DESIGNS, FACTORS AND CODES IN GRAPHS

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1. Introduction

THE idea of a perfect code in a finite vector space has recently been extended by Delsarte (3) and the present author (2). The extension in (2) is concerned with the notion of a perfect code in a graph, and it is shown that the important theorem of Lloyd [(7), 111] can be generalised when the graph is distance-transitive. Delsarte's extension, where an analogous theorem is proved, takes place in the context of association schemes; in his terminology, a distance-transitive graph is a metric, symmetric association scheme. He also investigates the analogue of a design in an association scheme, but his definition of a design seems slightly unnatural in view of its algebraic, rather than combinatorial, nature.

In this paper we generalise simultaneously, in a graph-theoretical context, the notions of a design and a perfect code. In particular, this leads to a combinatorial definition of a design in a graph. When the graph is distance-transitive we shall state, and give elementary proofs of, the extensions of some basic theorems of (2) and (3).

2. q-coverings and designs

Let q be a mapping of the non-negative integers into themselves, with the property that $q(i) = 0$ implies $q(j) = 0$ for all $j > i$. Denote by σ the largest integer for which q takes a non-zero value.

Let Γ be a connected finite graph, with distance function ∂ and diameter d.

The above notation will be fixed throughout this paper.

DEFINITION If q and Γ are given, such that $\sigma < d$, then a subset X of V will be called a *q-covering of* Γ when the numbers

$$\alpha(v) = \sum_{x \in X} q(\partial(x, v)) \qquad (v \in V\Gamma)$$

are all equal to a constant α. If $\alpha = q(0)$, we shall say that the q-covering is *sparse*.

With appropriate choices of q and Γ, this definition includes the classical theory of perfect codes and designs. We explain first the relationship with designs.

Let t be a positive integer less than d, and q the function

$$q(i) = \binom{d-i}{t},$$

where the binomial coefficient is, naturally, zero for $i > d\text{-}t$. Then a q-covering in Γ will be called a *t-design in* Γ. For the classical case we take Γ to be the graph $J(a, b)$ whose vertices $u, v, \ldots$ are the subsets of cardinality b (b-sets) chosen from a given a-set, and two vertices are adjacent when they have b-1 common elements. When $a > 2b$, $J(a, b)$ is a connected graph with distance function

$$\partial(u, v) = b - |u \cap v|,$$

and diameter $d = b$.

PROPOSITION. *The blocks of a t-(a, b, c) design in the usual sense* (4) *are the vertices of a t-design in* $J(a, b)$, *with* $\alpha = c\binom{b}{t}$. *If the given design is a Steiner system* ($c = 1$), *then the corresponding t-design in* $J(a, b)$ *is sparse.*

Proof. Let X denote the set of blocks of the t-(a, b, c) design, regarded as a set of vertices of $J(a, b)$. For any pair of vertices u, v in $J(a, b)$

$$q(\partial(u, v)) = \binom{|u \cap v|}{t},$$

which is the number of t-sets in $u \cap v$. Now, for a given v, there are $\binom{b}{t}$ t-sets in v and each one occurs just c times as a subset of a block. Thus

$$\alpha(v) = \sum_{x \in X} q(\partial(x, v)) = \sum_{x \in X} \binom{|x \cap v|}{t} = c\binom{b}{t},$$

and this is independent of v, as required. When $c = 1$, we have

$$\alpha = \binom{b}{t} = q(0),$$

so the t-design is sparse.

