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EXPANSIONS OF THE CHROMATIC POLYNOMIAL

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Abstract. The chromatic polynomial (or *chromial*) of a graph was first defined by Birkhoff in 1912, and gives the number of ways of properly colouring the vertices of the graph with any number of colours. A good survey of the basic facts about these polynomials may be found in the article by Read [3].

It has recently been noticed that some classical problems of physics can be expressed in terms of chromials, and papers by Nagle [2], Baker [1], Temperley and Lieb [4], are concerned with methods of expanding the chromial for use in such problems. In this note we shall unify, simplify, and generalise their treatments, confining our attention to the theoretical basis of the methods.

1. Subgraphs and separability

We shall be concerned with finite graphs Γ, with edge-set $E\Gamma$ and vertex-set $V\Gamma$; loops and multiple edges are allowed. For any subset $S \subseteq E\Gamma$ we define the *edge-generated subgraph* $\langle S\rangle_\Gamma$ to be the graph whose edges are the edges of Γ in S and whose vertices are the vertices of Γ incident with edges in S. If the context is clear, $\langle S\rangle_\Gamma$ is abbreviated to $\langle S\rangle$.

The set of all edge-generated subgraphs of Γ is denoted by $A(\Gamma)$; thus $A(\Gamma)$ has $2^{|E\Gamma|}$ members in bijective correspondence with the subsets of $E\Gamma$. We note the existence of the empty subgraph $\langle \emptyset \rangle$.

There is no loss of generality in restricting attention to those graphs which have no isolated vertices; that is, graphs Γ for which $\langle E\Gamma\rangle = \Gamma$. With this restriction in mind, we define a graph Γ to be *separable* if $E\Gamma$ is the disjoint union of two non-empty subsets E_1, E_2, such that

$$V\langle E_1\rangle \cup V\langle E_2\rangle = V\Gamma\,,$$

$$|V\langle E_1\rangle \cap V\langle E_2\rangle| = 0 \text{ or } 1\,.$$

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The empty graph, and a graph with a single edge, are not separable.

The *blocks* of a graph are its maximal non-separable edge-generated subgraphs. The symbol $B(\Gamma)$ will denote the set of blocks of Γ; thus, if Γ is non-separable, $B(\Gamma) = \{\Gamma\}$. In order to avoid confusion, it is worth remarking that a separable graph (in our terminology), may be disconnected. If this is the case, each component is either a block or a union of blocks joined at cut-vertices.

2. The logarithmic transformation

This section simplifies the work of Tutte [5, pp. 307–317].

Let τ be any function defined for each graph of the kind we are considering, and taking non-negative real values such that

$$\tau(\Gamma) = \begin{cases} 1\,, & \text{if } E\Gamma = \emptyset\,, \\ \prod_{\Delta \in B(\Gamma)} \tau(\Delta)\,, & \text{otherwise}\,. \end{cases} \tag{2.1}$$

Then we define a new function T by

$$T(\Gamma) = \sum_{S \subseteq E\Gamma} \tau\langle S\rangle\,. \tag{2.2}$$

We note that if $E\Gamma = \emptyset$, $T(\Gamma) = 1$.

Lemma 2.1. *If Γ is separable via the partition $E\Gamma = E_1 \cup E_2$, then*

$$T(\Gamma) = T\langle E_1\rangle\, T\langle E_2\rangle\,.$$

Proof. For each $S \subseteq E\Gamma$, let $S_1 = S \cap E_1$, $S_2 = S \cap E_2$, and write $\Gamma_1 = \langle E_1\rangle$, $\Gamma_2 = \langle E_2\rangle$. Then if S_1 and S_2 are non-empty, the blocks of $\langle S\rangle_\Gamma$ are precisely $\langle S_1\rangle_{\Gamma_1}$ and $\langle S_2\rangle_{\Gamma_2}$, and so

$$\tau\langle S\rangle_\Gamma = \tau\langle S_1\rangle_{\Gamma_1}\, \tau\langle S_2\rangle_{\Gamma_2}\,.$$

