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Cayley maps and symmetrical maps

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Abstract. In this paper we shall show how combinatorial methods can be applied to the study of maps on orientable surfaces. Our main concern is with maps which possess a certain kind of symmetry, called vertex-transitivity. We show how an extension of the well-known method of Cayley can be used to construct such maps, and we give conditions which suffice for the automorphism groups of these maps to have non-trivial vertex-stabilizers. Finally, we investigate the special case when the skeleton of the map is a complete graph; a classical theorem of Frobenius then implies that all vertex-transitive maps are given by our extension of Cayley's construction.

1. *Introductory definitions.* Let us denote the two-element set $\{+, -\}$ by Δ. Our basic definition is that a *map* consists of a finite set E of *edges*, together with a permutation ρ of the set $S = E \times \Delta$. The elements of S are called *sides*, and we abbreviate the sides $(e, +)$, $(e, -)$ to e^+, e^- respectively. If M denotes the map (E, ρ) we shall say that ρ is the *rotation* in M.

The involution $\tau: S \to S$ which interchanges the sides e^+ and e^- for each $e \in E$ will be said to *reverse* sides, and the permutation $\rho^* = \rho\tau$ (τ followed by ρ) of S gives rise to a *dual* map $M^* = (E, \rho^*)$. We refer to ρ^* as the *dual* rotation on M.

Underlying the map M is its *skeleton graph* $\Gamma(M)$ whose *vertices* are the orbits of ρ on S and whose *edges* are the edges of M. An edge e is *incident* with a vertex u if and only if either e^+ or e^- belongs to the orbit u of ρ on S. We also say that the side s is incident with the vertex (orbit of ρ) to which it belongs. Thus an edge is incident with either two vertices or one vertex (in which case it is a *loop*), whereas a side is incident with precisely one vertex.

The set of sides incident with a given vertex of $\Gamma(M)$ is an orbit of ρ, and so has a cyclic ordering determined by ρ. This corresponds to our intuitive picture of the situation when a graph is drawn on an orientable surface, and explains the motivation behind our terminology. If $\Gamma = \Gamma(M)$ we often speak of the rotation ρ as providing an *imbedding* of Γ.

In the case when the graph $\Gamma(M)$ is a simple graph, that is, when it has no loops or multiple edges, there is an alternative description of ρ in terms of the vertex set V of $\Gamma(M)$. For each vertex $u \in V$ we let the *star* of u, written $\mathrm{st}(u)$, denote the set of vertices v for which there is an edge of M incident with u and v. The hypothesis that $\Gamma(M)$ is simple means that $u \notin \mathrm{st}(u)$ and that for each $v \in \mathrm{st}(u)$ there is a unique side s such that s is incident with u and $\tau(s)$ with v. Thus S may be identified with a subset of $V \times V$, and E with the corresponding set of unordered pairs. Now the rotation ρ in M may equally well be described in terms of its cycles, of which there is one, say ρ_u, for each vertex u of $\Gamma(M)$, and ρ_u can be interpreted as a cyclic permutation of st(u). We shall speak of the *vertex-description* of ρ as being this set $\{\rho_u\}$ of cyclic permutations, given by $\rho(u,v) = (u, \rho_u(v))$.

We notice that the dual map $M^* = (E, \rho^*)$ also has a skeleton graph $\Gamma(M^*)$ whose vertices we call the *faces* of M. Thus the vertices V, edges E, and faces F, of M are respectively the vertices of $\Gamma(M)$, edges of M, and vertices of $\Gamma(M^*)$. It can be shown (1), by purely combinatorial arguments, that there is a non-negative integer $g = g(M)$, called the *genus* of M, such that $|V| - |E| + |F| = 2 - 2g$.

Finally, we define an *automorphism* of a map $M = (E, \rho)$ to be a permutation α of S such that $\alpha\rho = \rho\alpha$ and $\alpha\tau = \tau\alpha$. Clearly $\alpha\rho^* = \rho^*\alpha$ also. The group of all such automorphisms of M is the *automorphism group*, Aut M, which is by definition a permutation group on the set S.

In the case when $\Gamma(M)$ is simple, Aut M is also a permutation group on the vertex set V. For, since $\alpha\rho = \rho\alpha$, an automorphism α induces a permutation (also called α) of V, and when S is a subset of $V \times V$ the two are related by $\alpha(u,v) = (\alpha u, \alpha v)$. The condition $\alpha\tau = \tau\alpha$ is automatically fulfilled in this notation.