3. An example from graph theory

Let q be the function defined by

$$q(0) = 2,\ q(1) = 1,\ q(i) = 0 \qquad (i > 1).$$

PROPOSITION. *A graph Δ has a 1-factor if and only if its line graph $L(\Delta)$ has a sparse q-covering, with the particular function q defined above.*

Proof. Suppose Δ has a 1-factor and let X be the corresponding set of vertices of $L(\Delta)$. Then, for a vertex v of $L(\Delta)$, we have

$$\alpha(v) = \sum_{x \in X} q(\partial(x, v)) = 2\theta_0(v) + \theta_1(v),$$

where $\theta_i(v) = |\{x \in X \mid \partial(v, x) = i\}|$. If v is in X, then $\theta_0(v) = 1$ and $\theta_1(v) = 0$, whereas if v is not in X, then $\theta_0(v) = 0$ and $\theta_1(v) = 2$ (since each edge of Δ which is not in the 1-factor is adjacent to just two edges belonging to the 1-factor). Thus $\alpha(v) = 2$ for each v, and so X is a sparse q-covering of $L(\Delta)$.

Conversely, suppose that there is a subset X of the vertices of $L(\Delta)$ such that $2\theta_0(v) + \theta_1(v) = 2$ for each vertex v of $L(\Delta)$. Then either v is in X and $\theta_0(v) = 1$, $\theta_1(v) = 0$, or v is not in X and $\theta_0(v) = 0$, $\theta_1(v) = 2$. This implies that the edges of Δ corresponding to X form a 1-factor in Δ.

4. q-coverings in distance-transitive graphs

The theory of distance-transitive graphs is developed in (1). We shall need only the results and notation as given in (2), which will be used without further explanation. It should be noted that $J(a, b)$ is a distance-transitive graph when $a > 2b$.

We shall suppose that Γ is distance-transitive with valency k, diameter d, and $n = |V\Gamma|$. Let q be a function of the kind introduced in Section 2, and define Φ to be the $n \times n$ matrix, with rows and columns corresponding to the vertices of Γ, whose entries are given by

$$(\Phi)_{uv} = q(\partial(u, v)).$$

LEMMA. *Φ belongs to the adjacency algebra $\mathscr{A}(\Gamma)$.*

Proof. If $\{A_0, A_1, \ldots, A_d\}$ denotes the basis for $\mathscr{A}(\Gamma)$ defined in [(2), 290], then we have the equation

$$\Phi = \sum_{i=0}^{\sigma} q(i) A_i.$$

Since A_i is a polynomial $v_i(A)$ and v_i has degree $i (0 \leqslant i \leqslant d)$, it follows that Φ is a polynomial $\phi(A)$ and that the degree of ϕ is σ.

LEMMA. *Let X be a q-covering in Γ and $\mathbf{x}$ its representative column vector. Then, if $\mathbf{u}$ is the column vector each entry of which is 1,*

$$\Phi\mathbf{x} = \alpha\mathbf{u}.$$

Proof. This is just the matrix equation corresponding to the definition of a q-covering.

Now let z be a given vertex of Γ and $\mathbf{w}$ the *weight vector* of X with respect to z, that is

$$(\mathbf{w})_i = |\{x \in X \mid \partial(z, x) = i\}| \qquad (0 \leqslant i \leqslant d).$$

As explained in [(2), 291] there is an algebra $\hat{\mathscr{A}}(\Gamma)$ of $(d+1)\times(d+1)$ matrices isomorphic with $\mathscr{A}(\Gamma)$. Let $\hat{\Phi}$ be the matrix corresponding to Φ in $\hat{\mathscr{A}}(\Gamma)$, and let T be the $(d+1)\times n$ matrix defined on page 292 of (2), so that $T\Phi = \hat{\Phi}T$.

LEMMA. *Let* $\mathbf{w}$ *be the weight vector of a q-covering in* Γ. *Then*

$$\hat{\Phi}\mathbf{w} = \alpha\mathbf{k},$$

where $(\mathbf{k})_i$ *is the number of vertices of* Γ *at distance* i *from a given vertex.*

Proof. Applying T to the result of the preceding lemma and using the fact that $T\Phi = \hat{\Phi}T$, we get $\hat{\Phi}(T\mathbf{x}) = \alpha(T\mathbf{u})$. Then the definition of T implies that $T\mathbf{x} = \mathbf{w}$, $T\mathbf{u} = \mathbf{k}$, so we have the result.

DEFINITION. The *nullity* of a q-covering in Γ is the dimension of the kernel of $\hat{\Phi}$.

