ON TRIVALENT GRAPHS

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1. *Introduction and results*

In this note we determine completely a class of highly symmetrical trivalent graphs. The existence of a similar classification for graphs with valency greater than 3 remains an open problem (compare the remarks of Thompson [5]).

We deal with finite, undirected, connected graphs, and denote the distance between two vertices u and v of such a graph Γ by $d(u, v)$.

Definition. Γ is a *distance-transitive* graph if, for any vertices u, v, x, y, of Γ for which $d(u, v) = d(x, y)$ there is an automorphism h of Γ such that $uh = x$, $vh = y$.

Γ is an *automorphic* graph if it is distance-transitive and the group $H(\Gamma)$ of all automorphisms of Γ acts primitively but not doubly transitively on the vertices of Γ.

THEOREM. *There are exactly* 12 *trivalent distance-transitive graphs, and just* 3 *of them are automorphic.*

The twelve graphs are listed in the third section of this note. A quick description of the three automorphic graphs (iii), (vii) and (xi) is as follows. Let the symbol $(n|s)$, or $(x, n|s)$ if the names of the vertices are important, denote the polygon with vertices $x_0, \ldots, x_{n-1}$ and edges $x_i x_{i+s}$ for $i = 0, 1, \ldots n-1$ where the suffixes are taken modulo n. Then the smallest trivalent automorphic graph (Petersen's graph) may be constructed by taking five edges $x_i y_i$, $i = 0, 1, \ldots 4$, together with the polygons $(x, 5|1)$ and $(y, 5|2)$; the usual representation of this graph is shown in Fig. 1, together with a hopefully self-explanatory symbol for it.

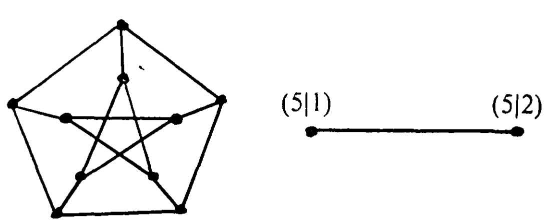

Fig. 1

Similarly, the remaining two automorphic trivalent graphs can be described by the symbols shown in Fig. 2, the first of which (for example) implies that we take seven copies of the graph shown and form a trivalent graph by joining the "free" ends by the polygons indicated.

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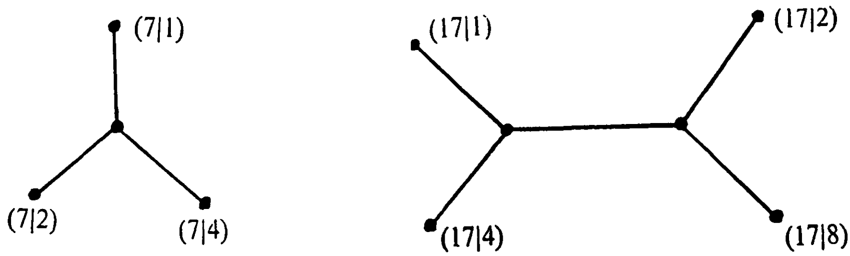

Fig. 2

2. *Theory of distance-transitive graphs*

Associated with every distance-transitive graph Γ is a tridiagonal matrix of size $(d+1) \times (d+1)$

$$P = P(\Gamma) = \begin{bmatrix} 0 & 1 & & & & & \\ k & a_1 & c_2 & & & 0 & \\ & b_1 & a_2 & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & c_{d-1} & \\ & & & & \cdot & a_{d-1} & c_d \\ & 0 & & & & b_{d-1} & a_d \end{bmatrix}$$

in which k and d are the valency and diameter of Γ and the integers c_i, a_i, b_i are defined as follows: let u and v be two vertices of Γ for which $d(u, v) = i$, then the number of vertices w of Γ such that $d(w, v) = 1$ and $d(w, u) = i-1, i, i+1$ respectively is defined to be c_i, a_i, b_i respectively. These numbers are easily seen to be independent of the apparent choices. We shall need to denote the columns of P by $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_d$.

Now not every tridiagonal matrix of integers can arise from a distance-transitive graph, since, for example, we must have $c_i + a_i + b_i = k$. Another elementary constraint comes from counting the sets of vertices

$$\Gamma_i(u) = \{w | d(u, w) = i\}.$$

Setting $k_i = |\Gamma_i(u)|$ we find $k_0 = 1$, $k_1 = k$, $k_2 = kb_1/c_2$, and generally

$$k_i = k_{i-1} b_{i-1}/c_i.$$

Thus the entries of P must be such that the numbers k_i so defined are integers.

