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Rectangulations

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1. *Introduction.* The subject of squared rectangles and squared squares was developed by Brooks, Smith, Stone and Tutte (1) using techniques based on the theory of electrical networks. In this note we shall treat the (apparently) more general topic of rectangulations using purely algebraic techniques. The starting point is the observation that the incidence matrices of the pair of dual networks associated with a squared rectangle in (1) can be more naturally derived as incidence matrices of the squaring (or rectangulation) itself. Further, the topological information contained in the networks is a simple consequence of Euler's formula applied directly to the rectangulation.

In the first two sections following, we develop the matrix algebra necessary for our purpose. Some of the early results here are well-known. Then we investigate rectangulations and the matrices associated with them, obtaining conditions for squared rectangles and squared squares. The main result (Theorem III) is a direct proof of the fact that to every rectangulation there corresponds an essentially unique squaring; however, it is not possible to describe this correspondence geometrically in the way that might be expected. (The author is grateful to the referee for some remarks on this subject.) Finally, there is a note on the application of the algebra to network theory.

2. *Some matrix algebra.* We are interested in the properties of matrices which arise as incidence matrices of networks; in such a matrix every column contains precisely two non-zero entries, one of them being $+1$ and the other -1. These matrices, called *i-matrices* for short, have some rather remarkable properties, which we proceed to investigate. First, notice that if A is an $m \times r$ i-matrix, and u is the m-rowed column vector with $+1$ in each position, then $A^t u = 0$ (A^t denoting the transpose of A). It follows that the nullity ($= m -$ rank of A) of A is at least 1, and that its rank is at most $m-1$. The nullity of AA^t is the same as that of A^t since

$$\begin{aligned} AA^t x = 0 &\Rightarrow x^t AA^t x = 0 \\ &\Rightarrow \|A^t x\|^2 = 0 \\ &\Rightarrow A^t x = 0. \end{aligned}$$

and $A^t x = 0 \Rightarrow AA^t x = 0$ inmediately. Consequently, denoting the rank of A by $\rho(A)$, we have

$$\rho(A) = \rho(A^t) = \rho(AA^t) \leqslant m-1. \tag{2·1}$$

In particular, if $\rho(AA^t) = m-1$, then its nullity is 1 and since $AA^t u^t = 0$ we have

$$\rho(A) = m-1 \Rightarrow \ker AA^t = \{x : x = \lambda u\}, \tag{2·2}$$

where $\ker AA^t$ denotes $\{z : AA^t z = 0\}$.

We now consider adj AA^t, that is, the matrix of $(m-1)\times(m-1)$ cofactors of AA^t. If the rank of A is less than $m-1$ then every cofactor is zero; adj $AA^t = 0$. Suppose A has rank $m-1$; since $AA^t \operatorname{adj} AA^t = (\det AA^t)I = 0$, each column of adj AA^t belongs to the kernel of AA^t and so is a multiple of u. Since AA^t is symmetric it follows that all the multipliers are equal and thus adj $AA^t = \mu U$ where U is the $m\times m$ matrix each of whose entries is $+1$. We can also show that μ is non-negative, as follows: for any x, $x^tAA^tx = \|A^tx\|^2 \geqslant 0$ so that AA^t is a non-negative symmetric matrix; such a matrix has all its principal cofactors non-negative. To sum up, for each i-matrix A we have a non-negative integer $\mu(A)$ such that

$$\operatorname{adj} AA^t = \mu(A)\,U \tag{2·3}$$

and $\mu(A) = 0$ if and only if $\rho(A) < m-1$. We note for future use that if A_0 denotes A with row k deleted then $A_0A_0^t$ is AA^t with row k and column k deleted so that

$$\det A_0A_0^t = \mu(A). \tag{2·4}$$

We shall always take the deleted row to be the last.

Another interesting and useful property of i-matrices is that every submatrix of an i-matrix A has determinant $+1$, -1, or 0. To see this we consider the three possibilities for a square submatrix C of A:

(1) C has a column consisting entirely of zeros;

(2) C has two non-zero entries in each column;

(3) C has a column with precisely one non-zero entry.

