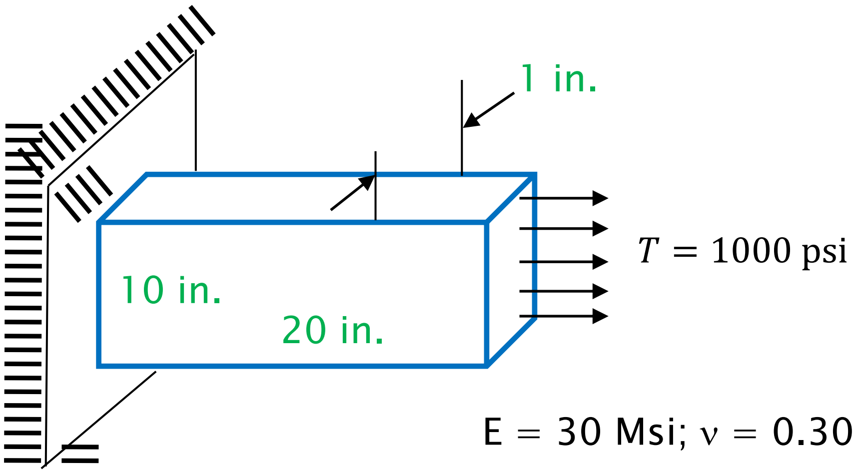
$$\{f_p\} = \begin{Bmatrix} f_{b1x} \\ f_{b1y} \\ f_{b2x} \\ f_{b2y} \\ f_{b3x} \\ f_{b3y} \end{Bmatrix} = t\begin{Bmatrix} 0 \\ 0 \\ \dfrac{pb}{2} \\ 0 \\ \dfrac{pb}{2} \\ 0 \end{Bmatrix}$$

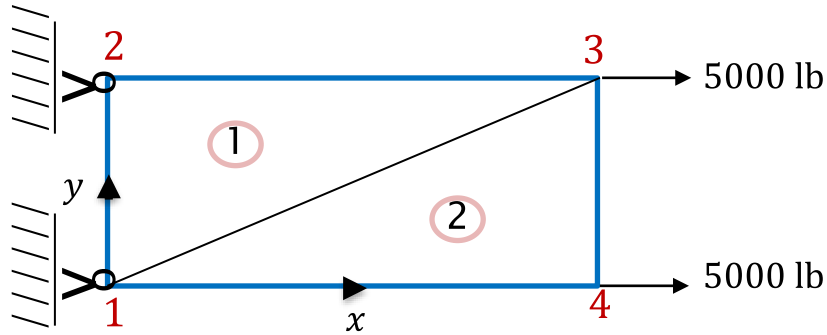
In general:

$$\{f_p\} = \begin{Bmatrix} f_{p1x} \\ f_{p1y} \\ f_{p2x} \\ f_{p2y} \\ f_{p3x} \\ f_{p3y} \end{Bmatrix} = t\begin{Bmatrix} 0 \\ \dfrac{p_y a}{2} \\ \dfrac{p_x b}{2} \\ 0 \\ \dfrac{p_x b}{2} \\ \dfrac{p_y a}{2} \end{Bmatrix}$$

$$\{p\} = \begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

Example 11 - CST Element





Plane Stress Problem - Discretized plate

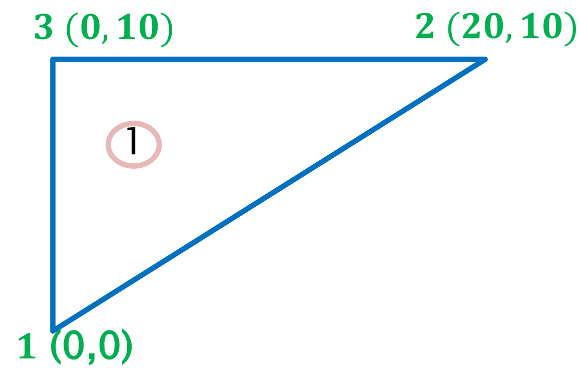
Element 2 Force Vector:

$$\begin{Bmatrix} f_{p1x} \\ f_{p1y} \\ f_{p2x} \\ f_{p2y} \\ f_{p3x} \\ f_{p3y} \end{Bmatrix} = 1.0 \begin{Bmatrix} 0 \\ 0 \\ \frac{1000X10}{2} \\ 0 \\ \frac{1000X10}{2} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 5000 \\ 0 \\ 5000 \\ 0 \end{Bmatrix} \text{lb} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix}$$

Example 11 - CST Element

Element 1 [K]:

$$\beta_1 = y_2 - y_3 = 0 \qquad \gamma_1 = x_3 - x_2 = -20$$
$$\beta_2 = y_3 - y_1 = 10 \qquad \gamma_2 = x_1 - x_3 = 0$$
$$\beta_3 = y_1 - y_2 = -10 \qquad \gamma_3 = x_2 - x_1 = 20$$



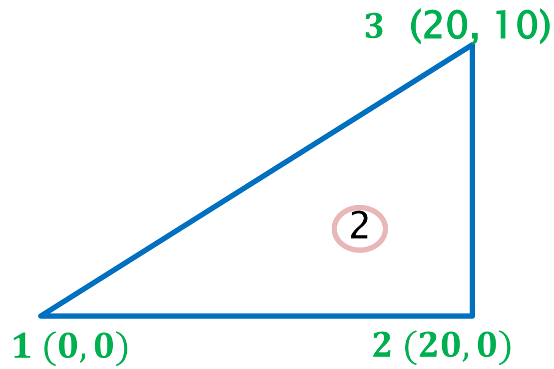
$$2A = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 200$$

$$C_{11} = C_{22} = \frac{E}{1-\nu^2} \qquad C_{12} = C_{22} = \frac{\nu E}{1-\nu^2} \qquad C_{66} = \frac{E}{2(1+\nu)}$$

$$[k^{(1)}] = \frac{75{,}000}{0.91} \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_3 & v_3 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 140 & 0 & 0 & -70 & -140 & 70 \\ 0 & 400 & -60 & 0 & 60 & -400 \\ 0 & -60 & 100 & 0 & -100 & 60 \\ -70 & 0 & 0 & 35 & 70 & -35 \\ -140 & 60 & -100 & 70 & 240 & -130 \\ 70 & -400 & 60 & -35 & -130 & 435 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}}$$

Example 11 - CST Element

<u>Element 2 [K]:</u>



$$\beta_1 = y_2 - y_3 = -10 \qquad \gamma_1 = x_3 - x_2 = 0$$
$$\beta_2 = y_3 - y_1 = 10 \qquad \gamma_2 = x_1 - x_3 = -20$$
$$\beta_3 = y_1 - y_2 = 0 \qquad \gamma_3 = x_2 - x_1 = 20$$

$$2A = 200$$

$$C_{11} = C_{22} = \frac{E}{1 - v^2} \qquad C_{12} = C_{22} = \frac{vE}{1 - v^2} \qquad C_{66} = \frac{E}{2(1+v)}$$

$$\left[k^{(2)}\right] = \frac{75{,}000}{0.91} \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_4 & v_4 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 100 & 0 & -100 & 60 & 0 & -60 \\ 0 & 35 & 70 & -35 & -70 & 0 \\ -100 & 70 & 240 & -130 & -140 & 60 \\ 60 & -35 & -130 & 435 & 70 & -400 \\ 0 & -70 & -140 & 70 & 140 & 0 \\ -60 & 0 & 60 & -400 & 0 & 400 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}}$$

Example 11 - CST Element

Global System:

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = [K] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \rightarrow \begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 5000 \\ 0 \\ 5000 \\ 0 \end{Bmatrix} = [K] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 5000 \\ 0 \\ 5000 \\ 0 \end{Bmatrix} = \frac{375{,}000}{0.91} \begin{bmatrix} 48 & 0 & -28 & 14 & 0 & -26 & -20 & 12 \\ 0 & 87 & 12 & -80 & -26 & 0 & 14 & -7 \\ -28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\ 14 & -80 & -26 & 87 & 12 & -7 & 0 & 0 \\ 0 & -26 & -20 & 12 & 48 & 0 & -28 & 14 \\ -26 & 0 & 14 & -7 & 0 & 87 & 12 & -80 \\ -20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\ 12 & -7 & 0 & 0 & 14 & -80 & -26 & 87 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Example 11 - CST Element

Reduced System and Solution:

$$\begin{Bmatrix} 5000 \\ 0 \\ 5000 \\ 0 \end{Bmatrix} = \frac{375{,}000}{0.91} \begin{bmatrix} 48 & 0 & -28 & 14 \\ 0 & 87 & 12 & -80 \\ -28 & 12 & 48 & -26 \\ 14 & -80 & -26 & 87 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 609.6 \\ 4.2 \\ 663.7 \\ 104.1 \end{Bmatrix} \times 10^{-6} \text{ in.}$$

Invert 4 X 4 reduced [K]

$$\delta\ (= u_3 = u_4) = \frac{PL}{AE} = \frac{10{,}000 \times 20}{10 \times 30 \times 10^6} = 670 \times 10^{-6} \text{ in.}$$

Bar solution

Example 11 - CST Element

Element Stresses:

$$\{\sigma\} = [\mathrm{C}][\mathrm{B}]\{u\}$$

$$\{\sigma\}^{Element\ 1} = \frac{30(10^6)(10^{-6})}{0.91(200)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 609.6 \\ 4.2 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^1 = \begin{Bmatrix} 1005 \\ 301 \\ 2.4 \end{Bmatrix} \text{ psi}$$

$$\{\sigma\}^{Element\ 2} = \frac{30(10^6)(10^{-6})}{0.91(200)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 663.7 \\ 104.1 \\ 609.6 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^2 = \begin{Bmatrix} 995 \\ -1.2 \\ -2.4 \end{Bmatrix} \text{ psi}$$

Example 11 - CST Element

Principal Stresses:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}\right]^{1/2}$$

$$\sigma_1 = \frac{995 + (-1.2)}{2} + \left[\left(\frac{995 - (-1.2)}{2}\right)^2 + (-2.4)^2\right]^{1/2}$$

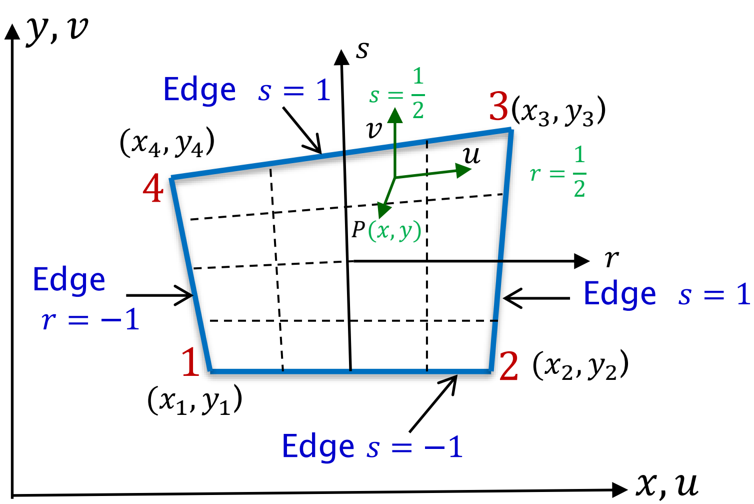
$$\sigma_1 = 497 + 498 = 995 \text{ psi}$$

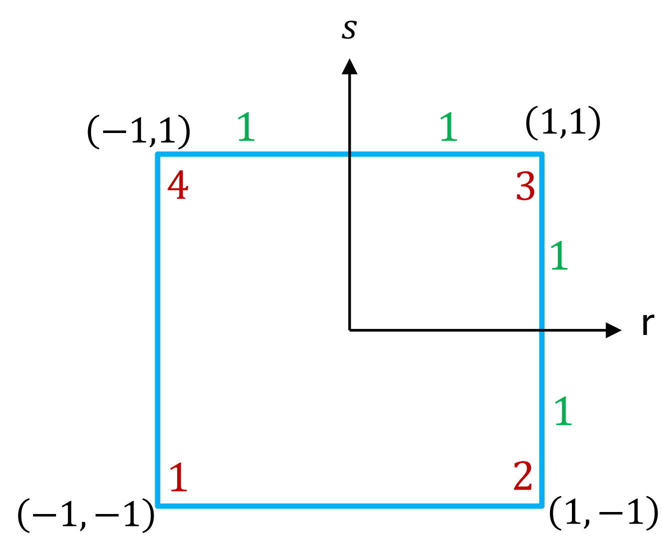
$$\sigma_2 = \frac{995 + (-1.2)}{2} - 498 = -1.1 \text{ psi}$$

$$\theta_P = \frac{1}{2}\tan^{-1}\left[\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right]$$

$$\theta_P = \frac{1}{2}\tan^{-1}\left[\frac{2(-2.4)}{995 - (-1.2)}\right] = 0^0$$

