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Recursive Families of Graphs

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A recursive family of graphs is defined as a sequence of graphs whose Tutte polynomials satisfy a homogeneous linear recurrence relation. Some necessary conditions for a family to be recursive are proved, and the theory is applied to the families of graphs known as the prisms and the Möbius ladders to give the chromatic polynomials and complexity of these graphs. It is conjectured that there is some function $B(k)$ such that the chromatic roots of all regular graphs of valency k lie in the disc $|u| \leqslant B(k)$. For the prisms and Möbius ladders it is shown that the chromatic roots lie in the disc $|u| \leqslant 3$.

1. Introduction

The Tutte polynomial of a graph, originally constructed by Tutte in 1954 [5], and extended to the more general setting of matroid theory by Crapo in 1969 [2], is a generating function which gives a very great deal of important information about the graph. In particular, the chromatic polynomial of a graph is a partial evaluation of the Tutte polynomial. In this paper we deal with families of graphs for which there is an especially simple procedure for calculating this polynomial. As an application we calculate the Tutte polynomials for the prisms and Möbius ladders [3], and employ our results to verify in these cases a conjecture concerning the location of chromatic roots [7] of regular graphs.

2. The Tutte Polynomial

We deal with finite, undirected, connected graphs Γ, which may have loops and multiple edges; the vertex set of Γ will be denoted by $V(\Gamma)$ and the edge set by $E(\Gamma)$. If $E(\Gamma)$ is given some total ordering $<$, then we may associate with each spanning tree T of Γ two non-negative integers

$i(T, <)$ and $j(T, <)$ called, respectively, the *internal activity* and *external activity* of T in the ordering $<$. The definitions are given in [5, p. 85]. Now if t_{ij} denotes the number of spanning trees T of Γ with $i(T, <) = i$ and $j(T, <) = j$ then we define a polynomial in (x, y) by

$$\Gamma(x, y; <) = \sum t_{ij} x^i y^j.$$

We have the remarkable result [2, p. 222] that

$$\Gamma(x + 1, y + 1; <) = \sum r_{ij} x^i y^j,$$

where r_{ij} is the number of subgraphs S of Γ for which, if we let $c(S)$ denote the set of connected components of S, we have:

$$| c(S)| - | V(S)| + | V(\Gamma)| - 1 = i;$$
$$| c(S)| - | V(S)| + | E(S)| = j.$$

The most important consequence of this is that the ordering $<$ is immaterial, and we write accordingly $\Gamma(x, y) = \sum t_{ij} x^i y^j$ for the *Tutte polynomial* of Γ.

If we define

$$C(\Gamma; u) = \pm u\Gamma(1 - u, 0),$$
$$C^*(\Gamma; u) = \pm u\Gamma(0, 1 - u),$$

where the signs are taken to make the leading coefficient positive, then $C(\Gamma; u)$ is the *chromatic polynomial* of Γ which enumerates, when u is a natural number the proper u-colorings of Γ, and $C^*(\Gamma; u)$ is a polynomial which enumerates the proper flows modulo u on Γ. If Γ is a planar graph and Γ^* is its dual then

$$C^*(\Gamma; u) = C(\Gamma^*; u).$$

We note also that $\kappa(\Gamma) = \Gamma(1, 1)$ is the number of spanning trees, or *complexity* of Γ.

The basic formulae which enable us to calculate the polynomial $\Gamma(x, y)$ for any connected graph Γ are:

P1. $\Gamma(x, y) = \Gamma_e'(x, y) + \Gamma_e''(x, y)$;

P2. $(\Gamma_1 \circ \Gamma_2)(x, y) = \Gamma_1(x, y)\, \Gamma_2(x, y)$;

P3. $L(x, y) = y$;

P4. $E(x, y) = x$.

In these formulae the notation is as follows: If $e \in E(\Gamma)$ then Γ_e' denotes Γ with e removed, and Γ_e'' denotes Γ with e contracted; formulae P1 then applies provided Γ_e' is connected. If Γ_1 and Γ_2 are disjoint graphs then $\Gamma_1 \circ \Gamma_2$ is the graph formed by identifying one vertex of Γ_1 with one vertex of Γ_2, and L is a graph consisting of a single loop while E is the graph consisting of a single edge which is not a loop.

