THE ICOSIAN CALCULUS OF TODAY

By Norman Biggs
London School of Economics, Houghton St., London

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Abstract

Hamilton's Icosian Calculus is less well known than his algebra of quaternions, but it is nevertheless an interesting system of non-commutative algebra. He used it to study complete cycles on the dodecahedron, a subject which so fascinated him that he attempted to popularise a game based on it. The game was not a success in commercial terms, but it resulted in the term 'Hamiltonian cycle' being used for a complete cycle on any graph. This was not only inappropriate, because Hamilton was not the first to study such things, but also inauspicious, because Hamilton's methods have little relevance to the study of 'Hamiltonian' cycles in general.

A modern development which can be linked more positively with the Icosian Calculus is the use of generators and relations to study graphs which have certain symmetry properties. In particular there is a remarkable theorem of W.T. Tutte, concerning graphs of degree three, the proof of which involves calculations like those of Hamilton. Similar ideas were also used by J.H. Conway in unpublished, but seminal, investigations on the same subject. This is the legacy of the Icosian Calculus which is discussed in the paper.

1. Introduction

Hamilton discovered the Icosian Calculus in 1856, thirteen years after his discovery of quaternions. He was led to it by considerations involving polyhedra, and he pursued it because he saw it as a system of non-commutative algebra which is capable of a geometric interpretation. Much of his work was unpublished at the time, but it is now available in Volume 3 of the Collected Papers, items LIV–LVIII (to which we shall refer by the Roman numbering).

In modern terminology, the geometric interpretation involves paths and cycles on the graph formed by the vertices and edges of a regular dodecahedron (see Fig. 1). In particular, the Icosian Calculus can be used to describe a cycle which passes just once through every vertex. Hamilton's association with this particular problem resulted in the name *Hamiltonian cycle* being used generally for a cycle with this property, in any graph. Unfortunately the Icosian Calculus has little relevance to the general problem, and subsequent work on Hamiltonian cycles has used quite different methods. Equally unfortunate is the fact that the general problem was first studied not by Hamilton but by Kirkman in 1855 [17].

It is not the purpose of this paper to discuss the work of Kirkman and Hamilton on 'Hamiltonian' cycles, although I shall say a little more about their relationship

FIG. 1—A dodecahedron and its graph.

at the end, referring the reader to another paper for details [1]. My aim here is to describe how the interplay between algebraic and geometrical ideas foreshadowed by the Icosian Calculus has led to interesting developments in another area of Discrete Mathematics.

2. The algebraic framework

The Icosian Calculus was the name Hamilton gave [LIV, LV] to the algebraic structure in which there are three basic symbols ι, κ, λ, satisfying the equations

$$\iota^2 = 1, \quad \kappa^3 = 1, \quad \lambda^5 = 1, \quad \lambda = \iota\kappa.$$

The choice of symbols was clearly influenced by the use of i, j, k for the quaternion units. By the same analogy, Hamilton referred to the new symbols as *roots of unity*, although he was quick to point out that they are not fourth roots like the quaternion units. He was careful to say explicitly that the symbols should satisfy the associative law $a(bc) = (ab)c$, but not the commutative law $ab = ba$: indeed, he noted that assuming the commutative law makes $\iota = \kappa = \lambda = 1$. Also he noted that his calculus was based on a single 'multiplicative' operation, unlike the quaternion algebra which has addition and multiplication. Thus, in modern terms, Hamilton was concerned with the group I defined by the presentation

$$\langle \iota, \kappa, \lambda \mid \iota^2 = \kappa^3 = \lambda^5 = 1, \ \lambda = \iota\kappa \rangle.$$

This fact led Miller [19] to cite Hamilton as one of the founders of abstract group theory.

Today, an algebraist faced with this presentation for the first time might argue as follows. Since $\lambda = \iota\kappa$, there are really only two generators, ι and κ, so we should look first at the group

$$M = \langle \iota, \kappa \mid \iota^2 = \kappa^3 = 1 \rangle.$$

This is very well known: it is an infinite group known as the *modular group*, which arises in many areas of mathematics. Adjoining the relation $(\iota\kappa)^5 = 1$ defines

I as a quotient group of M. But is the quotient finite or infinite? The answer is on page 67 of the second edition of the standard work by Coxeter and Moser [11], where we find that I, denoted there by the symbol $(2, 3, 5)$, is the alternating group A_5, of order 60. Those who like to work things out for themselves should study David Johnson's book [16], in which this is an exercise on page 105.