In case (1), expanding in terms of the column of zeros shows $\det C = 0$; in case (2) adding all the rows to the first gives a row of zeros so that $\det C = 0$ similarly. In case (3) we expand $\det C$ in terms of the column containing the one non-zero entry getting $\det C = \pm\det C'$, where C' is a square submatrix of A of order one less than C. We have the same three possibilities for C' and if we continue the process we eventually arrive at a zero determinant or a single element of A, whence the result.

This last result enables us to give another interpretation of the number $\mu(A)$, as follows. Let $Q_{k,l}$ denote the set of collections of k distinct integers which are subsets of the first l positive integers. For any $t\times l$ matrix M and any $\alpha\in Q_{k,l}$ we write $M(.\,|\alpha)$ for the matrix obtained from M by considering only the columns $\alpha_1, \alpha_2, \ldots, \alpha_k$ of M; for each $\beta\in Q_{p,t}$ we have similarly a matrix $M(\beta|\,.)$. Using this notation, the Binet–Cauchy theorem states that

$$\det A_0A_0^t = \sum_{\omega\in Q_{m-1,r}} \det A_0(.\,|\omega)\det A_0^t(\omega|\,.).$$

Comparing with (2·4) we get

$$\mu(A) = \sum_{\omega\in Q_{m-1,r}} [\det A_0(.\,|\omega)]^2. \tag{2·5}$$

Now, each square submatrix of A has determinant $+1$, -1, or 0 so that the non-zero summands on the right-hand side of (2·5) are all $+1$, and we have the result that $\mu(A)$ is the number of non-singular $(m-1)$-square submatrices of A_0.

The next object of study is the $(m-1)$-square matrix

$$E(A) = \operatorname{adj} A_0 A_0^t.$$

We shall restrict the investigation to the case $\rho(A) = m-1$, and then we shall prove that each entry e_{ij} of $E(A)$ is a non-negative integer, and that the largest entry in any row or column occurs on the main diagonal. That is, for each i and j between 1 and $m-1$,

$$e_{ii} \geqslant e_{ji} = e_{ij} \geqslant 0. \tag{2·7}$$

To prove this, we augment $E(A)$ by a final row of zeros, getting an $m \times (m-1)$ matrix $E^+(A)$, and let e denote the jth column of $E^+(A)$, $c = A^t e$. Then, partitioning AA^t so that we can multiply by $E^+(A)$ and performing this calculation we obtain

$$Ac = AA^t e = \mu(A)\, w, \tag{2·8}$$

where w has $+1$ in the jth place, -1 in the mth place and zeros elsewhere. (At this point the reader may find it helpful to refer to the remarks of section 7; the network interpretation will illuminate the lemmas which follow.)

Continuing with the proof of (2·7), we define a set S_j by putting $p \in S_j$ if and only if there is some k for which $a_{pk} c_k \neq 0$. From (2·8) we see that j and m are in S_j. Two lemmas give the required result.

LEMMA 1. *If $p \in S_j$ then*

either (1) $p = j$ *or* $p = m$,

or (2) *there are t and u in S_j such that* $e_{tj} < e_{pj} < e_{uj}$.

Proof. Suppose $p \in S_j$ and $p \neq j$, $p \neq m$. Then $\Sigma a_{pk} c_k = 0$, and the existence of one non-zero summand implies the existence of two: that is, there are x, y such that

$$a_{px} c_x > 0, \quad a_{py} c_y < 0.$$

Since $a_{px} \neq 0$, we can find just one other non-zero entry in column x of A, say a_{tx}. There is a similarly defined a_{uy}. Then

$$0 < a_{px} c_x = a_{px}(a_{px} e_{pj} + a_{tx} e_{tj}) = e_{pj} - e_{tj},$$

$$0 > a_{py} c_y = a_{py}(a_{py} e_{pj} + a_{uy} e_{uj}) = e_{pj} - e_{uj},$$

and both t and u are in S_j since, for instance,

$$a_{tx} c_x = -a_{px} c_x \neq 0.$$

LEMMA 2. *If $p \notin S_j$ then there is a $z \in S_j$ such that $e_{pj} = e_{zj}$.*

Proof. For each $z = 1, 2, \ldots, m$ let

$$Q(z) = \{v : \exists\, w \quad \text{with} \quad a_{zw} a_{vw} \neq 0\};$$

that is, $Q(z)$ is the set of rows of A which have a non-zero entry in the same column as a non-zero element of row z.