Now in this case the rotation ρ can be given by what we have called its vertex-description; in terms of this, the condition that α should be an automorphism becomes $\rho_{\alpha u}(\alpha v) = \alpha\rho_u(v)$. In other words,

$$\rho_{\alpha u} = \alpha \rho_u \alpha^{-1} \quad \text{for each} \quad u \in V. \tag{1·1}$$

2. *Vertex-transitive maps*

PROPOSITION 1. *If $\alpha \in$ Aut M and $\alpha(s) = s$ for some $s \in S$, then α is the identity. (In other words no non-identity automorphism of M can fix a side of M.)*

Proof. If α fixes s, since $\alpha\rho = \rho\alpha$ we see that α fixes all sides with the same vertex as s. Also, since $\alpha\tau = \tau\alpha$, it follows that α fixes $\tau(s)$, and consequently all sides with the same vertex as $\tau(s)$. Proceeding in this way, since M is connected, we deduce that α fixes all sides and so is the identity.

We now consider Aut M as a permutation group on the vertex set V of M. In this context the preceding proposition states that if the automorphism α fixes a vertex u then it does not fix any vertex adjacent to u, unless of course α is the identity. Thus if $A =$ Aut M, and A_u denotes the stabilizer of the vertex u, we see that an element α of A_u is determined by its action on st(u).

Our next proposition is expressed in terms of the vertex-description of ρ for maps M with $\Gamma(M)$ a simple graph.

PROPOSITION 2. *If M is a map with $\Gamma(M)$ simple, and u is a vertex of M, then the group of automorphisms of M which fix u is isomorphic to a subgroup of the cyclic group generated by the permutation ρ_u of* st(u). *Thus the order of the stabilizer group is a divisor of the valency of u.*

Proof. Let α be an automorphism of M fixing u. By (1·1) $\rho_{\alpha u} = \alpha\rho_u\alpha^{-1}$, that is $\alpha\rho_u = \rho_u\alpha$.

For convenience let β denote the permutation of $\mathrm{st}(u)$ induced by α, and label the vertices of $\mathrm{st}(u)$ as $x_1, x_2, \ldots, x_k$ where $\rho_u = (x_1 x_2 \ldots x_k)$. Suppose $\beta(v_1) = v_j$; then a simple calculation shows that $\beta(v_i) = \rho_u^{j-1}(v_i)$ for all $i = 1, 2, \ldots, k$, and so $\beta = \rho_u^{j-1}$. Thus each automorphism α fixing u is equal to a power of ρ_u on the vertices adjacent to u. But α is determined by its action on $\mathrm{st}(u)$ and so the subgroup of $\mathrm{Aut}\, M$ which fixes u is isomorphic to a group of powers of ρ_u, that is, a subgroup of the cyclic group of order k generated by ρ_u.

We now define a map M to be *vertex-transitive* if $\mathrm{Aut}\, M$ is transitive in its action on the vertex set V, and *symmetrical* if $\mathrm{Aut}\, M$ is transitive on the set S of sides. If M is symmetrical then $\mathrm{Aut}\, M$ must in fact be regular on S, from proposition 1, and the order of $\mathrm{Aut}\, M$ is $|S| = 2|E| = nk$, where $n = |V|$ and k is the valency of $\Gamma(M)$.

If M is vertex-transitive, then the vertex-stabilizers are all conjugate in $\mathrm{Aut}\, M$, and have order d, a divisor of k, so that $\mathrm{Aut}\, M$ has order nd.

3. *Cayley maps.* For any group G we denote by $G^\#$ the set of non-identity elements of G.

Suppose G is a given abstract group and $X \subset G^\#$ is a set of generators for G with the property that $X^{-1} = X$ (that is, $x \in X \Rightarrow x^{-1} \in X$). For any cyclic permutation $r: X \to X$ we define the *Cayley map* $M(G, X, r)$ as follows: M has edge set E which is the subset of the set of unordered pairs $\{g_1, g_2\}$ for which g_1 and g_2 are in G and $g_1^{-1} g_2 \in X$. (Since $X = X^{-1}$ this definition is precise.) We may take the set S of sides to be the corresponding set of ordered pairs, and define the rotation $\rho: S \to S$ by

$$\rho(g_1, g_2) = (g_1, g_1 . r(g_1^{-1} g_2)).$$

Since r is cyclic, it is clear that the orbits of ρ (vertices of M) are in bijective correspondence with the elements of G, and we shall write them as such. Thus $\Gamma(M)$ is a simple graph, with vertex set G, and it is easy to see that it is connected as a consequence of the fact that X generates G. The vertex-description of ρ is given by $\rho_g(gx) = g . r(x)$ for each $g \in G$ and $x \in X$.