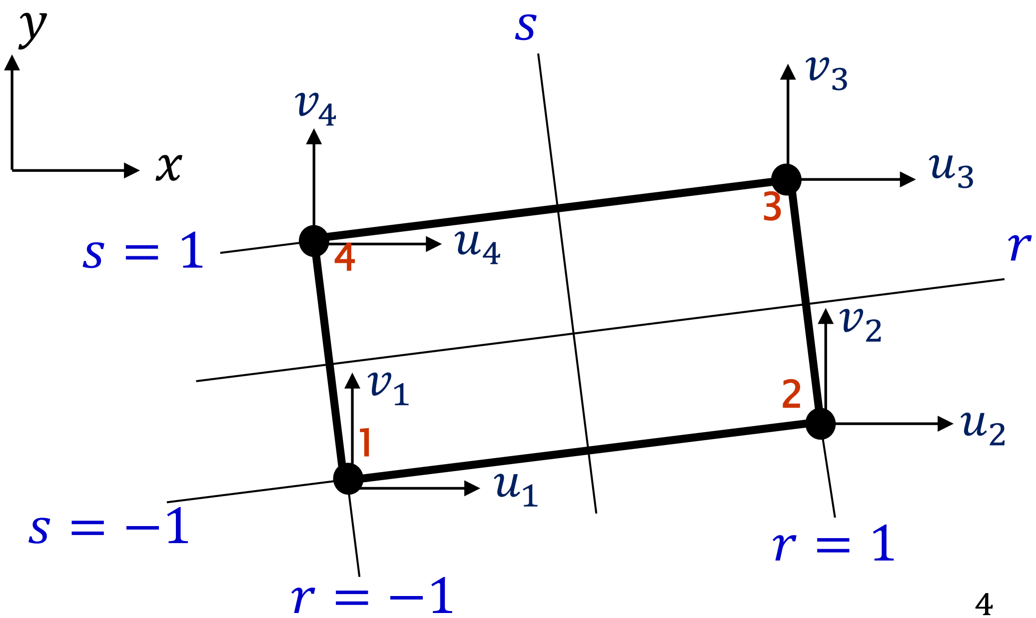
Linear Quadrilateral Element





$$x = \sum_{i=1}^{4} N_i x_i \qquad y = \sum_{i=1}^{4} N_i y_i \qquad u = \sum_{i=1}^{4} N_i u_i \qquad v = \sum_{i=1}^{4} N_i v_i$$

Linear Quadrilateral Element



$$N_1 = \frac{1}{4}(1-r)(1-s)$$

$$N_2 = \frac{1}{4}(1+r)(1-s)$$

$$N_3 = \frac{1}{4}(1+r)(1+s)$$

$$N_4 = \frac{1}{4}(1-r)(1+s)$$

$$\sum_{i=1}^{4} N_i = 1$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ v_4 \end{Bmatrix}$$

Linear Quadrilateral Element

$$\{\varepsilon\} = [\mathrm{B}]\{u\}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|[J]|} \begin{bmatrix} \frac{\partial y}{\partial s}\frac{\partial (\,)}{\partial r} - \frac{\partial y}{\partial r}\frac{\partial (\,)}{\partial s} & 0 \\ 0 & \frac{\partial x}{\partial r}\frac{\partial (\,)}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial (\,)}{\partial r} \\ \frac{\partial x}{\partial r}\frac{\partial (\,)}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial (\,)}{\partial r} & \frac{\partial y}{\partial s}\frac{\partial (\,)}{\partial r} - \frac{\partial y}{\partial r}\frac{\partial (\,)}{\partial s} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix}$$

$$|[J]| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-s & s-r & r-1 \\ s-1 & 0 & r+1 & -r-s \\ r-s & -r-1 & 0 & s+1 \\ 1-r & r+s & -s-1 & 0 \end{bmatrix} \{Y_c\} = \frac{\mathrm{A}}{4} \text{ (for rectangles)}$$

where $\{X_c\}^T = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]$ and $\{Y_c\} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$

Linear Quadrilateral Element

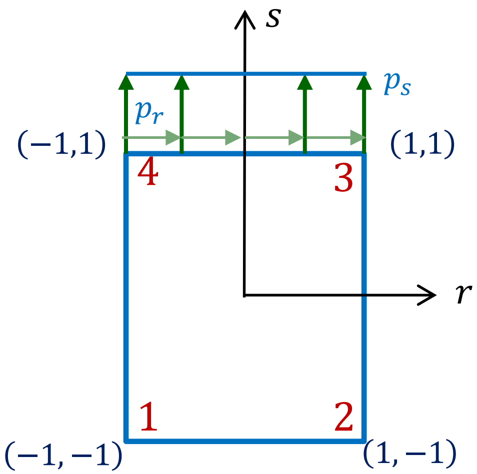
$$\boxed{\{\sigma\} = [\mathrm{C}]\,\{\varepsilon\}}$$

$$[K] = \iint\limits_A [B]^T[D][B]t\,dx\,dy$$

$$\iint\limits_A f(x,y)\,dxdy = \iint\limits_A f(r,s)|[J]|drds$$

$$\boxed{[K] = \int_{-1}^{1}\int_{-1}^{1}[B]^T\,[D][B]t|[J]|dr\,ds}$$

Linear Quad Element – Surface Forces



Surface traction : p_r and p_s acting at edge $s = 1$

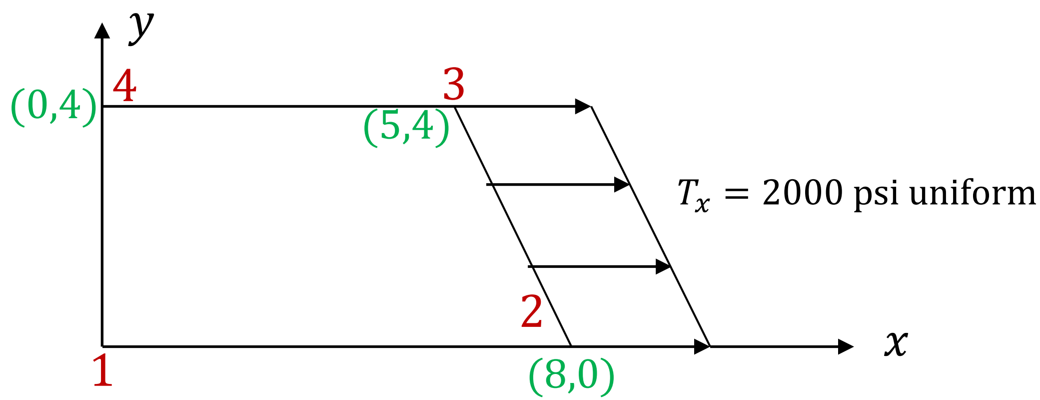
The surface−force matrix, along edge s=1 with overall length a is

$$\underset{(4 \times 1)}{\{f_s\}} = \int_{-1}^{1} \underset{(4 \times 2)}{[N_{s=1}]^T} \quad \underset{(2 \times 1)}{\{T\}} \quad h\frac{a}{2}dr \qquad \text{or} \qquad \begin{Bmatrix} f_{s3r} \\ f_{s3s} \\ f_{s4r} \\ f_{s4s} \end{Bmatrix} = \int_{-1}^{1} \begin{bmatrix} N_3 & 0 & N_4 & 0 \\ 0 & N_3 & 0 & N_4 \end{bmatrix}^T \begin{Bmatrix} p_r \\ p_s \end{Bmatrix} t\frac{a}{2}dr$$

Because $N_1 = N_2 = 0$ along edge $s = 1$, and hence, no nodal forces exist at nodes 1 and 2.For the case of uniform (constant) p_r & p_s along edge $s = 1$, the total surface−force matrix is

$$\boxed{\{f_s\} = t\frac{a}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & p_r & p_s & p_r & p_s \end{bmatrix}^T}$$

Example 12 - Linear Quad Element



Length of side 2-3 is given by

$$L = \sqrt{(5-8)^2 + (4-0)^2} = \sqrt{9+16} = 5$$

N_2 & N_3 must be used along side $2-3$ (at $r = 1$).

$$\{f_s\} = \int_{-1}^{1} [N_s]^T \{T\} t \frac{L}{2} ds = \int_{-1}^{1} \begin{bmatrix} N_2 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_3 \end{bmatrix}^T \begin{Bmatrix} p_r \\ p_s \end{Bmatrix} t \frac{L}{2} ds$$

evaluated along $s = 1$.

Example 12 - Linear Quad Element

$$N_2 = \frac{(1+r)(1-s)}{4} = \frac{(1-s)}{2} \quad \text{and} \quad N_3 = \frac{(1+r)(1+s)}{4} = \frac{(1+s)}{2} \;\; at\, r = 1$$

$$\{T\} = \begin{Bmatrix} p_r \\ p_s \end{Bmatrix} = \begin{Bmatrix} 2000 \\ 0 \end{Bmatrix}; \quad \mathrm{t} = 0.1\ \mathrm{in}$$

$$\{f_s\} = \int_{-1}^{1} [N_s]^T \{T\} t \frac{L}{2} dt = \int_{-1}^{1} \begin{bmatrix} N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{Bmatrix} 2000 \\ 0 \end{Bmatrix} 0.1 \frac{5}{2} ds$$

$$\{f_s\} = 0.25 \int_{-1}^{1} \begin{bmatrix} 2000N_2 \\ 0 \\ 2000N_3 \\ 0 \end{bmatrix} ds = 500 \int_{-1}^{1} \begin{bmatrix} \frac{1-s}{2} \\ 0 \\ \frac{1+s}{2} \\ 0 \end{bmatrix} ds = 500 \begin{bmatrix} 0.50s - \frac{s^2}{4} \\ 0 \\ 0.50s + \frac{s^2}{4} \\ 0 \end{bmatrix}_{-1}^{1} = 500 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} lb$$

$$\begin{Bmatrix} f_{s2r} \\ f_{s2s} \\ f_{s3r} \\ f_{s3s} \end{Bmatrix} = \begin{bmatrix} 500 \\ 0 \\ 500 \\ 0 \end{bmatrix} lb$$

Quadratic Triangular Element

$$N_1 = r(2r-1)$$

$$N_2 = s(2s-1)$$

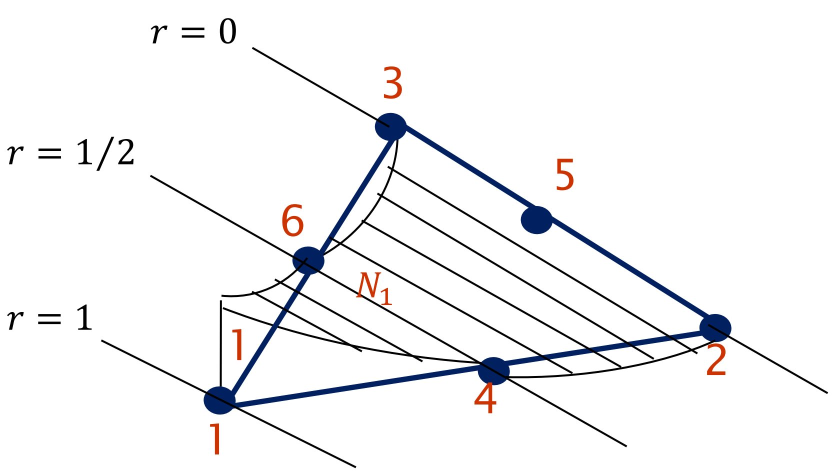
$$N_3 = (1-r-s)(1-2(r+s))$$

$$N_4 = 4rs$$

$$N_5 = 4s(1-r-s)$$

$$N_6 = 4r(1-r-s)$$

Shape Function N_1



$$u = \sum_{i=1}^{6} N_i u_i, \quad v = \sum_{i=1}^{6} N_i v_i \quad x = \sum_{i=1}^{6} N_i x_i, \quad y = \sum_{i=1}^{6} N_i y_i,$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ v_6 \end{Bmatrix}$$

- Quadratic triangular element is also known as Linear Strain Triangular (LST) Element

Linear Strain Triangular Element

$$\begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial x}{\partial y} \end{Bmatrix}$$

$\mathbf{J} = Jacobian\ matrix$

$$\frac{\partial x}{\partial r} = (4r-1)x_1 + (0)x_2 + (4r+4s-3)x_3 + (4s)x_4 + (-4s)x_5 + (4-8r-4s)x_6$$

$$\frac{\partial x}{\partial s} = (0)x_1 + (4s-1)x_2 + (4r+4s-3)x_3 + (4r)x_4 + (4-4r-8s)x_5 + (-4r)x_6$$

…

Jacobian is NOT independent of r and s anymore !

