Intersection Matrices for Linear Graphs

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1. Distance-transitive Graphs

Let Γ be a finite, connected, undirected, linear graph without loops or parallel edges. If V is the vertex set of Γ, an *automorphism* of Γ is a permutation g of V such that two vertices are joined by an edge of Γ if and only if their images under g are joined by an edge. The set of all such g forms the *automorphism group* G of Γ. Also, recall that the *distance* $d(x, y)$ between two vertices $x, y \in V$ is the number of edges in the shortest path joining x and y; and the *diameter* d of Γ is the maximum of such distances.

We shall be concerned with the transitivity properties of the permutation group (V, G). If (V, G) is transitive in the usual sense, then we say that Γ is a *vertex-transitive* graph; this is a useful notion, but not a very strong one, for we cannot expect any classification of vertex-transitive graphs. On the other hand, if we require the permutation group (V, G) to be 2-fold transitive in the usual sense, this implies (since an automorphism must preserve distances) that each pair of vertices of Γ must be at the same distance, and so Γ must be a complete graph K_n. The present paper is concerned with a concept midway between the two just mentioned, which turns out to have interesting consequences. In fact, we shall sneak in at the back door of a theory developed by J. S. Frame, H. Wielandt and D. G. Higman, for which a basic reference is [5].

DEFINITION. *A graph Γ with vertex set V and automorphism group G will be said to be distance-transitive if, for any two ordered pairs (u, v), $(x, y) \in V \times V$ such that $d(u, v) = d(x, y)$ there is an automorphism $g \in G$ such that*

$$g(u) = x \ \textit{and}\ g(v) = y.$$

Briefly, this says that (V, G) is as 2-fold transitive as possible, given that automorphisms preserve distances.

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2. The Incidence Algebra

For any graph Γ with $|V| = n$ and diameter d we may define $d + 1$ *incidence matrices* $A_0, A_1, \ldots A_d$ in $M_n(\mathbf{C})$ by

$$(A_\alpha)_{ij} = \begin{cases} 1 & \text{if } d(i,j) = \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where we have labelled the rows and columns of the matrices with the vertices of Γ in some way. It is clear that $A_0 = I$, and

$$A_0 + A_1 + \ldots + A_d = J$$

where each entry of J is 1. Also $\{A_0, A_1, \ldots A_d\}$ is a linearly independent set over $M_n(\mathbf{C})$ and so forms a basis for a vector space which will be denoted by $U(\Gamma)$. We shall show that if Γ is distance-transitive, $U(\Gamma)$ is an algebra.

LEMMA. *$(A_\alpha A_\beta)_{ij}$ depends only on α, β and $\gamma = d(i,j)$, not on i and j individually, provided that Γ is distance-transitive.*

Proof. From the definition it follows that $(A_\alpha A_\beta)_{ij}$ is the number of vertices k such that $d(i,k) = \alpha$ and $d(j,k) = \beta$. If Γ is distance-transitive, for any other pair i', j', such that $d(i,j) = d(i',j') = \gamma$ there is some $g \in G$ taking i to i' and j to j'. Thus for each vertex k at the correct distances from i and j, gk is at the correct distances from i' and j'.

THEOREM. *If Γ is distance-transitive $U(\Gamma)$ is an algebra with "structure constants"*

$$\sigma^{\gamma}{}_{\alpha\beta} = (A_\alpha A_\beta)_{ij}, \text{ where } d(i,j) = \gamma,$$

with respect to the basis $\{A_0, A_1, \ldots A_d\}$.

Proof. If $d(i,j) = \gamma$ the matrix A_γ is the only one of the basic set with 1 in position ij. Thus

$$(A_\alpha A_\beta)_{ij} = \sigma^{\gamma}_{\alpha\beta}(A_\gamma)_{ij} = \left(\sum_\gamma \sigma^{\gamma}_{\alpha\beta} A_\gamma\right)_{ij},$$

that is,

$$A_\alpha A_\beta = \sum_\gamma \sigma^{\gamma}_{\alpha\beta} A_\gamma,$$

which is the required result.