Proposition 3. *Any Cayley map $M(G, X, r)$ is vertex-transitive.*

Proof. For each $g \in G$ define $\tilde{g}: S \to S$ by $\tilde{g}(g_1, g_2) = (gg_1, gg_2)$. It is straightforward to check that $\tilde{g}$ does in fact take values in S, and it is an automorphism of M since

$$\begin{aligned} \tilde{g}\rho(g_1, g_2) &= \tilde{g}(g_1, g_1 . r(g_1^{-1} g_2)) = (gg_1, gg_1 . r((gg_1)^{-1} gg_2)) \\ &= \rho(gg_1, gg_2) = \rho\tilde{g}(g_1, g_2). \end{aligned}$$

It follows that $\mathrm{Aut}\, M$ is transitive on the vertex set G of M.

It is not true that a Cayley map is symmetrical – in many cases the automorphism group of $M(G, X, r)$ will consist only of the automorphisms $\tilde{g}$ constructed in the proof of Proposition 3, so that in these cases $\mathrm{Aut}\, M \approx G$.

However, it is possible to be more specific about the conditions which suffice for the automorphism group of $M(G, X, r)$ to be larger than G. Since M is in any case vertex-transitive, all vertex-stabilizers will be isomorphic, and we need only describe one of them, say the stabilizer A_1, where 1 is the identity in G and $A = \mathrm{Aut}\, M$.

PROPOSITION 4. *With the notation above, suppose that r^a: $X \to X$ is a power of r which extends to a group automorphism γ: $G \to G$. Then γ, regarded as a permutation of the vertex set G of $M(G, X, r)$, belongs to the stabilizer A_1.*

Proof. Since γ is a group automorphism, $\gamma(1) = 1$. We must show that γ is an automorphism of M. First, if $(g_1, g_2) \in S$ then $g_1^{-1}g_2 \in X$ so that

$$\gamma(g_1)^{-1}\gamma(g_2) = \gamma(g_1^{-1}g_2) = r^a(g_1^{-1}g_2) \in X,$$

and consequently $\gamma(g_1, g_2) = (\gamma(g_1), \gamma(g_2))$ is in S. Secondly, we have

$$\begin{aligned}\gamma\rho(g_1, g_2) &= \gamma(g_1, g_1 . r(g_1^{-1}g_2)) \\ &= (\gamma(g_1), \gamma(g_1) . \gamma(r(g_1^{-1}g_2))) \\ &= (\gamma(g_1), \gamma(g_1) . r^{a+1}(g_1^{-1}g_2)) \\ &= (\gamma(g_1), \gamma(g_1) . r(\gamma(g_1^{-1}g_2))) \\ &= (\gamma(g_1), \gamma(g_1) . r(\gamma(g)^{-1}\gamma(g_2))) \\ &= \rho(\gamma(g_1), \gamma(g_2)) = \rho\gamma(g_1, g_2).\end{aligned}$$

Thus $\gamma \in A_1$, as required.

It follows from this proposition that if r itself extends to a group automorphism of G, then $M(G, X, r)$ is a symmetrical map.

We conclude this section by showing how the genus of a Cayley map may be calculated. For a given $M(G, X, r)$ we let r^*: $X \to X$ be defined by $r^*(x) = r(x^{-1})$, and note that r^* is not necessarily cyclic. Suppose r^* has t cycles $\omega_1, \omega_2, \ldots, \omega_t$ in its action on X, and if $\omega_i = (x_1 x_2 \ldots x_j)$ let m_i denote the order of the element $x_1 x_2 \ldots x_j$ in G. This m_i is well defined, since a change in the initial member of the cycle ω_i results in a conjugate element of G. The set $m_1, m_2, \ldots, m_t$ is the set of *periods* of M and standard arguments show that the number of faces of M is

$$|F| = n\sum_{i=1}^{t} m_i^{-1} \tag{3·1}$$

where $n = |V|$. Since $|E| = \frac{1}{2}nk$, where $k = |X|$, we deduce that the genus of M is given by

$$4(g-1) = n(k-2-2\Sigma m_i^{-1}) \tag{3·2}$$

which is the form taken by the classical Riemann–Hurwitz formula in our present context. The arguments which are given, for example, in ((3), p. 398) adapt readily to give the necessary proofs.