Then there is a sequence $f, g, \ldots, l$, of distinct rows of A, such that

$$f \in Q(p), \quad g \in Q(f) \ldots m \in Q(l);$$

for if not we could partition the rows of A into two sets, each of which contained either two or zero non-zero entries in each column, and this would imply $\rho(A) < m-1$.

Now if $p \notin S_j$, $a_{pk} c_k = 0$ for each k. But by definition of f, $a_{pk}\, a_{fk} \neq 0$ for some k; thus $c_k = 0$, and we have

$$0 = c_k = a_{pk} e_{pj} + a_{fk} e_{fj} = \pm (e_{pj} - e_{fj}).$$

So if $f \in S_j$ we are finished, since $e_{pj} = e_{fj}$. If $f \notin S_j$ the same argument shows that $e_{fj} = e_{gj}$, and so if $g \in S_j$ we are finished. That the process must finish somewhere is ensured by the fact that $m \in S_j$.

Lemmas 1 and 2 lead immediately to the required result (2·7). We have only to recall that $e_{mj} = 0$ by definition, and that $e_{jj} > 0$ since it is a principal cofactor of the positive symmetric matrix $A_0 A_0^t$.

We note for future reference that (2·8) tells us that a solution of the equation $AA^t x = \alpha w$, where α is a real number, is

$$x = [\alpha/\mu(A)]\, e \tag{2·9}$$

where e is the jth column of $E^+(A)$. From the result (2·3) on the kernel of AA^t we see that this is the unique solution with $x_m = 0$, provided $\rho(A) = m-1$.

3. *Dual pairs of i-matrices.* We shall define the pair $\{A, B\}$ of i-matrices of size $m \times r$ and $n \times r$ respectively, to be a *dual pair* if and only if

(1) $\rho(A) + \rho(B) = (m-1) + (n-1) = r$,

(2) $AB^t = 0$.

It follows from the definition that $\rho(A) = m-1$, $\rho(B) = n-1$, and that the rank and size of B are determined by those of A.

THEOREM I. *If $\{A, B\}$ is a dual pair then*

$$\mu(A) = \mu(B). \tag{3·1}$$

Proof. Recall that $\mu(A)$ is the number of non-singular $(m-1)$-square submatrices of A_0. Suppose first that A_0 is partitioned as

$$A_0 = (A_0' | A_0'')$$

where A_0' has $m-1$ columns and is non-singular. If B_0 is partitioned in the same way it follows from condition (2) of the definition that B_0'' is an $(n-1)$-square matrix. Since $A_0 B_0^t = 0$ we have

$$A_0' B_0'^t + A_0'' B_0''^t = 0$$

and since A_0' is non-singular we can write

$$B_0' = B_0'' R,$$

where $R = [-(A_0')^{-1} A_0'']^t$. But this states that the columns of B_0' are linearly dependent on the columns of B_0'', so that the $n-1$ columns of B_0'' span the column space of B_0.

Further, these columns must be linearly independent since $\rho(B_0) = n-1$. Thus B_0'' is non-singular. A parallel argument shows that if B_0'' is non-singular then so is A_0', and it is immediate that for any $\gamma \in Q_{m-1,r}$ the argument extends to show that $A_0(.\,|\gamma)$ is non-singular if and only if $B_0(.\,|\gamma')$ is non-singular, where $\gamma' \in Q_{n-1,r}$ is the complement of γ. So we have a one-one correspondence between non-singular submatrices giving

