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Automorphisms of Imbedded Graphs

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If a linear graph is imbedded in a surface to form a map, then the map has a group of automorphisms which is a subgroup (usually, a proper subgroup) of the automorphism group of the graph. In this note it will be shown that, for any imbedding of K_n in an orientable surface, the order of the automorphism group of the resulting map is a divisor of $n(n-1)$, and that the order equals $n(n-1)$ if and only if n is a prime power. The explicit construction of imbeddings of K_q, $q = p^m$ with map automorphism group of order $q(q-1)$ gives rise to new types of regular map. There are also tenuous connections with the theory of Frobenius groups.

1. AUTOMORPHISMS AND IMBEDDINGS

Our basic definitions will be as in [1]. Thus a *graph* is a quadruple $G = (E, V, \lambda, \tau)$ where E and V are finite sets, $\lambda: E \to V$ is a surjection, and $\tau: E \to E$ is an involution. A *smooth rotation* on G is a permutation $\rho: E \to E$ such that $\lambda\rho = \lambda$ and the orbits of ρ are the sets $\lambda^{-1}(v)$, $v \in V$. Thus ρ gives a cyclic permutation of each $\lambda^{-1}(v)$. A pair (G, ρ) where G is a graph and ρ is a smooth rotation on G gives rise to a *dual* pair (G^*, ρ^*) in which $E^* = E$, $V^* = E/\rho\tau$, λ^* is the quotient map $E \to E/\rho\tau$, $\tau^* = \tau$, and $\rho^* = \rho\tau$.

DEFINITION 1. An *automorphism* of G is a pair $\alpha = (\alpha_1, \alpha_2)$ of permutations $\alpha_1 : V \to V$, $\alpha_2 : E \to E$ such that

$$\alpha_1\lambda = \lambda\alpha_2 \qquad \text{and} \qquad \alpha_2\tau = \tau\alpha_2 .$$

An *automorphism* of (G, ρ) is an automorphism of G such that

$$\alpha_2\rho = \rho\alpha_2 .$$

Such automorphisms clearly form groups, which will be written aut G

and aut (G, ρ), the latter being a subgroup of the former. It is also quite straightforward to prove:

LEMMA 1. *If ρ is a smooth rotation on G then*

$$\mathrm{aut}(G, \rho) \approx \mathrm{aut}(G^*, \rho^*).$$

For an intuitive explanation of these definitions we recall that a smooth rotation ρ describes an imbedding of G in an orientable surface, in which the edges of a face of the resulting map form an orbit of $\rho\tau$. Thus an automorphism of G preserves only the edge-vertex incidences, while an automorphism of (G, ρ) preserves also the edge-face incidences.

2. THE ORDER OF aut (K_n, ρ)

DEFINITION 2. K_n is the graph with $V = \{0, 1,..., n-1\}$, E the set of ordered pairs of distinct elements of V, and with $\lambda: E \to V$, $\tau: E \to E$ defined by

$$\lambda(ij) = i, \qquad \tau(ij) = (ji).$$

We see that if $\alpha \in \mathrm{aut}\, K_n$, $\alpha_2(ij) = (\alpha_1(i)\, \alpha_1(j))$; hence for each permutation α_1 of $\{0, 1,..., n-1\}$ the permutation α_2 of E is determined, so that aut $K_n \approx S_n$, the symmetric group of degree n. Henceforth, we shall drop the subscripts on α.

Thus for any smooth rotation ρ on K_n, aut(K_n, ρ) is a permutation group of degree $\leqslant n$. Now for each $i \in V$, ρ gives a cyclic permutation of $\lambda^{-1}(i)$, that is, the set

$$\{(i0), (i1),..., (i, i-1)(i, i+1),..., (in)\},$$

which is clearly equivalent to a cyclic permutation of $\{0, 1,..., i-1, i+1,..., n\}$. If this has cycle representation $(c_1 c_2 \cdots c_{n-1})$ then we write ρ_i for the permutation of V, whose cycle representation is

$$(i)(c_1 c_2 \cdots c_{n-1}).$$

If $\alpha \in \mathrm{aut}\, K_n$, that is, α is a permutation of V, then the condition $\alpha\rho = \rho\alpha$ just says that for each $i \in V$ we have

$$\alpha\rho_i\alpha^{-1} = \rho_{\alpha(i)}, \qquad 0 \leqslant i \leqslant n-1.$$

Thus, replacing c by $\alpha(c)$ in the cycle representation of ρ_i gives $\rho_{\alpha(i)}$; this is a necessary and sufficient condition for a given permutation α of $\{0, 1,..., n-1\}$ to belong to $\mathrm{aut}(K_n, \rho)$.