$$\boxed{\{\sigma\} = [C][B]\{u\}}$$

$$\boxed{[K] = \int_V [B]^T [C][B]\, dV}$$

Quadratic Quadrilateral Element

$$N_1 = \frac{1}{4}(1-r)(s-1)(r+s+1)$$

$$N_2 = \frac{1}{4}(1+r)(s-1)(s-r+1)$$

$$N_3 = \frac{1}{4}(1+r)(1+s)(r+s-1)$$

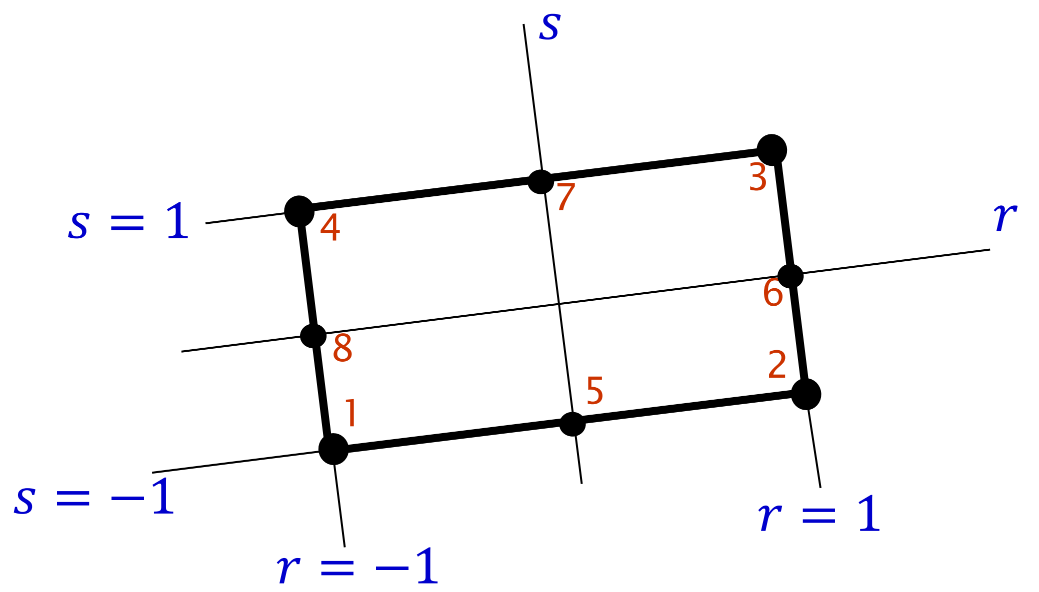
$$N_4 = \frac{1}{4}(r-1)(s+1)(r-s+1)$$

$$N_5 = \frac{1}{2}(1-s)(1-r^2)$$

$$N_6 = \frac{1}{2}(1+r)(1-s^2)$$

$$N_7 = \frac{1}{2}(1+s)(1-r^2)$$

$$N_8 = \frac{1}{2}(1-r)(1-s^2)$$

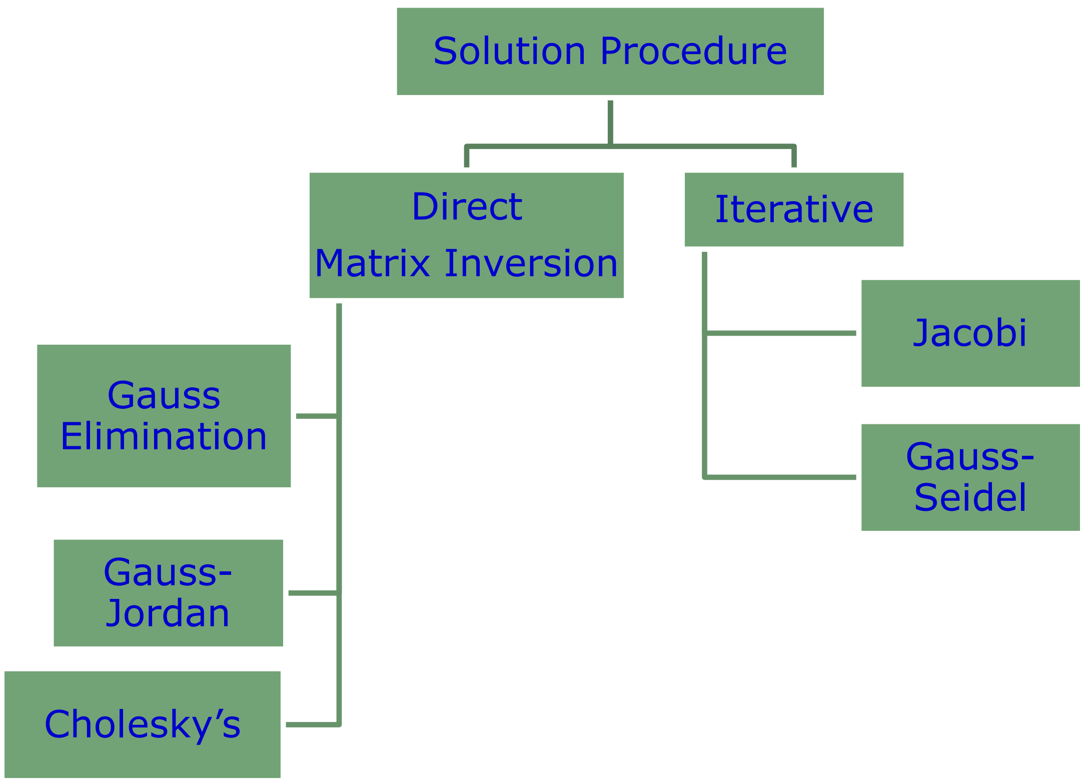


$$\sum_{i=1}^{8} N_i = 1 \qquad u = \sum_{i=1}^{8} N_i u_i\,, \qquad v = \sum_{i=1}^{8} N_i v_i$$

Solving [K]{q}={f}

Solving [K]{q}={f}

- Solution to system of equations [K]{q}={f} requires inversion of global stiffness matrix since {q} =$[K]^{-1}$ {f}. The inverse of a Hermitian positive-definite matrix exists when its determinant is non-zero. Finding the $[K]^{-1}$ is computationally expensive. Hence, sometimes, iterative methods are preferred over the direct inversion methods



Gauss Elimination

- Gauss elimination is a form of LU factorization refers to the factorization of A, with proper row and/or column orderings or permutations, into two factors – a lower triangular matrix L and an upper triangular matrix U. Hence,

  A = LU

- Gauss elimination involves producing an upper or lower triangular matrix as first step, and then use backward or forward substitution to solve for the unknowns

$$\begin{bmatrix} A11 & A12 & A13 \\ A21 & A22 & A23 \\ A31 & A32 & A33 \end{bmatrix} \begin{bmatrix} X1 \\ X2 \\ X3 \end{bmatrix} = \begin{bmatrix} B1 \\ B2 \\ B3 \end{bmatrix}$$

- Divide the first row by (pivot element a11) to get

$$\begin{bmatrix} 1 & A'12 & A'13 \\ A21 & A22 & A23 \\ A31 & A32 & A33 \end{bmatrix} \begin{bmatrix} X1 \\ X2 \\ X3 \end{bmatrix} = \begin{bmatrix} B'1 \\ B2 \\ B3 \end{bmatrix}$$

Gauss Elimination

- Pivot element A22 to get

1	A'12	A'13	X1		B'1
0	1	A"23	X2	=	B"2
0	A'32	A'33	X3		B'3

- Pivot element A'33 to get

1	A'12	A'13	X1		B'1
0	1	A"23	X2	=	B"2
0	0	1	X3		B'"3

- Backward substitution to solve for the components of X. One can produce a lower triangular matrix and use a forward substitution

Gauss – Jordan Elimination

- Gauss Jordan elimination is an adaptation of Gauss elimination in which both elements above and below the pivot element are manipulated to zero values; Hence, the entire column except the pivot (or the diagonal) elements attain zero values

- Advantage is that no backward/forward substitution is necessary anymore

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X1 \\ X2 \\ X3 \end{bmatrix} = \begin{bmatrix} B'''1 \\ B'''2 \\ B'''3 \end{bmatrix}$$

Cholesky's Method

- Cholesky's method involves LDL decomposition of a Hermitian positive-definite matrix A as

  $$A = LDL^*$$

  where L is a lower triangular matrix with real and positive diagonal entries, and L^* denotes the conjugate transpose of L. D is a diagonal matrix

- The solution to $[K]\{q\} = \{f\}$ is obtained by first computing the Cholesky decomposition $[K] = [L][D][L]^*$. Then solving $[L]\{y\} = \{f\}$ for $\{y\}$, and finally, solving $[D][L]^*\{q\} = \{y\}$ for $\{q\}$

- For linear systems that can be put into symmetric form, the Cholesky decomposition (or its LDL variant) is the method of choice, for superior efficiency and numerical stability. Compared to the LU decomposition, it is roughly twice as efficient

Iterative Methods - Jacobi

- The first iterative technique to obtain solutions of a strictly diagonally dominant system of linear equations is called the Jacobi method, after Carl Gustav Jacobi . This method makes two assumptions:

  (1) that the system given by has a unique solution

  a11 x1 + a12 x2 + an xn = b1

  a21 x1 + a22 x2 + a2n xn = b2

  …

  an1 x1 + an2 x1 + ann xn = bn

  (2) that the diagonal coefficient matrix A has no-zero values. If any of the diagonal entries are zero, then rows or columns must be interchanged to obtain a coefficient matrix that has nonzero entries on the main diagonal

- For the given linear equations to solve using the Jacobi method the computation of $x_i^{(k+1)}$ requires each element in $\mathbf{x}^{(k)}$ except itself. The minimum amount of storage is two vectors of size n. The iterations continue until the values converges to specified tolerance per chosen convergence criteria

Iterative Methods – Gauss Seidel

- With the Jacobi method, the values of x_i obtained in the nth approximation remain unchanged until the entire (n+1) th approximation has been calculated. With the Gauss- Seidel method, on the other hand, you use the new values of each x_i as soon as they are known. That is, once you have determined x_1 from the first equation, its value is then used in the second equation to obtain the new x_2. Similarly, the new x_1 and x_2 are used in the third equation to obtain the new x_3 and so on

- Though Gauss-Seidel method can be applied to any matrix with non-zero elements on the diagonals, convergence is only guaranteed if the matrix is either diagonally dominant, or symmetric and positive definite

- Unlike the Jacobi method, only one storage vector is required as elements can be overwritten as they are computed, which can be advantageous for very large problems

Thermal Analysis

Thermal Analysis

$$U = \int_V \frac{1}{2}(\{\varepsilon\} - \{\varepsilon_T\})^T[C](\{\varepsilon\} - \{\varepsilon_T\})qV$$

$$U = \frac{1}{2}\int_V ([B]\{q\} - \{\varepsilon_T\})^T[C]([B]\{q\} - \{\varepsilon_T\})dV$$

$$U = \frac{1}{2}\int_V (\{q\}^T[B]^T[C][B]\{q\} - \{q\}^T[B]^T[C]\{\varepsilon_T\} - \{\varepsilon_T\}^T[C][B]\{q\} + \{\varepsilon_T\}^T[C]\{\varepsilon_T\}dV)$$

$$U_M = \frac{1}{2}\int_V \{q\}^T[B]^T[C][B]\{q\}dV \qquad U_T = \int_V \{q\}^T[B]^T[C]\{\varepsilon_T\}dV \qquad U_C = \int_V \{\varepsilon_T\}^T[C]\{\varepsilon_T\}dV$$

$$\frac{\partial U}{\partial\{q\}} = 0 \rightarrow \frac{\partial(U_M - U_T + U_C)}{\partial\{q\}} = 0 \qquad \frac{\partial U}{\partial\{q\}} = \int_V [B]^T[C][B]\{q\}dV - [B]^T[C]\{\varepsilon_T\}dV = 0$$

$$[\mathrm{K}]\{q\} = \{f_M\} + \{f_T\}$$

$$[K] = \int_V [B]^T[C][B]\,dV$$

$$\int_V [B]^T[C]\{\varepsilon_T\}dV = \{f_T\}$$

Element Thermal Force Vector

Bar Element:

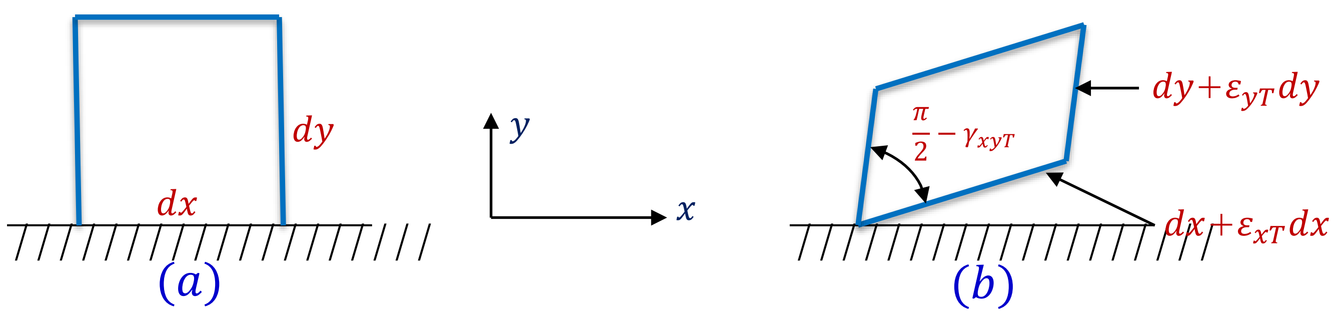
$\{\varepsilon_T\} = \{\varepsilon_{xT}\} = \{\alpha T\}$ where the units on α are typically (in./in.)/^{o}F or (mm/mm)/^{o}C.

$$\{f_T\} = A\int_0^L B^T[C]\{\alpha T\}dx \qquad [C] = E \qquad [B] = \left[-\frac{1}{L}\ \ \frac{1}{L}\right]$$

$$\boxed{\{f_T\} = \begin{Bmatrix} f_{T1} \\ f_{T2} \end{Bmatrix} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}}$$

Plane stress and Plane Strain

$$\{\varepsilon_T\} = \begin{Bmatrix} \varepsilon_{xT} \\ \varepsilon_{yT} \\ \gamma_{xyT} \end{Bmatrix}$$



Element Thermal Force Vector

For isotropic materials:

$$\{\varepsilon_T\} = \begin{Bmatrix} \alpha T \\ \alpha T \\ 0 \end{Bmatrix}$$

$$\{\varepsilon_T\} = (1+v)\begin{Bmatrix} \alpha T \\ \alpha T \\ 0 \end{Bmatrix}$$