A useful way of interpreting the structure constants is as follows. Choose one vertex $v^{(0)} \in V$ and arrange the remaining vertices according to distance from $v^{(0)}$:

$$v^{(0)}; v_1^{(1)}, \dots v_{k_1}^{(1)}; v_1^{(2)}, \dots v_{k_2}^{(2)}; \dots; v_1^{(d)}, \dots v_{k_d}^{(d)};$$

where

$$d(v^{(0)}, v_j^{(i)}) = i \text{ for } j = 1, \dots k_i.$$

We may call the set of vertices $\{v_j^{(i)}\}$ for a fixed i, the ith *circle* centred on $v^{(0)}$. If we label the rows and columns of A_α in this order then it is partitioned into $(d+1)^2$ blocks, and the entries in block (i, j) tell us which vertices of circle i are distant α from which vertices of circle j. The hypothesis of distance-transitivity now implies that the sum of a column of block (i, j) is independent of the particular column chosen and of the initial choice of $v^{(0)}$; the column sum is $\sigma_{\alpha i}^{j}$.

3. Intersection Matrices

Henceforth Γ will always denote a distance-transitive graph, and the rows and columns of the matrices A_α will be arranged as in the previous paragraph. It follows that every matrix in the algebra $U(\Gamma)$ has the property that in each block the column sum is constant. For $X \in U(\Gamma)$ define $\phi(X) \in M_{d+1}(\mathbf{C})$ to be the $(d+1) \times (d+1)$ matrix obtained by replacing each block in X by its column sum.

LEMMA. *$\phi: U(\Gamma) \to M_{d+1}(\mathbf{C})$ is an algebra monomorphism.*

Proof. It is clear that ϕ is a vector-space homomorphism; that $\phi(XY) = \phi(X)\,\phi(Y)$ may be proved directly with a little labour. To see that ϕ is a monomorphism consider the images of the basic set; we found in the previous section that if $B_\alpha = \phi(A_\alpha)$ then

$$(B_\alpha)_{ij} = \sigma_{\alpha i}^{j}.$$

(Note that the rows and columns of B_α are labelled 0, 1, ... d.)
Now, directly from the definition of the structure constants,

$$\sigma_{\alpha 0}^{j} = \delta_{\alpha j} \text{ (Kronecker delta)},$$

and so the top row of B_α is non-zero only in column α. Thus $\{B_0, B_1, \dots B_d\}$ is a linearly independent set, so that ϕ is a monomorphism.

THEOREM. *For each $\alpha = 0, 1, \dots d$, A_α and B_α have the same set of eigenvalues.*

Proof. By the lemma, A_α and B_α have the same minimum polynomial.

The matrices B_α are called intersection matrices; they reduce the study of distance-transitive graphs from a problem of size n to one of size $d+1$, where $d + 1 \ll n$.

We concentrate on A_1 and B_1, which will be abbreviated to A and B. B takes the *tridiagonal* form

$$B = \begin{bmatrix} 0 & 1 & & & & & \\ k_1 & a_1 & b_2 & & & \text{O} & \\ & c_1 & a_2 & \cdot & & & \\ & & c_2 & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & \text{O} & & & \cdot & \cdot & b_d \\ & & & & & c_{d-1} & a_d \end{bmatrix}$$

which we shall write

$$B = \begin{bmatrix} * & 1 & b_2 \dots b_{d-1} & b_d \\ 0 & a_1 & a_2 \dots a_{d-1} & a_d \\ k_1 & c_1 & c_2 \dots c_{d-1} & * \end{bmatrix}.$$

Recalling the last paragraph of the previous section, this tells us that each vertex in circle i is adjacent to b_i, a_i, c_i vertices in circles $i - 1$, i, $i + 1$, respectively.

Example. Let Q_n be the edge-vertex graph of the n-dimensional cube. Precisely, the vertices of Q_n are the 2^n symbols $(\varepsilon_1, \dots \varepsilon_n)$ where each $\varepsilon_j = 0$ or 1, and two vertices are joined by an edge wherever their symbols differ in just one coordinate. It is straightforward to prove that Q_n is distance-transitive; if $v^{(0)} = (0, 0, \dots 0)$ then the vertices in circle i are those for which $\Sigma\, \varepsilon_j = i$, and B is

$$\begin{bmatrix} * & 1 & 1 & \dots & n-1 & n \\ 0 & 0 & 0 & \dots & 0 & 0 \\ n & n-1 & n-2 & \dots & 1 & * \end{bmatrix}.$$

It is not quite obvious that the eigenvalues of B are

$$n, n-2, n-4, \dots, -n+2, -n,$$

but this fact gives immediately the "Hoffmann polynomial" of Q_n, the first few of which were worked out in [7].

4. The Characterisation Problem

We consider the question: which matrices can be the "B" matrix of some distance-transitive graph? Several necessary conditions are plain.