For example, let ρ be the following smooth rotation on K_4:

$$\rho_1 = (1)(230)$$
$$\rho_2 = (2)(103)$$
$$\rho_3 = (3)(201)$$
$$\rho_0 = (0)(132)$$

Then $\alpha = (12)(30)$ is an automorphism of (K_4, ρ) since, for instance, $\alpha\rho_1\alpha^{-1} = (2)(103) = \rho_2$. It should be remarked that the permutation $\gamma = (12)$ is not an automorphism of the map (K_4, ρ) on our definition; in fact $\gamma\rho_i\gamma^{-1} = \rho_{\gamma(i)}^{-1}$ for each $i = 0, 1, 2, 3$, and so we may say that γ reverses orientation. Some authors, such as H. S. M. Coxeter, and F. Harary and W. T. Tutte [4], allow such permutations to be automorphisms of maps, but with the present definitions it is more natural to exclude them. The distinction is a trivial one, since including orientation-reversing automorphisms merely extends the group of automorphisms by an involution, and so multiplies its order by 2.

The first lemma following is crucial, while the second merely serves to throw some light on the constructions which follow.

LEMMA 2. *For any smooth rotation ρ on K_n, a permutation $\alpha \in \mathrm{aut}(K_n, \rho)$ which has two fixed points must be the identity.*

Proof. Suppose $\alpha(i) = i$, $\alpha(j) = j$, $i \neq j$. Then

$$\alpha\rho_i\alpha^{-1}(j) = \rho_{\alpha(i)}(j),$$

i.e.,

$$\alpha\rho_i(j) = \rho_i(j).$$

Thus $\rho_i(j)$ is fixed by α. Similarly $\rho_i^2(j),..., \rho_i^{n-2}(j)$ are fixed by α. But ρ_i is a cyclic permutation on $V - \{i\}$, so that everything is fixed by α.

LEMMA 3. *If $\alpha \in \mathrm{aut}(K_n, \rho)$ has one fixed point i, then α is a power of ρ_i.*

Proof. We have $\alpha\rho_i = \rho_i\alpha$, so that regarding α and ρ_i as permutations of the $n-1$ objects in $V - \{i\}$ we may say that α belongs to the centralizer $C(\rho_i)$ of ρ_i in S_{n-1}. But (see, for example, [6, p. 39]),

$$\begin{aligned}[S_{n-1} : C(\rho_i)] &= \text{number of conjugates of } \rho_i \text{ in } S_{n-1}\\ &= (n-2)!\end{aligned}$$

Thus $|C(\rho_i)| = n - 1$. However it is immediate that $C(\rho_i)$ contains the cyclic group $Z_{n-1}\{\rho_i\}$ generated by ρ_i. Thus

$$C(\rho_i) = Z_{n-1}\{\rho_i\}.$$

In the language of permutation groups [7], Lemma 2 implies that $A = \mathrm{aut}(K_n, \rho)$ is of minimal degree at least $n - 1$. If no element of A except the identity has a fixed point then A is of degree n and minimal degree n, and is called regular of degree n; if each element of A fixes a given point then A is regular of degree $n - 1$. In these cases, the order of A divides n, $n - 1$, respectively. More interesting is the case in which A is of degree n and minimal degree $n - 1$. We first prove by an independent method the following result:

LEMMA 4. *If $A = \mathrm{aut}(K_n, \rho)$ then $|A|$ divides $n(n-1)$. If $|A| = n(n-1)$ then A is a Frobenius group, that is, a transitive group of degree n and minimal degree $n - 1$.*

Proof. The action of A on V induces an action on E as explained at the beginning of this section, and Lemma 1 ensures that the identity is the only operation in this action which has a fixed point. Thus the number of A-orbits in E is (for example, by the well-known Burnside lemma) $|E|/|A| = n(n-1)/|A|$. So $|A|$ divides $n(n-1)$, and if $|A| = n(n-1)$ then there is just one A-orbit in E. This means that, for any i, j, k, l, there is an $\alpha \in A$ such that $\alpha(ij) = (kl)$. Finally, we recall that this is the definition of A being transitive on E, and in fact doubly transitive on V.

LEMMA 5. *If $|\mathrm{aut}(K_n, \rho)| = n(n-1)$ for some smooth rotation ρ on K_n, then n is the power of a prime.*

Proof. This is a consequence of the result [2, p. 172] that a Frobenius group of degree n and order $n(n-1)$ can exist only if n is a prime power.

In the next section we shall present the detailed construction of a Frobenius group of degree q and order $q(q-1)$ for a given prime power q, and show how to construct a smooth rotation ρ on K_q such that $|\mathrm{aut}(K_q, \rho)| = q(q-1)$.

