The Life Annuity Puzzles?

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Abstract

The annuity and life puzzles exist because current economic theory shows that the individual should fully annuitize and not insure. This analysis introduces life insurance as a third asset in a portfolio model. The model is developed with and without a bequest motive. Without the bequest motive, the model reveals not only the annuity puzzle but also the life insurance puzzle. With a bequest motive, the model reveals the existence of an arbitrage direction which shows that the annuity only becomes part of an optimal portfolio in one of three possible cases. The results are consistent with market observations and so do not yield puzzles.

> *"It can scarcely be denied that the supreme goal of all theory is to make the irreducible basic elements as simple and as few as possible without having to surrender the adequate representation of a single datum of experience."* Albert Einstein

I. The Puzzles

Consumption-saving behavior has been studied since Marshall (1920) and Fisher (1930). Still, until Yaari (1965) the question of how a consumer should optimally allocate her limited resources over an uncertain lifetime had not been carefully addressed. Yaari extended the analysis of optimal consumption plans by maximizing an investor's expected utility over a random time horizon and showed that investors without bequest motives would find it optimal to completely annuitize their savings. Despite the full annuitization prediction, the demand for life annuities is thin; this disparity has become known as the "annuity puzzle."[1] The current economic theory generates two predictions and puzzles, i.e., strong annuity markets and no life insurance markets. The predictions yield puzzles, i.e., why do we see anemic annuity markets and robust life insurance markets when current theory predicts the opposite.

Work by Davidoff, Brown et al. (2005) relaxed restrictive assumptions in Yaari (1965) and showed that the annuity puzzle remains. According to Davidoff *et al.*, even with incomplete annuity markets, more risk factors, and more general utility functions, complete or substantial annuitization is still optimal since the annuities generate a mortality credit that cannot be captured otherwise. Numerous authors have attempted to resolve the annuity puzzle by exploring various rational

[1] Also, see Johnson, Burman et al. (2004) and (James and Song 2001)

factors and behavioral biases[2]; for a review, see Brown (2007) and Alexandrova and Gatzert (2019).[3] A consensus has been elusive. Some contradictory predictions have been noted in the literature. Some research suggests that people with bequest motives should annuitize part of their wealth, *e.g.*, Davidoff, Brown et al. (2005) and Bernheim (1991). On the other hand, Friedman and Warshawsky (1990) and Lockwood (2012) argue that even with modest bequest motives, the existing annuity loads may be sufficient to eliminate any annuitization.[4] In sum, the annuity puzzle remains.

The current economic theory assumes that individuals are motivated by self-interest alone. We note that this assumption generates not only an annuity puzzle but also a life insurance puzzle. The sole pursuit of self-interest yields a strong demand for fairly priced life annuities and no demand for life insurance. Both predictions conflict even with armchair empiricism; the life annuity market is anemic while the life insurance market is robust.

This analysis includes other interests in addition to self-interest. For simplicity, the other interests is introduced with an old model. The other interests is limited to bequests. The bequest motive has been used by Yaari (1965), Bernheim (1991), Lockwood (2012) and others. While Yaari first provided the analysis for full annuitization, that was only one of the cases he considered. In another case, he introduced the bequest motive and showed that full annuitization was not optimal. In the bequest model, the individual's utility function depends on the well-being of another through the

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[2] Among the many factors studied are pricing (Mitchell et al. 1999), inflation and risk premia (Koijen, Nijman et al., 2011), uncertain healthcare expenses (Sinclair and Smetters (2004);(Pang and Warshawsky 2010;Ameriks, Caplin et al. 2011;Reichling and Smetters 2015;Peijnenburg, Nijman et al. 2017), bequests (Friedman and Warshawsky 1990;Bernheim 1991;Lockwood 2012), incomplete annuity menus (Horneff, Maurer et al. 2008;Horneff, Maurer et al. 2008;Koijen, Nijman et al. 2011) and psychological and behavioral factors (Brown, Casey et al. 2008;Benartzi, Previtero et al. 2011;Brown, Kapteyn et al. 2017).

[3] Also see the three books that provide commentary on the annuity markets and puzzles: Brown, Mitchell et al. (2001); Cannon and Tonks (2008); Sheshinski (2008).

[4] Similar to Lockwood (2012), Inkmann, Lopes et al. (2011) run simulations and find that reasonable calibrations generate low annuity demand.

bequest. This definition is consistent with the altruism formulated by Gary Becker in his book "A Treatise on the Family" where Becker states that "If I am correct that altruism dominates family behavior perhaps to the same extent as selfishness dominates market transactions, then altruism is much more important in economic life than is commonly understood."[5]

We further incorporate life insurance into the bequest framework to analyze the demand for both life insurance and life annuities. The framework may explain both the annuity puzzle and the life insurance puzzle. Annuities and life insurance are opposite investments in one's lifespan. A model that explains life insurance demand, however, is also essential to understanding the anemic annuity market. The annuity puzzle literature is largely silent on the role of life insurance. The work by Bernheim (1991) is an exception. Unlike so many authors after Yaari, Bernheim discussed life insurance and empirically addressed its relationship with annuity demand. We have taken the next step by explicitly introducing the life asset in a portfolio model rather than generating it using a long position in a bond and a short position in an annuity. The separate introduction of life insurance is essential because it reveals an arbitrage opportunity that was not apparent in the two asset models (Pang and Warshawsky) that dominate the literature. The life or annuity assets only become part of optimal portfolios when the non-negativity constraints become binding. Hence it can no longer be said that the theory predicts fully investing in annuities. The arbitrage opportunity created by the three assets (life insurance, annuity, and bond) yields an optimal portfolio. The form of the optimal portfolio depends on the relative strength of the bequest motive.