In addition to such simple conditions derived from combinatorial considerations there is a much deeper (and more restrictive) condition which must be satified. This is due, in a different form, to D. G. Higman [3]. We note first that P has $d+1$ distinct real eigenvalues $k = \lambda_0 > \lambda_1 > \ldots > \lambda_d$, and so for each λ_i there is thus a unique left eigenvector $\mathbf{u}_i$ with first entry 1, and a unique right eigenvector $\mathbf{v}_i$ with first entry 1. The condition we require is that, for $i = 0, 1, \ldots, d$ the numbers

$$\frac{(\mathbf{u}_0, \mathbf{v}_0)}{(\mathbf{u}_i, \mathbf{v}_i)}$$

must be integers (where (,) denotes the usual inner product). For a proof in these terms, see [**1**].

3. *Trivalent graphs*

The columns of the matrix P (except the first and last column) have just three entries which are not identically zero and if Γ is a trivalent ($k = 3$) distance-transitive graph then these must be of one of the following types:

$$A: \begin{matrix} 1 \\ 0 \\ 2 \end{matrix} \qquad B: \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \qquad C: \begin{matrix} 2 \\ 0 \\ 1 \end{matrix}$$

It is thus a fairly easy matter to enumerate all such matrices of a given size and to apply the tests of the previous section on an electronic computer. These criteria turn out to be very restrictive. There remain two steps before the classification of trivalent distance-transitive graphs is complete. One step is to decide if a matrix which passes the tests does in fact correspond to a distance-transitive graph, and if so, whether or not the graph is unique. The uniqueness is quite easy in these cases, and there are very few " redundant " matrices, so we shall not pursue these details further here. See [**4**].

The other step is crucial. We shall show that there is an upper bound for the diameter d of a trivalent distance-transitive graph. In the terminology of Tutte [**6, 9**] such a graph is edge-transitive and so is s-regular for $1 \leqslant s \leqslant 5$.

This means that the order of the stabilizer of any given vertex is 3.2^{s-1} and we deduce that each integer k_i is a divisor of 48. From the recurrence $k_i = k_{i-1}\, b_{i-1}/c_i$ it follows that $\mathbf{p}_1, \ldots, \mathbf{p}_6$, cannot all be of type A, and that $\mathbf{p}_1, \ldots, \mathbf{p}_5$ cannot all be of type A with $\mathbf{p}_6$ of type B. Thus there are two cases to consider: letting j be the smallest integer such that $\mathbf{p}_j$ is of type B or C, we have

Case 1: $\mathbf{p}_j$ is of type C with $1 \leqslant j \leqslant 6$.

Case 2: $\mathbf{p}_j$ is of type B with $1 \leqslant j \leqslant 5$.

LEMMA. 1. *In case* 1, $d < 2j$.

Proof. Suppose $d \geqslant 2j$. Then we can choose $w \in \Gamma_{2j}(u)$ and $v \in \Gamma_j(u)$ such that $d(v, w) = j$. Now because $\mathbf{p}_j$ is of type C, u and v are on a circuit of length $2j$. But $d(v, w) = d(u, v) = j$, so v and w must likewise be on a circuit of length $2j$; this is impossible since there is only one vertex of $\Gamma_{j+1}(u)$ adjacent to v. Hence $d < 2j$.

LEMMA 2. *In case* 2, $d < 3j$.

Proof. Suppose $d \geqslant 3j$. The following schematic diagram may illuminate this proof (Fig. 3).

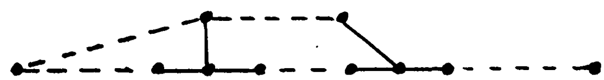

Fig. 3

We choose vertices $v \in \Gamma_j(u)$, $w \in \Gamma_{2j}(u)$, $x \in \Gamma_{3j}(u)$ such that $d(v, w) = d(w, x) = j$. Then because $\mathbf{p}_j$ is of type B, u and v lie on a circuit of length $2j+1$, and hence so must v and w. Thus $\mathbf{p}_{2j}$ must be of type C. Now w and x must also lie on a circuit of length $2j+1$ and this is impossible. Hence $d < 3j$.

We conclude that $d < 15$ in all cases, and so the computational criteria need only be applied to a finite number of matrices of relatively small size.

4. *Table of trivalent distance-transitive graphs*

	Diameter	Number of vertices	Name, description, reference.
(i)	1	4	K_4.
(ii)	2	6	$K_{3,3}$.
(iii)	2	10	Petersen's graph.
(iv)	3	8	Cube.
(v)	3	14	Heawood's graph, [**8**, p. 61].
(vi)	4	18	Pappus's graph, [**2**, p. 434].
(vii)	4	28	See section 1, also [7].
(viii)	4	30	Tutte's 8-cage [**8**, p. 76].
(ix)	5	20	Dodecahedron.
(x)	5	20	Desargues's graph [**2**, p. 435].
(xi)	7	102	See Section 1.
(xii)	8	90	Related to (viii). See [**4**].

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