$$\mu(A) = \mu(B).$$

Suppose now that $Ax = 0$. Since $Ab = 0$ where b^t is any row of B, and the kernel of A has dimension $r-(m-1) = (n-1)$, it follows that the $n-1$ rows of B_0 form a basis for the kernel of A. That is, $x = B_0^t y$. If now $Bx = 0$ also, then $B_0 B_0^t y = 0$ so that $y = 0$, since $B_0 B_0^t$ is non-singular. Thus we have

$$\ker A \cap \ker B = \{0\}. \tag{3·2}$$

4. *Rectangulations of rectangles.* A set of points in the plane of the form

$$\mathscr{R} = \{(x_1, x_2) : a < x_1 < b, c < x_2 < d\}$$

will be called an *open rectangle*. A point set of the form

$$\mathscr{V} = \{(x_1, x_2) : x_1 = f, \quad g \leqslant x_2 \leqslant h\}$$

will be called a *vertical cut*, and a *horizontal cut* is defined similarly. A finite set of pairwise disjoint horizontal cuts and a finite set of pairwise disjoint vertical cuts, with union $\mathscr{C}$ will be called a *rectangulation* of a given open rectangle $\mathscr{R}$ if

(*a*) $\mathrm{fr}\mathscr{R} \subseteq \mathscr{C} \subset \mathrm{cl}\mathscr{R}$,

(*b*) $\mathrm{cl}\mathscr{R} - \mathscr{C}$ is a set of pairwise disjoint open rectangles.

Any rectangulation $\mathscr{C}$ of a rectangle $\mathscr{R}$ has an *underlying network* in which the *vertices* are the points of intersection of cuts, and two vertices are joined by an *edge* if and only if they lie on the same cut and have no vertices between them. Since this network is planar we define its *regions* to be the open rectangles of $\mathrm{cl}\mathscr{R} - \mathscr{C}$ together with the complement of $\mathrm{cl}\mathscr{R}$. We shall use the notation that there are m_0 horizontal cuts, n_0 vertical cuts, $c_0 = m_0 + n_0$, and that the underlying network has v vertices, e edges and r regions.

The integers c_0, s, v, e, r, satisfy certain relations, combinatorial and topological. Suppose that there are ϕ_i cuts consisting of i edges, then

$$e = \phi_1 + 2\phi_2 + 3\phi_3 + \ldots,$$

and since each vertex lies on precisely two cuts, and a cut with i edges contains $i+1$ vertices,

$$2v = 2\phi_1 + 3\phi_2 + 4\phi_3 + \ldots.$$

But

$$c_0 = \phi_1 + \phi_2 + \phi_3 + \ldots,$$

whence

$$c_0 = 2v - e. \tag{4·1}$$

A topological constraint on a rectangulation is given by Euler's theorem

$$v - e + r = 2. \tag{4·2}$$

From (4·1) and (4·2) we obtain

$$r - c_0 + 2 = 2e - 3v + 4. \tag{4·3}$$

Now consider the valences of the vertices of the underlying network. There are four vertices of valence 2 (the corners), and no vertices of valence greater than 4. Suppose there are ψ_3, ψ_4, vertices of valence 3, 4 respectively; by usual counting arguments

$$v = 4 + \psi_3 + \psi_4,$$

$$2e = 4.2 + \psi_3 . 3 + \psi_4 . 4,$$

giving
$$2e - 3v + 4 = \psi_4,$$

and comparing with (4·3)
$$\psi_4 = r - c_0 + 2. \tag{4·4}$$

In what follows it will be necessary that the cuts are counted in such a way that the right-hand side of (4·4) is zero; in the present system this is so if and only if $\psi_4 = 0$. Accordingly we introduce the convention that at each vertex of valence 4 the *horizontal* cut involved will be regarded as two cuts, one lying to the left of the vertex and the other to the right. If now, m, n, c denote the number of cuts counted in this way, $m = m_0 + \psi_4$, $n = n_0$, $c = c_0 + \psi_4$ so that always