If either or both of S_1, S_2 are empty, this equation remains valid for trivial reasons. Hence

$$T(\Gamma) = \sum_{S \subseteq E\Gamma} \tau\langle S\rangle = \sum_{S_1 \subseteq E_1} \sum_{S_2 \subseteq E_2} \tau\langle S_1\rangle\, \tau\langle S_2\rangle = T(\Gamma_1)\, T(\Gamma_2)\,.$$

We now define $L(\Gamma) = \log T(\Gamma)$ and

$$\bar{L}(\Gamma) = (-1)^{|E\Gamma|} \sum_{S \subseteq E\Gamma} (-1)^{|S|} L\langle S\rangle_\Gamma .$$

Again we note first that if $E\Gamma = \emptyset$, then $\bar{L}(\Gamma) = \log T(\Gamma) = 0$. The importance of $\bar{L}$ stems from the fact that this conclusion remains true if Γ is separable.

Lemma 2.2. *If* Γ *is separable,* $\bar{L}(\Gamma) = 0$.

Proof. We shall make use of the elementary identity:

$$\sum_{X \subseteq Y} (-1)^{|X|} = \begin{cases} 0 & \text{if } Y \text{ is non-empty ,} \\ 1 & \text{if } Y \text{ is empty .} \end{cases}$$

Suppose Γ is separable via the partition $E\Gamma = E_1 \cup E_2$ and recall the notation of the previous proof. By the result of Lemma 2.1, we know that $T\langle S\rangle_\Gamma = T\langle S_1\rangle_{\Gamma_1} T\langle S_2\rangle_{\Gamma_2}$ for all $S \subseteq E\Gamma$, and so

$$L\langle S\rangle_\Gamma = L\langle S_1\rangle_{\Gamma_1} + L\langle S_2\rangle_{\Gamma_2} .$$

Consequently,

$$\begin{aligned}
\bar{L}(\Gamma) &= (-1)^{|E\Gamma|} \sum_{S \subseteq E\Gamma} (-1)^{|S|} L\langle S\rangle \\
&= (-1)^{|E\Gamma|} \sum_{S_1 \subseteq E_1} \sum_{S_2 \subseteq E_2} (-1)^{|S_1| + |S_2|} (L\langle S_1\rangle + L\langle S_2\rangle) \\
&= (-1)^{|E\Gamma|} \Big[\sum_{S_1 \subseteq E_1} (-1)^{|S_1|} L\langle S_1\rangle \sum_{S_2 \subseteq E_2} (-1)^{|S_2|} \\
&\qquad + \sum_{S_2 \subseteq E_2} (-1)^{|S_2|} L\langle S_2\rangle \sum_{S_1 \subseteq E_1} (-1)^{|S_1|} \Big] .
\end{aligned}$$

Since E_1 and E_2 are both non-empty, our elementary identity shows that the whole sum is zero.

Our next result is an inversion formula.

Lemma 2.3. $L(\Gamma) = \sum_{S \subseteq E\Gamma} \bar{L}\langle S\rangle$.

Proof.

$$\sum_{S \subseteq E\Gamma} \bar{L}\langle S\rangle = \sum_{S \subseteq E\Gamma} (-1)^{|S|} \sum_{R \subseteq S} (-1)^{|R|} L\langle R\rangle$$

from the definition of $\bar{L}$, and using the fact that $\langle R\rangle$ as a subgraph of $\langle S\rangle$ is the same as $\langle R\rangle$ as a subgraph of Γ. The right-hand side is, writing $Q = S \setminus R$,

$$\sum_{R \subseteq E\Gamma} \sum_{Q \subseteq E\Gamma\setminus R} (-1)^{|R|+|Q|}(-1)^{|R|} L\langle R\rangle =$$

$$= \sum_{R \subseteq E\Gamma} L\langle R\rangle \sum_{Q \subseteq E\Gamma\setminus R} (-1)^{|Q|} = L\langle E\Gamma\rangle = L(\Gamma)\,,$$

since the inner sum is nonzero only when $R = E\Gamma$.