Plane Strain

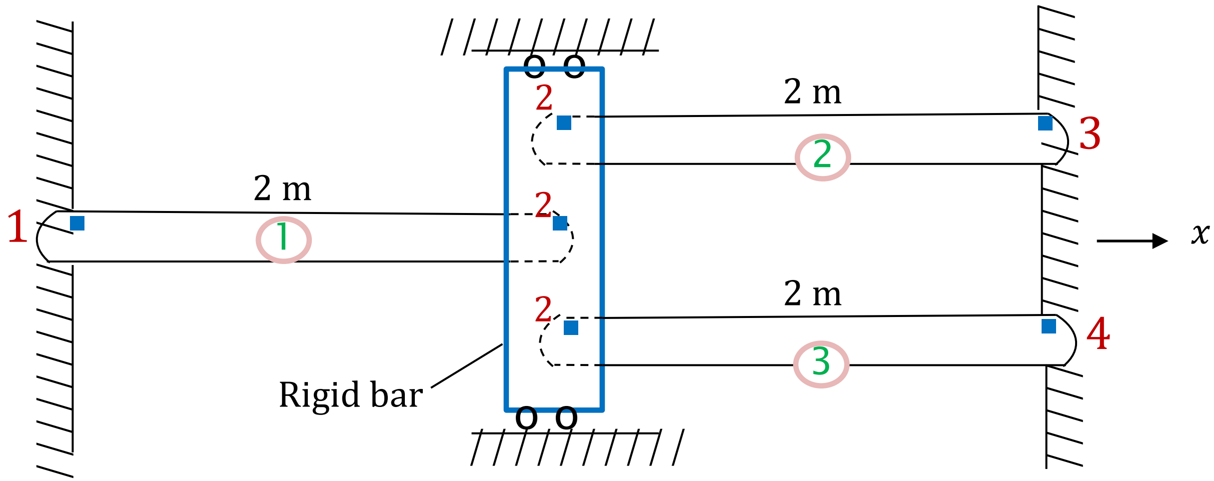
$$\{f_T\} = [B]^T[C]\{\varepsilon_T\}tA$$

$$\{f_T\} = \begin{Bmatrix} f_{T1x} \\ f_{T1y} \\ \vdots \\ f_{T3y} \end{Bmatrix} = \frac{\alpha EtT}{2(1-v)}\begin{Bmatrix} \beta_i \\ \gamma_i \\ \beta_j \\ \gamma_j \\ \beta_m \\ \gamma_m \end{Bmatrix}$$

Plane Stress

$$\{\sigma\} = [C](\{\varepsilon\} - \{\varepsilon_T\}) = [C][B]\{q\} - [C]\{\varepsilon_T\}$$

Example 13 – Thermal Analysis



For the bar assemblage shown in figure, determine the reactions at the fixed ends and the axial stress in each bar. Bar 1 is subjected to a temperature drop of 10^0 C . Let bar 1 be aluminum with $E = 70$ GPa, $\alpha = 23 \text{ x } 10^{-6} (\text{mm/mm})/^0$ C, $A = 12 \text{ x } 10^{-4} \text{m}^2$, and $L = 2$ m. Let bars 2 and 3 be brass with $E = 100$GPa, $\alpha = 20 \text{ x } 10^{-6} (\text{mm/mm})/^0$C, $A = 6 \text{ x } 10^{-4} \text{m}^2$, and $L = 2$ m.

Example 13 – Thermal Analysis

Element 1 [K]:

$$\left[k^{(1)}\right] = \frac{(12 \text{ x } 10^{-4})(70 \text{ x } 10^{6})}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 42{,}000 \begin{array}{c} \begin{matrix} 1 & \quad 2 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \frac{\text{kN}}{\text{m}}$$

Element 2 and 3 [K]:

$$\left[k^{(1)}\right] = \left[k^{(3)}\right] = \frac{(6 \text{ x } 10^{-4})(100 \text{ x } 10^{6})}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 30{,}000 \begin{array}{c} \begin{matrix} 2 & \quad 3 \end{matrix} \\ \begin{matrix} 2 & \quad 4 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \frac{\text{kN}}{\text{m}}$$

Thermal Force Vector [f]:

$$-E\alpha TA = -(70 \text{ x } 10^{6})(23 \text{ x } 10^{-6})(-10)(12 \text{ x } 10^{-4}) = 19.32\text{kN}$$

$$\left\{f^{(1)}\right\} = \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} 19.32 \\ -19.32 \end{Bmatrix}\text{kN} \qquad \left\{f^{(2)}\right\} = \left\{f^{(3)}\right\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

No temperature change in elements 2 and 3

Example 13 – Thermal Analysis

Global System:

$$1000\begin{bmatrix} 42 & -42 & 0 & 0 \\ -42 & 42+30+30 & -30 & -30 \\ 0 & -30 & 30 & 0 \\ 0 & -30 & 0 & 30 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 19.32 \\ -19.32 \\ 0 \\ 0 \end{Bmatrix}$$

where the right−side thermal forces are considered to be equivalent nodal forces. Using the boundary conditions.

$$u_1 = 0 \qquad u_3 = 0 \qquad u_4 = 0$$

$$1000(102)u_2 = -19.32$$ from the second equation,

$$u_2 = -1.89 \text{ x } 10^{-4}\text{m}$$

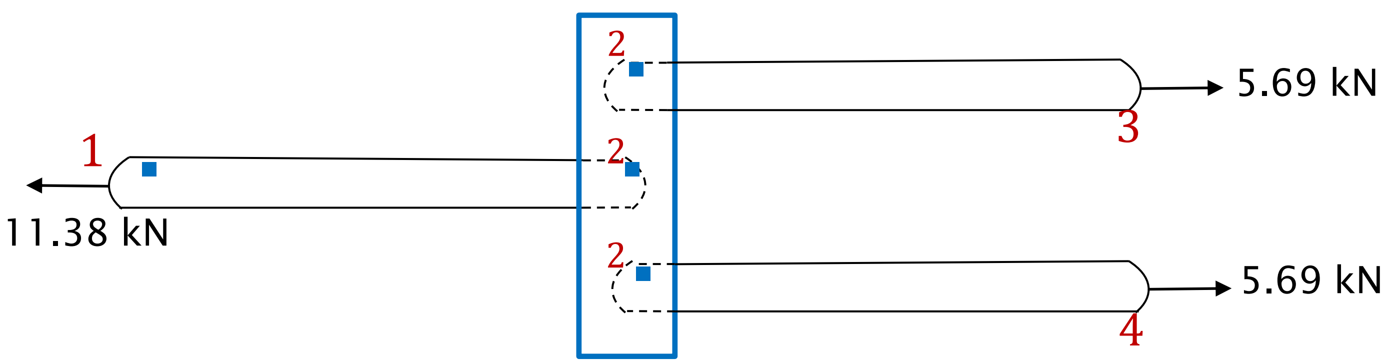
Example 13 – Thermal Analysis

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 1000 \begin{bmatrix} 42 & -42 & 0 & 0 \\ -42 & 102 & -30 & -30 \\ 0 & -30 & 30 & 0 \\ 0 & -30 & 0 & 30 \end{bmatrix} \begin{Bmatrix} 0 \\ -1.89 \text{ x } 10^{-4} \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 19.32 \\ -19.32 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{Bmatrix} -11.38 \\ 0 \\ 5.69 \\ 5.69 \end{Bmatrix} \text{kN}$$

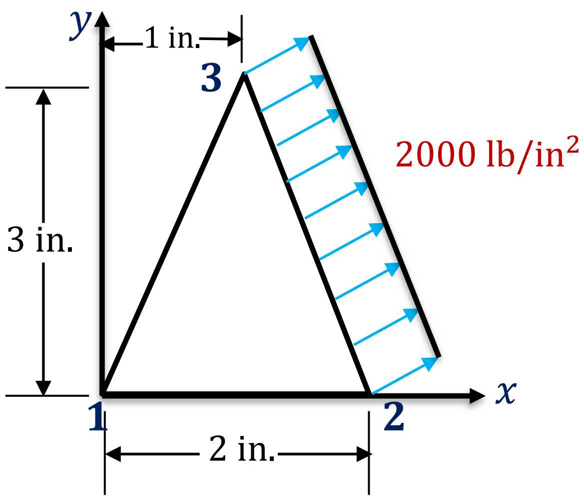
$$\sigma^{(1)} = \frac{11.38}{12 \text{ x } 10^{-4}} = 9.48 \text{ x } 10^{3} \text{ kN/m}^2 \qquad (9.48 \text{ MPa})$$

$$\sigma^{(2)} = \sigma^{(3)} = \frac{5.69}{6 \text{ x } 10^{-4}} = 9.48 \text{ x } 10^{3} \text{ kN/m}^2 \qquad (9.48 \text{ MPa})$$



Free-body diagram of the bar assemblage

Example 14 – Thermal Analysis



$$t = 1 \text{ in.}$$
$$E = 30 \text{ x } 10^6 \text{psi}$$
$$\alpha = 7 \text{ x } 10^{-6} (\text{in./in.})^0\text{F}$$
$$v = 0.25$$
$$T = 30^0\text{F}$$

$$A = \frac{(3)(2)}{2} = 3\text{in}^2$$

$$\beta_1 = y_1 - y_3 = -3 \qquad \gamma_1 = x_3 - x_1 = -1$$

$$\beta_2 = y_3 - y_1 = 3 \qquad \gamma_2 = x_1 - x_3 = -1$$

$$\beta_3 = y_1 - y_2 = -3 \qquad \gamma_3 = x_2 - x_1 = 2$$

$$[B] = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

Example 14 – Thermal Analysis

$$[C] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix} = \frac{30\text{ x}10^6}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} = (4\text{ x }10^6)\begin{bmatrix} 8 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix}\text{psi}$$

$$[B]^T[C] = \frac{1}{6}\begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}(4\text{ x }10^6)\begin{bmatrix} 8 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \frac{4\text{ x }10^6}{6}\begin{bmatrix} -24 & -6 & -3 \\ -2 & -8 & -9 \\ 24 & 6 & -3 \\ -2 & -8 & 9 \\ 0 & 0 & 6 \\ 4 & 16 & 0 \end{bmatrix}$$

$$[K] = (1\text{ in.})\frac{3\text{in.}^2}{6}\frac{4\text{ x }10^6}{6}\begin{bmatrix} -24 & -6 & -3 \\ -2 & -8 & -9 \\ 24 & 6 & -3 \\ -2 & -8 & 9 \\ 0 & 0 & 6 \\ 4 & 16 & 0 \end{bmatrix}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

Example 14 – Thermal Analysis

$$[K] = \frac{1 \text{ x } 10^6}{3}\begin{bmatrix} 75 & 15 & -69 & -3 & -6 & -12 \\ 15 & 35 & 3 & -19 & -18 & -16 \\ -69 & 3 & 75 & -15 & -6 & 12 \\ -3 & -19 & -15 & 35 & 18 & -16 \\ -6 & -18 & -6 & 18 & 12 & 0 \\ -12 & -16 & 12 & -16 & 0 & 32 \end{bmatrix}\frac{\text{lb}}{\text{in.}}$$

$$\{f_T\} = \frac{\alpha E t T}{2(1-v)}\begin{Bmatrix} \beta_i \\ \gamma_i \\ \beta_j \\ \gamma_j \\ \beta_m \\ \gamma_m \end{Bmatrix} = \frac{(7 \text{ x } 10^{-6})(30 \text{ x } 10^6)(1)(30)}{2(1-0.25)}\begin{Bmatrix} -3 \\ -1 \\ 3 \\ -1 \\ 0 \\ 2 \end{Bmatrix} = 4200\begin{Bmatrix} -3 \\ -1 \\ 3 \\ -1 \\ 0 \\ 2 \end{Bmatrix}$$

$$\{f_T\} = \begin{Bmatrix} -12{,}600 \\ -4200 \\ 12{,}600 \\ -4200 \\ 0 \\ 8400 \end{Bmatrix}\text{lb}$$

Example 14 – Thermal Analysis

The force matrix due to the pressure applied alongside $2-3$ is determined as follows:

$$L_{2-3} = [(2-1)^2 + (3-0)^2]^{1/2} = 3.163 \text{ in.}$$

$$p_x = p\cos\theta = 2000\left(\frac{3}{3.163}\right) = 1896 \text{ lb/in}^2 \qquad p_y = p\sin\theta = 2000\left(\frac{1}{3.163}\right) = 632 \text{ lb/in}^2$$

where θ is the angle measured from the x axis to the normal to surface $2-3$

$$\{f_p\} = \iint_{S_{j-m}} [N_s]^T \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} dS = \iint_{S_{j-m}} \begin{bmatrix} N_i & 0 \\ 0 & N_i \\ N_j & 0 \\ 0 & N_j \\ N_m & 0 \\ 0 & N_m \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} dS = \frac{tL_{j-m}}{2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

evaluated alongside $j-m$

$$\{f_p\} = \frac{(1 \text{ in.})(3.163 \text{ in.})}{2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1896 \\ 632 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 3000 \\ 1000 \\ 3000 \\ 1000 \end{Bmatrix} \text{ lb}$$