The analysis here first shows that the annuity puzzle is a relic of the model with only self-interest and two assets. When other interests are introduced, and the decision-making is formed as a portfolio problem with three assets. i.e., annuities, bonds, and life insurance, the theory makes predictions consistent with empirical observations. Other models include life, albeit indirectly, e.g., see Yaari (1965), Bernheim (1991) and Lockwood (2012). Life insurance is in the Yaari model and

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[5] See Becker (2009)

in the Bernheim model, where it is formed by taking a long position in the bond and a short position in the annuity. Here, life insurance is added as a separate asset. That addition of life insurance and a bequest motive clarifies the role of the bequest in generating the demand for life insurance and determining the composition of the optimal portfolio. The model here shows that the annuity is included in an optimal portfolio if the marginal value of the bequest is sufficiently small relative to the marginal value of future consumption. Similarly, the analysis shows that the optimal portfolio includes life insurance if there is a sufficiently strong bequest motive. Finally, the analysis indicates that the bond portfolio is optimal if the marginal bequest value is less than the marginal value of consumption and greater than a fraction of the marginal value of consumption.[6]

Second, the analysis adds to the literature with and without the bequest motive. Without the bequest motive, Davidoff, Brown et al. (2005) show that full annuitization is optimal with weaker assumptions than Yaari but assuming a death probability greater than the annuity loading. With the portfolio model developed here and without the bequest motive, this analysis shows that full annuitization is optimal given a death probability greater than the annuity loading but also that a bond portfolio with no annuities is optimal given a death probability less than the annuity loading. With the bequest motive, Yaari (1965), Lockwood (2009), Bernheim (1991), have shown that no or partial annuitization is optimal. With the bequest motive and life insurance in the portfolio model developed here, the analysis shows that the optimal portfolio can take only one of three possible forms, *i.e.*, an all-bond portfolio, a bond, and life insurance portfolio, or a bond and annuity portfolio, and the form depends on the strength of the bequest motive. Bernheim notes that

> *strong bequest motives are evidently quite common. For close to 30 percent of households with children, desired bequests substantially exceeded the value of conventional assets. A surprising number of childless households (roughly 16-18 percent) also acted to augment their bequests. Bernheim (1991), p. 924.*

[6] The fraction depends on the death probability and the loading factor on annuities.

This analysis shows that life insurance will be included in the portfolio if the marginal bequest value exceeds that of future consumption in the survival event.[7] Using Bernheim's empirical evidence and the analysis here, a strong case for individuals holding the bond and life insurance portfolio can be made. This then suggests the existence of a robust life insurance market. The proportion of the population with marginal bequest values smaller than that of future consumption determines the proportion that purchases a bond portfolio or a bond and annuity portfolio.

Third, when the Government supplies annuities and places a lower limit on the annuities in the portfolio, the composition of the optimal portfolio changes. In this case, one form of an optimal portfolio includes all three assets.

The paper is organized as follows. Section II presents the portfolio model. The bequest motive is included in the preferences of the investor as well as three assets, *i.e.*, life insurance and life annuities, and bonds. The analysis demonstrates three potential forms for an optimal portfolio, *i.e.*, a portfolio consisting of annuities and bonds, a portfolio of only bonds, or a portfolio of bonds and life insurance. The optimal portfolio choice depends on the strength of the marginal bequest value relative to the marginal utility of consumption. As a corollary, the analysis also demonstrates the potential form of the optimal portfolio without the bequest motive. Section III demonstrates that when an annuity is provided by the Government and puts a floor on the annuities held, then the optimal portfolio may contain all three assets. Section IV concludes.

II. The Portfolio Model

The classic approach in the literature to the theory of consumer choice given an uncertain lifetime has been the choice of bonds or annuities to cover consumption expenditures in the context of a life cycle model, *e.g.*, see Yaari (1965) and Davidoff, Brown et al. (2005). Due to the annuity puzzle,

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[7] This prediction is consistent with the finding of Inkmann and Michaelides (2012), who provide evidence supporting that demand for term life insurance is largely driven by a bequest motive.

the model has frequently been extended to include a bequest motive, *e.g.*, see Yaari (1965)[8], Bernheim (1991), and Lockwood (2009, 2012). Life insurance has been included in some models, *i.e.*, Yaari (1965) and Bernheim (1991); that inclusion, however, has been through short sales since a life policy may be artificially created by going long in a bond and short in an annuity.[9] The analysis here departs from the existing literature by considering the consumer choice problem from the perspective of a portfolio model in which the consumer with an uncertain life selects a portfolio of bonds, annuities and life insurance to cover future consumption expenditures. The difference here is that the life insurance is included as a separate asset and its price by a competitive life insurance market.

Consider an individual with an uncertain life making decisions *now*. The individual makes decisions at t = 0 that yield dollars for consumption at t = 0 and the payoffs from a portfolio that yield dollars for consumption at t = 1. For simplicity, the dates t = 0 and 1 are referred to as *now* and *then*, respectively. The decisions determine the dollars available for consumption *now* and a portfolio of assets that transfer money from *now* to *then* and determine dollars available for consumption *then* if the individual survives. The individual survives until *then* with probability $1 - q$. The portfolio includes life annuities, bonds, and life insurance. Suppose λ_a, λ_b and λ_l represent the number of life annuities, bonds, and life insurance contracts, respectively. Let r denote the interest rate. Suppose the bond provides a one dollar return *then* for each bond purchased *now* and let $p_b = 1/(1+r)$ denote the bond price *now* in the competitive bond market.

[8] While one version of the Yaari model showed that fairly priced annuities would dominate bonds had not become known as the annuity puzzle yet, Yaari introduced the bequest motive in another version of his model to show that

[9] The artificial introduction of life insurance may create a problem if not all prices are fair. Given fair prices the annuity price would be pa = (1 – q) pb and so long in a bond and short in an annuity wo uld be at a price pl = pb – pa = q pb but with loading the artificial life insurance price is

$$p_l = \frac{q-\delta}{1-\delta} p_b > q\, p_b$$

Similarly, suppose the annuity provides a one dollar return *then* if the individual survives and zero otherwise; let

$$p_a = \frac{(1-q)}{(1-\delta)(1+r)} = \frac{1-q}{(1-\delta)} p_b$$ [10]

denote the annuity price *now*, where $\delta \in [0, 1]$ is a loading factor, *e.g.*, see Brown (2007), Mitchell, Poterba et al. (1999) or Brown, Mitchell et al. (2001), Mitchell and Poterba (2000), Warshawsky (1988) and Friedman and Warshawsky (1988). The life annuity differs from the bond instrument because it has a survival trigger. Also, consider a life insurance contract. Suppose λ_l represents the number of life contracts. Let the life insurance provides a one dollar return *then* for each life contract purchased *now*. Let $\lambda = (\lambda_a, \lambda_b, \lambda_l)$ denote the portfolio and $p = (p_a, p_b, p_l)$ denote the asset price vector. The individual investor selects a portfolio of annuities, bonds, and life insurance that determine consumption *now* and *then*, *i.e.*, (c_0, c_1), as follows:

$$c_0 = w - p\lambda \tag{1}$$

where w represents the wealth *now* and

$$c_1 = \begin{cases} c_{11} & q \\ c_{12} & 1-q \end{cases} = \begin{cases} 0 & q \\ \lambda_a + \lambda_b & 1-q \end{cases} \tag{2}$$

where consumption *then* depends upon the realized state; the states are death with probability q and survival with probability $(1-q)$. The life insurance contract yields λ_l dollars in the event of death and zero otherwise. The individual, however, cannot consume the λ_l dollars; it is a payment

