THE SYMPLECTIC REPRESENTATION OF MAP AUTOMORPHISMS

NORMAN BIGGS

1. *Statement of results*

Let M be a map of genus $g = g(M)$ with automorphism group $A = \operatorname{Aut} M$. (These terms are defined in §2.) This note is concerned with a connecting link between g and A, which can be expressed by saying that (provided g is at least 2) there is a faithful representation of A in the group of integral symplectic matrices of size $2g \times 2g$.

The theory of maps will be developed in a purely combinatorial setting, independent of extraneous topological results. A simple application of the main result will be described in the final section.

2. *The homology of maps*

The intuitive idea of a map, consisting of vertices, edges, and faces drawn on an orientable surface, has been formalised in purely combinatorial terms by several authors, as for example in [1], [2], [3], [6], [9]. The fundamental observation is that the edges which are incident with any given vertex are cyclically ordered by the orientation on the surface, so that we have a permutation of the directed edges (or sides) of the map. This permutation assigns to each side that side which is next in the cyclic ordering at the relevant terminal vertex. We crystallize this idea in the following definitions.

Let T denote a two-element set, $T = \{+, -\}$. A *map* M is a pair (E, ρ) where E is a non-empty finite set and ρ is a permutation of $S = E \times T$. The elements of E are the *edges* of M, the elements of S are the *sides* of M, and ρ is the *rotation* in M. We let e^+, e^- respectively, stand for the sides $(e, +), (e, -)$, and e^0 denote either e^+ or e^-.

Suppose $\tau : S \to S$ is the involution which interchanges e^+ and e^- for each $e \in E$ (we say that τ *reverses* sides), and let ρ^* be the permutation $\rho\tau$ of S. Then $M^* = (E, \rho^*)$ is the *dual* of M, and the set of orbits of ρ is the *vertex* set (V) of M, while the set of orbits of ρ^* is the *face* set (F) of M.

The *skeleton* graph $\Gamma(M)$ is the graph whose vertices and edges are the vertices and edges of M, the edge e being incident with the vertex v if either e^+ or e^- belongs to the orbit v of ρ on S.

It can be shown that $\Gamma(M)$ is connected if and only if $\Gamma(M^*)$ is connected, and this case there is an integer $g \geqq 0$ such that $|V| - |E| + |F| = 2 - 2g$. This integer is called the *genus* of M.

Henceforth in this note, *all maps considered are connected.*

Received 18 November, 1971.

[Bull. London Math. Soc., 4 (1972), 303–306]

As an example to elucidate this battery of definitions consider the map $M = (E, \rho)$, where $E = \{a, b, c, d, e, f,\}$ and $\rho = (a^+ d^- c^-)(a^- b^+ f^-)(b^- c^+ e^-)(d^+ f^+ e^+)$. The vertices may be labelled 1, 2, 3, 4, corresponding to the four orbits of ρ in the order given, and we see that $\Gamma(M)$ is the complete graph K_4. For the faces, we find

$$\rho^* = (a^+ b^+ c^+)(a^- d^- f^+)(b^- f^- e^+)(c^- e^- d^+).$$

Since $|V|-|E|+|F| = 2$ the map is planar ($g = 0$); it is depicted in Fig. 1, where an arrow on the edge x points from the vertex of x^- to the vertex of x^+.

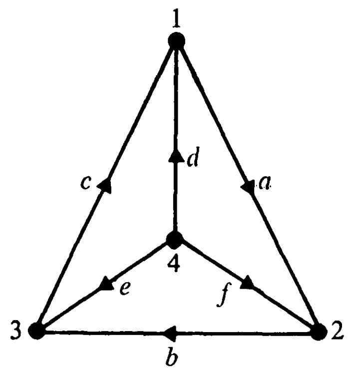

Fig. 1

For each map M there is an associated *chain complex* $C_2(M) \xrightarrow{\partial_2} C_1(M) \xrightarrow{\partial_1} C_0(M)$ defined as follows. $C_0(M)$, $C_1(M)$, $C_2(M)$ are the free abelian groups on the sets V, E, F, respectively. We note for later use that $C_1(M)$ is isomorphic in an obvious way to the abelian group generated by S with relations $e^+ + e^- = 0$ for each $e \in E$. The homomorphism ∂_2 is defined on F by $\partial_2(f) = \sum \pm e$, where $+e$ $(-e)$ appears in the sum if e^+ (e^-) is in the face f, and ∂_2 is extended linearly to $C_2(M)$. The homomorphism ∂_1 is defined on S by $\partial_1(e^0) = v_1 - v_2$ where v_1 is the vertex of e^0 and v_2 is the vertex of $\tau(e^0)$; ∂_1 is compatible with the relations $e^+ + e^- = 0$ and extends linearly to $C_1(M)$. That $\partial_1 \partial_2 = 0$ is easily checked.