3. Recursive Families

A sequence of graphs $\{\Gamma_n\}$ will be called a *recursive family* if there is a homogeneous recurrence relation

$$\Gamma_{n+\rho} + \alpha_1 \Gamma_{n+\rho-1} + \cdots + \alpha_\rho \Gamma_n = 0 \qquad (n \geqslant 1)$$

in which $\Gamma_i = \Gamma_i(x, y)$ is the Tutte polynomial of Γ_i and $\alpha_j = \alpha_j(x, y)$ is a polynomial with integral coefficients independent of n $(1 \leqslant j \leqslant \rho)$. The smallest number ρ for which there is such a relation will be called the *recursiveness* of $\{\Gamma_n\}$.

As an example, we mention the family of wheels $\{W_n\}$ [6, p. 102], which is a family with recursiveness 3:

$$W_{n+3} - (x + y + 2)\, W_{n+2} + (xy + x + y + 1)\, W_{n+1} - xyW_n = 0.$$

The auxiliary equation for this recurrence is

$$(t - 1)(t^2 - \{x + y + 1\}\, t + xy) = 0,$$

so that we can write down an explicit formula for $W_n(x, y)$, involving the roots of this equation and some fixed functions of (x, y). A computer program has been written to calculate $\Gamma(x, y)$ for small graphs Γ, and this enables us to determine the fixed functions and derive:

$$\begin{aligned} W_n(x, y) &= xy - x - y - 1 \\ &\quad + 2^{-n}[(x + y + 1 + \sqrt{\beta})^n + (x + y + 1 - \sqrt{\beta})^n], \end{aligned}$$

where $\beta = \beta(x, y) = (x + y + 1)^2 - 4xy$. Putting $y = 0$ and $x = 1 - u$ we find the well-known chromatic polynomial

$$C(W_n\, ; u) = u(u - 1)^n + (-1)^n\, (u - 1),$$

and since $W_n^* = W_n$ we note as expected that $C^*(W_n; u) = C(W_n; u)$. Also putting $x = y = 1$ we obtain the result

$$\kappa(W_n) = 2^{-n}[(3 + \sqrt{5})^n + (3 - \sqrt{5})^n] - 2$$

given by Sedlacek [4].

There are elementary considerations which suffice to show that some well-known sequences of graphs are not recursive families. These are contained in the following theorem:

THEOREM I. *Let $\phi(\Gamma)$ denote the mean-valency of Γ, and $\kappa(\Gamma)$ its complexity. Then, if $\{\Gamma_n\}$ is a recursive family for which $|V(\Gamma_n)| \to \infty$ as $n \to \infty$,*

(a) *$\phi(\Gamma_n)$ does not tend to infinity as $n \to \infty$;*

(b) *$\sqrt[n]{\kappa(\Gamma_n)}$ does not tend to infinity as $n \to \infty$.*

Proof. (a) Let $v_n = |V(\Gamma_n)|$, $e_n = |E(\Gamma_n)|$. Then the degree of the polynomial $\Gamma_n(x, y)$ is $v_n - 1$ in x and $e_n - v_n + 1$ in y, and from the linearity of the recurrence we deduce that $v_n - 1$ and $e_n - v_n + 1$ are linear functions of n. Consequently $\phi(\Gamma_n) = 2e_n/v_n$ cannot tend to infinity as $n \to \infty$, unless v_n is bounded (that is, constant).

(b) $\kappa(\Gamma_n)$ satisfies a linear recurrence with constant integer coefficients and for large n it behaves like $An^\theta\tau^n$ where A is a constant, θ is a fixed number less than the recursiveness ρ, and τ is the largest root of the auxiliary equation. Thus $\sqrt[n]{\kappa(\Gamma_n)}$ cannot tend to infinity as $n \to \infty$.

This theorem shows that the families of complete graphs $\{K_n\}$, bipartite complete graphs $\{K_{n,n}\}$, cubes $\{Q_n\}$, and many others, are not recursive in our sense.