[10] The asset prices are specified here so that the load factors are transparent. The functional form of the asset prices, however, may be left unspecified without altering the results.

that goes to the beneficiary.[11] Hence, consumption *then* in the death event is zero for the investor but a function φ of the number of life contracts and bonds for the beneficiary. The life annuity yields λ_a dollars if the individual survives and zero dollars otherwise. The bond yields λ_b dollars whether the individual survives or not, but death precludes the consumption of those dollars by the individual but not the beneficiary. Let (u_0, u_1) denote the utility of consumption *now* and *then* and let the utilities be increasing, concave, and satisfy the Inada conditions, i.e., see Inada, Shōda et al. (1992) and Bavre (2005). The value of the bequest in the death event is increasing and satisfies the Inada conditions.

The bequest is one seemingly obvious but sometimes neglected motivation for life insurance and has implications for both life annuities and life insurance. The individual decision-maker here has preferences defined on (c, φ) . The bequest value $\varphi(\lambda_b + \lambda_l)$ is based on the value of the assets passed on to the individual's beneficiaries.

The addition of bequests does add an obvious element and motivation for bonds and life insurance since both payoff in the death event while the annuity does not. However, given the existence of a single life annuity and other instruments such as bonds and life insurance, the question becomes whether the individual has an incentive to include life annuities, bonds, and life insurance in a portfolio.

Yaari used an intertemporally independent utility function to introduce the bequest motive. We use the same type of utility for simplicity.[12] A bequest function φ is introduced here to allow the investor to value assets that can be passed to beneficiaries in the death event; those assets are life

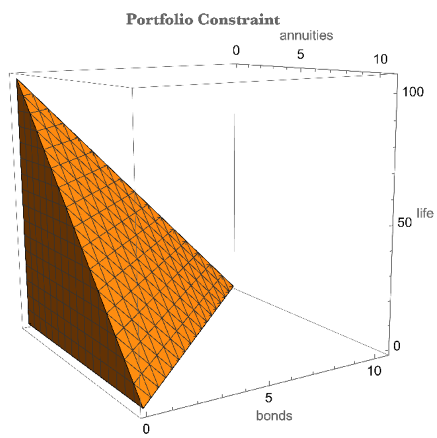
[11] The beneficiary is not specified here. It may be a spouse, child, or other.

[12] Intertemporal independence is not critical for the results reported here but it does simplify the analysis.

insurance and bonds. Life insurance is not included explicitly as a separate asset[13] in the literature on the annuity puzzle, but it will be introduced as a third asset here. Let $H(\lambda)$ be the expected utility. Then

$$H(\lambda) = u_0(w - p\lambda) + q\,\varphi(\lambda_b + \lambda_l) + (1 - q)\,u_1(\lambda_a + \lambda_b) \tag{3}$$

where u_0 and u_1 represent the utility of consumption *now* and *then,* and φ is the bequest function that expresses how the individual feels about the dollar amount paid to the beneficiary in the death event. The individual investor selects the portfolio of assets to maximize expected utility subject to the non-negativity constraints for the assets, *i.e.*,



$$\begin{aligned}&\textit{maximize}\ \mathrm{H}(\lambda)\\&\text{subject to } \lambda_j \geq 0 \text{ for } j = a,b,l\end{aligned} \tag{4}$$

The portfolio determines savings, or wealth transferred from *now* to *then* for consumption in retirement and the wealth transferred by bequests from *now* to *then* in the event of death. It is possible to consider how the optimum conditions here compare with the current paradigm, *i.e.*, the self-interested paradigm by letting $\varphi \equiv 0$ be a special case.

The expected marginal utilities with respect to the annuity, bond, and life insurance are:

$$D_1H = -\,p_a Du_0 + (1 - q)Du_1 \tag{5}$$

[13] Yaari (1965) and Bernheim (1991) did include life insurance but only via short sales, i.e., long in a bond and short in an annuity yields the insurance payoff. The problem with going short is that with loading that cost will be different than simply going long in the life insurance. In addition, annuities cannot be sold short.

$$D_2H = -p_b Du_0 + q\,D\varphi + (1-q)Du_1 \tag{6}$$

$$D_3H = -\,p_l\,Du_0 + q\,D\varphi \tag{7}$$

The derivative in (7) shows that a positive marginal bequest value provides a basis for the demand for life insurance. The portfolio theorem specifies the conditions for life insurance to be part of an optimal portfolio.

The bequest value provides some incentive to reduce annuity holdings. In addition, it provides the motivation to potentially include life insurance in an optimal portfolio, as shown in the expected marginal utility in (7). The introduction of the life insurance policy and the bequest motive also reveals an arbitrage opportunity since the life annuity and insurance contracts generate the same payoff as the bond for the investor and beneficiary but at different costs.

Claim: Given the markets for annuities, bonds and life insurance and trading in all three assets, **an arbitrage opportunity exists**.