Alternatively, the result is a moment's work for the widely used computational group theory package CAYLEY. The coset enumeration algorithm will tell us that the order is 60, and, with a bit of help, CAYLEY will confirm that there is a faithful representation of I defined by

$$\iota \mapsto (12)(34), \quad \kappa \mapsto (135), \quad \lambda \mapsto (12345).$$

Since the generators map to even permutations of degree 5, this establishes that I is isomorphic to the alternating group A_5.

3. The geometric framework

It is when we turn to the geometrical interpretation of the Icosian Calculus that we begin to see the richness of its possibilities. Hamilton himself observed that it could be generalised [LVI; LVII, p. 623], although he concentrated for the most part on the icosian system.

Hamilton's initial geometrical interpretation was in terms of the faces of a regular icosahedron. However, he saw immediately that there is a 'dual' interpretation in terms of vertices of the dodecahedron (Fig. 1), and this soon became his preferred mode of expression. Roughly speaking, his idea was to regard ι, κ, and λ as operations on the set of oriented edges of the dodecahedron. He took ι as the operation which 'reverses' any oriented edge, κ as the operation which 'rotates' it around one end, and λ as the operation which 'shunts' it along to one of its successors. The sense of the rotation and the choice of successor must be defined consistently, and this can be done in terms of the plane drawing. Following Hamilton, we shall choose λ to be the operation which gives the successor obtained by 'turning right'. Explicitly, suppose we fix our attention on a pair of adjacent vertices (P, Q), and let U, V be the other neighbours of Q, as in Fig. 2. Then

$$\iota(P, Q) = (Q, P), \quad \kappa(P, Q) = (V, Q), \quad \lambda(P, Q) = (Q, V).$$

It is worth emphasising that Hamilton's operations are defined as permutations of the set of oriented edges: they are not permutations of the vertex-set, and they are not automorphisms of the graph. However, as we shall explain in section 4, they are closely related to automorphisms.

The basic relations are obvious consequences of the geometry. Clearly $\iota^2 = 1$, since reversing twice is equivalent to doing nothing. Similarly, $\kappa^3 = 1$ since there are three edges at each vertex. The relation $\lambda = \iota\kappa$ can be checked as follows:

$$\iota\kappa(P, Q) = \iota(V, Q) = (Q, V) = \lambda(P, Q).$$

Finally, the relation $\lambda^5 = 1$ expresses the fact that if we walk along the edges of the dodecahedron, turning right five times in succession, then we get back to the starting position.

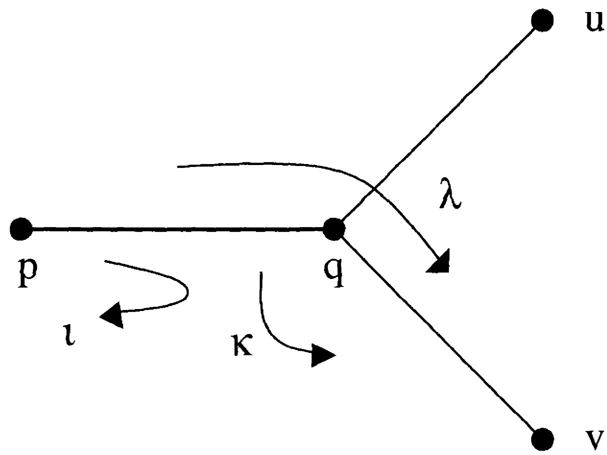

FIG. 2—The basic operations.

In this light, it clearly makes sense to consider the operation which takes an oriented edge to its 'other' successor, obtained by turning left instead of right. Hamilton used the symbol μ for this operation, noting that it can be defined in terms of the generators as follows:

$$\mu = \lambda\kappa = \iota\kappa^2.$$

Let us say that any sequence of λ's and μ's (but *not* their inverses) is a *positive word.* Clearly a positive word can be interpreted as a route on the graph, turning right or left at each vertex according as the appropriate symbol is λ or μ. The fact that a positive word reduces to the identity in I means that the route ends in the same place as it started. For example, consider the positive word $\mu\lambda^3\mu\lambda^3$. Using the definition $\mu = \lambda\kappa$, we have

$$\mu\lambda^3\mu\lambda^3 = \lambda\kappa.\lambda^3.\lambda\kappa.\lambda^3 = \lambda\kappa\lambda^4\kappa\lambda^3.$$