Example 14 – Thermal Analysis

Alternately:

$$\{f_p\} = t\begin{Bmatrix} 0 \\ 0 \\ \dfrac{p_x b}{2} \\ \dfrac{p_y a}{2} \\ \dfrac{p_x b}{2} \\ \dfrac{p_y a}{2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 3000 \\ 1000 \\ 3000 \\ 1000 \end{Bmatrix} \text{ lb}$$

$$p_x = 1896 \text{ lb/in}^2$$

$$p_y = 632 \text{ lb/in}^2$$

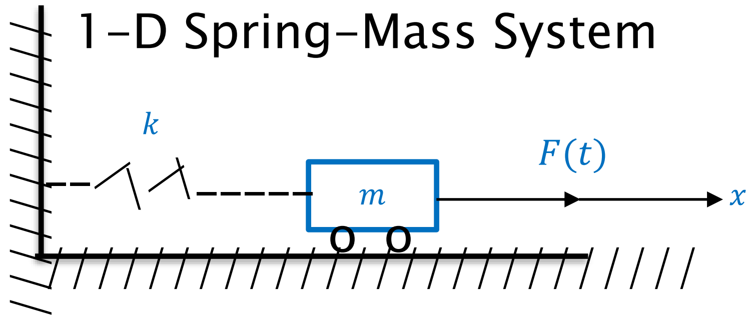
$$a = b = 3.163\ in$$

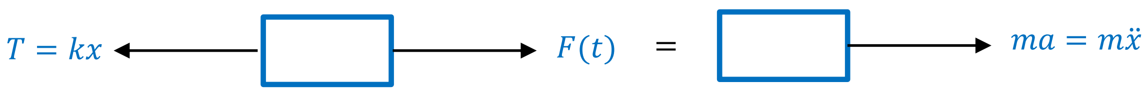
$$\frac{1 \text{ x } 10^6}{3}\begin{bmatrix} 75 & 15 & -69 & -3 & -6 & -12 \\ & 35 & 3 & -19 & -18 & -16 \\ & & 75 & -15 & -6 & 12 \\ & & & 35 & 18 & -16 \\ & & & & 12 & 0 \\ symm & & & & & 32 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} -12{,}600 \\ -4200 \\ 15{,}600 \\ -3200 \\ 3000 \\ 9400 \end{Bmatrix}$$

where the force matrix is $\{f_T\} + \{f_p\}$

Dynamic Analysis

Dynamic Analysis



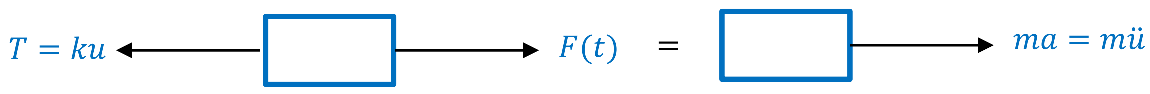


$$F(t) - kx = m\ddot{x} \rightarrow \quad m\ddot{x} + kx = F(t)$$

$$Free\ Vibrations\colon\ F(t) = 0; x(t) = Xe^{i\omega t}$$

$$\omega^2 = \frac{k}{m} \text{ or } \omega = \quad Natural\ Frequency$$

Dynamic Analysis – Bar Element



E, A, L Mass $= \rho AL$

$$m\ddot{u} + ku = F(t)$$

$$\boxed{[m] = \frac{\rho AL}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}$$

Mass lumped equally at the two nodes

$$\boxed{[m] = \iiint_v \rho[N]^T[N]dV}$$

Consistent Mass Matrix

$$\{f_b\} = \iiint_V [N]^T\{X\}dV$$

$$\{X\} = \rho\ddot{q} = \rho[N]\ddot{u}$$

$$\boxed{[m] = \frac{\rho AL}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}}$$

Consistent Mass Matrix

Dynamic Analysis – Beam Element

$$\begin{array}{cc} & \begin{matrix} v_1 & \theta_1 & v_2 & \theta_2 \end{matrix} \\ [m] = \dfrac{\rho A L}{2} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Mass lumped equally at the two nodes corresponding to translational dof

$$[m] = \iiint_v \rho [N]^T [N] dV$$

$$[m] = \int_0^L \iint_A \rho \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} [N_1 \;\; N_2 \;\; N_3 \;\; N_4] dA dx$$

$$[m] = \frac{\rho A L}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

Consistent Mass Matrix

Dynamic Analysis – Frame Element

Consistent Mass Matrix

$$[m'] = \rho A L \begin{bmatrix} 2/6 & 0 & 0 & 1/6 & 0 & 0 \\ & 156/420 & 22L/420 & 0 & 54/420 & -13L/420 \\ & & 4L^2/420 & 0 & 13L/420 & -3L^2/420 \\ & & & 2/6 & 0 & 0 \\ & & & & 156/420 & -22L/420 \\ Symmetry & & & & & 4L^2/420 \end{bmatrix}$$

Lumped Mass Matrix

$$[m'] = \frac{\rho A L}{2} \begin{array}{c} \begin{array}{cccccc} u_1' & v_1' & \theta_1' & u_2' & v_2' & \theta_2' \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

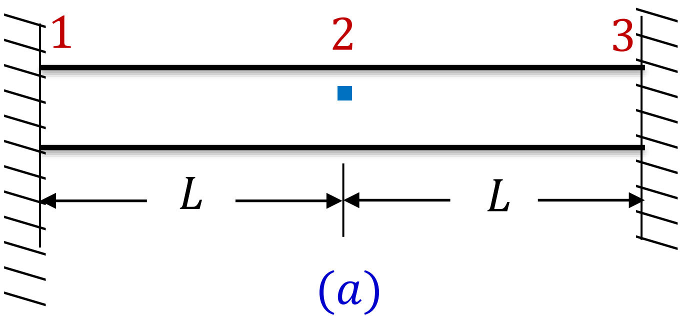
$$[m] = [T]^T [m'] \ [T]$$

Dynamic Analysis – CST Element

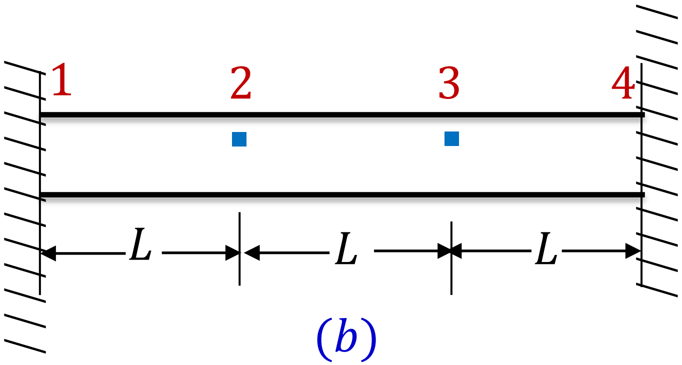
Consistent Mass Matrix

$$[m] = \frac{\rho t A}{12}\begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ & 2 & 0 & 1 & 0 & 1 \\ & & 2 & 0 & 1 & 0 \\ & & & 2 & 0 & 1 \\ & & & & 2 & 0 \\ SYM & & & & & 2 \end{bmatrix}$$

Example 15 - Dynamic Analysis



Two elements



Three elements

Two element Solution:

(Using boundary conditions $v_1 = 0, \theta_1 = 0, v_3 = 0$, and $\theta_3 = 0$ to reduce the matrices) as

$$[K] = \frac{EI}{L^3}\begin{array}{cc} v_2 & \theta_2 \end{array}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} \qquad [M] = \frac{\rho AL}{2}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left| \frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^3 \end{bmatrix} - \omega^2 \rho AL \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right| = 0 \qquad \omega^2 = \frac{24EI}{\rho AL^4} \quad \text{OR} \quad \boxed{\omega = \frac{4.90}{L^2}\left(\frac{EI}{A\rho}\right)^{1/2}}$$

Example 15 - Dynamic Analysis

Three element Solution:

$$[m^{(1)}] = \frac{\rho AL}{2}\begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad [m^{(2)}] = \frac{\rho AL}{2}\begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad [m^{3}] = \frac{\rho AL}{2}\begin{array}{c} \begin{array}{cccc} v_3 & \theta_3 & v_4 & \theta_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(Using boundary conditions $v_1 = 0, \theta_1 = 0, v_4 = 0,$ and $\theta_4 = 0$ to reduce the matrices) as

$$[M] = \frac{\rho AL}{2}\begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 15 - Dynamic Analysis

$$\left[k^{(1)}\right] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \qquad \left[k^{(2)}\right] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

$$\left[k^{(3)}\right] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_3 & \theta_3 & v_4 & \theta_4 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

$$[K] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12-12 & 6L+6L & -12 & 6L \\ 6L-6L & 4L^2+2L^2 & -6L & 2L^2 \\ -12 & -6L & 12+12 & -6L+6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 \end{bmatrix} \end{array} = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 0 & 12L & -12 & 6L \\ 0 & 6L^2 & -6L & 2L^2 \\ -12 & -6L & 24 & 0 \\ 6L & 2L^2 & 0 & 8L^2 \end{bmatrix} \end{array}$$

Example 15 - Dynamic Analysis

$$\left|\frac{EI}{L^3}\begin{bmatrix} 0 & 12L & 12 & 6L \\ 0 & 6L^2 & -6L & 2L^2 \\ -12 & -6L & 24 & 0 \\ 6L & 2L^2 & 0 & 8L^2 \end{bmatrix} - \omega^2\rho AL \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right| = 0$$

$$\rightarrow \begin{vmatrix} -\omega^2\rho AL & 12\,EI/L^2 & -12\,EI/L^3 & 6EI/L^2 \\ 0 & 6\,EI/L & -6\,EI/L^2 & 2\,EI/L \\ -12\,EI/L^3 & -6\,EI/L^2 & 24\,EI/L^3 - \omega^2\rho AL & 0 \\ 6\,EI/L^2 & 2\,EI/L & 0 & 8\,EI/L \end{vmatrix} = 0$$

$$\rightarrow \begin{vmatrix} -\omega^2\beta & 12\,EI/L^2 & -12\,EI/L^3 & 6EI/L^2 \\ 0 & 6\,EI/L & -6\,EI/L^2 & 2\,EI/L \\ -12\,EI/L^3 & -6\,EI/L^2 & 24\,EI/L^3 - \omega^2\beta & 0 \\ 6\,EI/L^2 & 2\,EI/L & 0 & 8\,EI/L \end{vmatrix} = 0$$

where $\beta = \rho AL$

Example 15 - Dynamic Analysis

$$\omega_1^2 = \frac{29.817254EI}{\beta L^3}$$

OR

$$\omega_1 = \sqrt{\frac{29.817254EI}{A\rho L^4}} = \frac{5.46}{L^2}\sqrt{\frac{EI}{A\rho}}$$

$$\boxed{\omega_1 = \frac{4.90}{L^2}\left(\frac{EI}{A\rho}\right)^{1/2}}$$

Two element Solution

$$\boxed{\omega_1 = \frac{5.46}{L^2}\sqrt{\frac{EI}{A\rho}}}$$

Three element Solution

$$\boxed{\omega_1 = \frac{5.59}{L^2}\left(\frac{EI}{A\rho}\right)^{1/2}}$$

Exact Solution

Time Integration Methods for Dynamic Analysis

Time Integration Methods

- For a given dynamic system, time integration methods enables us to determine the nodal displacements, element strains & stresses and many other quantitates of interest at different time increments

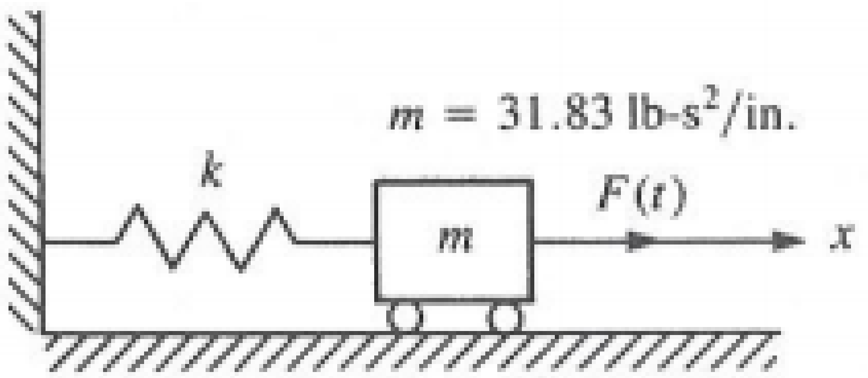
- Time integration methods using Direct Integration Techniques
  - Explicit Method
    - Central Difference Method
  - Implicit Method
    - Newmark-Beta
    - Wilson-Theta

Time Integration Methods

- In explicit method the state of a dynamic system at a later time is calculated using the its state at the current time
  - Y(t) is the current state of the system at time ‘t’
  - Y(t + Δt) is the state at a later time ‘t+Δt’
  - Δt is a small time step
  - $Y(t + \Delta t) = F(Y(t))$

- In implicit method the state of a dynamic system at a later time is calculated using the its state at the current time as well as its state at the later time
  - Y(t) is the current state of the system at time ‘t’
  - Y(t + Δt) is the state at a later time ‘t+Δt’
  - Δt is a small time step
  - $G(Y(t), Y(t + \Delta t)) = 0$, to find Y(t + Δt)

Example of Time Integration Methods



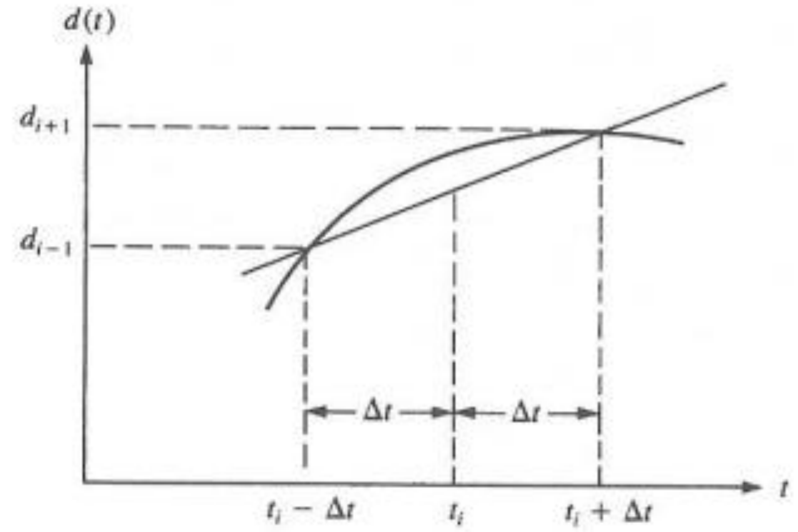
Given:
F(t) = 2000 lb (decreases to 0 lb at .2s)
k = 100 lb/in
Δt = 0.05 s (upto .2s)

Required:
Determine the displacement, velocity, and acceleration throughout the time interval