The *homology groups* $H_0(M)$, $H_1(M)$, $H_2(M)$ are the homology groups of this chain complex. Since M is connected, it follows that these groups depend only on the genus g of M, and that they are free abelian groups of rank 1, $2g$, 1, respectively. The proof of this fact is essentially the proof of the reduction of a " surface symbol " to its "normal form ", as given in [4].

3. *Automorphisms of maps*

An *automorphism* of a map $M = (E, \rho)$ is a permutation α of S such that $\alpha\rho = \rho\alpha$ and $\alpha\tau = \tau\alpha$ (and consequently $\alpha\rho^* = \rho^*\alpha$). The group of all automorphisms of M will be written Aut M.

LEMMA 1. *A non-identity automorphism of a connected map $M = (E, \rho)$ fixes no sides of M.* (Compare Lemma 2 of [1].)

Proof. Suppose α is an automorphism of M fixing the side e^0, whose vertex (ρ-orbit) is v, and let d^0 with vertex u be any other side of M. Since M is connected there is a chain of vertices $v, v_1, \ldots, u$, joined by edges in $\Gamma(M)$. Now $\alpha\rho = \rho\alpha$ and so α fixes all sides at the vertex v, in particular, the side whose reverse has vertex v_1. Since $\alpha\tau = \tau\alpha$, α fixes this reverse side also, and consequently all sides at v_1. Continuing we find that α fixes all sides at u, including d^0. Thus, α is the identity.

LEMMA 2. *If α is an automorphism of M then α induces group automorphisms $C_i(\alpha) : C_i(M) \to C_i(M)$ $(i = 0, 1, 2)$ giving a commutative diagram*

$$\begin{array}{ccccc} C_2(M) & \xrightarrow{\partial_2} & C_1(M) & \xrightarrow{\partial_1} & C_0(M) \\ \downarrow C_2(\alpha) & & \downarrow C_1(\alpha) & & \downarrow C_0(\alpha) \\ C_2(M) & \xrightarrow{\partial_2} & C_1(M) & \xrightarrow{\partial_1} & C_0(M). \end{array}$$

Proof. Since $\alpha\tau = \tau\alpha$, α is compatible with the relations $e^+ + e^- = 0$ and extends linearly to $C_1(M)$; since $\alpha\rho = \rho\alpha$ and $\rho\alpha^* = \rho^*\alpha$, α induces permutations $\alpha_0 : V \to V$ and $\alpha_2 : F \to F$. Then α_i can be extended linearly to $C_i(\alpha)$ $(i = 0, 2)$ and it is straightforward to check that the diagram is commutative.

4. *The symplectic representation*

From Lemma 2 it follows that for each automorphism α of a map M there are group automorphisms $H_i(\alpha) : H_i(M) \to H_i(M)$, $(i = 0, 1, 2)$. Since M is connected $H_0(\alpha)$ and $H_2(\alpha)$ are identity automorphisms while $H_1(\alpha)$ is an automorphism of $H_1(M)$, a free abelian group on $2g$ generators; thus if we fix a basis for $H_1(M)$, $H_1(\alpha)$ is represented by a unimodular integral matrix of size $2g \times 2g$.

In fact we notice that $H_1(\alpha)$ is actually induced by an automorphism $\pi(\alpha)$ of the fundamental group $\pi(M)$, and $H_1(M)$ is just $\pi(M)$ made abelian. Hence [**8**, pp. 177–8] a matrix $\sigma(\alpha)$ representing $H_1(\alpha)$ is more than merely unimodular; it is symplectic.