Since $\lambda^5 = 1$ we have $\lambda^4 = \lambda^{-1} = \kappa^{-1}\iota$, so

$$\mu\lambda^3\mu\lambda^3 = \lambda\kappa.\kappa^{-1}\iota.\kappa\lambda^3 = \lambda\lambda\lambda^3 = 1.$$

In other words, $\mu\lambda^3\mu\lambda^3$ is a positive word which reduces to the identity in I. Remembering that the operations are to be carried out in reverse order, this means that turning right three times, then left, then right three times again, and finally left, will ensure that we return to the starting position.

Hamilton made an extensive table of words of this kind [LVII, pp 619–20]. He was particularly interested in the word

$$(\lambda^3\mu^3(\lambda\mu)^2)^2,$$

which has twenty operations in all. Indeed, it represents a closed route with twenty steps, passing through each of the twenty vertices of the dodecahedron just once. As we have already remarked, this is the origin of the name 'Hamiltonian cycle' for a cycle passing through all the vertices of any graph.

When we try to generalise Hamilton's method, we soon realise that it must depend on the existence of some symmetry in the graph, because the instructions for following a route do not specify the choice of the initial edge. Furthermore, as we shall see, there is a very specific sense in which it can succeed only in the *cubic* case, that is, when there are three edges meeting at each vertex. In the twentieth century, many mathematicians have worked on the symmetry properties of cubic graphs, and three (in particular) have discovered significant extensions of Hamilton's method. They are R.M. Foster, W.T. Tutte, and J.H. Conway. In all three cases the circumstances of the discoveries have picturesque features, and we shall digress briefly to describe them.

R.M. Foster worked as an electrical engineer, and in that context he became interested in the use of cubic graphs as electrical circuits. In 1932 he published a paper [12] containing drawings of twelve such graphs, all having special symmetry properties. Throughout his long life he continued to devote his leisure to the study of such graphs, producing a remarkable *Census* which was eventually published in 1988 [8]. This is an invaluable listing of symmetric cubic graphs with up to 512 vertices and their properties. It contains a useful summary of Foster's methods, which makes the link with Hamilton very clear.

W.T. Tutte was originally destined to be a chemist, but by a happy chance he met with three other Cambridge undergraduates who became fascinated by the problem of 'squaring the square'. This story will be familiar to those interested in mathematical recreations (see [13]). After a brief interlude spent in the service of his country, he became a serious mathematician, and soon published several important papers, among them one entitled 'A family of cubical graphs' [20]. In that paper he used ideas like Hamilton's to study a general problem on the symmetry of cubic graphs, and he proved a very surprising theorem, the content of which we shall explain in the next section.

J.H. Conway is another leading mathematician who is well known for his work on mathematical recreations. His contribution to our subject began in an unlikely way. In his youth, a relative presented Conway with a multi-volume *Junior World Encyclopaedia*, which turned out to be deficient, in that the volume devoted to topics beginning with F and G was entirely blank, owing to a printing error. The fact that he grew up ignorant of such things as Fish and Fingerprints is not our concern. We have to thank this mischance for providing him with a notebook for his investigations on the symmetry of cubic graphs, in which, like Foster and Tutte, he used 'Hamiltonian' methods. His contribution is less well known than theirs, because Conway himself has published little on the subject. But it deserves to be better known, and because of its clear link with the Icosian Calculus we shall take the opportunity to use it in the next section.

4. Modern developments

In this section we shall outline a general framework for the study of symmetry in cubic graphs, which can be regarded as a generalisation of the Icosian Calculus. The topic is still very much alive, and we shall mention some of the open questions. More details may be found in [5].

An *automorphism* of a graph with vertex-set V and edge-set E is a permutation π of V such that $\{\pi(v), \pi(w)\}$ is in E whenever $\{v, w\}$ is in E. Clearly, the set of all automorphisms of a given graph forms a group under the operation of composition, and any group of automorphisms of the graph is a subgroup of the full group. For example, the full group of the dodecahedron has order 120, and there is a subgroup of order 60 which, as we shall see, corresponds to Hamilton's group I.