Sketch of Proof: The arbitrage opportunity may be seen by considering the following trade in the markets. An investor with a portfolio that includes life insurance and annuities sells one annuity and one life insurance contract and uses the cash to invest in bonds. The sale of annuity and life contracts yield $p_a + p_l$ dollars *now* and this cash allows the investor to purchase $(p_a + p_l)/p_b$ bonds where

$$\frac{p_a + p_l}{p_b} = \frac{\frac{1-q}{1-\delta}\,p_b + q\,p_b}{p_b} = \frac{1-q}{1-\delta} + q > 1$$

Let v denote the trading direction where

$$v = (v_1, v_2, v_3) = \left(-1,\ \frac{1-q}{1-\delta} + q\ ,\ -1\right) \tag{8}$$

The derivative of expected utility in the direction v is

$$
\begin{aligned}
D_v H &= v_1 D_1 H + v_2 D_2 H + v_3 D_3 H \\
&= -D_1 H + \left(\frac{1-q}{1-\delta} + q\right) D_2 H - D_3 H \\
&= -\left(-p_a Du_0 + (1-q) Du_1\right) + \left(\frac{1-q}{1-\delta} + q\right)\left(-p_b Du_0 + q\,D\varphi + (1-q) Du_1\right) \\
&\quad - \left(-p_l Du_0 + q\,D\varphi\right) \\
&= \left[\left(\frac{1-q}{1-\delta} + q\right) - 1\right]\left(q\,D\varphi + (1-q) Du_1\right) \\
&= (1-q)\left(\frac{1}{1-\delta} - 1\right)\left(q\,D\varphi + (1-q) Du_1\right) \\
&> 0
\end{aligned}
\tag{9}
$$

Hence, an arbitrage opportunity exists, and v as defined in (8) is the arbitrage direction. QED

This remark has several implications, but two observations are in order. First, it should be noted that the arbitrage direction exists whether there is a bequest motive or not; this follows since (9) is positive even if $D\varphi \equiv 0$. Second, it should be observed that this arbitrage opportunity has not been identified in the literature because the life insurance market had not been introduced. One might expect that artificially creating the life contract would produce the same result but that does not follow because the life insurance prices are different. If the investor artificially created a life contract by going long in a bond and short in an annuity, the price would be p_b - p_a. Then selling an annuity and this artificial life contract would generate p_a + (p_b - p_a) = p_b dollars and allow the investor to purchase one bond. The direction of the trading would be u = (-1, 1, -1) and the derivative of the expected utility in that direction is

$$
\begin{aligned}
D_u H &= -D_1 H + D_2 H - D_3 H \\
&= -\left(-p_a Du_0 + (1-q) Du_1\right) + \left(-p_b Du_0 + q\,D\varphi + (1-q) Du_1\right) \\
&\quad - \left(-\left(p_b - p_a\right) Du_0 + q\,D\varphi\right) \\
&= 0
\end{aligned}
$$

Hence, an arbitrage opportunity does not exist for this trading direction.

The most important implication of the remark is that in providing the existence of an arbitrage direction it implies that no portfolio on the interior of the constraint set can represent an optimal portfolio. Any interior position represents a portfolio with less expected utility than one on the boundary of the constraint set. Similarly, no portfolio that consists of annuities and life can be optimal due to the arbitrage direction. This model also implies annuities do not dominate bonds or other assets without sufficient qualifications; given $q > \delta$, the annuity portfolio $\lambda^a = \left(\lambda_a, 0, 0\right)$ can only be optimal in the absence of a bequest motive. The analysis shows that the annuity portfolio cannot be optimal in the presence of a bequest motive and so lends support to (Lockwood 2012).

The following theorem uses this three-asset model to demonstrate the investor's choice of financial instruments in retirement.

Portfolio Theorem. Given complete financial markets, the investor maximizing expected utility subject to non-negativity constraints on the assets selects the bond portfolio

$$\lambda^b = \left(0, \lambda_b^b, 0\right) \text{ if } \frac{q-\delta}{q} < \frac{D\varphi\left(\lambda_b^b\right)}{Du_1\left(\lambda_b^b\right)} \leq 1.$$

The investor selects a bond and life insurance portfolio of the form

$$\lambda^{bl} \in E_2 \backslash \{\lambda^l, \lambda^b\} \text{ if } \frac{D\varphi\left(\lambda_b^b\right)}{Du_1\left(\lambda_b^b\right)} > 1$$

Or an annuity and bond portfolio of the form

$$\lambda^{ab} \in E_3 \backslash \{\lambda^a, \lambda^b\} \text{ if } \frac{D\varphi\left(\lambda_b^b\right)}{Du_1\left(\lambda_b^b\right)} < \frac{q-\delta}{q}.$$

Proof. See the appendix.[14]

Corollary. Given complete financial markets, an investor motivated by self-interest alone selecting a portfolio to maximize expected utility subject to non-negativity constraints on the assets, the optimal portfolio is the annuity portfolio λ^a if $q > \delta$ or the bond portfolio λ^b if $q < \delta$.

Proof. See the appendix.

The selection of the bond portfolio given the other interests of a bequest motive and loading follows in much the same way that the annuity portfolio did in the current paradigm with self-interests and a sufficiently small loading. Thus, the movement toward more bonds in the portfolio is only stopped by the non-negativity constraints, and even then, the movement continues if the bequest motive does not dominate[15] the value of consumption *then,* but the bequest value exceeds the opportunity cost of bonds.[16]

[14] Yaari (1965) and Bernheim (1991) did include life insurance but only via short sales, i.e., long in a bond and short in an annuity yields the insurance payoff. The problem with going short is that with loading that cost will be different than simply going long in the life insurance. In addition, annuities cannot be sold short.

[15] This means that the life policies which payoff in the death event will be traded for more bonds.

[16] This means that annuities will be traded for more bonds because the opportunity cost of the bonds relative to the annuities is less than the bequest value; the bonds, unlike the annuities, payoff in the death event.

Note that the bequest theorem says that if the marginal value of the bequest is less than the marginal utility of consumption *then*, the individual will increase expected utility by investing in bonds and finding the optimal portfolio at λ^b. Next, observe that

$$\frac{q-\delta}{q} = \frac{\frac{1-q}{1-\delta}(q-\delta)}{\frac{1-q}{1-\delta}q} = \frac{\frac{q-\delta}{1-\delta}(1-q)}{\frac{1-q}{1-\delta}q} = \frac{p_b\left(1-\frac{1-q}{1-\delta}\right)(1-q)}{p_b\frac{1-q}{1-\delta}q} = \frac{(p_b-p_a)(1-q)}{p_a\,q} \tag{10}$$

If $q > \delta$, the bond price exceeds the annuity price, the difference is the excess cost of investing in bonds. Hence, the numerator on the right-hand side of (10) is the expected opportunity cost of bonds. Similarly, since the annuity does not pay if the investor does not survive, the annuity price times the death probability is the opportunity cost of an annuity. The right-hand side may be interpreted as the expected opportunity cost of bonds relative to annuities. So, the condition for an optimal portfolio at λ^b may also be represented as:

$$\frac{(p_b-p_a)(1-q)}{p_a\,q} < \frac{D\varphi(\lambda^b)}{Du_1(\lambda^b)} < 1 \tag{11}$$

The bequest theorem then says that if the marginal value of the bequest relative to consumption exceeds the opportunity cost of bonds relative to annuities, then investment in bonds dominates. Roughly put, this says that the marginal value of bequests exceeds the marginal cost and so investing in bonds is optimal. This conclusion follows because the bond satisfies the bequest motive while the annuity does not. The inequalities in (11) characterize the conditions for the bond portfolio being optimal.