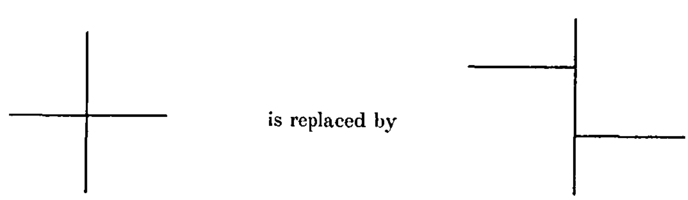
$$r - c + 2 = 0,$$

or, in the form we shall require it

$$(m-1) + (n-1) = r. \tag{4·5}$$

We could have introduced this convention at the start, but its motivation would then have been obscure, and in any event the previous work would still have been necessary to establish the vital statement (4·5).

Of course, in a 'general' rectangulation a vertex of degree four can always be abolished by the process illustrated:



without altering the configuration of the rectangles. But this is clearly not possible if there are further constraints—if, for example, the rectangles are required to be squares.

5. *The matrices associated with a rectangulation.* We shall assume, without loss of generality, that a given open rectangle $\mathscr{R}$ lies in the first quadrant of the plane and is bounded in part by the coordinate axes. Given a rectangulation $\mathscr{C}$ of $\mathscr{R}$ we label the m horizontal cuts in order of distance from the horizontal axis, the first horizontal cut being the upper boundary of $\mathscr{R}$, and so forth. If two horizontal cuts are at the same level we take the one to the left first. In a similar way we label the n vertical cuts. The $r-1$ interior rectangles are labelled in an arbitrary manner.

We define an $m \times (r-1)$ matrix $\bar{H} = (h_{ij})$ associated with the given rectangulation as follows:

$$h_{ij} = \begin{cases} +1 & \text{if the } i\text{th horizontal cut is the upper boundary of the } j\text{th rectangle,} \\ -1 & \text{if the } i\text{th horizontal cut is the lower boundary of the } j\text{th rectangle,} \\ 0 & \text{otherwise.} \end{cases}$$

An $n \times (r-1)$ i-matrix $\bar{V} = (v_{ij})$ is defined by replacing the words: horizontal, upper, lower; by: vertical, right-hand, left-hand. Our first remark is that the rank of $\bar{H}$ is $m-1$; for if it were less than $m-1$ there would be a proper subset Σ of the rows of $\bar{H}$ which is linearly dependent. This would mean that the two non-zero entries of each column are both in Σ or both in the complementary set of rows Σ', so that the interior rectangles are divided into two sets in such a way that any rectangle of the first set has no common horizontal boundary with any rectangle of the second set. But this is clearly impossible; and a similar argument holds for $\bar{V}$, hence

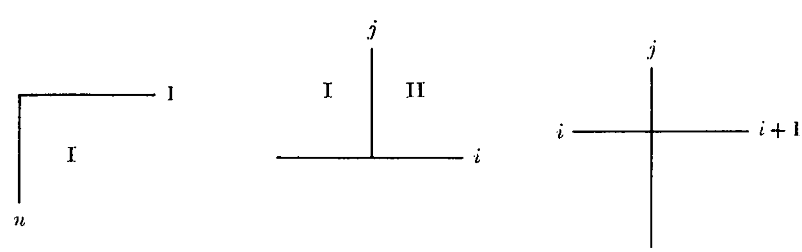
$$\rho(\bar{H}) = m-1, \quad \rho(\bar{V}) = n-1.$$

There is an important relation between the matrices $\bar{H}$ and $\bar{V}$. If $\bar{H}\bar{V}^t = X$ then we show that X is the $m \times n$ matrix with just four non-zero entries:

$$x_{11} = +1, \quad x_{1n} = -1, \quad x_{m1} = -1, \quad x_{mn} = +1.$$

Consider
$$x_{ij} = \sum_{k=1}^{r-1} h_{ik} v_{jk}.$$

A summand $h_{ik}v_{jk}$ is non-zero if and only if both h_{ik} and v_{jk} are non-zero, that is, if and only if horizontal cut i and vertical cut j are both boundaries of rectangle k. For a given i and j there can be either one or two rectangles with this property, some typical configurations being illustrated below.