(1) B is tridiagonal with non-negative integer entries.

(2) The entries above and below the main diagonal are strictly positive (otherwise Γ would be disconnected).

(3) Each column of B sums to k_1, the valency of Γ, (for distance-transitivity implies constant valency).

Further necessary conditions involve the theorem of the previous section stating that A and B have the same set of eigenvalues. First we notice some elementary facts.

LEMMA. *If A and B are the first incidence and intersection matrices of some distance-transitive graph Γ then:*

(*a*) *the eigenvalues of B all have multiplicity* 1;

(*b*) *the valency $k = k_1$ of Γ is an eigenvalue of A and B, the corresponding eigenvectors being $u = [1, 1, \dots 1]^t$ and $v = [1, k_1, k_2, \dots k_d]^t$, where k_i is the number of vertices in circle i;*

(*c*) *k has multiplicity* 1 *as an eigenvalue of A and all other eigenvalues λ satisfy $|\lambda| \leqslant k$.*

Proof. (a) This is a standard result on tridiagonal matrices [9, p. 155].

(b) $Au = ku$ immediately, and if we partition u in the same way as A then the corresponding vector of column sums is v, thus $Bv = kv$.

(c) This is the so-called Perron–Frobenius theorem [9, p. 124].

Finally we notice that each eigenvalue λ of B is also an eigenvalue of A and so occurs with some multiplicity $m(\lambda)$ as an eigenvalue of A. Remarkably, in the case under consideration, $m(\lambda)$ can be computed from B alone.

THEOREM. *If A and B correspond to same distance-transitive graph and have common eigenvalue set $\{k, \lambda_1, \dots \lambda_d\}$, and if the corresponding eigenvectors of B with initial entry* 1 *are $\{v, v_1, \dots v_d\}$ then the multiplicity of λ_i os an eigenvalue of A is*

$$m(\lambda_i) = \frac{u^t v}{u_i{}^t v_i}$$

where $(u_i)_j = (v_i)_j / k_j$. Explicitly

$$m(\lambda_i) = \sum k_j \cdot \left(\sum (v_i)_j{}^2 / k_j \right)^{-1}.$$

Proof. The only proof known to the author at the moment is algebraically tedious and will not be given here.

The foregoing results give two more necessary conditions for B to arise from some distance-transitive graph.

(4) The unique eigenvector v of B satisfying $Bv = kv$ and $v_0 = 1$ must have positive integer entries.

(5) The numbers $m(\lambda_i)$ computed from v and the other eigenvectors of B as in the theorem must be positive integers.

If B satisfies conditions 1–5 we shall say that it is *feasible*.

5. First Application: The Problem of Cages

Definition. *A graph is a (k, γ)-cage if it has valency k, girth γ, and no graph with fewer vertices has these properties. (We take $k \geqslant 3$, $\gamma \geqslant 3$ to avoid trivial cases.)*

Lemma. *A (k, γ)-cage has at least $v_0(k, \gamma)$ vertices, where*

$$v_0(k,\gamma) = \begin{cases} 1 + k + k(k-1) + \dots + k(k-1)^{(\gamma-3)/2} & \text{if } \gamma \text{ is odd,} \\ 2[1 + (k-1) + \dots + (k-1)^{(\gamma/2)-1}] & \text{if } \gamma \text{ is even.} \end{cases}$$

Proof. [13, p. 70].

If we define a *minimal* cage to be a (k, γ)-cage C with the smallest possible number $v_0(k, \gamma)$ of vertices, then although C may not be distance-transitive, the entries of the matrix B as interpreted in section 3 are defined and independent of the vertices chosen. We have

$$B = \begin{bmatrix} * & 1 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & k-1 \\ k & k-1 & \dots & k-1 & * \end{bmatrix} \quad \text{if } \gamma \text{ is odd,}$$

while

$$B = \begin{bmatrix} * & 1 & \dots & 1 & k \\ 0 & 0 & \dots & 0 & 0 \\ k & k-1 & \dots & k-1 & * \end{bmatrix} \quad \text{if } \gamma \text{ is even.}$$

The analysis of section 4 can be carried out under the slightly weaker conditions obtaining here, so that the feasibility of the matrix B is necessary for the existence of a minimal (k, γ)-cage. If can be shown that, when γ is even, only $\gamma = 4, 6, 8, 12$ are feasible, while $\gamma = 3$ is always possible (complete graphs) but for $\gamma = 5$, only $k = 3, 7$, and 57 are feasible. For these results, by various means, and some constructions of minimal cages see [1], [3], [8], [11]. We return briefly to the case $\gamma = 5$ in section 7.