For a map M of genus g we now have a homomorphism $\sigma : \operatorname{Aut} M \to \operatorname{Sp}(g, \mathbb{Z})$, where $\operatorname{Sp}(g, \mathbb{Z})$ is the group of integral symplectic matrices of size $2g \times 2g$. Our aim is to show that this representation is faithful if $g \geqq 2$, and to do this we shall use the character $\chi(\alpha) = \operatorname{tr} \sigma(\alpha)$, which is independent of our choice of basis in $H_1(M)$.

LEMMA 3. *Suppose α is a non-identity automorphism of the map M, and let ϕ_0, ϕ_2 denote the number of vertices and faces of M fixed by α_0 and α_2 respectively. Further, let ϕ_1 denote the number of sides of M reversed by α. Then*

$$\chi(\alpha) = 2-(\phi_0+\phi_1+\phi_2).$$

Proof. The Hopf trace theorem [**5**; p. 166] shows that

$$\operatorname{tr} H_2(\alpha)-\operatorname{tr} H_1(\alpha)+\operatorname{tr} H_0(\alpha) = \operatorname{tr} C_2(\alpha)-\operatorname{tr} C_1(\alpha)+\operatorname{tr} C_0(\alpha)$$

Now $\operatorname{tr} H_2(\alpha) = \operatorname{tr} H_0(\alpha) = 1$, and $\chi(\alpha) = \operatorname{tr} H_1(\alpha)$. Further $\operatorname{tr} C_2(\alpha) = \phi_2$ and $\operatorname{tr} C_0(\alpha) = \phi_0$, while the trace of $C_1(\alpha)$ is the number of sides fixed by α, minus the number of sides reversed by α, that is, by virtue of Lemma 1, $\operatorname{tr} C_1(\alpha) = -\phi_1$. Hence the result.

THEOREM. *If M is a map of genus $g \geqq 2$, there is a monomorphism*

$$\sigma : \operatorname{Aut} M \to \operatorname{Sp}(g, \mathbb{Z}).$$

Proof. Suppose $\alpha \in \operatorname{Aut} M$ is such that $\sigma(\alpha) = I$, the identity matrix of size $2g \times 2g$, so that $\chi(\alpha) = 2g \geqq 4$. Lemma 3 shows that $\chi(\alpha) \leqslant 2$ if α is not the identity, so we deduce that $\alpha = 1$ and σ is a monomorphism.

5. *Application of the theorem*

The fact that the automorphism group of a map of genus $g \geqq 2$ is a finite subgroup of $\operatorname{Sp}(g, \mathbb{Z})$ places certain restrictions on the group. For example, consider the order of elements of Aut M. The results of Kirby [7] show that there is a numerical function ψ such that an integral matrix of size $2g \times 2g$ can have order m only if $2g \geqq \psi(m)$. The function ψ dominates Euler's ϕ-function, and is equal to it in a few cases; in particular if p is a prime, $\psi(p) = \phi(p) = p-1$.

Thus if M is a map with p dividing the order of Aut M, the genus of M must be 0, 1, or at least $\frac{1}{2}(p-1)$.

References

1. N. L. Biggs, " Automorphisms of imbedded graphs ", *J. Combinatorial Theory*, 11 (1971), 132–138.
2. ———, " Cayley maps and symmetrical maps ", *Proc. Cambridge Philos. Soc.* (to appear in 1972).
3. J. R. Edmonds, " A combinatorial representation for polyhedral surfaces ", *Notices Amer. Math. Soc.*, 7 (1960), 646.
4. M. Frechet and Ky Fan, *Initiation to combinatorial topology*. (Prindle, Weber and Schmidt, Boston, 1967.)
5. P. J. Hilton and S. Wylie, *Homology theory* (Cambridge, C.U.P. 1960).
6. A. Jacques, " Sur le genre d'une paire de substitutions ", *C.R. Acad. Sci. Paris*, 267 (1968), 625–627.
7. D. Kirby, " Integer matrices of finite order ", *Re. Mat. Applic.*, 2 (VI) (1969), 1–6.
8. W. Magnus, A. Karass, D. Solitar, *Combinatorial group theory* (Wiley, New York, 1966).
9. T. R. S. Walsh and A. B. Lehman, " Counting rooted maps by genus ", *J. Combinatorial Theory* (to appear).

Royal Holloway College,
University of London,
Englefield Green,
Surrey.