The last lemma implies that, if Γ has a q-covering, a suitable linear combination of $\mathbf{w}$ and $\mathbf{u}$ belongs to the kernel of $\hat{\Phi}$, so that the nullity is at least one.

THEOREM 1. *If* ν *denotes the nullity of a q-covering in a distance-transitive graph* Γ, *then* $\nu \leqslant \sigma$ *and* ν *of the* σ *zeros of* ϕ *are eigenvalues of* Γ.

Proof. Since $\Phi = \phi(A)$ we have $\hat{\Phi} = \phi(B)$ where $B = \hat{A}$ is the intersection matrix of Γ. Now B is tridiagonal, that is $(B)_{ij} = 0$ for $|i-j| > 1$, and we also know that

$$(B)_{i,i+1} \neq 0 \qquad (0 \leqslant i \leqslant d-1).$$

The degree of ϕ is σ, so $(\hat{\Phi})_{ij} = 0$ for $|i-j| > \sigma$, and

$$(\hat{\Phi})_{i,i+\sigma} \neq 0 \qquad (0 \leqslant i \leqslant d-\sigma).$$

This means that the first $d-\sigma+1$ rows of $\hat{\Phi}$ are linearly independent, whence the rank of $\hat{\Phi}$ is at least $d-\sigma+1$ and its kernel has dimension at most σ.

The eigenvalues of $\hat{\Phi}$ are $\phi(\lambda_0), \phi(\lambda_1), \ldots, \phi(\lambda_d)$, where $\lambda_0 = k$, $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of Γ. Since ν eigenvalues of $\hat{\Phi}$ are equal to zero, we have the result.

5. Nullity and minimum distance

DEFINITION. The minimum distance of a q-covering X in Γ is

$$\delta = \min\{\partial(x, y) \mid x, y \in X \text{ and } x \neq y\}.$$

The minimum distance δ is related to the nullity ν, as we now show. Since δ is a more fundamental property than ν, we often use this relationship in the application of Theorem 1.

THEOREM 2. *Using the above notation, we have* $\nu \geqslant [\frac{1}{2}(\delta - 1)]$.

Proof. With respect to a distinguished vertex z belonging to X define

$$\Gamma_i(z) = \{v \in V\Gamma \mid \partial(v, z) = i\} \qquad (0 \leqslant i \leqslant d).$$

Let $r = [\frac{1}{2}(\delta - 1)]$. Then for $0 \leqslant i, j \leqslant r$ we have

$$(*) \qquad u \in \Gamma_i(z) \text{ and } v \in \Gamma_j(z) \Rightarrow \partial(u, v) \leqslant i + j < \delta.$$

Suppose that z is in X, and choose automorphisms $g_1, g_2, \ldots, g_d$ of Γ such that $\partial(z, g_i z) = i(1 \leqslant i \leqslant d)$. Then we have q-coverings $X_0 = X$, $X_1 = g_1(X), \ldots, X_d = g_d(X)$, and X_i contains at least one vertex, $g_i z$, belonging to $\Gamma_i(z)$ for $0 \leqslant i \leqslant d$ (if we take $g_0 z = z$). Since δ is the minimum distance for each of these q-coverings, the statement (*) implies that $g_i z$ is the only vertex of X_i belonging to any one of $\Gamma_0(z)$, $\Gamma_1(z), \ldots, \Gamma_r(z)$ for $0 \leqslant i \leqslant r$. Thus, if $\mathbf{w}^{(i)}$ is the weight vector of X_i with respect to z, then

$$(\mathbf{w}^{(i)})_j = \delta_{ij} \qquad (0 \leqslant i, j \leqslant r).$$

So the vectors $\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(r)}$ are linearly independent, and since

$$\hat{\Phi}(\mathbf{w}^{(0)} - \mathbf{w}^{(i)}) = \alpha(\mathbf{k} - \mathbf{k}) = \mathbf{0},$$

we have r linearly independent vectors in the kernel of $\hat{\Phi}$.