6. Second Application: Trivalent Graphs

If B arises from some trivalent distance-transitive graph then the three non-zero entries in any column (except the first and last) are

$$\begin{matrix}1\\0\\2\end{matrix} \quad \text{or} \quad \begin{matrix}1\\1\\1\end{matrix} \quad \text{or} \quad \begin{matrix}2\\0\\1.\end{matrix}$$

Thus it is a reasonably simple matter to apply the feasibility tests to such B of small size. There are ten feasible matrices for $d \leqslant 5$, and each one in fact corresponds to a graph, as shown in the following table.

d	B	$k_0, k_1, \dots k_d$	$\lvert V\rvert$	Eigenvalues and multiplicities	Graph, references
2	* 1 0 2 3 *	1, 3	4	3, −1 1, 3	K_4
3	* 1 3 0 0 0 3 2 *	1, 3, 2	6	3, 0, −3 1, 4, 1	$K_{3,3}$
3	* 1 1 0 0 2 3 2 *	1, 3, 6	10	3, 1, −2 1, 5, 4	Petersen's graph [13, p. 74]
4	* 1 2 3 0 0 0 0 3 2 1 *	1, 3, 3, 1	8	$3, 1, -1, -\sqrt{3}$ 1, 3, 3, 1	Q_3
4	* 1 1 3 0 0 0 0 3 2 2 *	1, 3, 6, 4	14	$3, \sqrt{2}, -\sqrt{2}, -3$ 1, 6, 6, 1	Heawood's graph [13, p. 61]
5	* 1 1 2 3 0 0 0 0 0 3 2 2 1 *	1, 3, 6, 6, 2	18	$3, \sqrt{3}, 0, -\sqrt{3}, -3$ 1, 6, 4, 6, 1	Pappus's graph [2, p. 434]
5	* 1 1 1 2 0 0 0 1 1 3 2 2 1 *	1, 3, 6, 12, 6	28	$3, 2, \sqrt{2}-1, -1, -\sqrt{2}-1$ 1, 8, 6, 7, 6	[12], [14, p. 237]
5	* 1 1 1 3 0 0 0 0 0 3 2 2 2 *	1, 3, 6, 12, 8	30	3, 2, 0, −2, −3 1, 9, 10, 9, 1	Tutte's 8-cage [13, p. 76]
6	* 1 1 2 2 3 0 0 0 0 0 0 3 2 2 1 1 *	1, 3, 6, 6, 3, 1	20	3, 2, 1, −1, −2, −3 1, 4, 5, 5, 4, 1	Desargues's graph [2, p. 435]
6	* 1 1 1 2 3 0 0 1 1 0 0 3 2 1 1 1 *	1, 3, 6, 6, 3, 1	20	$3, \sqrt{5}, 1, 0, -2, -\sqrt{5}$ 1, 3, 5, 4, 4, 3	Dodecahedron

7. Third Application: Graphs of Diameter 2

For a distance-transitive graph of valency k and diameter 2 we have

$$B = \begin{bmatrix} 0 & 1 & 0 \\ k & a & b \\ 0 & k-a-1 & k-b \end{bmatrix}.$$

The eigenvalues of B are k and the roots of a quadratic equation, so that the feasibility conditions can be formulated explicitly in terms of the parameters k, a, b. The author hopes to collect the known results on such graphs in a later paper; for the moment it suffices to notice that great interest attaches to this topic because many of the classical simple groups have representations as automorphism groups of graphs of this kind [4]. Further, several of the newly discovered simple groups occur in this way: that of Higman–Sims [6] corresponds to $k = 22$, $a = 0$, $b = 6$, while that of McLaughlin [10] corresponds to $k = 112$, $a = 30$, $b = 56$. To round off this topic we state without proof a trivial extension of the famous result of Hoffmann and Singleton [8].

THEOREM. *The matrix B is feasible with $a = 0$ (that is, the girth is 4 or 5) and a given $b \neq 2, 4, 6$ only for a finite list of k in each case. For example,*

$$\begin{aligned} b = 1&: \quad k = 3, 7, 57. \\ b = 3&: \quad k = 3, 21, 183. \\ b = 5&: \quad k = 5, 55, 155, 1155. \end{aligned}$$

If $b = 2, 4, 6$ there is an infinite list of feasible k in each case.

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