The bequest theorem also shows that if the marginal bequest value exceeds the marginal utility of consumption *then,* at the bond portfolio, then the bond-life portfolio is optimal. It takes the form $\lambda^{bl} \in E_2 \setminus \{\lambda^l, \lambda^b\}$ so that the optimal portfolio comprises bonds and life insurance. Finally, if the marginal bequest value relative to consumption is less than the opportunity cost of bonds relative to annuities, then investment in annuities and bonds is optimal. Roughly put, this says that the marginal value of bequests is less than the marginal cost. So, selling a portion of bonds and buying annuities is optimal until the marginal value of bequests is equal to the marginal cost. Such trading will reduce the marginal bequest value while increasing the marginal utility of consumption *then*.

While the bequest theorem shows that the optimal portfolio will take one of three possible forms, Bernheim's empirical work suggests that the bond and life insurance portfolio will be significant. Bernheim says, "The evidence in this paper documents the existence of powerful bequest motives for a large segment of the population." Our interpretation of "powerful bequest motives" is that the marginal bequest value exceeds the marginal utility of consumption *then.* For this case, the bond-life portfolio λ^{bl} is optimal. Another group of investors may have marginal bequest values less than the marginal utility of consumption *then,* but greater than the fraction $(q-\delta)/q$ of the marginal utility of consumption *then* and so find that the bond portfolio λ^{b} is optimal. If there are investors with marginal bequest values less than the fraction $(q-\delta)/q$ of the marginal utility of consumption *then,* they will find the annuity-bond portfolio λ^{ab} optimal. It may also be noted that if $\delta = q$ then, there will be no investors in this group. Hence, this analysis is consistent with thin annuity markets and robust life markets. The remaining question is empirical: How big is the group of investors that Bernheim describes as a large segment of the population? At one-point Bernheim describes this group as 46-48% of the population.

The theorem shows that the optimal portfolio will not include all three assets. Other considerations and constraints, however, can generate an optimal portfolio with all the assets. The following section provides one example.

III. Government Provision of Annuities

Next, suppose the individual has λ_a^g annuities provided by the Government. Suppose these annuities cannot be traded. How does this affect the optimal portfolio? The constrained optimization problem is altered and becomes the following:

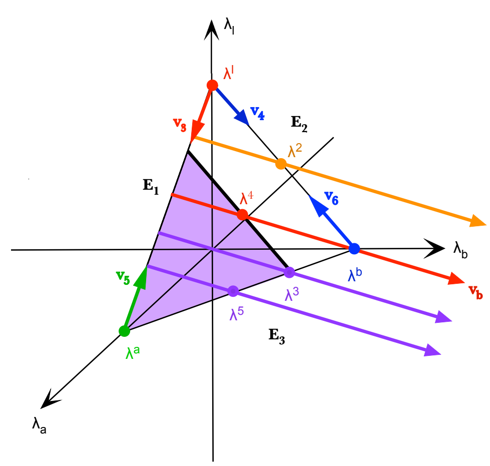
$$\begin{gathered} \textit{maximize } H(\lambda) \\ \textit{subject to } \lambda_b \geq 0,\ \lambda_l \geq 0 \textit{ and } \lambda_a \geq \lambda_a^g \end{gathered}$$

The constraint is shown in the following figure 3. It shows how the government provision of annuities limits the constraint. The portfolios $\lambda^4 = \left(\lambda_a^g, \lambda_b^4, \lambda_l^4\right)$ and $\lambda^3 = \left(\lambda_a^g, \lambda_b^3,\ 0\right)$ are on the boundary of the constraint on annuities. The condition for one of these portfolios to be optimal will be determined by considering the sales of a life contract and the purchase of bonds with the proceeds of the sale, i.e., $v_4 = (0, q,\ -1)$, and

$$\begin{aligned} D_{v_4}H &= q\, D_2 H - D_3 H \\ &= q(1-q)\left[Du_1\left(\lambda_a^g + \lambda_b^4\right) - D\varphi\left(\lambda_b^4 + \lambda_l^4\right)\right] \end{aligned} \tag{12}$$

Again, the derivative $D_{v_4}H$ is monotone decreasing. If the derivative of H in the direction v_4 is non-negative at λ^3 then $\lambda^3 = \left(\lambda_a^g, \lambda_b^3,\ 0\right)$ is optimal.

Figure 3: Government Annuity



$$D_{v_4}H\left(\lambda^3\right) \geq 0 \iff Du_1\left(\lambda_a^g + \lambda_b^3\right) - D\varphi\left(\lambda_b^3\right) \geq 0$$

Or equivalently,

$$\frac{D\varphi\left(\lambda_b^3\right)}{Du_1\left(\lambda_a^g + \lambda_b^3\right)} \leq 1$$

If this inequality was reversed, then a portfolio such as $\lambda^4 = \left(\lambda_a^g, \lambda_b^4, \lambda_l^4\right)$ would be optimal, and the individual would hold life insurance as well as the government annuities and bonds.[17] Finally, if the investor holds portfolio λ^3 and considers a move in the direction $v_1 = \left(\frac{1-\delta}{1-q},\ -\ 1,\ 0\right)$ then a portfolio of the form λ^5 is optimal if

$$\begin{aligned} D_{v_1}H &= \frac{1-\delta}{1-q}D_1H - D_2H \\ &= (q-\delta)\ Du_1\left(\lambda_a^g + \lambda_b^3\right) - q\ D\varphi\left(\lambda_b^3\right) \\ &> 0 \end{aligned} \tag{13}$$

Or equivalently, if

$$\frac{D\varphi\left(\lambda_b^3\right)}{Du_1\left(\lambda_a^g + \lambda_b^3\right)} < \frac{q-\delta}{q} \tag{14}$$

where $\lambda^5 = \left(\lambda_a^5, \lambda_b^5, 0\right)$ *with* $\lambda_a^5 > \lambda_a^g$. If the inequality (14) holds, then there exists a portfolio of the form λ^5 that is optimal.