In the classical case of a t-design in $J(a, b)$ it follows from the definition that δ is at least b-$t = \sigma$, since two blocks can have at most t common members. In the special case of a Steiner system we must have $\delta > \sigma$, since each t-set occurs only once in a block. In that case we notice that the corresponding q-covering is sparse, and in fact we have a general result.

LEMMA. *For any sparse q-covering we have* $\delta > \sigma$.

Proof. From the definitions

$$\alpha = \alpha(v) = \sum_{x \in X} q(\partial(x, v)) = q(0)\theta_0(v) + \ldots + q(\sigma)\theta_\sigma(v) \qquad (v \in V\Gamma),$$

where $\theta_i(v) = |\{x \in X \mid \partial(v, x) = i\}|$.

Now if v is in X then $\theta_0(v) = 1$ and so

$$q(1)\theta_1(v)+\ldots+q(\sigma)\theta_\sigma(v) = 0.$$

But $q(1),\ldots, q(\sigma)$ are all positive, so $\theta_1(v) = \ldots = \theta_\sigma(v) = 0$. That is, $\delta > \sigma$.

6. Conclusion

We have shown that a necessary condition for the existence of a q-covering having minimum distance δ, in a distance-transitive graph Γ, is that at least $[\frac{1}{2}(\delta-1)]$ of the σ zeros of ϕ are eigenvalues of Γ. In the case of a sparse q-covering the number $[\frac{1}{2}(\delta-1)]$ can be replaced by $[\frac{1}{2}\sigma]$.

Let q be the function given by $q(i) = 1$ $(0 \leqslant i \leqslant e)$, $q(i) = 0$ otherwise. Then a sparse q-covering of a graph Γ is just a perfect e-code in Γ, as defined in (**2**). We must have $\delta > \sigma = e$, and in fact it is easy to see that $\delta = 2e+1$, so that all zeros of ϕ are eigenvalues of Γ. This is the generalisation of Lloyd's theorem, a theorem which plays an important part in the non-existence proofs of van Lint (**7**) and Tietäväinen (**5**) concerning perfect codes in finite vector spaces.

At the other end of the utility scale we have the example of Section 3. For that particular function q, the polynomial $\phi(\lambda)$ is just $\lambda+2$, so we require that, for a 1-factor in Δ, -2 must be an eigenvalue of $L(\Delta)$. But it is well known that, provided Δ has more edges than vertices, -2 is necessarily an eigenvalue of $L(\Delta)$!

In the classical case of a t-design in $J(a, b)$ we encounter a similar phenomenon. The zeros of the relevant polynomial ϕ are all eigenvalues of $J(a, b)$, as may be shown by simple calculations involving the Eberlein polynomials [(**3**), 70]. This may be disappointing for design theorists, but from a more detached viewpoint it merely says that the graphs $J(a, b)$ are excellent candidates wherein to seek for designs. In any case, the methods of this paper are not without interest for design theory, for detailed consideration of the weight vector can lead to useful results, as Delsarte has shown.

To end on a more positive note, we give an example of a 3-design in a 'sporadic' distance-transitive graph. This is the graph Γ with intersection array

$$\iota(\Gamma) = \{7, 6, 4, 4; 1, 1, 1, 6\},$$

which may be constructed as follows from the blocks of the Steiner system S(5, 8, 24), given in (**6**). There are 330 blocks of this system which do not contain two given points; these are the vertices of Γ, and they are joined in Γ whenever they are disjoint. The polynomial $\phi(\lambda)$ for a 3-design in Γ is just $\lambda+4$ and -4 is an eigenvalue of Γ, so a 3-design is possible. In fact we may construct one by taking X to be the set of vertices of Γ which do contain one given point. There are 120 such vertices and we find, using the q function for a 3-design with $d = 4$,

$$\sum_{x \in X} q(\partial(x, v)) = 4\theta_0(v) + \theta_1(v) = 4 \qquad (v \in V\Gamma).$$

Thus X is a sparse 3-design in Γ.

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