The function T is defined as a sum over the set $A(\Gamma)$ of all subgraphs of Γ; Lemmas 2.2 and 2.3 give an expansion of $L = \log T$ in terms of the much smaller class $N(\Gamma)$ of non-separable subgraphs of Γ.

Theorem 2.4. *With the above definitions we have*

$$L(\Gamma) = \sum_{\Lambda \in N(\Gamma)} \bar{L}(\Lambda)\,.$$

Equivalently, we have a multiplicative expansion

$$T(\Gamma) = \prod_{\Lambda \in N(\Gamma)} \bar{T}(\Lambda)\,,$$

where $\bar{T} = \exp \bar{L}$.

Thus we can regard the logarithmic transformation as converting an additive expansion of T into a multiplicative one involving far fewer terms.

3. Additive expansions of the chromial

For each natural number u let $[u]$ denote the set $\{1, 2, ..., u\}$, and $[u]^X$ the set of all functions $x : X \to [u]$. Suppose we are given a graph Γ; then for each $x \in [u]^{V\Gamma}$ we have an associated *indicator* function

$\hat{x} : E\Gamma \to \{0, 1\}$ which is defined as follows: $\hat{x}(e) = 1$ if there are vertices v_1, v_2 incident with the edge e such that $x(v_1) \neq x(v_2)$ and $\hat{x}(e) = 0$ otherwise. In particular $\hat{x}(e) = 0$ if e is a loop.

Now for each $t \in \mathbf{R}$ and $u \in \mathbf{N}$ we have a *weight* function

$$W(\Gamma; t, u) = u^{-|V\Gamma|} \sum_{x \in [u]^{V\Gamma}} \prod_{e \in E\Gamma} (\hat{x}(e) - t) .$$

In other words, $W(\Gamma; t, u)$ is the mean value of the product in the right-hand side taken over all functions $x : V\Gamma \to [u]$.

Lemma 3.1. *If $C(\Gamma; u)$ is the chromial of Γ, then*

$$C(\Gamma; u) = u^{|V\Gamma|} W(\Gamma; 0, u) .$$

Proof. Consider the product $\Pi_{e \in E\Gamma} \hat{x}(e)$. This is zero unless x is a proper colouring of Γ with values in $\{1, 2, ..., u\}$, and then it takes the value 1. Thus $u^{|V\Gamma|} W(\Gamma; 0, u)$ is the number of proper colourings of Γ.

Lemma 3.2. (i). *If Γ is separable via the partition $E\Gamma = E_1 \cup E_2$ and $\Gamma_1 = \langle E_1 \rangle$, $\Gamma_2 = \langle E_2 \rangle$, then*

$$W(\Gamma; t, u) = W(\Gamma_1; t, u) \, W(\Gamma_2; t, u) .$$

(ii). *If e is an edge of Γ which is not a loop, and Γ', Γ'', denote the reduction and contraction of Γ with respect to e, then*

$$W(\Gamma; t, u) = (1 - t) \, W(\Gamma'; t, u) - u^{-1} \, W(\Gamma''; t, u) .$$

Proof. (i) This is proved by elementary manipulation of the sums of products involved.

(ii) As a temporary notation, let $X = u^{|V|} W$. Then if the vertices of e are a and b, split the sum over $x \in [u]^{V\Gamma}$ into two parts, one containing those terms for which $x(a) \neq x(b)$ and the other those terms for which $x(a) = x(b)$. The first sum is of the form $(1 - t) Y$ since $\hat{x}(e) = 1$ therein; and the second sum is $(-t) X(\Gamma'')$. That is,

$$X(\Gamma) = (1 - t) \, Y + (-t) \, X(\Gamma'') .$$

Proceeding in the same way with Γ', we find

$$X(\Gamma') = Y + X(\Gamma'') .$$

Eliminating Y gives

$$X(\Gamma) = (1-t)\,X(\Gamma') - X(\Gamma'') ,$$

and since $|V\Gamma| = |V\Gamma'| = |V\Gamma''| + 1$, we have the result on substituting for W.