Solve using three different time integration methods

Explicit Method: Central Difference

- Finite (Central) difference expressions for velocity and acceleration are used in an iterative process to obtain the solution for a dynamic system



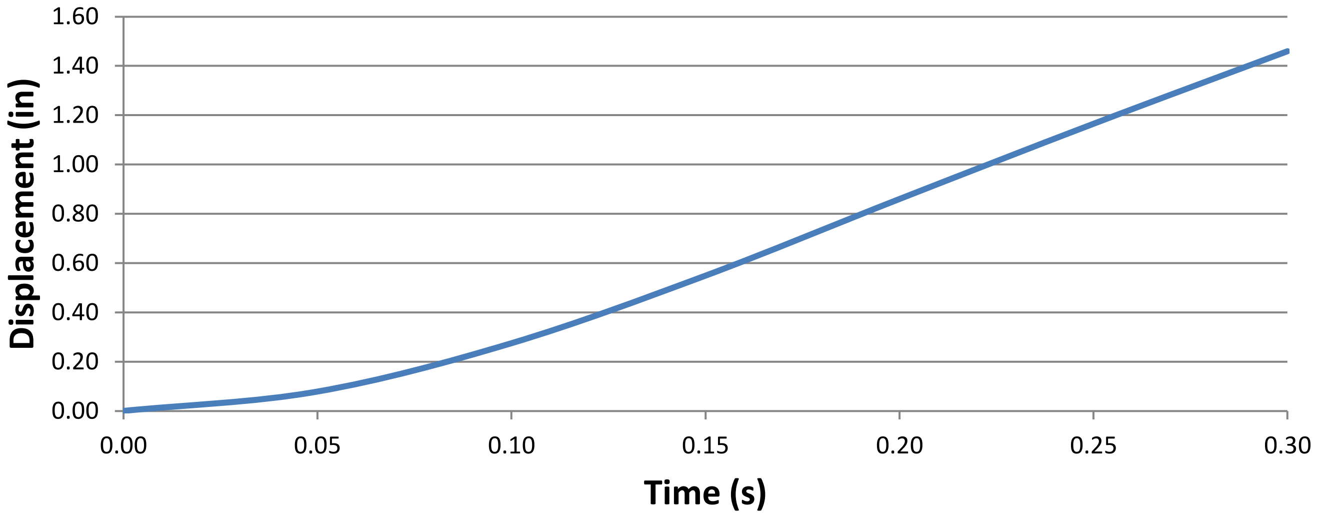
$\{d_i\}$ Nodal Displacements

$$\{d_i\}' = \frac{\{d_{i+1}\} - \{d_{i-1}\}}{2(\Delta t)}$$ Velocity

$$\{d_i\}'' = \frac{\{d_{i+1}\} - 2\{d_i\} + \{d_{i-1}\}}{(\Delta t)^2}$$ Acceleration

$$md_i'' + kd_i \ = F \rightarrow m \times \left[\frac{d_{i+1} - 2d_i + d_{i-1}}{\Delta t^2}\right] + kd_i = F$$

Explicit Method: Central Difference



	Time (s)	F(t) (lb)	d_i (in)	Q (lb)	d_i'' (in/s^2)	d_i' (in/s)
d_{-1}	-	-	-	-	-	-
d_0	0.00	2000.00	0.00	0.00	62.83	0.00
d_1	0.05	1500.00	0.08	7.85	46.88	2.75
d_2	0.10	1000.00	0.27	27.48	30.55	4.71
d_3	0.15	500.00	0.55	54.91	13.98	5.85
d_4	0.20	0.00	0.86	86.03	-2.70	6.17
d_5	0.25	0.00	1.17	116.56	-3.66	6.00
d_6	0.30	0.00	1.46	146.03	-4.59	-11.66

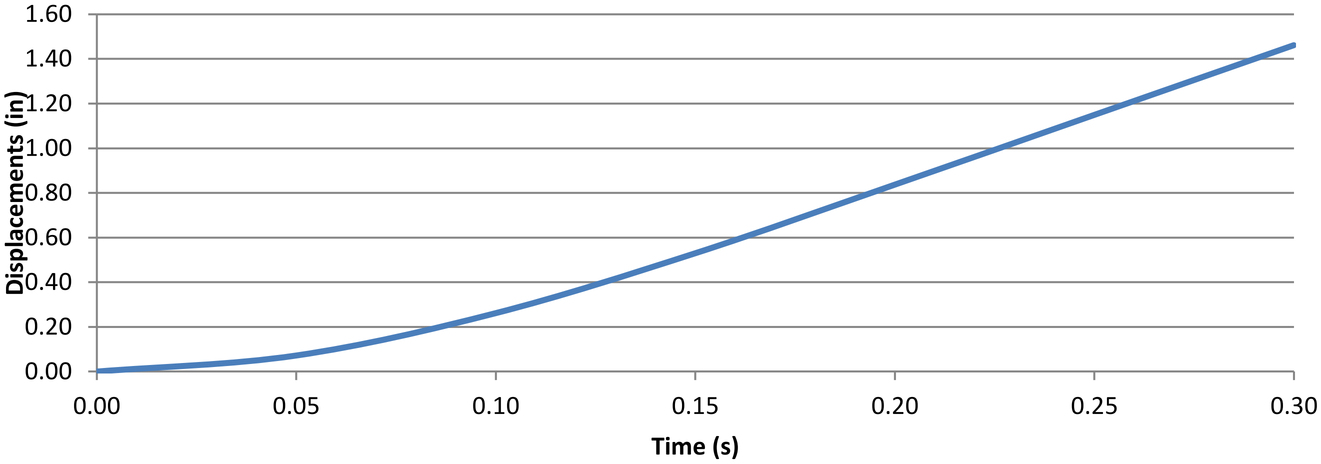
Implicit Method: Newmark-Beta

- Acceleration varies linearly
- Utilizes two “adjustable” arguments β and γ to “tune” the calculations to specific applications
- Method is unconditionally stable for $\gamma \geq \frac{1}{2}$ and
  $\beta \geq \frac{1}{4}(\gamma + \frac{1}{2})^2$

$$\{d'_{i+1}\} = \{d'_i\} + (\Delta t)[(1-\gamma)\{d''_i\} + \gamma\{d''_{i+1}\}]$$

$$\{d_{i+1}\} = \{d_i\} + (\Delta t)\{d'_i\} + (\Delta t^2)[(\frac{1}{2} - \beta)\{d''_i\} + \beta\{d''_{i+1}\}]$$

Implicit Method: Newmark-Beta



	Time (s)	F(t) (lb)	di (in)	Q (lb)	di" (in/s^2)	di' (in/s)
d_{-1}	-	-	-	-	-	-
d_0	0.00	2000.00	0.00	0.00	62.83	0.00
d_1	0.05	1500.00	0.07	7.20	47.06	2.75
d_2	0.10	1000.00	0.26	26.16	31.38	4.71
d_3	0.15	500.00	0.53	52.97	15.69	5.89
d_4	0.20	0.00	0.84	83.68	-0.74	6.26
d_5	0.25	0.00	1.15	114.91	0.00	6.24
d_6	0.30	0.00	1.46	146.11	0.00	6.27

Implicit Method: Wilson-Theta

❖ Acceleration varies linearly

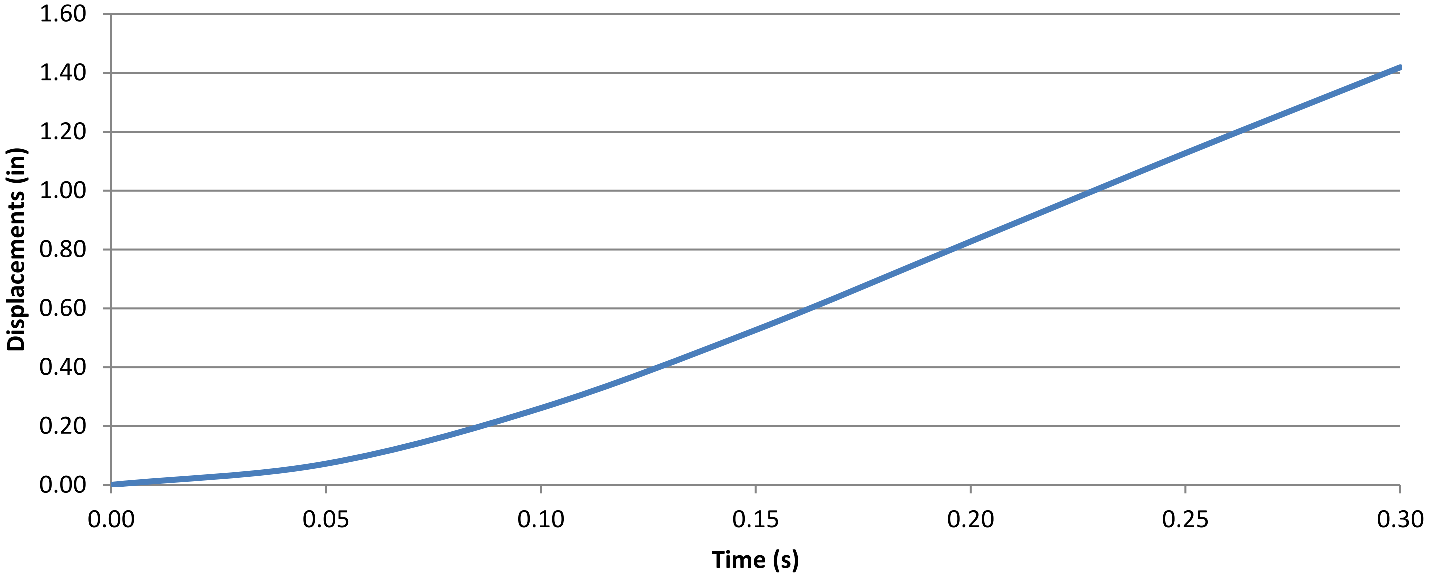
❖ Utilizes an adjustable parameter Θ as multiplier to the time step Δt where Θ ≥ 1.0

❖ Method is unconditionally stable for linear systems for Θ ≥ 1.37 and for non-linear at Θ ≥ 1.4

$$\{d'_{i+1}\} = \{d'_{i}\} + \frac{\Theta \Delta t}{2}(\{d''_{i+1}\} + \{d''_{i}\})$$

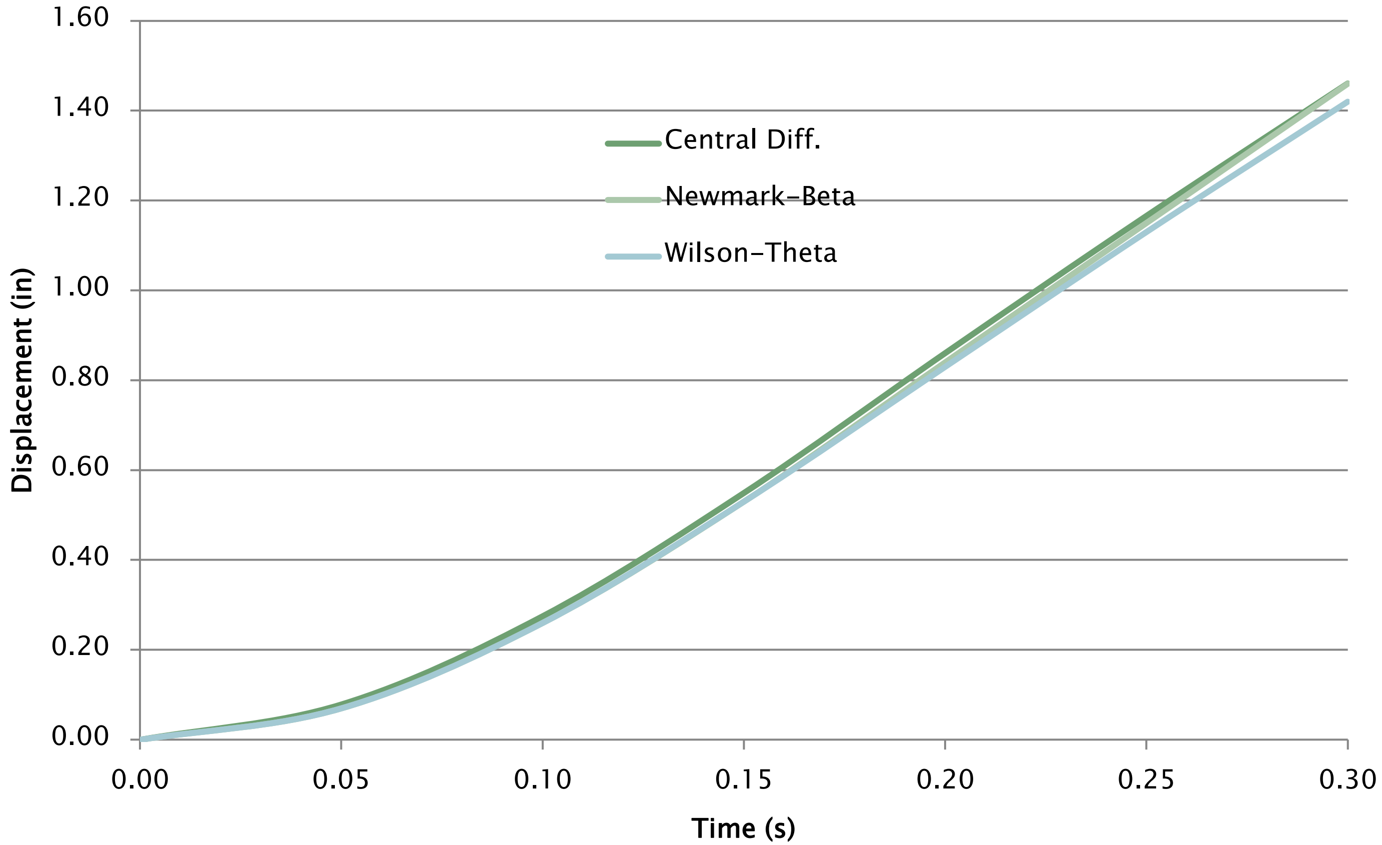
$$\{d_{i+1}\} = \Theta \Delta t \{d'_{i}\} + \frac{\Theta^2 (\Delta t)^2}{6}(\{d''_{i+1}\} + 2\{d''_{i}\}$$