3. A CONSTRUCTION IN FINITE FIELDS

The construction which follows is based on Burnside [2, pp. 182–4], who attributes it to Mathieu [5], although neither of these authors refers explicitly to finite fields. We give the method in detail as it is required in the construction of a suitable smooth rotation.

For a given prime power $q = p^m$ let F denote the finite field of order q. Then $F^* = F - \{0\}$ is, under multiplication, a cyclic group of order $q - 1$, and a generator x of this group is called a primitive element of F. Fix a primitive element of F and define two mappings $a, b: F \to F$ by

$$a(t) = t + x, \qquad b(t) = tx.$$

Then the elements $\{0, x, x^2, \ldots, x^{q-2}, 1\}$ are permuted by a and b, and we have induced permutations α, β of the set $V = \{0, 1, 2, \ldots, q-2, q-1\}$ using the canonical bijection between F and V in the order shown. It is immediate that β has cycle representation

$$\beta = (0)(1\ 2 \cdots q-1)$$

while for $\alpha: V \to V$ we have $\alpha(0) = 1$ and the relation

$$x^k + x = x^{\alpha(k)}, \qquad k \geqslant 1.$$

Now a and b generate a group D of transformations of F of the form

$$d(t) = ty + z, \qquad \text{where} \quad \begin{cases} y = x^r, & 1 \leqslant r \leqslant q-1, \\ z = 0, & \text{or} \quad z = x^s, \quad 1 \leqslant s \leqslant q-1; \end{cases}$$

and there are just $q(q-1)$ distinct transformations of this kind. The obvious direct arguments show that D is of degree q, minimal degree $q - 1$, and is doubly transitive on F. Thus the subgroup Δ of S_q generated by α and β has these properties also.

We now construct a smooth rotation ρ on K_q such that $\alpha, \beta \in \operatorname{aut}(K_q, \rho)$. Put

$$\rho_0 = \beta, \qquad \rho_k = \beta^{k-1}\alpha\beta\alpha^{-1}\beta^{1-k}, \qquad \text{for} \quad k \geqslant 1.$$

Then

$$\beta\rho_0\beta^{-1} = \rho_0\,, \qquad \beta\rho_k\beta^{-1} = \rho_{k+1} = \rho_{\beta(k)}\,, \qquad \text{for} \quad k \geqslant 1,$$

so that $\beta \in \operatorname{aut}(K_q, \rho)$. Also $\alpha\rho_0\alpha^{-1} = \rho_1 = \rho_{\alpha(0)}$. In order to show that $\alpha\rho_k\alpha^{-1} = \rho_{\alpha(k)}$ for $k \geqslant 1$ we compute in F as follows:

$$ab^{k-1}aba^{-1}b^{1-k}a^{-1}(t) = tx + (1 - x)(x + x^k),$$

$$b^{\alpha(k)-1}aba^{-1}b^{1-\alpha(k)}(t) = tx + (1 - x)\, x^{\alpha(k)}.$$

Thus, for $k \geqslant 1$, since $x^k + x = x^{\alpha(k)}$, the two permutations of F on the left-hand side are equal; the same holds for the induced permutations of V, so that $\alpha \in \operatorname{aut}(K_q, \rho)$. We summarize the results obtained:

THEOREM. *For any smooth rotation* ρ *on* K_n *the order of* $\mathrm{aut}(K_n, \rho)$ *is a divisor of* $n(n-1)$. *There is a* ρ *for which* $|\mathrm{aut}(K_n, \rho)| = n(n-1)$ *if and only if n is a prime power.*

4. Illustration: A Map of Type {8, 8}

The tables of Bussey [3] enable us to compute in finite fields. For example, if $q = 3^2$ we may take the elements of F:

$$0, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8,$$

to be

$$0, x, x+1, 2x+1, 2, 2x, 2x+2, x+2, 1,$$

where we compute modulo 3 and $x^2 - x - 1$. We find, in the notation of the previous section,

$$\alpha = (015)(238)(476), \qquad \beta = (0)(12345678).$$

Thus a smooth ρ on K_9 for which $|\mathrm{aut}(K_9, \rho)| = 72$ is:

$$\begin{gathered}
(0)(12345678)\\
(1)(53870462)\\
(2)(64180573)\\
(3)(75210684)\\
(4)(86320715)\\
(5)(17430826)\\
(6)(28540137)\\
(7)(31650248)\\
(8)(42760351)
\end{gathered}$$

We find that there are 9 orbits of $\rho\tau$, each of length 8, so that we have K_9 imbedded in a surface of genus 10 to form a self-dual map with 9 8-valent vertices and 9 octagonal faces.

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