Our basic result about the weight functions now follows. It can be regarded as a simple translation theorem.

Theorem 3.3. *For any graph* Γ, *real numbers* s, t, *and natural number* u, *we have*

$$W(\Gamma; s, u) = \sum_{S \subseteq E\Gamma} W(\langle S\rangle; s+t, u)\, t^{|E\Gamma| - |S|} .$$

Proof. From the definition we have

$$\begin{aligned} W(\Gamma; s, u) &= u^{-|V\Gamma|} \sum_{x \in [u]^{V\Gamma}} \prod_{e \in E\Gamma} (\hat{x}(e) - s) . \\ &= u^{-|V\Gamma|} \sum_{x \in [u]^{V\Gamma}} \prod_{e \in E\Gamma} (\hat{x}(e) - (s+t) + t) \\ &= u^{-|V\Gamma|} \sum_{x \in [u]^{V\Gamma}} \sum_{S \subseteq E\Gamma} \prod_{e \in S} \{\hat{x}(e) - (s+t)\}\, t^{|E\Gamma| - |S|} . \end{aligned}$$

Now if $V_0 = V\langle S\rangle$, each function in $[u]^{V\Gamma}$ is the extension of $u^{|V\Gamma| - |V_0|}$ functions in $[u]^{V_0}$; hence we may rewrite our double sum in the following way

$$\begin{aligned} W(\Gamma; s, u) &= u^{-|V\Gamma|} \sum_{S \subseteq E\Gamma} u^{|V\Gamma| - |V_0|} \\ &\quad \times \sum_{x \in [u]^{V_0}} \prod_{e \in S} \{\hat{x}(e) - (s+t)\}\, t^{|E\Gamma| - |S|} \\ &= \sum W(\langle S\rangle; s+t, u)\, t^{|E\Gamma| - |S|} . \end{aligned}$$

Corollary 3.4. *Putting $s = 0$, we obtain*

$$u^{-|V\Gamma|}\, C(\Gamma; u) = \sum_{S \subseteq E\Gamma} W(\langle S\rangle; t, u)\, t^{|E\Gamma| - |S|} .$$

This corollary is a generalisation of the expansions of Birkhoff and Whitney [6] and Nagle [2]. If we put $t = 1$, $W(\langle S\rangle; 1, u)$ is the mean value of the product $\Pi_{e\in S}\,(\hat{x}(e) - 1)$ and this product is $(-1)^{|S|}$ if x is constant on each component of $\langle S\rangle$, and zero otherwise. Hence if C_0 denotes the set of components of $\langle S\rangle$, we have

$$W(\langle S\rangle; 1, u) = u^{-|V_0|}\, (-1)^{|S|}\, u^{|C_0|} ,$$

and consequently

$$C(\Gamma, u) = \sum_{S \subseteq E\Gamma} (-1)^{|S|}\, u^{|V\Gamma| - |V_0| + |C_0|} .$$

This is the Birkhoff–Whitney expansion; all subgraphs have to be considered, but their weights are easily found.

Nagle's expansion is obtained by putting $t = 1 - u^{-1}$ In this case many subgraphs have weight zero, for Lemma 3.2 shows that this is so for any graph with a bridge (and consequently for any graph with a monovalent vertex). Further, another application of Lemma 3.2 shows that divalent vertices can be ignored, so that the weight in this case is a homeomorphism invariant. Thus Nagle's expansion depends upon a restricted class of subgraphs, but the determination of the relevant weights is a relatively difficult matter.