IV. Concluding Remarks

The models constructed in the literature have been used to integrate the bequest motive, adverse selection, aggregate mortality risk, and incomplete markets in attempts to resolve the annuity puzzle, but no consensus has emerged. The current paradigm of self-interested behavior is used here to demonstrate the annuity and life puzzles; it is also used to show the conditions under which the individual will select the all-annuity portfolio or the all-bond portfolio. Given self-interested behavior, the portfolio choice is determined by a first-order stochastic dominance condition. It

[17] Here it is also possible that the optimal portfolio takes the form $\lambda^4 = \left(\lambda_a^g, 0, \lambda_l^4\right)$

should be noted that the annuity portfolio is only selected due to the binding non-negativity constraints.

Since the current paradigm does not resolve the life or annuity puzzles, a simple theoretical construct is presented that incorporates the often-used bequest motive in addition to the inequality constraints. The portfolio model introduced here includes life insurance in addition to the other financial contracts. While life insurance has been introduced in the literature by authors including Yaari (1965) and Bernheim (1991), it has been through short sales, i.e., long in a bond and short in an annuity yields a life contract. The introduction of life as a third asset provides a motivation for life insurance demand that is missing in the current economic paradigm and the basis for selecting it as part of an optimal portfolio. The bequest model with three financial assets is used to show that the boundary portfolio of all annuities is not and cannot be optimal. The model shows that bonds play an essential role in each of three possible optimal portfolios and the conditions under which one of the three will be optimal. First, the analysis shows that a marginal bequest value less than the marginal utility of consumption *then* and greater than relative cost of bonds to that of annuities makes the bond portfolio optimal. Under these conditions, the bond portfolio dominates the bond-life and bond-annuity portfolios. Second, if the marginal value of the bequests is greater than the marginal utility of consumption *then*, the bond-life portfolio is the optimal portfolio. Third, if the marginal bequest value relative to consumption is less than the opportunity cost of bonds relative to annuities, then investment in the bond-annuity portfolio is optimal. If the loading factor, however, goes to the probability of death, then the annuity-bond portfolio is forced out. The analysis of Warshawsky (1988), Friedman and Warshawsky (1990), and others suggests the loading may easily approach the death probability and so make the market for annuities markets anemic. The three cases taken together do suggest thin annuities markets.

One other model extension is of interest. The optimal portfolio can take different forms if an annuity is provided by the Government and cannot be traded. The analysis with the provision of a government annuity shows that the optimal portfolios take the same form except that one of the optimal portfolios may include all three assets due to the constraint on the sale of the Government provided annuity.

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Appendix

Portfolio Theorem Proof. The Kuhn-Tucker Lagrange Function for the constrained maximization problem in (4) is

$$L(\lambda, \mu) = H(\lambda) - \mu\,(p\lambda - w)$$

where μ is the Lagrange multiplier. The Kuhn-Tucker conditions, i.e., see Wilde (2013), for a maximum are

$$\lambda_j \frac{\partial L}{\partial \lambda_j} = 0, \frac{\partial L}{\partial \lambda_j} \leq 0, j = a, b, l,$$
$$and \tag{A.1}$$
$$\mu \frac{\partial L}{\partial \mu} = \mu (\mathrm{p} \lambda - \mathrm{w}) = 0, \frac{\partial L}{\partial \mu} = \mathrm{p} \lambda - \mathrm{w} \leq 0$$

The derivatives of the Kuhn-Tucker Lagrange function with respect to the life annuity, bond, and life insurance contracts are the following:

$$\frac{\partial L}{\partial \lambda_a} = \frac{\partial H}{\partial \lambda_a} - \mu p_a = -\mathrm{p}_a Du_0 + (1-\mathrm{q})\, Du_1 - \mu p_a \leq 0 \tag{A.2}$$

$$\frac{\partial L}{\partial \lambda_b} = \frac{\partial H}{\partial \lambda_b} - \mu p_b = -\mathrm{p}_b Du_0 + q\, D\varphi + (1-\mathrm{q})\, Du_1 - \mu p_b \leq 0 \tag{A.3}$$

$$\frac{\partial L}{\partial \lambda_l} = \frac{\partial H}{\partial \lambda_l} - \mu p_l = -\mathrm{p}_l Du_0 + q\, D\varphi - \mu p_l \leq 0 \tag{A.4}$$

It may be noted that if a portfolio was selected such that $p\lambda = w$ then consumption *now* would be zero and so, by the Inada conditions, the expected marginal utility of consumption *now* would be arbitrarily large making equations (A.2) through (A.4) negative; by the first order conditions it would follow that all the asset holdings would be zero contradicting the initial assertion that the portfolio value equaled the wealth *now*. Hence, $p\lambda < w$ and $\mu = 0$.

There are seven generic positions identified in Figure 4, that may be optimal and so satisfy the conditions in (A.1). The first order conditions may be restated in standard form as the following:

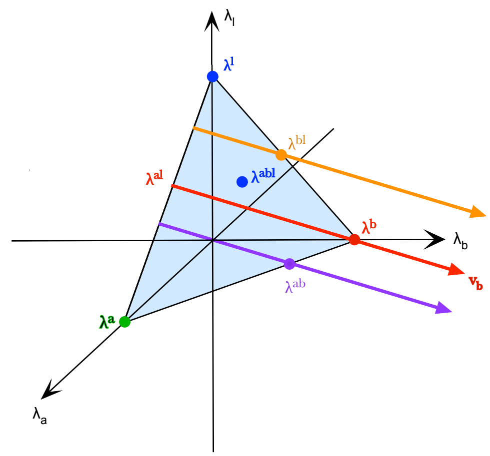
$$mrs_a \equiv \frac{(1-\mathrm{q})\, \mathrm{Du}_1}{Du_0} \leq p_a \tag{A.5}$$

$$mrs_b \equiv \frac{q\, D\varphi + (1-\mathrm{q})\, Du_1}{Du_0} \leq p_b \tag{A.6}$$

$$mrs_l \equiv \frac{q\, D\varphi}{Du_0} \leq p_l \tag{A.7}$$

The left-hand side of each represents the marginal rate of substitution, *i.e.*, the rate at which the investor is willing to trade consumption *now* for more consumption *then* and remain indifferent. If the marginal rate of substitution equals the price or opportunity cost, then the investor takes a positive position in that asset, while if the marginal rate of substitution is less than the price, then the investor takes a zero position in the asset. If the three conditions cannot be satisfied, the portfolio cannot be optimal.