It is now immediate that $x_{ij} = 0$ except in the four cases detailed above.

We now augment $\bar{H}$ and $\bar{V}$ by writing $H = (\bar{H} \vdots w)$, $V = (\bar{V} \vdots -w)$ where w is the vector $(1, 0, \ldots, 0, -1)^t$, it being understood that w has the apt number of rows whenever it occurs. (We may think of the extra column in H and V as representing the whole rectangle $\mathscr{R}$). It follows from the fact that $\bar{H}\bar{V}^t = X$, and (4·5), that $\{H, V\}$ is a dual pair, so from Theorem I we have

$$\mu(H) = \mu(V). \tag{5·1}$$

THEOREM II. *If $\bar{H}$, $\bar{V}$ are the matrices associated with any rectangulation then*

$$\mu(\bar{H}) + \epsilon(\bar{H}) = \mu(\bar{V}) + \epsilon(\bar{V}), \tag{5·2}$$

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where $\epsilon(\bar{H})$ *is the determinant of the submatrix of* $\bar{H}\bar{H}^t$ *formed by deleting the first and last rows and columns.*

Proof. Since

$$H_0 H_0^t = \bar{H}_0 \bar{H}_0^t + \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

expanding determinants by the first row gives

$$\mu(H) = \mu(\bar{H}) + \epsilon(\bar{H}), \tag{5·3}$$

whence (5·1) implies the result.

6. *Squared rectangles and squared squares.* Suppose we are given a rectangulation with associated matrices $\bar{H}$ and $\bar{V}$. Let the height of the ith horizontal cut be h_i and define v_j similarly; the convention stated at the beginning of section 5 implies that

$$h_1 > 0, \quad v_1 > 0, \quad h_m = v_n = 0.$$

We have column vectors $h = (h_i)$ and $v = (v_j)$.

Let $y = \bar{H}^t h$. For a fixed i, h_{ij} is non-zero only when $j = p$ and $j = q$ say, p and q being the upper and lower horizontal cuts bounding the ith rectangle. So y_i is the height of the ith rectangle and the condition for the interior rectangles to be squares is that $\bar{H}^t h = \bar{V}^t v$. We now show that any rectangulation of a rectangle determines an essentially unique squared rectangle.

THEOREM III. *If* $\bar{H}$ *and* $\bar{V}$ *are the matrices associated with a rectangulation, then:*

(1) *There are vectors* h^* *and* v^* *such that*

$$\bar{H}^t h^* = \bar{V}^t v^*,$$

$$h_1^* \geqslant h_i^* \geqslant h_m^* = 0, \quad v_1^* \geqslant v_j^* \geqslant v_n^* = 0.$$

(2) *If* h, v *are any vectors such that* $\bar{H}^t h = \bar{V}^t v$, $h_1 > 0$, $v_1 > 0$, $h_m = v_n = 0$, *then* $h = \gamma h^*$, $v = \gamma v^*$ *for some number* $\gamma > 0$.

(3) $\epsilon(\bar{H}) = \mu(\bar{V}), \quad \epsilon(\bar{V}) = \mu(\bar{H}).$ (6·1)

Proof. Let $E^+(\bar{H})$ denote $(\operatorname{adj} \bar{H}_0 \bar{H}_0^t \vdots 0)^t$ as in section 2 and let h^* be the first column of $E^+(\bar{H})$; define v^* to be the first column of $E^+(\bar{V})$. These definitions and the result (2·7) give the inequalities in part (1) directly. Let

$$x = \begin{pmatrix} \bar{H}^t h^* \\ \hline -\mu(\bar{H}) \end{pmatrix}, \quad x' = \begin{pmatrix} \bar{V}^t v^* \\ \hline -\epsilon(\bar{V}) \end{pmatrix}.$$

Then using (2·8), (5·3), and the fact that $\bar{H}\bar{V}^t = X$, a simple calculation shows that

$$Hx = Hx' = 0,$$

$$Vx = \mu(H)\,w, \quad Vx' = \mu(V)\,w.$$

Since $\{H, V\}$ is a dual pair, $\mu(H) = \mu(V)$ and $\ker H \cap \ker V = \{0\}$; since $H(x-x') = 0$, $V(x-x') = 0$. it thus follows that $x = x'$, giving parts (1) and (3) of the theorem.