Implicit Method: Wilson-Theta



	Time (s)	F(t) (lb)	di (in)	Q (lb)	di" (in/s^2)	di' (in/s)
d_{-1}	-	-	-	-	-	-
d_0	0.00	2000.00	0.00	0.00	62.83	0.00
d_1	0.05	1500.00	0.07	7.19	46.90	2.74
d_2	0.10	1000.00	0.26	26.09	30.60	4.68
d_3	0.15	500.00	0.53	52.63	14.06	5.80
d_4	0.20	0.00	0.83	82.68	-2.60	6.08
d_5	0.25	0.00	1.13	112.73	-3.54	5.93
d_6	0.30	0.00	1.42	141.90	-4.46	5.73

Comparison of Three Methods



Numerical Integration Techniques in FEA

Integration Techniques

- The computation of the stiffness matrix and load vectors requires the evaluation of one or more integrals depending on the dimension of the requested analysis

- Analytical solution for integrals is not always feasible, can be cumbersome and can take too much computational time

- Division by zero and machine precision induced errors can be issues affecting accuracy and convergence

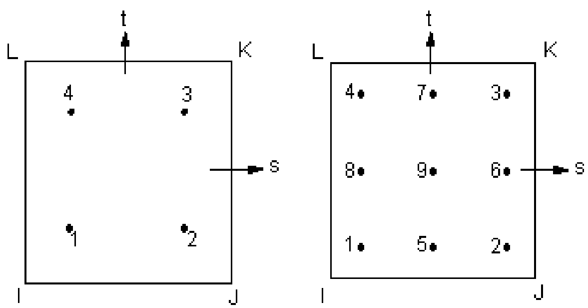
- Numerical Integration techniques such as Trapezoidal rule, Simpsons rule, Newton-Cotes quadrature rules, and Gaussian Quadrature are commonly used

- Reduced integration requires using fewer integration points than a full conventional Gaussian quadrature. This has the effect of using a lower degree of polynomial in the integration process. This can be beneficial when encountering shear locking as in for example the Timoshenko beam since we assume the same order of polynomial for displacements and rotations, even though we know they are related by derivative, using reduced integration numerically simulates the use of a lower polynomial. Thus the inherent relations between displacements and rotations can be better accounted for

Integration Points

- An integration point is the point within an element at which integrals are evaluated numerically. These points are chosen in a way so that the results for a particular numerical integration scheme are the most accurate

- **Gauss points** are also called integration **points** because at these **points** numerical integration is carried out. Stresses are generally the most accurate at **Gauss points** and thus instead of calculating them at nodes, we do it at integration **points** and then extrapolate to the rest of the element



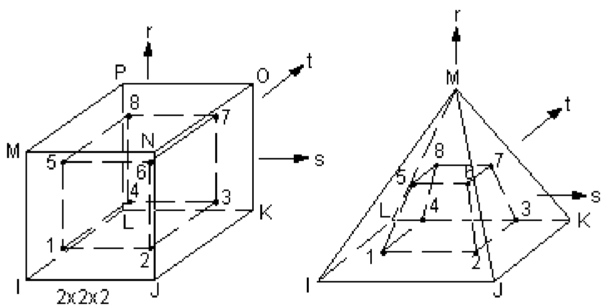
$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy = \sum_{j=1}^{m}\sum_{i=1}^{\ell} H_j H_i f(x_i, y_j)$$

f(x) = function to be integrated

H_i = weighting factor

x_i = locations to evaluate function

ℓ = number of integration (Gauss) points



$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(x,y,z)\,dx\,dy\,dz = \sum_{k=1}^{n}\sum_{j=1}^{m}\sum_{i=1}^{\ell} H_k H_j H_i f(x_i, y_j, z_k)$$

h- and p-Type Elements in FEA

h- and p- type Finite Elements

- Two main types of elements used in FEA
  - h-type
    - Classical lower order element
    - Linear or quadratic shape functions
    - Increase number of elements to achieve convergence
    - Local refinement to achieve accuracy
    - Completeness & Compatibility
    - Longer computing time
  - p-type
    - No mesh refinement and change needed
    - Adaptable polynomial order

Advantages and Disadvantages

p-type element

+ Faster, more accurate, easier to use

+ No remeshing for greater accuracy

+ Good for fatigue and fracture where local accuracy is required

+ Local or global error estimates

- Used only for linear structural and nonlinear solutions applications

- Required significantly greater computational resources

h-type element

+ Solution times know in advance

+ Adaptive Meshing to automatically achieve desired accuracy

+ Can be used for Dynamics, CFD, Coupled field and Magnetics

- Mesh refinement for precise accuracy control could be tedious

Computer Softwares for Finite Element Analysis

FE Analysis Programs

Computer Programs

Commercial programs

Advantages	Disadvantages
Special knowledge not required	Initial cost high
Can solve variety of problems	Low efficiency

Special purpose

Advantages	Disadvantages
Low development cost	Inability to solve variety of problems
Can run on small computers	
Can be easily revised	

Commercial FE Analysis Programs

Algor	Abaqus	ANSYS
General purpose FEA software	Routine and sophisticated engineering problems	Widely used FEA software
Bricks, shells, beams and trusses	Extensive range of material models	Structural, thermal, mechanical, electrical, electromagnetic
Bending, mechanical, thermal, fluid dynamics, coupled or uncoupled multiphysics	Automotive industry	Performs global structural assessment
Easy to use features	Coupled acoustic-structural, piezoelectric, and structural-pore capabilities	Automation with flexibility to customize
Multiple view windows	Generates report, image, animation, etc. from the output file	Parametric geometry creation
Contours or plots, image formats, animation, report wizard	4 core software products	Preparing existing geometry for analysis

Commercial FE Analysis Programs

COSMOS/M	GR-STRUDL
Complete, modular, self-contained finite element system	Architectural Engineering, offshore, civil works
Elastic beam elements, curved and straight pipe element, spar/truss, plates, shells	Plane truss, frame, grid, triangular prisms, bricks, shells
Static and dynamic structural problems, heat transfer, fluid mechanics, electromagnetics and optimization, buckling	Frame and finite static, dynamic, and nonlinear analysis, finite element analysis, structural frame design
Completely modular	Graphical modeling and result display
Powerful, intuitive, easy to learn and use	Different material properties
Reduce solution time and disk spacee	Joint loads, displacements, concentrated, uniformly and linearly distributed loads, temperature loads, element loads

Commercial FE Analysis Programs

MARC	MSC/NASTRAN	NISA
Nonlinear FEA solver	World's most widely used FEA solver	State-of-the-art GUI, seamless interoperability
Segment-segment contact method: smoother results contours	Comprehensive element library	Completely integrated pre/post- processing environment
Automatically replaces a distorted mesh	Offers a complete set of implicit and explicit nonlinear analysis	Extremely User Friendly
Static, Dynamic, Multi physics and Coupled Analysis	Unparalleled support for super elements	Analysis type: Redundancy, Static equilibrium, Quasi-static equilibrium, Dynamic Kinematic, Inverse dynamic
Models a broad range of materials	Nonlinear and contact analysis	Extensive finite element library
Creates bolt models easily	Real and complex eigenvalues in vibration analysis	No restriction on the lamination
Customizes databases	Dynamic response to transient loads including random excitation	Edge effects and delamination can be predicted
Creates images and movies for reports and presentations	Solves large, complex assemblies more efficiently.	Power spectral density (PSD) for random load

Commercial FE Analysis Programs

Pro/MECHANICA	SAP 2000
Broad Range of Analysis Capabilities	Sophisticated and Versatile
Powerful Design Intent functionality	Creates any arbitrary shape and any user defined material
Material properties shared with the design model	Elements: Frame, Tendon, Cable, Shell, Solid, Link
One file stores all simulation and design data	Advanced SAPFire Analysis Engine
Captures actual model geometry as designed, not as an approximation	Multiple 64-Bit Solvers for analysis optimization
Compare model iterations side-by-side	Automatically generates: wind loads, seismic loads, wave-loads
Automate results-creation using templates	Bi-directional direct link to MS Excel for editing, Moment, Shear and Axial Force Diagrams

Text Book Reference

- **Logan ,Daryl L**., *A FIRST COURSE IN THE FINITE ELEMENT METHOD,* Cengage Learning, Stamford, CT, 2012, Fifth Edition

Practice Problems

Problem 1

$$\frac{d^2\varphi}{dx^2} = \mathrm{x} + 1; 0 < \mathrm{x} < 1$$

$$\varphi\,(0) = 0$$

$$\varphi\,(1) = 1$$

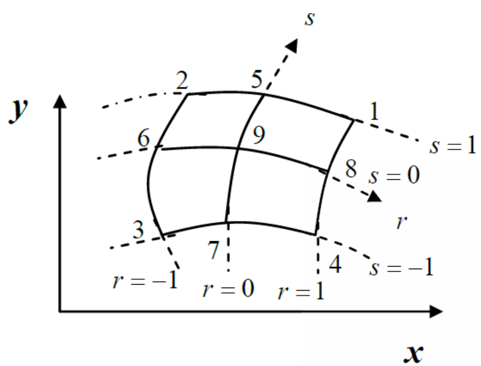
$$F(\bar{\varphi}) = \frac{1}{2}\int_0^1 \left(\frac{d\bar{\varphi}}{dx}\right)^2 dx + \int_0^1 (x+1)\bar{\varphi}\,dx$$

- Find Exact Solution
- Use Ritz Solution Method and find approximate solution
- Use Galerkin method and find approximate solution
- Compare the three solutions

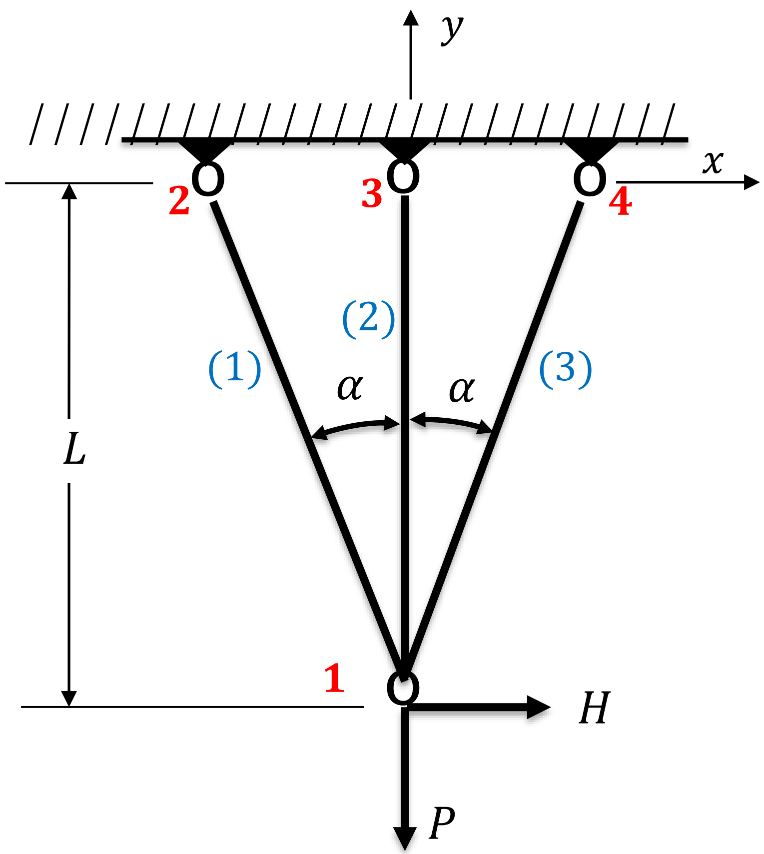
Note that the functional is given above.

Problem 2

- Write shape functions for a 4-node, 8-node and 9-node 2-D quadrilateral element in (r, s) system
- Write shape functions for a 8-node brick element in (r, s, t) system



Problem 3

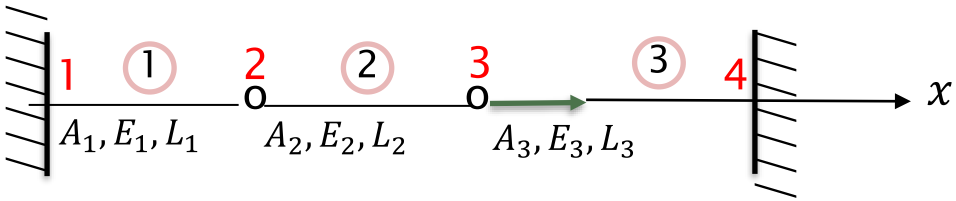


E and A same for all three bars

Consider the truss problem defined here. Geometric and material properties are: $L, \alpha \neq 0, E$ and A, as well as the applied forces P and H, are to be kept as variables.