4. PRISMS AND MÖBIUS LADDERS

The *prism* is a well-known object in geometry and we use the symbol T_n to denote the graph formed by the vertices and edges of a prism with $2n$ vertices. This graph is regular of valency 3, and may be visualized as a "ladder" with its free ends joined. This visualization explains the name *Möbius ladder* for the graph considered in [3], which we picture as a ladder with the ends joined after a twist, so that the graph is naturally embedded in a Möbius band. We use the symbol $T_n^{\#}$ for a Möbius ladder with $2n$ vertices, which is again a regular graph of valency 3. Both T_n and $T_n^{\#}$ can be unambiguously defined for all $n \geqslant 1$; to avoid confusion we state that $T_1(x, y) = xy^2$, $T_1^{\#}(x, y) = x + y + y^2$, $T_2(x, y) = (x + y)^2 (x + 1) + x + y + y^2 + y^3$, and we note the identities $T_2^{\#} = K_4$, $T_3^{\#} = K_{3,3}$, $T_4 = Q_3$.

If we apply the formulae P1, P2, P3, P4 to T_n and $T_n^{\#}$ in a systematic fashion, introducing suitable notation for the graphs which occur, we derive the following result:

THEOREM II. *The families* $\{T_n\}$ *and* $\{T_n^{\#}\}$ *are recursive, with recursiveness* 6. *The coefficients* $\alpha_1, \alpha_2, \ldots, \alpha_6$, *in the recurrence relation are, in both cases:*

$$-\alpha_1 = x^2 + 3x + 2y + 4$$

$$\begin{aligned}\alpha_2 = {} & 2x^3 + 6x^2 + 9x + 2x^2y + 5xy \\ & + y^2 + 5y + 5\end{aligned}$$

$$\begin{aligned}-\alpha_3 = {} & x^4 + 5x^3 + 8x^2 + 8x + 4x^3y + 8x^2y \\ & + 9xy + 3y + x^2y^2 + 2xy^2 + y^2 + 2\end{aligned}$$

$$\begin{aligned}\alpha_4 = {} & x^4 + 3x^3 + 3x^2 + 2x + 2x^4y + 7x^3y + 7x^2y \\ & + 4xy + 2x^3y^2 + 2x^2y^2 + 2xy^2\end{aligned}$$

$$-\alpha_5 = 2x^4y + 3x^3y + x^2y + x^4y^2 + 2x^3y^2 + x^2y^2$$

$$\alpha_6 = x^4y^2$$

The details of the proof will not be given, as it is hoped that eventually a more general result will subsume this theorem.

The auxiliary equation is

$$\begin{aligned}(t - 1)(t - x)(t^2 - \{x^2 + x + y + 1\}\, t + x^2y) \\ \times (t^2 - (x + y + 2\}\, t + xy) = 0,\end{aligned}$$

so that we could, if required, write down an explicit formula for $T_n(x, y)$ and $T_n^{\#}(x, y)$. It will suffice here to confine ourselves to determining chromatic polynomials.

Thus, putting $y = 0$ we find an expression of the following form, in which A, B, C, D, are fixed functions of x:

$$T_n(x, 0) = A + Bx^n + C(x + 2)^n + D(x^2 + x + 1)^n,$$

and a similar form for $T_n^{\#}(x, 0)$, leading to the chromatic polynomials:

$$C(T_n\,; u) = (u^2 - 3u + 3)^n + (u - 1)\{(3 - u)^n + (1 - u)^n\} + u^2 - 3u + 1;$$
$$C(T_n^{\#}; u) = (u^2 - 3u + 3)^n + (u - 1)\{(3 - u)^n - (1 - u)^n\} - 1.$$

Both these results are thought to be new.

Putting $x = 0$ we find

$$T_n(0, y) = A + C(y + 1)^n + D(y + 2)^n,$$
$$C^*(T_n\,; u) = u(u - 1)(u - 3)^n + u(u - 2)^n + (-1)^n\, u(u^2 - 3u + 1).$$

Since T_n is planar and $T_n{}^*$ is a double pyramid we recognise this as the well-known chromatic polynomial $C(T_n{}^*; u)$. The result for $C^*(T_n^{\#}; u)$ is similar but in this case there is no dual graph and we cannot interpret this polynomial as a chromatic polynomial. Finally, putting $x = y = 1$ the auxiliary equation for $\kappa(T_n) = T_n(1, 1)$ is

$$(t - 1)^2\,(t^2 - 4t + 1)^2 = 0,$$

leading to the results

$$\kappa(T_n) = (n/2)[(2 + \sqrt{3})^n + (2 - \sqrt{3})^n] - n,$$
$$\kappa(T_n^{\#}) = (n/2)[(2 + \sqrt{3})^n + (2 - \sqrt{3})^n] + n,$$

the second of which is given in [4].