An *s-arc* is a sequence of vertices $v_0, v_1, \ldots, v_s$ such that $\{v_i, v_{i+1}\}$ is an edge for $0 \leq i \leq s-1$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. A group G of automorphisms is *transitive* on the s-arcs if, given any two s-arcs, there is an element of G which transforms the first into the second. The special feature of cubic graphs is that any group G of automorphisms which is transitive on 1-arcs must be *regular* on s-arcs for some $s \geq 1$. This means that there is a *unique* automorphism in G which transforms one given s-arc into another one. (The proof is not difficult: see [5, chapter 18].) It follows that if we choose a 'reference' s-arc R arbitrarily, then there is a bijective correspondence $S \leftrightarrow g_S$ between s-arcs S and automorphisms in G, where g_S is the unique automorphism such that

$$g_S(R) = S.$$

This observation enables us to make the link between automorphisms and Hamilton's icosian operations explicit. We associate with each automorphism g a corresponding permutation $\bar{g}$ of the set of s-arcs, defined by

$$\bar{g}(S) = g_S g(R) = g_S g g_S^{-1}(S).$$

It is clear that Hamilton's operations are permutations of the form $\bar{g}$, rather than automorphisms g. Specifically, in the case of the dodecahedron there is a well-known group of automorphisms which is regular on the 1-arcs: it is the group of rotational symmetries of the solid dodecahedron in $\mathbb{R}^3$. The corresponding group of permutations of the 1-arcs is Hamilton's group I, as previously described. For example, the operation ι which inverts each 1-arc is the $\bar{g}$ corresponding to the unique rotational automorphism g which inverts the reference arc R. We note that the number of 1-arcs is $20 \times 3 = 60$, as we should expect, since they are in bijective correspondence with a group of order 60. We also note that the full automorphism group of the graph (and the group of all symmetries of the solid) is of order 120, and is regular on the 2-arcs. This observation provided the context for a second wave of calculations by Hamilton, done in 1863 [LVIII], to which we shall refer below.

From now on we shall work with automorphisms, remembering that everything can be translated into 'Hamiltonian' language by means of the bijective correspondence described above. For cubic graphs in general there are the obvious questions:

what values of s are possible, and what are the corresponding groups and graphs? The first question was answered in 1947 by Tutte [20]. Remarkably, it turns out that for finite cubic graphs only the values $s = 1, 2, 3, 4, 5$ can occur.

We shall use Conway's approach to discuss this result. Suppose we have a group which is regular on the s-arcs of a cubic graph. Conway defines three automorphisms σ, a, b (Fig. 3), which act on the reference s-arc R in ways analogous to Hamilton's ι, λ, μ respectively. That is, σ reverses R, while a and b are the two 'shunt' automorphisms which move each one of the first s vertices of R on to the next vertex.

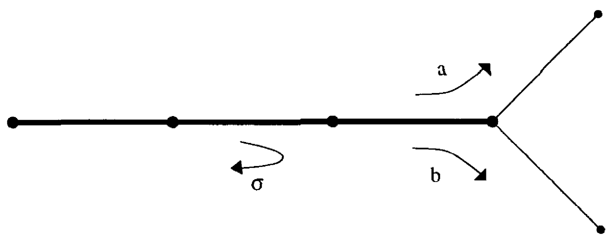

FIG. 3—The 'local' operations in Conway's notation.

We begin by considering the 'local' relations which these automorphisms must satisfy, remembering that we must not assume any sense of left or right like that provided by our drawing of the dodecahedron. First, there is the obvious relation $\sigma^2 = 1$, corresponding to the fact that two reversals are the same as doing nothing. Then we note that the operations $\sigma a \sigma$ and $\sigma b \sigma$ have same effect on R as the operations a^{-1} and b^{-1}. But, in the absence of an orientation-convention, we cannot tell which is which. Thus, for any value of s, there are two sets of relations to consider:

$$\mathcal{R}^+ = \{\sigma^2 = 1,\ \sigma a \sigma = a^{-1},\ \sigma b \sigma = b^{-1}\};$$

$$\mathcal{R}^- = \{\sigma^2 = 1,\ \sigma a \sigma = b^{-1},\ \sigma b \sigma = a^{-1}\}.$$

Note that in the $\mathcal{R}^-$ case the last two relations are equivalent, and could be replaced by the single relation $\sigma a \sigma b = 1$.

