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Spanning Trees of Dual Graphs

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It is a well-known result that if G and G^* are dual planar graphs and T is a spanning tree for G, then the complement of the edges dual to T is a spanning tree for G^*. The purpose of this note is to show how ideas of Edmonds [1], Gustin [2], and Youngs [3] can be used to formulate precisely the generalization of this result to graphs imbedded in any orientable surface. In the course of the work several new interpretations of standard graph-theory concepts will be presented.

1. DUALITY

In the usual sense, an undirected linear graph consists of a set V of vertices and a set E of edges together with a mapping which assigns to each edge an unordered pair of vertices. This formulation is difficult to handle in abstract terms, and so we use here the following notion.

DEFINITION 1. A *graph* is an ordered quadruple (E, V, λ, τ) where E and V are sets (finite for our purposes), $\lambda: E \to V$ is a surjection, and $\tau: E \to E$ an involution.

Intuitively, E consists of the edges taken twice, once in each direction, V is the vertex set, λ assigns to each directed edge its initial vertex, and τ reverses directions. Thus $\lambda\tau$ assigns to each directed edge its final vertex, and an edge $e \in E$ for which $e = \tau(e)$ will be called a *loop*.

Now if a graph (in the geometric sense) is imbedded in a closed orientable 2-manifold, then the orientation at each vertex determines a cyclic permutation of the incident edges; conversely, given such a set of cyclic permutations we can construct such a 2-manifold in which the graph is imbedded, following Edmonds [1]. Accordingly we introduce this idea formally. (Throughout this note, $\alpha\beta$ denotes the composite $X \xrightarrow{\beta} Y \xrightarrow{\alpha} Z$.)

DEFINITION 2. A *rotation* ρ on a graph $G = (E, V, \lambda, \tau)$ is a permutation $\rho: E \to E$ such that $\lambda\rho = \lambda$.

The condition $\lambda\rho = \lambda$ ensures that ρ induces, for each $v \in V$, a permutation of $\lambda^{-1}(v)$, the set of directed edges starting from v. It will clarify the situation if ρ is not yet required to be cyclic on each $\lambda^{-1}(v)$, but it will be necessary to have a word to signify when this is the case.

DEFINITION 3. A rotation ρ on G is *smooth* if ρ is cyclic on $\lambda^{-1}(v)$, for each $v \in V$.

Suppose now that a graph G and a rotation ρ on it are given. Then let

$E^* = E$;

$V^* = E/\rho\tau$, the set of orbits of $\rho\tau: E \to E$;

$\lambda^* : E^* \to V^*$ be the quotient mapping $E \to E/\rho\tau$;

$\tau^* = \tau$;

$\rho^* = \rho\tau$.

It is immediate that $G^* = (E^*, V^*, \lambda^*, \tau^*)$ is a graph and that ρ^* is a smooth rotation on it. Intuitively, the orbits of $\rho\tau$ are the edges belonging to a "face" of G when it is imbedded using the rotation ρ.

DEFINITION 4. The *dual* of a pair (G, ρ) consisting of a graph G with a rotation ρ on it, is the pair (G^*, ρ^*) defined above.

The duality construction can be iterated to give a double dual (G^{**}, ρ^{**}) which turns out to be in natural one-one correspondence with (G, ρ) provided ρ is smooth.

2. PERMUTATIONS

If $\pi: X \to X$ is a permutation of a finite set X we shall write $c(\pi) = | X/\pi |$ for the number of orbits of π; when the cyclic ordering of the orbit sets induced by π is relevant we shall refer to the *cycles* of π. (Some care is necessary in using the standard notation which ignores orbits of length 1: thus if $X = \{1, 2, 3\}$ and $\pi = (12)$ then $c(\pi) = 2$, not 1.) A *transposition* is a permutation with just one orbit of length 2, all others being of length 1. Thus the involution $\tau: E \to E$ in the definition of a graph is the composite of a number of transpositions, and if there are no loops then $c(\tau)$ is the number of undirected edges.

The following trivial observation is basic.

LEMMA 1. *If* $\pi: X \to X$ *is a permutation and* $\sigma: X \to X$ *is a transposition written* $\sigma = (xx')$ *then*

$$c(\pi\sigma) = \begin{cases} c(\pi) + 1, & \text{if } x \text{ and } x' \text{ are in the same orbit of } \pi, \\ c(\pi) - 1, & \text{if } x \text{ and } x' \text{ are in different orbits of } \pi. \end{cases}$$

3. THE CHARACTERISTIC AND ITS APPLICATIONS

We begin the main argument with a definition that generalizes the Euler characteristic.