4. The logarithmic expansion of the chromial

If we take $t > 0$ in the previous section and write

$$\tau(\Gamma; t, u) = W(\Gamma; t, u)\, t^{-|E\Gamma|} , \tag{4.1}$$

then τ is a function satisfying the conditions (2.1) and the function T defined by (2.2) is

$$T(\Gamma; t, u) = \sum_{S \subseteq E\Gamma} \tau(\langle S \rangle; t, u)$$

$$= \sum_{S \subseteq E\Gamma} W(\langle S \rangle; t, u)\, t^{-|S|}$$

$$= u^{-|V\Gamma|}\, t^{-|E\Gamma|}\, C(\Gamma; u)\,.$$

The last equality follows from Corollary 3.4.

We now use the logarithmic transformation to get an expansion of the chromial in terms of non-separable subgraphs. We use the notation of Section 2, with the functions T, L, $\overline{L}$, $\overline{T}$, defined in terms of the particular τ function given above. Thus from Theorem 2.4 we have

$$(4.2) \qquad u^{-|V\Gamma|}\, t^{-|E\Gamma|}\, C(\Gamma; u) = T(\Gamma; t, u) = \prod_{\Lambda \in N(\Gamma)} \overline{T}(\Lambda; t, u)\,.$$

At first sight it would appear that different values of t will give different expansions, as in the additive case. Remarkably, this is not so.

Lemma 4.1. (i) *If I denotes the graph with one edge joining two distinct vertices, then* $\overline{T}(I; t, u) = t^{-1}\,(1 - u^{-1})$.

(ii) *If Γ is any non-separable graph with more than one edge, then $\overline{T}(\Gamma; t, u)$ is independent of t.*

Proof. (i) Explicit calculations starting from the definition of W give

$$W(I; t, u) = (1 - t) - u^{-1}\,, \qquad \tau(I; t, u) = (t^{-1} - 1) - u^{-1}\, t^{-1}\,,$$

$$T(I; t, u) = t^{-1}\,(1 - u^{-1})\,,$$

and finally $\overline{T}(I; t, u) = t^{-1}\,(1 - u^{-1})$ as required.

(ii) We use the formula for $\overline{L} = \log \overline{T}$ in terms of the chromial. We have

$$\overline{L}(\Gamma) = (-1)^{|E\Gamma|} \sum_{S \subseteq E\Gamma} (-1)^{|S|}\, L\langle S \rangle$$

$$= (-1)^{|E\Gamma|} \sum_{S \subseteq E\Gamma} (-1)^{|S|} \log[u^{-|V_0|}\, t^{-|S|}\, C(\langle S \rangle; u)]\,.$$

The part of this which depends (apparently) on t is

$$\sum_{S \subseteq E\Gamma} (-1)^{|S|} \log t^{-|S|} = (\log t) \sum_{S \subseteq E\Gamma} (-1)^{|S|+1} |S| .$$

Now this sum is zero unless $|E\Gamma| = 1$, so we have the result of part (ii), and incidentally verification of part (i).

If we let $N^*(\Gamma)$ denote the class of non-separable subgraphs of Γ with more than one edge, the multiplicative expansion of the chromial in terms of $N^*(\Gamma)$ is explicitly independent of t:

Theorem 4.2. *If we write* $\overline{T}(\Lambda; t, u) = Q(\Lambda; u)$ *when* $|E\Lambda| > 1$

$$C(\Gamma; u) = u^{|V\Gamma|} (1 - u^{-1})^{|E\Gamma|} \prod_{\Lambda \in N^*(\Gamma)} Q(\Lambda; u) .$$

Proof. This follows from (4.2) and the results of Lemma 4.1.

The expansion of Theorem 4.2 is a version of the one found by Baker [1] by a method based on ideas of Rushbrooke which involve limiting processes. Tutte's derivation of a similar result [5, p. 317] uses the complicated notion of a "tree-mapping". The present method shows the essentially finite lattice-theoretic nature of the result, and also that it has a uniqueness not shared by the additive expansions.

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