For part (2), suppose $\bar{H}^t h = \bar{V}^t v$, then

$$\bar{H}\bar{H}^t h = \bar{H}\bar{V}^t v = Xv = v_1 w.$$

so by (2·9) we see that if $h_m = 0$, then

$$h = [v_1/\mu(\bar{H})]h^*, \tag{6·2}$$

and similarly $v = [h_1/\mu(\bar{V})]\,v^*$. Equating the first entries in (6·2) gives

$$h_1 = \frac{v_1}{\mu(\bar{H})} h_1^* = \frac{v_1}{\mu(\bar{H})} \epsilon(\bar{H}) = \frac{v_1}{\mu(\bar{H})} \mu(\bar{V}),$$

that is

$$\frac{h_1}{\mu(\bar{V})} = \frac{v_1}{\mu(\bar{H})} = \gamma > 0. \tag{6·3}$$

Finally, if we have a *squared square* then there is the additional requirement that $h_1 = v_1$, so that from (6·1) and (6·3) we deduce:

THEOREM IV. *If $\bar{H}$, $\bar{V}$ are the matrices associated with a squared square then*

$$\mu(\bar{H}) = \mu(\bar{V}) = \epsilon(\bar{H}) = \epsilon(\bar{V}).$$

It turns out that although any rectangulation of a rectangle will determine a squared rectangle (Theorem III) the additional condition for a squared *square* (Theorem IV) is very restrictive.

It remains to remark on the interpretation of Theorem III. One would like to be able to say that the h^* and v^* vectors describe a squaring with incidence matrices $\bar{H}$ and $\bar{V}$, whose underlying network is the same as that of the original rectangulation determining $\bar{H}$ and $\bar{V}$. Unfortunately, when we come to associate a squaring with the system $\{\bar{H}, \bar{V}, h^*, v^*\}$, we find that this can be done only via a normalization process (for instance, by using the algebraic counterparts of the geometric procedures of ((1), p. 320–322)). This process implies, for example, that if $\bar{H}^t h^*$ has some zero entries then the corresponding rectangles will not appear in the squaring. However, our Theorem III does provide a precise and constructive statement of as much as is generally true, and in this respect it represents an improvement on the methods of (1).

7. *Applications to network theory.* A directed network $\mathscr{N}$ with m vertices and r edges has an incidence matrix $A = A(\mathscr{N})$ which is an $m \times r$ i-matrix in our sense. The network is connected if and only if $\rho(A) = m-1$. The edges corresponding to a non-singular $(m-1)$-square submatrix of A_0 can easily be seen to form a spanning tree in $\mathscr{N}$, so that the integer $\mu(A)$ can be interpreted as the number of spanning trees in $\mathscr{N}$. If $\mathscr{N}$ is a planar network and $\mathscr{M}$ is its dual then the incidence matrices of $\mathscr{N}$ and $\mathscr{M}$ are dual in the sense of section 3, so that we have given a proof of the well-known fact that $\mathscr{N}$ and $\mathscr{M}$ have the same number of spanning trees.

Any m-vector x can be thought of as assigning a 'potential' to each vertex of $\mathscr{N}$ and $A^t x$ is then an r-vector which assigns a corresponding 'flow' to each edge of $\mathscr{N}$. The m-vector $AA^t x$ represents the accumulation of flow at the vertices; since $u^t AA^t x = 0$ the algebraic sum of these accumulations is zero. If the network is connected, and a flow α enters at vertex j and leaves at vertex m, the accumulations at other vertices being zero, then (2·8) tells us that a potential vector for this flow is $[\alpha/\mu(A)]\,e$, where e is the jth column of $E^{+}(A)$. Thus the work of that section may be regarded as proving the existence of a positive potential vector for any flow of this sort.

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