Next we introduce relations specific to the given value of s. These are obtained by finding pairs of words in σ, a, b which have the same action on the reference arc R, and which must therefore be equal. For example, in the case $s = 3$ one possible set of such relations is

$$ab^{-1}a = b,\ a^2b^{-2}a^2 = b^2,\ a^2b\sigma a^3 = b^3.$$

Together with $\mathcal{R}^+$ these define a group G_3^+ with presentation

$$\langle \sigma, a, b \mid \sigma^2 = 1, \sigma a \sigma = a^{-1}, \sigma b \sigma = b^{-1}, ab^{-1}a = b, a^2b^{-2}a^2 = b^2, a^2b\sigma a^3 = b^3 \rangle.$$

This is an infinite group, which we shall refer to as a *type*: it captures the local action of a group of automorphisms of a cubic graph which is regular on 3-arcs. In fact it can be shown that it is the only type for the case $s = 3$. This follows from arguments which establish that very few sets of putative relations are feasible. In many cases the group collapses, for example when we try to define a group G_3^- using $\mathcal{R}^-$ instead of $\mathcal{R}^+$. The result of Tutte, together with a detailed analysis of the possibilities, leads to the conclusion that there are only seven essentially distinct types (with respect to a technical notion of equivalence). These we denote by

$$G_1^-, G_2^+, G_2^-, G_3^+, G_4^-, G_4^+, G_5^+.$$

In particular, we see that in the case $s = 1$ only the $\mathcal{R}^-$ relations are relevant. In fact the group G_1^- is given by the presentation

$$\langle \sigma, a, b \mid \sigma^2 = 1,\ \sigma a \sigma b = 1,\ (a^{-1}b)^3 = 1 \rangle,$$

where the first two relations are equivalent to $\mathcal{R}^-$, and the third follows from the fact that $a^{-1}b$ fixes the initial vertex of the reference arc. This is, in fact, the modular group M in disguise. If we define a function $G_1^- \to M$ by

$$\sigma \mapsto \iota, \quad a \mapsto \iota\kappa, \quad b \mapsto \iota\kappa^2,$$

then it is easy to check that we have an isomorphism. For example, the relation $\sigma a \sigma b = 1$ is preserved because

$$\sigma a \sigma b \mapsto \iota.\iota\kappa.\iota.\iota\kappa^2 = \kappa^3 = 1.$$

Given the local 'symmetry-structure', as determined by the type, Conway then turns to the question of constructing finite cubic graphs which have the given structure. Roughly speaking, this means looking for a positive word (or words) $w = w(a, b)$, such that adjoining the relation $w = 1$ to the infinite group G_s^ϵ results in a finite group $F = G_s^\epsilon(w)$. A finite cubic graph can then be constructed in the following way. Let K be the subgroup containing those automorphisms which, in the intuitive description, fix the initial vertex of the reference arc. For a given value of s, K is generated by $a^{-1}b, a^{-2}b^2, \ldots, a^{-s}b^s$. The vertices of the graph are the left cosets of K, and the edges join a typical coset gK to $ga^{-1}K$, $gb^{-1}K$ and gaK. (Note that $gaK = gbK$.) Thus the construction becomes an exercise in the technique of *coset enumeration* [11]. Nowadays this is usually done on a computer, although there are inescapable difficulties about the termination of the process.

For example, consider G_1^- and the relation $a^5 = 1$. Returning for a moment to Hamilton's notation, the relation becomes $\lambda^5 = 1$ and, as we have seen, adjoining it to the modular group produces a group of order 60. Thus the coset enumeration technique will produce 20 cosets of $K = \langle a^{-1}b \rangle$ in the group

$$G_1^-(a^5) = \langle \sigma, a, b \mid \sigma^2 = 1,\ \sigma a \sigma b = 1,\ (a^{-1}b)^3 = 1,\ a^5 = 1 \rangle,$$

and we can construct a cubic graph in the manner just described. Of course, it is the dodecahedron. By the same method, the groups $G_1^-(a^3)$ and $G_1^-(a^4)$ define the tetrahedron and the cube respectively. On the other hand, coset enumeration will fail in the case of the relation $a^6 = 1$ because this defines an infinite graph, the plane hexagonal lattice. In this case we can try to find additional relations which, together with $a^6 = 1$, make the group and the graph finite. Conway discovered that in this case the most general solution is

$$(ab)^{mn} = 1, \quad (ab)^{mp}(ba)^m = 1,$$

where m, n, p are positive integers such that $p^3 \equiv 1 \pmod{n}$. Versions of this result have been found independently by other people in other contexts: for example, the corresponding graphs can be regarded as toroidal quotients of the hexagonal lattice, as described by Coxeter and Moser [11].