Figure 4: Optimal Portfolios



First, consider four portfolios of the form λ^a, λ^l, λ^{al}, and λ^{abl}. For $\lambda^a \equiv (\lambda_a, 0, 0)$ it may be noted that $D\varphi \rightarrow \infty$ *as* $\lambda_b, \lambda_l \rightarrow 0$ by the Inada conditions. Hence, the first order conditions cannot be satisfied for λ^a. Similarly, for $\lambda^l \equiv (0, 0, \lambda_l)$ note that

$$\mathrm{Du}_1 \rightarrow \infty \ as\ \lambda_a, \lambda_b \rightarrow 0$$

by the Inada conditions, and so the first order conditions cannot be satisfied for λ^l. Next, consider a portfolio of annuities and life insurance such as λ^{al} shown in Figure 4. $\lambda^{al} \equiv (\lambda_a, 0, \lambda_l)$ and (A.5) and (A.7) are satisfied with equalities while the $mrs_b < p_b$ in (A.7); it follows that

$$mrs_b = \frac{q\,\mathrm{D}\varphi + (1-q)\,\mathrm{Du}_1}{Du_0} = p_a + p_l = \left(\frac{1-q}{1-\delta} + q\right)p_b > p_b \tag{A.9}$$

which contradicts $mrs_b < p_b$. Hence the first order conditions cannot be satisfied at λ^{al}. Next, consider a portfolio that consists of all the assets such as λ^{abl} shown in Figure 4. Here, if $mrs_b = p_b$ then

$$mrs_a + mrs_l = mrs_b = p_b < \left(\frac{1-q}{1-\delta} + q\right)p_b = mrs_a + mrs_l \tag{A.10}$$

Hence there is a contradiction and λ^{abl} cannot be an optimal portfolio.

Second, consider the three portfolios of the form λ^b, λ^{bl}, and λ^{ab}. For $\lambda^b \equiv (0, \lambda_b, 0)$ we have

$$mrs_a < p_a, mrs_b = p_b, \textit{ and } mrs_l < p_l \tag{A.11}$$

and

$$mrs_a + mrs_l = mrs_b = p_b < p_a + p_l \tag{A.12}$$

Hence the first order conditions may be satisfied at λ^b. To refine the conditions required to make $\lambda^b \equiv (0, \lambda_b, 0)$ optimal note that

$$\frac{mrs_b}{mrs_l} = \frac{mrs_a}{mrs_l} + 1 > \frac{p_b}{p_l}$$
$$\Leftrightarrow \frac{(1-q)\,Du_1}{q\,D\varphi} > \frac{p_b}{p_l} - 1$$
$$\Leftrightarrow \frac{D\varphi}{Du_1} < 1$$

Similarly

$$\frac{mrs_b}{mrs_a} = \frac{mrs_l}{mrs_a} + 1 > \frac{p_b}{p_a}$$

$$\Leftrightarrow \frac{q\,D\varphi}{(1-q)Du_1} > \frac{1-\delta}{1-q} - 1 \tag{A.13}$$

$$\Leftrightarrow \frac{D\varphi}{Du_1} > \frac{q-\delta}{q}$$

Hence the conditions for the boundary bond portfolio to be optimal are

$$\frac{q-\delta}{q} < \frac{D\varphi}{Du_1} < 1$$

Next, consider a portfolio of the form $\lambda^{bl} \equiv (0, \lambda_b, \lambda_l)$. For this portfolio, we have

$$mrs_a < p_a,\ mrs_b = p_b,\ and\ mrs_l = p_l \tag{A.14}$$

Here note that if the investor moves from the portfolio λ^l to λ^b then the investor is moving in the direction $v_4 = (0, q, -1)$ and the derivative of expected utility in the direction v_4 is

$$\begin{aligned}
D_{v_4}H &= q\,D_2H - D_3G \\
&= q\left[-p_b\left(q\frac{\partial u(c_0,c_{11})}{\partial c_0} + (1-q)\frac{\partial u(c_0,c_{12})}{\partial c_0}\right) + q\frac{\partial u(c_0,c_{11})}{\partial c_{11}} + (1-q)\frac{\partial u(c_0,c_{12})}{\partial c_{12}}\right] - \left[-p_l\left(q\frac{\partial u(c_0,c_{11})}{\partial c_0} + (1-q)\frac{\partial u(c_0,c_{12})}{\partial c_0}\right) + q\frac{\partial u(c_0,c_{11})}{\partial c_{11}}\right] \\
&= q\left[q\frac{\partial u(c_0,c_{11})}{\partial c_{11}} + (1-q)\frac{\partial u(c_0,c_{12})}{\partial c_{12}}\right] - q\frac{\partial u(c_0,c_{11})}{\partial c_{11}} \\
&= -(1-q)\,q\frac{\partial u(c_0,c_{11})}{\partial c_{11}} + q\,(1-q)\frac{\partial u(c_0,c_{12})}{\partial c_{12}} \\
&= q\,(1-q)\left[\frac{\partial u(c_0,c_{12})}{\partial c_{12}} - \frac{\partial u(c_0,c_{11})}{\partial c_{11}}\right]
\end{aligned}$$

Note that $D_{v_4}H(\lambda^l) > 0$ by the Inada conditions and $D_{v_4}H(\lambda^b) < 0$ if

$$\frac{\dfrac{\partial u(c_0,c_{11})}{\partial c_{11}}\varphi'}{\dfrac{\partial u(c_0,c_{12})}{\partial c_{12}}} > 1 \tag{A.15}$$

Given the continuity of $D_{v_4}H$ it follows by the Intermediate Value Theorem that there exists a portfolio λ^{bl} such that $D_{v_4}H(\lambda^{bl}) = 0$. Hence, an optimal portfolio of the form λ^{bl} exists if (A.15) holds.