DEFINITION 5. If ρ is a rotation on a graph G then the *characteristic* of (G, ρ) is the number

$$\chi(G, \rho) = |V| - c(\tau) + c(\rho\tau).$$

A trivial consequence is:

LEMMA 2. *If* ρ *is a smooth rotation on* G *then*

$$\chi(G, \rho) = \chi(G^*, \rho^*) = c(\rho) - c(\tau) + c(\rho\tau).$$

Thus, in order to compute the characteristic, it is necessary to discover how the number of orbits of ρ changes as ρ is composed with successive transpositions of the involution τ. .It will be convenient, in order to clarify the exposition, to suppose henceforth that G has no loops.

Let S be a symmetric subset of E, by which we mean that $\tau(S) = S$, and let τ_S denote the permutation of E consisting of the transpositions of the elements of S; that is

$$\tau_S(e) = \begin{cases} \tau(e), & \text{if } e \in S, \\ e, & \text{if } e \notin S. \end{cases}$$

In our study of the process in which ρ becomes $\rho\tau$ through composition with transpositions we try first to reduce ρ to a single orbit. If there is a symmetric subset S of E such that $c(\rho\tau_S) = 1$ then we shall call S a spanning subset of (G, ρ). Looking back to Lemma 1 we see that such a τ_S must consist of at least $c(\rho) - 1$ transpositions, and that, since the effect of composition depends only on the orbit sets and not their cyclic structure, if S spans (G, ρ) then it also spans (G, ρ') where ρ and ρ' have the same orbit sets. In particular, this is the case if ρ and ρ' are two smooth rotations on G.

DEFINITION 6. A graph G is *connected* if, for some smooth rotation ρ on G (and thus for all such), there is a τ_S consisting of $c(\rho) - 1$ transpositions such that $c(\rho\tau_S) = 1$. S is called a *spanning tree* for G.

It is clear that any spanning subset of G contains a spanning tree for G. It is an instructive exercise to show that these definitions are equivalent to the usual, more intuitively meaningful, ones, but this is not essential for our present purposes.

Now if G is a connected graph without loops and S is a spanning tree for G, put $R = E - S$ so that R is symmetric and τ_R consists of $c(\tau) - c(\rho) + 1$ transpositions. In order to find $c(\rho\tau)$ it is sufficient, since $\rho\tau = \rho\tau_S\tau_R$, to examine the changes which take place when the successive transpositions of τ_R are composed with the single cycle $\rho\tau_S$. Suppose that, when this is done, in some order, the number of orbits increases by 1 at p stages and decreases by 1 at m stages. Then

$$p + m = c(\tau) - c(\rho) + 1,$$
$$p - m + 1 = c(\rho\tau),$$

whence

$$\chi(G, \rho) = c(\rho) - c(\tau) + c(\rho\tau) = 2 - 2m,$$

where m is an integer $\geqslant 0$. Thus we recover the classical result that the Euler characteristic of a graph imbedded in orientable surface must be an even integer $\leqslant 2$; we also have a new interpretation of the genus (m) of an orientable surface. It is also immediate that p and m do not depend on the spanning tree S or the order of composition of the transpositions in τ_R. Furthermore, since

$$c(\rho\tau\tau_R) = c(\rho\tau_S\tau_R^2) = c(\rho\tau_S) = 1,$$

we see that R spans (G^*, ρ^*) and so R contains a spanning tree for G^* which must have

$$c(\rho\tau) - 1 = p - m = c(\tau_R) - 2m$$

edges. Our final result is:

THEOREM. *If G is a connected graph without loops and ρ is a smooth rotation on G, S any spanning tree for G, then the complement R of S spans (G^*, ρ^*) and contains a spanning tree of G^* which consists of all but $2m$ undirected edges of R, where m is the genus of (G, ρ).*

References

1. J. R. Edmonds, A combinatorial representation for polyhedral surfaces, *Notices Amer. Math. Soc.* **7** (1960), 646.
2. W. Gustin, Orientable embedding of Cayley graphs, *Bull. Amer. Math. Soc.* **69** (1963), 272–275.
3. J. W. T. Youngs, Minimal imbeddings and the genus of a graph, *J. Math. Mech.* **12** (1963), 303–315.