Although we have satisfactory results in such special cases, much work remains to be done on the problem of determining the finite quotients of the seven types and the corresponding graphs. Conway was the first to show that infinitely many graphs of each type exist; his proof (see [5, chapter 19]) required astronomically large graphs. For example, the second G_5^+ graph in his family (the last one in the table below) has about two million vertices, and the next one has about 2^{100000} vertices. A somewhat stronger result is also true: it can be shown by non-constructive means [4] that there are graphs of each type with arbitrarily large girth. (The *girth* is the length of the shortest cycle.)

The type G_5^+ is especially interesting because it exhibits the 'maximum' symmetry. The complete definition of the group is

$$G_5^+ = \langle \sigma, a, b \mid \sigma^2 = (\sigma a)^2 = (\sigma b)^2 = 1, (a^{-1}b)^2 = (a^{-2}b^2)^2 = (a^{-3}b^3)^2 = 1,$$
$$a^4 b^{-4} a^4 = ba^2 b, a^4 b \sigma a^5 = ba^3 b \rangle.$$

It might be thought that we should be able to classify finite quotients of this group, given that it has so many relations. However, we know very few graphs of this type, and correspondingly little is known about the classification of quotients of G_5^+. Most of the known examples are members of an infinite family, the *sextet* graphs $S(p)$, p a prime congruent to 3 or 5 modulo 8, or coverings of these graphs [6]. The sextet graph $S(p)$ has $p^2(p^4 - 1)/24$ vertices, and the automorphism group is $\mathrm{P\Gamma L}(2, p^2)$. These graphs have some interesting graph-theoretical properties, such as the fact that their girth is exceptionally 'large', in a technical sense.

The simplest sextet graph $S(3)$, which has 30 vertices and girth 8, was known to Foster, Tutte, and Conway, and is usually known as *Tutte's* 8*-cage.* Its group is $\mathrm{P\Gamma L}(2, 9)$, of order 1440, which is isomorphic to Aut (S_6), and has the presentation $G_5^+(a^8)$. There is a threefold covering of $S(3)$, also known to Foster and Conway, whose group has the presentation $G_5^+(b^{10})$. The next sextet graph $S(5)$ has 650 vertices and girth 12: Foster constructed it by hand (!), and Conway discovered it via the presentation $G_5^+(a^{12})$, using coset enumeration on a computer. The next three graphs in the sequence are $S(11)$, $S(13)$, and $S(19)$, with girths 20, 24 and 28 respectively, the last of which has nearly two million vertices.

A list of the known cubic 5-regular graphs with less than 2×10^6 vertices follows. There are reasons for thinking that the list may be complete up to 5000 vertices, but beyond that we have no information.

Size	*Reference*	*Group*	*Presentation*
30	$S(3)$	Aut S_6	$G_5^+(a^8)$
90	$3.S(3)$		$G_5^+(b^{10})$
234	Wong [21]	Aut $PSL(3,3)$	$G_5^+(a^{13})$
468	2.234		$G_5^+(b^{12})$
650	$S(5)$	$P\Gamma L(2,25)$	$G_5^+(a^{12})$
2352	Biggs [2]	2.Hol $PSL(2,7)$	$G_5^+(a^{14})$
4704	2.2352		$G_5^+((ab)^8)$
73810	$S(11)$	$PSL(2,121)$	
75600	Conder [10]	S_{10}	
201110	$S(13)$	$PSL(2,169)$	
249696	Biggs [3]	2.Hol $PSL(2,17)$	
1960230	$S(19)$	$PSL(2,361)$	
1966080	$2^{16}.S(3)$	$\mathbb{Z}_2^{16}$.Aut S_6	

Perhaps the most interesting recent discovery is the Conder graph. Conder was able to show that there are permutations of degree ten which satisfy the defining relations for G_5^+; in other words, the symmetric group S_{10} is a quotient of G_5^+, and so there is a corresponding graph, as listed above.