Next, consider a portfolio of the form $\lambda^{ab} \equiv (\lambda_a, \lambda_b, 0)$. For this portfolio, we have

$$mrs_a = p_a,\ mrs_b = p_b,\ and\ mrs_l < p_l \tag{A.16}$$

Note that if the investor moves from the portfolio λ^a to λ^b then the investor is moving in the direction $v_2 = \left(-1, \frac{1-q}{1-\delta}, 0\right)$ and the derivative of expected utility in the direction v_2 is

$$\begin{aligned}
D_{v_2}H &= -D_1H + \frac{1-q}{1-\delta} D_2H \\
&= -\left[-p_a\left(q\,\frac{\partial u(c_0, c_{11})}{\partial c_0} + (1-q)\,\frac{\partial u(c_0, c_{12})}{\partial c_0}\right) + (1-q)\,\frac{\partial u(c_0, c_{12})}{\partial c_{12}}\right] + \frac{(1-q)}{1-\delta}\left[-p_b\left(q\,\frac{\partial u(c_0, c_{11})}{\partial c_0} + (1-q)\,\frac{\partial u(c_0, c_{12})}{\partial c_0}\right) + q\,\frac{\partial u(c_0, c_{11})}{\partial c_{11}}\varphi' + (1-q)\,\frac{\partial u(c_0, c_{12})}{\partial c_{12}}\right] \\
&= -(1-q)\,\frac{\partial u(c_0, c_{12})}{\partial c_{12}} + \frac{(1-q)}{1-\delta}\left[q\,\frac{\partial u(c_0, c_{11})}{\partial c_{11}}\varphi' + (1-q)\,\frac{\partial u(c_0, c_{12})}{\partial c_{12}}\right] \\
&= -\left(1 - \frac{(1-q)}{1-\delta}\right)(1-q)\,\frac{\partial u(c_0, c_{12})}{\partial c_{12}} + \frac{(1-q)}{1-\delta}\,q\,\frac{\partial u(c_0, c_{11})}{\partial c_{11}}\varphi' \\
&= -\frac{(q-\delta)}{1-\delta}(1-q)\,\frac{\partial u(c_0, c_{12})}{\partial c_{12}} + \frac{(1-q)}{1-\delta}\,q\,\frac{\partial u(c_0, c_{11})}{\partial c_{11}}\varphi' \\
&= \frac{(1-q)}{1-\delta}\left[q\,\frac{\partial u(c_0, c_{11})}{\partial c_{11}}\varphi' - (q-\delta)\,\frac{\partial u(c_0, c_{12})}{\partial c_{12}}\right]
\end{aligned} \tag{A.17}$$

Note that $D_{v_2}H(\lambda^a) > 0$ by the Inada conditions and $D_{v_2}H(\lambda^b) < 0$ if

$$\frac{\frac{\partial u(c_0, c_{11})}{\partial c_{11}}\varphi'}{\frac{\partial u(c_0, c_{12})}{\partial c_{12}}} < \frac{(q-\delta)}{q}. \tag{A.18}$$

Given the continuity of $D_{v_2}H$ it follows by the Intermediate Value Theorem that there exists a portfolio λ^{ab} such that $D_{v_2}H(\lambda^{ab}) = 0$. Hence, an optimal portfolio of the form λ^{ab} exists if

$$\frac{\frac{\partial u(c_0, c_{11})}{\partial c_{11}}\varphi'}{\frac{\partial u(c_0, c_{12})}{\partial c_{12}}} < \frac{(q-\delta)}{q} \quad . \tag{A. 20}$$

QED

Corollary Sketch of Proof: Given the sole pursuit of self-interest, the marginal utility of consumption then in the death state is zero, i.e., $\varphi' \equiv 0$.

No portfolio of the form λ^{al} *or* λ^{abl} is optimal. To show this consider moving in the arbitrage direction from such a portfolio. These yields

$$\begin{aligned} D_v H &= -D_1 H + \frac{p_a + p_l}{p_b} D_2 H - D_3 H \\ &= -(1-q)\, Du_1 + \frac{p_a + p_l}{p_b}(1-q)\, Du_1 \\ &= \left[\frac{p_a + p_l}{p_b} - 1\right](1-q)\, Du_1 \\ &= (1-q)\left[\frac{1}{1-\delta} - 1\right](1-q)\, Du_1 \\ &> 0 \end{aligned} \tag{A. 21}$$

and the inequality confirms that expected utility can be increased by moving in the direction *v*.

Similarly, consider the portfolios of the form λ^l and λ^{bl}. The arbitrage direction is not feasible so consider moving along the constraint boundary in the direction $v_4 = (0, q, -1)$. Then

$$
\begin{aligned}
D_{v_4}H &= q\,D_2H - D_3H \\
&= q\left(-p_bDu_0 + (1-q)Du_1\right) - \left(-\,p_l\,Du_0\right) \\
&= q\;(1-q)\,Du_1 \\
&> 0
\end{aligned}
\tag{A. 22}
$$

Hence, these portfolios are not optimal.

Finally, consider the portfolios λ^b, λ^{ab}, *and* λ^a. Suppose the investor moves in the direction $v_1 = \left(\frac{1-\delta}{1-q}, -1, 0\right)$. This is movement from bonds to annuities. Then

$$
\begin{aligned}
D_{v_1}H &= \frac{1-\delta}{1-q}\,D_1H - D_2H \\
&= \frac{1-\delta}{1-q}\left(-\,p_a\,Du_0 + (1-q)Du_1\right) - \left(-p_bDu_0 + (1-q)Du_1\right) \\
&= \frac{1-\delta}{1-q}\,(1-q)Du_1 - \;(1-q)Du_1 \\
&= (q-\delta)\ \mathrm{Du}_1 \\
&\gtreqless\ 0\ as\ q \gtreqless \delta
\end{aligned}
\tag{A. 23}
$$

Therefore, the portfolios λ_a *and* λ_b are optimal as $q \gtrless \delta$, respectively. QED