4. *Complete maps.* In this section we shall investigate maps M for which $\Gamma(M)$ is a complete graph, that is, those maps in which each pair of distinct vertices is adjacent. This special case is the subject of the classical Heawood Problem (7), and the basic work of Gustin (4) in that context is founded on a generalization of Cayley's construction which is related to that described in the previous section. However, our concern is with maximizing symmetry, rather than minimizing genus, and we shall explore ideas not directly relevant to Heawood's Problem.

The result of proposition 1 implies that, if $\Gamma(M)$ is a complete graph, then a non-identity automorphism of M can fix at most one vertex of M. Thus, if M is a vertex-transitive map, the permutation group Aut M acting on the vertex set V is a *Frobenius group* ((6), p. 10). The remarkable result of Frobenius states that such a group has precisely $n-1$ (where $n = |V|$) elements which fix no point, and that these, together with the identity form a regular normal subgroup of Aut M. This vital fact has the following consequence.

PROPOSITION 5. *Any vertex-transitive map M with $\Gamma(M)$ a complete graph K_n, is in fact a Cayley map $M(G, G^\sharp, r)$ with G a group of order n.*

Proof. Let the vertex set V of M be $\{0, 1, 2, \ldots, n-1\}$, and let G be the regular normal subgroup of Aut M guaranteed by Frobenius's theorem. There is a bijection β: $V \to G$, written $i \to \beta_i$, defined by the statement that $\beta_i(0) = i$. We note that β_0 is the identity in G.

Suppose the given map M has rotation ρ; we may then define a cyclic permutation $r: G^\sharp \to G^\sharp$ by insisting that $r(\beta_i) = \beta_{\rho_0(i)}$ for $i \neq 0$. (Here ρ_0 denotes the cyclic permutation of $V - \{0\}$ induced by ρ at the vertex 0.) We shall show that with these definitions $M(G, G^\sharp, r)$ is equivalent in an obvious way to the original map M.

That is, if σ denotes the rotation in $M(G, G^\sharp, r)$ derived from r as in (3·1), we show that σ and ρ correspond under the bijection β.

Now $\beta\rho(i,j) = \beta(i, \rho_i(j)) = (\beta_i, \beta_{\rho i(j)})$, whereas $\sigma\beta(i,j) = \sigma(\beta_i, \beta_j) = (\beta_i, \beta_i . r(\beta_i^{-1}\beta_j))$, so that we have to show that $\beta_i . r(\beta_i^{-1}\beta_j)$ is the same as $\beta_{\rho_i(j)}$.

Let $\beta_h = \beta_i^{-1}\beta_j$ and use the definition of r:

$$\beta_i . r(\beta_i^{-1}\beta_j) = \beta_i . r(\beta_h) = \beta_i . \beta_{\rho_0(h)}.$$

Since β_i is an automorphism of M

$$\beta_i\rho_0 = \rho_{\beta_i(0)}\beta_i = \rho_i\beta_i,$$

and so

$$(\beta_i . \beta_{\rho_0(h)})(0) = \beta_i\rho_0(h) = \rho_i\beta_i(h) = \rho_i\beta_j\beta_h^{-1}(h) = \rho_i\beta_j(0) = \rho_i(j).$$

In other words, $\beta_i . \beta_{\rho_0(h)}$ is the element of G (which is regular on V) which takes 0 to $\rho_i(j)$. Thus it is $\beta_{\rho_i(j)}$, as required.

The converse to proposition 5 is clearly true, for any Cayley map $M(G, G^\sharp, r)$ is vertex-transitive, by Proposition 3, and its skeleton is a complete graph. Thus we have a classification of vertex-transitive imbeddings of complete graphs.

It is possible to go further, since in this particular case the converse of Proposition 4 holds, and we can determine the full automorphism group of such a map in terms of G and r alone.