5. A Conjecture about Chromatic Roots

The zeros of the chromatic polynomial $C(\Gamma; u)$ are sometimes called the *chromatic roots* of Γ, and their location in the complex plane is a matter of some historical importance in graph theory. Concerning regular graphs, there is evidence to support the following conjecture, which can be regarded as a generalization of the theorem of Brooks [1].

Conjecture. *There is a function $B(k)$ such that for all regular graphs Γ of valency k we have*

$$C(\Gamma; u) = 0 \Rightarrow |\,u\,| \leqslant B(k).$$

In particular, computations show that, for all trivalent graphs Γ with $|V(\Gamma)| \leqslant 10$, all chromatic roots lie in the disc $|u| \leqslant 3$. However, the essence of the conjecture is the fact that $B(k)$ is to be independent of the number of vertices, and in this spirit the following theorem is significant:

THEOREM III. *For all $n \geqslant 1$, the chromatic roots of the prism T_n and the Möbius ladder $T_n^{\#}$ lie in the disc $|u| \leqslant 3$.*

Proof. We give the proof for T_n; the proof for $T_n^{\#}$ is substantially the same. The idea is to apply Rouché's theorem to

$$C(T_n\,;u) = f(u) + g(u),$$

where

$$f(u) = (u^2 - 3u + 3)^n,$$
$$g(u) = (u-1)\{(3-u)^n + (1-u)^n\} + (u^2 - 3u + 1).$$

For this we need the following inequalities which hold on the circle $|u| = 3$:

$$|u^2 - 3u + 3| > (1.4)\,|3 - u|,$$
$$|u^2 - 3u + 3| > (1.4)\,|1 - u|,$$
$$|u^2 - 3u + 3| \geqslant 3,$$
$$|u - 1| \leqslant 4.$$

Consequently:

$$\begin{aligned}|u^2 - 3u + 1| &\leqslant |u^2 - 3u + 3| + 2 \leqslant \frac{5}{3}|u^2 - 3u + 3| \\ &\leqslant \frac{5}{3^n}|u^2 - 3u + 3|^n,\end{aligned}$$

and we have

$$\begin{aligned}|g(u)| &\leqslant |u-1|\,(|3-u|^n + |1-u|^n) + |u^2 - 3u + 1| \\ &\leqslant \left\{\frac{8}{(1.4)^n} + \frac{5}{3^n}\right\}|u^2 - 3u + 3|^n.\end{aligned}$$

Now for $n > 7$ the numerical coefficient is strictly less than 1, and so $|g(u)| < |f(u)|$. Thus, since all zeros of f lie inside the circle $|u| = 3$, the same is true for $f + g$, which is the chromatic polynomial of T_n. The proof is completed for $n \leqslant 7$ by direct computation of the zeros. (We note that, for $n = 4$, $T_n^{\#}$ has a chromatic root $u = 3$.)

The zeros of $C(T_n\,; u)$ have been computed for all $n \leqslant 15$; it appears that as n increases the zeros lie very close to the line $\mathscr{R}u = 2$ or to a curve whose equation has not been determined. This configuration is shown in Figure 1. The zeros of $C(T_n^{\#}; u)$ behave in a similar fashion.

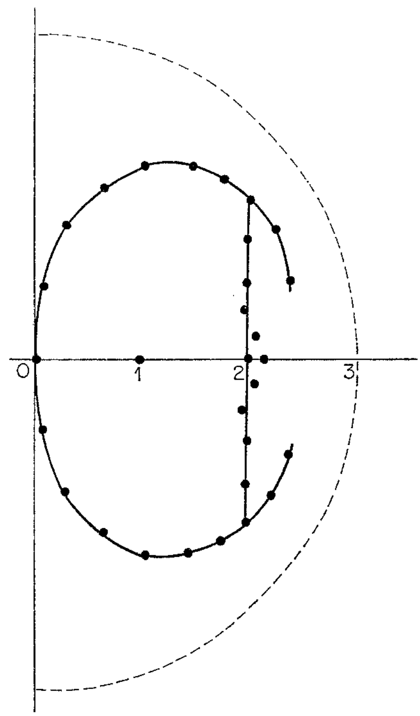

FIG. 1. Apparent limiting behavior of zeros of $C(T_n\,; u)$ as $n \to \infty$. The points marked are the zeros of $C(T_{15}\,; u)$.

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