(i) Derive Global Stiffness K and system of equations
(ii) Apply BC , find reduced equations and solve for unknown nodal displacements
(iii) Find nodal forces and check for equilibrium

Problem 4



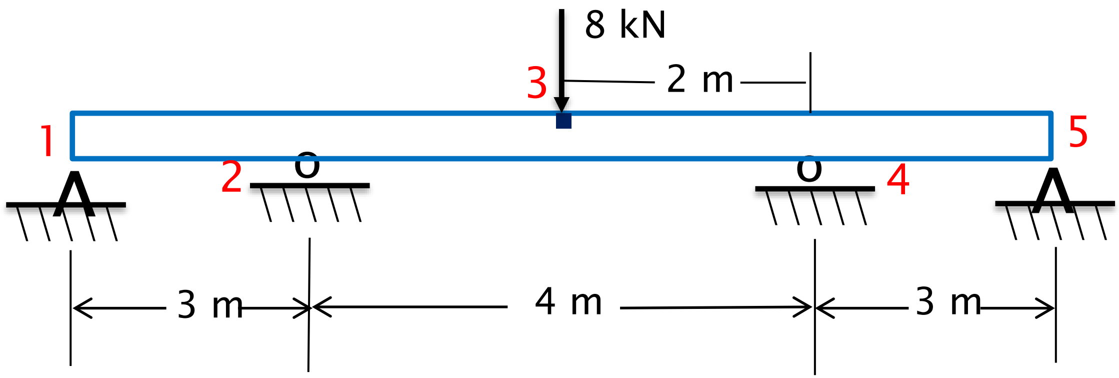
A. Compute the global stiffness matrix $[K]$ of the assemblage shown in figure by superimposing the stiffness matrices of the individual bars. Note that $[K]$ should be in terms of $A_1, A_2, A_3, E_1, E_2, E_3, L_1, L_2,$ and L_3. Here $A, E,$ and L are generic symbols used for cross-sectional area, modulus of elasticity, and length, respectively.

B. Now let $A_1 = A_2 = A_3 = A, E_1 = E_2 = E_3 = E$ and $L_1 = L_2 = L_3 = L$. If nodes 1 and 4 are fixed and a force P acts at node 3 in the positive x direction, find expressions for the displacement of nodes 2 and 3 in terms of A, E, L and P.

C. Now let $A = 1\ \text{in}^2, E = 10 \text{ x } 10^6$ psi, $L = 10$ in., and $P = 1000$ lb.
   i. Determine the numerical values of the displacement of nodes 2 and 3.
   ii. Determine the numerical values of the reactions at nodes 1 and 4.
   iii. Determine the stresses in element 1-3.

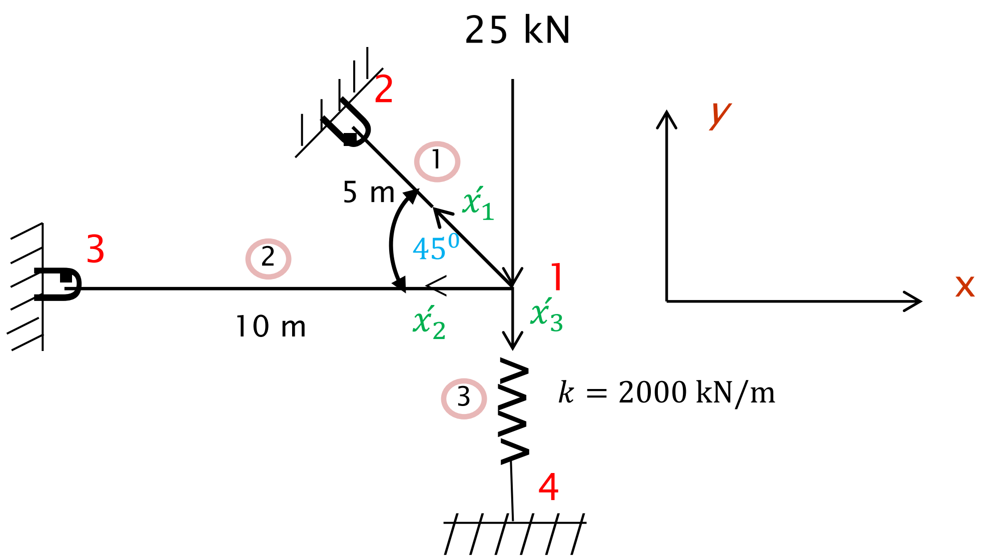
Problem 5

For the beam shown in Figure, determine the displacements and the Slopes at the nodes, the forces in each element, and the reactions. Use symmetry at Node 3 to reduce the problem.



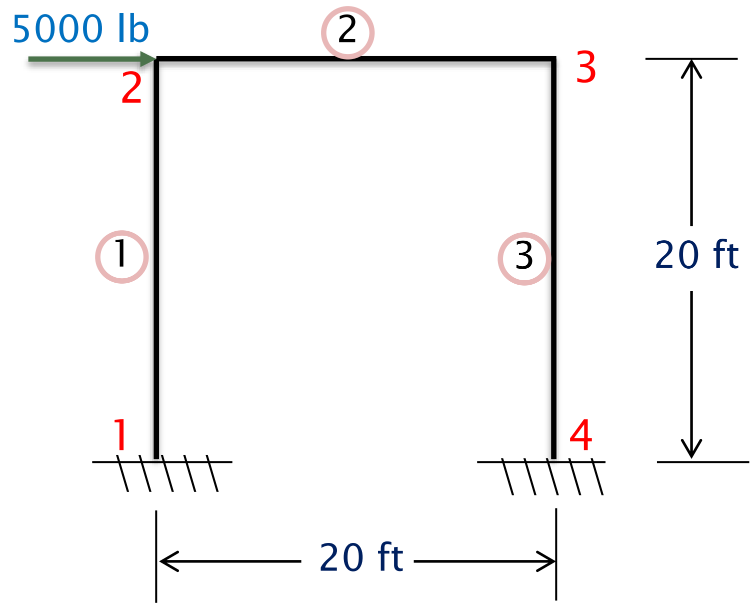
$E = 70$ GPa
$I = 1 \text{ x } 10^{-4}\text{m}^4$

Problem 6



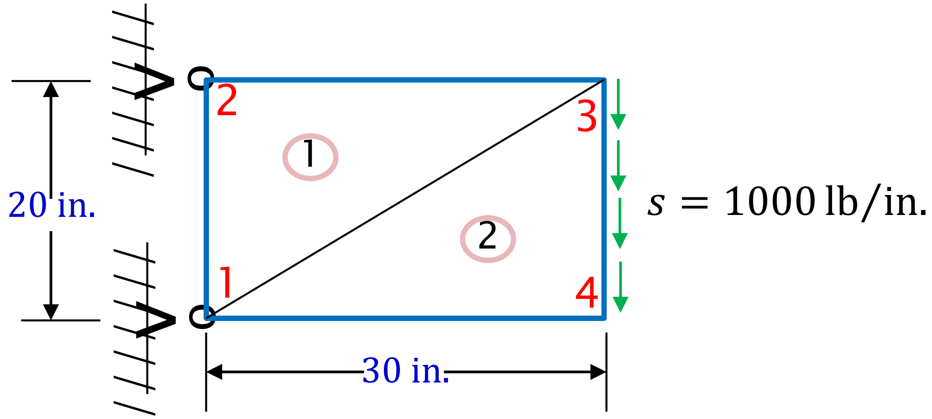
For the problem shown in Figure determine the (1) nodal displacements and (2) stresses in bar elements.
Let $E = 210\ GPa,$ and $A = 5.0\ \times 10^{-4}\ \mathrm{m}^2$ for both bar elements.

Problem 7



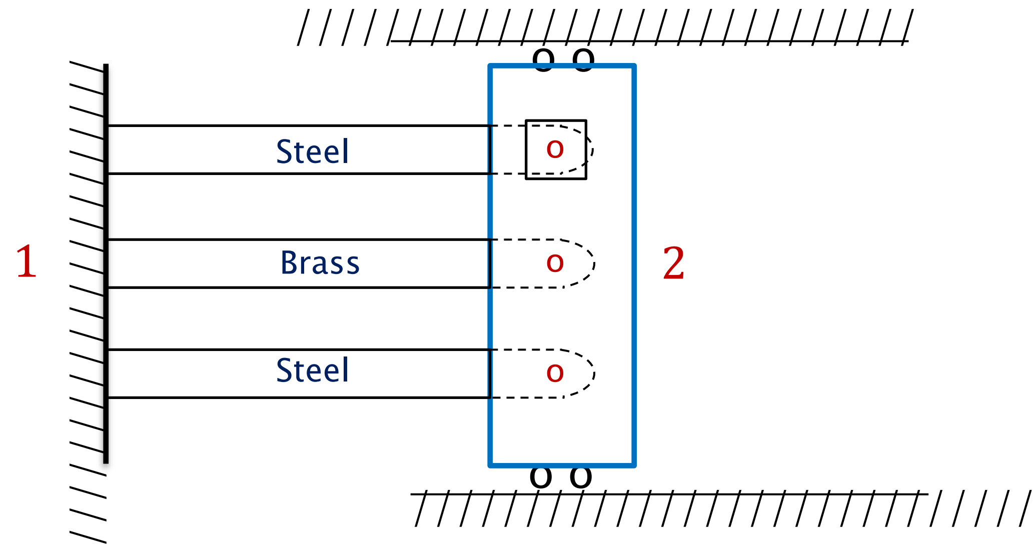
For the rigid frame shown in Figure determine (1) the nodal displacement components and rotations, (2) the support reactions, and (3) the forces in each element. Let $E = 30 \text{ x } 10^6 \text{ psi}$, $A = 10 \text{ in}^2$, and $I = 200 \text{ in}^4$ for all elements.

Problem 8



Determine the (i) nodal displacements and (ii) element stresses for the thin plate subjected to a uniform shear load acting on the right edge as shown in Figure.
Use $E = 30 \times 10^6$psi, $v = 0.30$, and $t = 1$ *inch*

Problem 9



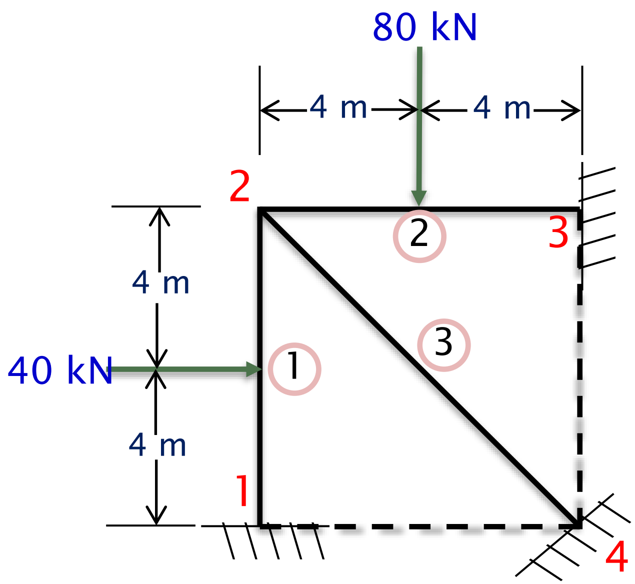
A bar assemblage consists of two outer steel bars and an inner brass bar. The three−bar assemblage is then heated to raise the temperature by an amount $T = 40^0$F. Let all cross-sectional areas be $A = 2$ in^2 and $L = 60$ in., $E_{\text{steel}} = 30 \times 10^6$psi, $E_{\text{brass}} = 15 \times 10^6$psi, $\alpha_{\text{steel}} = 6.5 \times 10^{-6}/^0$F, $\alpha_{\text{brass}} = 10 \times 10^{-6}/^0$F. Determine (a) the displacement of node 2 and (b) the stress in the steel and brass bars.

Problem 10

For the beam shown in Figure, determine the natural frequencies using
(i) Three elements and lumped mass matrix , and
(ii) Two elements and consistent mass matrix

Let E, ρ, and A be constant for the beam.

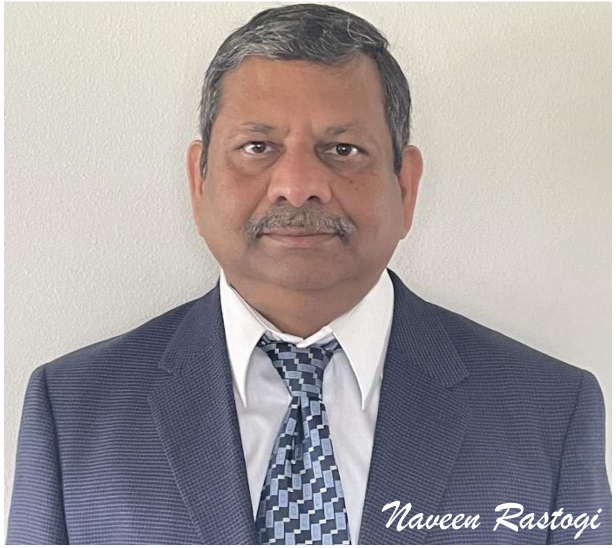
Problem 11



$$E = 210\ GPa$$
$$A = 1.0 \text{ x } 10^{-2}\text{m}^2$$
$$I = 1.0 \text{ x } 10^{-4}\text{m}^4$$

For the rigid frame in the Figure, determine the displacements and rotations of the nodes, the element forces, and the reactions.

About the Author



- Founder and Principal Consultant, 3P Composites, LLC
- Earned Ph.D. from Virginia Tech, Blacksburg, USA, M. Tech. from Indian Institute of Technology, Madras and B.E. from Punjab Engineering College, Chandigarh, India in Aerospace Engineering
- More than forty years of work experience in industry, research and academia
- Associated with composites since 1981 and witnessed tremendous growth in the application of composite materials touching all aspects of human lives
- Authored, co-authored or presented more than 75 technical papers in international conferences
- Subject Matter Expert in composite materials and structures