PROPOSITON 6. *Let A be the automorphism group of the Cayley map $M(G, G^\sharp, r)$, and A_1 the stabilizer of the vertex 1 (the identity in G). Then A_1 consists of those powers of r: $G \to G$ which when extended by putting $r(1) = 1$, are group automorphisms of G.*

Proof. We need not distinguish between r: $G^\sharp \to G^\sharp$ and its extension to G which fixes 1. From Proposition 4 we know that if r^a is an automorphism of G it is an automorphism of $M(G, G^\sharp, r)$.

For the converse, we require the fact that in the case of $M(G, G^\sharp, r)$ the subgroup $\tilde{G}$ of A consisting of the automorphisms $\tilde{g}$ (Proposition 3) is normal in A. Thus if

$r^a \in A_1$, $r^a \tilde{g} r^{-a}$ is in $\tilde{G}$, and since it takes the vertex 1 to $r^a(g)$, it is $\widetilde{r^a(g)}$. Now for any g, $h \in G$ we have

$$\widetilde{r^a(gh)} = r^a \widetilde{gh} r^{-a} = r^a \tilde{g} r^{-a} . r^a \tilde{h} r^{-a} = \widetilde{r^a(g)} . \widetilde{r^a(h)}$$

whence $r^a(gh) = r^a(g)\, r^a(h)$ so that r^a is an automorphism of G.

We may now summarise our results on vertex-transitive imbeddings of complete graphs. Such a map is a Cayley map $M(G, G^\sharp, r)$ and its automorphism group is a Frobenius group whose Frobenius kernel is G and Frobenius complement is the cyclic group given by Proposition 6. Furthermore, the genus of the map is given by the formula (3·2). Thus everything is determined by G and the cyclic permutation r of $G^\sharp$.

In particular we notice that $M(G, G^\sharp, r)$ is a symmetrical map if and only if r itself is an automorphism of G. In order to see for which values of $n = |G|$ it is possible for G to have automorphisms with the cycle structure of r^a, we proceed as follows.

Since r^a $(a < n-1)$ acts without fixed points on $G^\sharp$ we know ((5), p. 501) that for each prime p dividing n there is a Sylow p-subgroup of G which is setwise fixed by r^a. Since r^a consists of cycles of length $d = (n-1)/(n-1, a)$ we have

$$d < \min\{p_1^{e_1}, p_2^{e_2}, \ldots, p_s^{e_s}\}$$

where $n = p_1^{e_1} . p_2^{e_2} \ldots p_s^{e_s}$ is the prime factorisation of n. As extreme cases we see that: (i) if r itself is an automorphism of G, so that M is symmetrical and $d = n-1$, we must have $n = p^e$, a prime power; (ii) if $n \equiv 2 \pmod 4$, we can only have $d = 1$, and so any vertex-transitive map with skeleton K_n for such n has a trivial vertex-stabilizer.

In (2) the present author showed that symmetrical imbeddings of K_n do exist for all prime power values of $n = p^e$. In our present notation these are constructed by taking G to be the elementary Abelian group $(\mathbb{Z}_p)^e$ underlying the field $GF(p^e)$ and r to be the cyclic permutation given by $r(g) = gu$ where u is a primitive element of $GF(p^e)$. The genus of these maps is easily calculated, for the formula (3·1) shows that there are n faces if $p = 2$ or $n \equiv 1 \pmod 4$, but $2n$ faces if $n \equiv 3 \pmod 4$.

REFERENCES

(1) BIGGS, N. L. Spanning trees of dual graphs. *J. Combinatorial Theory*, **11** (1971), 127–131.

(2) BIGGS, N. L. Automorphisms of imbedded graphs. *J. Combinatorial Theory*, **11** (1971), 132–138.

(3) BURNSIDE, W. *Theory of groups of finite order* (Cambridge, 1911).

(4) GUSTIN, W. Orientable imbeddings of Cayley graphs. *Bull. Amer. Math. Soc.* **69** (1963), 272–275.

(5) HUPPERT, B. *Endliche gruppen* (Springer, 1967).

(6) WIELANDT, H. *Finite permutation groups* (Academic Press. N.Y. 1964).

(7) YOUNGS, J. W. T. The Heawood map colouring conjecture. In *Graph theory and theoretical physics* (Academic Press, London, 1967).