Finally, we recall that Hamilton himself saw the possibility of generalisations of the kind we have described. In particular, in his memorandum of 1863 [LVIII] (not published until 1967) he discussed the case $s = 2$ on the dodecahedron. With commendable foresight, he used the symbols α and β for the shunts which we have called a and b. He observed that α has order 5, whereas β has order 10, and he carried out extensive calculations involving positive words in α and β. In our terms he was working in the finite quotient of G_2^+ obtained by adjoining the relations $a^5 = 1$ and $b^{10} = 1$. Explicitly, we have the group $G_2^+(a^5, b^{10})$ with presentation

$$\langle \sigma, a, b \mid \sigma^2 = a^5 = b^{10} = 1, \sigma a \sigma = a^{-1}, \sigma b \sigma = b^{-1}, ab^{-1}a = b, ab\sigma a^2 = b^2 \rangle,$$

which turns out to be a group of order 120 isomorphic to the full group of the dodecahedron.

5. The Icosian Game

The earliest recorded reference to the Icosian Calculus in the Hamilton papers is a memorandum to J.T. Graves dated 7 October 1856 [LVI]. Soon afterwards he sent a brief note on the subject to the *Philosophical Magazine* [LV], and wrote a similar outline in a letter to Rev. Charles Graves [LIV]. All three accounts mention the geometrical interpretation, but none of them provides any details.

Hamilton wrote again to J.T. Graves on 17 October 1856 [LVII]. This is a long letter with 23 folio pages, the mathematical content of which has been outlined above. In the preamble he suggested that earlier contacts with Graves may have been responsible for rekindling his interest in polyhedra. Apparently Graves replied to the effect that he himself had no claim, but he thought that the recently published work of Kirkman might be responsible. On 1 November Hamilton wrote that he had been ignorant of Kirkman's work. He said that the first part of the *Philosophical Transactions* for 1856 had just arrived, and he had immediately studied the two papers on polyhedra by Kirkman which were contained therein. He acknowledged that on the geometrical side Kirkman had dealt with far greater generalities, but observed that Kirkman's general result [17, theorem A] did not appear to cover the case of the dodecahedron. But he admitted that Kirkman could easily have obtained that special case independently.

The last part of the letter of 17 October contains the germ of another idea.

> I have found that some young persons have been much amused by trying a new mathematical game which the Icosian furnishes, one person sticking five pins in any five consecutive points . . . and the other player then aiming to insert, which by the theory in this letter can always be done, fifteen other pins, in cyclical succession, so as to cover all the other points, and end in immediate proximity to the pin wherewith his antagonist had begun.

He was so taken with this idea that in due course he sold it to Jacques and Son, a firm which made toys and games. (The firm still exists, producing croquet sets among other things. Unfortunately, all their records from the 1850s have long since been destroyed.) The form in which the 'Icosian Game' was sold was a circular wooden board marked with the skeleton of the dodecahedron, and a set of twenty pegs. There was a leaflet of instructions, with hints from the inventor himself. The leaflet is printed in full in [7], and more details of the negotiations between Hamilton and Jacques are given in [15].

The mathematical content of the game, in the form outlined in Hamilton's letter to Graves, depends on the fact that 'five consecutive points' determine a sequence of three λ or μ operations. In other words, the starting configuration is a positive word of length 3 in λ and μ. Now Hamilton's word of length 20,

$$h(\lambda, \mu) = \lambda\lambda\lambda\mu\mu\mu\lambda\mu\lambda\mu\lambda\lambda\lambda\mu\mu\mu\lambda\mu\lambda\mu,$$

contains each of the eight possible words of this kind, and so by starting at the correct place in h, and noting that any cyclic rotation of h also reduces to the identity, it is possible to write down the required cycle.

There is no evidence that the Icosian Game was a popular success, and the copy which is in the Royal Irish Academy appears to be the only one known to have survived. There must have been others in existence in the nineteenth century, since the game is referred to in several contemporary records. For example, we know that Hamilton visited Kirkman in 1861, and Kirkman later wrote [18] that Hamilton had presented him with a copy of the game. No direct descendants of Kirkman have been traced, although he was the father of seven children. Perhaps somewhere there is a mine of fascinating material, including an Icosian Game, just

Icosian Calculus is a continuing stream of mathematical research and discovery, some of which has been described above.

References

Hamilton's papers on the Icosian Calculus appear in H. Halberstam and S.J. Ingram (eds), *Mathematical papers of Sir William Rowan Hamilton* (Cambridge, 1967), vol. 3. The papers are identified by Roman numerals, and these are used for reference in the text above.

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